Open problems involving super edge-magic labelings and related topics

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Abstract

Graph labelings has experimented a fast development during the last four decades. Two books dedicated to this topic, a very complete survey on the subject and over 1000 papers in the literature constitute a good proof of this fact. In this paper we explore some open problems on super edge-magic labelings, and about related topics. We are particularly interested on super edge-magic labelings due to the large amount of relations existing among super edge-magic labelings and other types of labelings, mainly graceful and harmonious labelings.

Keywords: super edge-magic, deficiency, magic model, product magic

1 Introduction

A labeling of a graph is a function with domain being either the set of vertices, the set of edges, or both, the set of vertices and edges of a graph and ranges taken from some set (usually the integers), that meets some properties. In 1967 Rosa [20] introduced graceful labelings as an alternative way to attack Ringel's conjecture [19], that states that any tree T of size p decomposes the complete graph K_{2p+1} into $2p + 1$ copies of T. Nowadays this conjecture remains open and only particular families of trees have been proved to admit such labeling. Motivated by graceful labelings, many other labelings have appeared in the literature. Among these labelings, probably the most important ones, if we mesure importance by the number of papers devoted to them, are the harmonious labelings introduced by Graham and Sloane [10] in 1980. Enomoto et al. [3] introduced the concept of super edge-magic labeling. However, we must mention that in a previous paper Acharya and Hegde [1] had introduced an equivalent labeling using arithmetic progressions. In this paper we will mainly concentrate on super edge-magic labelings due to the close relations that they have with graceful and harmonious labelings. Next, in order to make this paper reasonably self-contained, we define the concepts of graceful, harmonious and super edge-magic graphs.

Let $G = (V, E)$ be a graph of order p and size q. A *graceful labeling* of G is an injective function $f : V \longrightarrow \mathbb{Z}_{q+1}$ such that when each edge xy is assigned the label $|f(x)-f(y)|$ then the resulting edge labels are all distinct. Graphs that admit such labelings are called graceful. The graph G is called *harmonious* if there exists an injective function $f : V \longrightarrow \mathbb{Z}_q$ such that if we assign to each edge xy the label $f(x) + f(y) \bmod(q)$ then the resulting labels are all distinct. When G is a tree, exactly one label can be used on two distinct vertices. Such a function f is called a harmonious labeling of G. The graph G is *super edge-magic* if there exists a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that $f(V) = \{i\}_{i=1}^{p}$ and $f(x)+f(xy)+f(y) = k$ for every edge xy.

The goal of this paper is to present some open problems related with super edge-magic labelings. First of all, we consider the problem of the *maximum density* of super edge-magic graphs. That is to say, let $G = (V, E)$ be a super edge-magic graph of order p and size q. Is it possible to obtain an upper bound for q in terms of p?. Enomoto et al., showed in [3] that $q \leq 2p - 3$, and in [7] it was shown that super edge-magic graphs that reach the bound must contain triangles. This suggests that if we enlarge the girth of the graphs then the size of super edge-magic graphs gets smaller. In Section 2, we discuss this idea and we introduce some possible lines of research.

Let $G = (V, E)$ be a graph and let $M(G)$ be a set of non-negative integers. An integer n is in $M(G)$ if $G \cup nK_1$ is super edge-magic. The super edge-magic deficiency of G is defined to be the minimum of $M(G)$ when $M(G)$ is non-empty, otherwise it is defined to be infinity. In Section 3 we introduce what we feel that are the most significant results about super edge-magic deficiency and we introduce some questions that suggest different lines of

research.
The concept of super edge-magic labeling has been recently generalized to the concept of super edge-magic labelings with respect to a model [15]. This generalization shows that the concept of super edge-magicness and the concept of proper coloring of the edges of a graph are strongly related. In Section 4, we introduce some questions that rise from this new approach. In Section 5, we consider the relation existing between the well known Queen's problem and the labeling that is named after this problem. In Section 6, we study a type of labeling that falls in the category of product magic and product antimagic labelings. Such labelings have been so far analyzed either using probabilistic methods, or using powerful results in Number Theory.

2 Bounds for the size of super edge-magic graphs.

In [3] Enomoto et al. proved the following result.

Lemma 2.1 *Let $G = (V, E)$ be a super edge-magic graph of order p and size q. Then*

$$q \leq 2p - 3.$$

It is easy to find super edge-magic graphs that reach the bound, see for instance the graph shown in Figure 1.

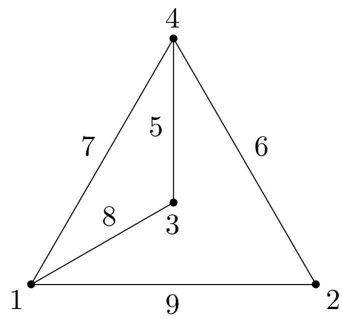

Figure 1: A super edge-magic graph with $q = 2p - 3$.

Figueroa-Centeno et al. proved in [7] that if a graph G of order p and size $2p - 3$ is super edge-magic, then G must contain at least one triangle as a subgraph. In fact, they prove the following.

Theorem 2.1 *[7] Let $G = (V, E)$ be a super edge-magic graph of order p and size q with $p \geq 4$ and $q \geq 2p - 4$. Then G contains triangles.*

Corollary 2.1 *[7] Let $G = (V, E)$ be a super edge-magic graph of order $p \geq 4$ and size q with girth greater than 3. Then $q \leq 2p - 5$.*

It is not hard to find examples of super edge-magic bipartite graphs of order $p \geq 8$ and size $q = 2p-5$, which shows that the bound provided in Corollary 2.1 is tight. These results suggest that super edge-magic graphs with larger girth must have size upper bounded by a smaller quantity. Hence we ask the following question.

Question 2.1 *Is it possible to find an infinite family of super edge-magic graphs of order p, size $q = 2p - 5$ and girth 5?*

In [12] Ichishima, Muntaner and Rius propose a family of graphs with such properties.
Let $\mathfrak{P} = \{P_n : n \in \mathbb{N}\}$ be the family of graphs where each P_n has order $5n$ and size $10n-5$. Let P_1 be the cycle of order 5. For $n \in \mathbf{N}\backslash\{1\}$, let $[0, 5n-1]$ be the vertex set of P_n, where $[i, j]$ denotes the set $\{i, i+1, \ldots, j\}$. The graph P_n consists of n cycles, each one of them called the level L_k, where $k \in [1, n]$. The vertices of the level L_k are $V(L_k) = [5k-5, 5k-1]$. Each vertex in L_k is adjacent with exactly one vertex of the level L_{k-1} and with exactly one vertex of the level L_{k+1}, for each $k \in [2, n-1]$. Consequently the vertices of $L_2, \ldots, L_{n-1}$ have all degree 4 and the vertices of L_1 and L_n have degree 3. Next we describe these adjacencies.
Let $a, b \in V(L_k)$; $k \in [1, n]$. We denote by $\bar{a}$ and $\bar{b}$ the remainders of a and b modulo 5. Then $(a, b) \in E(P_n)$ if and only if either $\bar{a} = \bar{b} +_5 2$ or $\bar{b} = \bar{a} +_5 2$. Next, let $a \in V(L_k)$ and $b \in V(L_{k+1})$, $k \in [1, n-1]$ then $ab \in E(P_n)$ if and only if $\bar{b} = \pi(\bar{a})$ when k is odd or $\bar{b} = \pi^{-1}(\bar{a})$ when k is even, where π is the following permutation of elements of $\mathbb{Z}_5$ in cycle notation: $(0, 4, 1, 2)(3)$.

Figure 2 shows the graph P_3. Note that P_2 is the Petersen graph.
It is easy to show that for all $n \in \mathbb{N}$ the girth of P_n is 5 and that P_n is super edge-magic. For instance, we obtain that for all $n \in \mathbb{N}$ the function $f : [0, 5n-1] \longrightarrow [1, 5n]$ defined by the rule $f(i) = i + 1$ provides us with a super edge-magic labeling of P_n (see Lemma 4.1 in Section 4). In Figure 3, it is shown a super edge-magic labeling of P_1
We ask the followings questions about the relation order-girth-size.

Question 2.2 *Let $p \in \mathbb{N}$. Does there exist a super edge-magic graph of order p, size $q = 2p - 5$ and girth 5?*

Question 2.3 *Find tight bounds for the size of super edge-magic graphs with girth $g \geq 6$.*

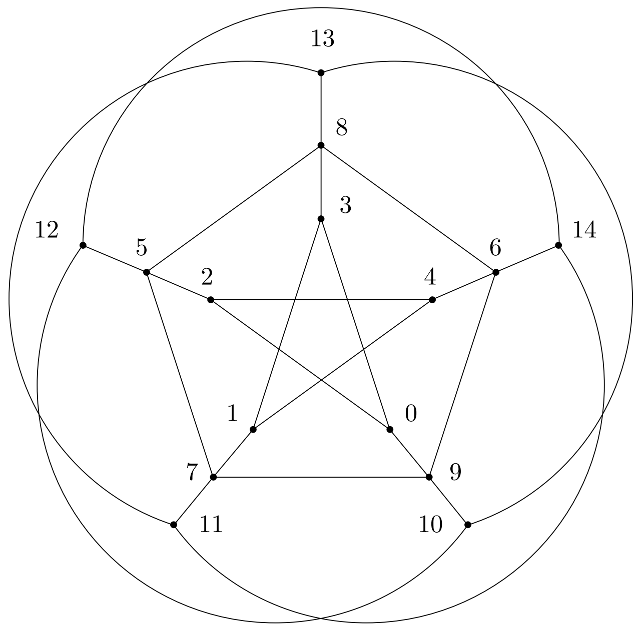

Figure 2: The graph P_3.

3 The problem of the super edge magic deficiency

Motivated by the concept of edge-magic deficiency, introduced by Kotzig and Rosa in [14], Figueroa-Centeno et al. introduced in [8] the super edge-magic deficiency of a graph. Let $G = (V, E)$ be a graph and let

$$M(G) = \{n \in \mathbb{N} \cup \{0\} : \ G \cup nK_1 \text{ is super edge-magic}\}.$$

The super edge-magic deficiency of G, denoted by $\mu_s(G)$, is defined as:

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset, \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

The problem to determine the super edge-magic deficiency interested by many researchers, and in spite of being hard to find general results on the topic, this definition has been the focus even of some Ph.D. thesis, see for instance [18]. Next we present some results that we believe are representative.

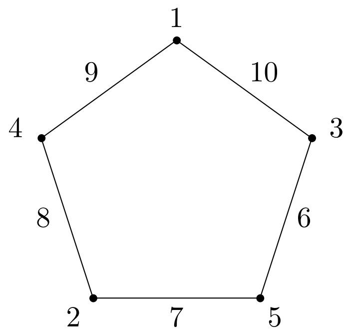

Figure 3: Super edge-magic labeling of P_1.

Theorem 3.1 *[8] Let K_n be the complete graph of order n. Then,*

$$\mu_s(K_n) = \begin{cases} 0, & \textit{if } n \in \{1,2,3\}, \\ 1, & \textit{if } n = 4, \\ +\infty, & \textit{if } n \geq 5. \end{cases}$$

Theorem 3.2 *[8] Let C_n be the cycle of order n. Then,*

$$\mu_s(C_n) = \begin{cases} 0, & \textit{if } n \textit{ is odd}, \\ 1, & \textit{if } n \equiv 0 \ (mod\ 4), \\ +\infty, & \textit{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

Theorem 3.3 *[6] For every integer $n \geq 3$,*

$$\mu_s(2C_n) = \begin{cases} 1, & \textit{if } n \textit{ is even}, \\ +\infty, & \textit{if } n \textit{ is odd}. \end{cases}$$

Theorem 3.4 *[6] For every $n \geq 3$,*

$$\mu_s(3C_n) = \begin{cases} 0, & \textit{if } n \textit{ is odd}, \\ 1, & \textit{if } n \equiv 0 \ (mod\ 4), \\ +\infty, & \textit{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

Theorem 3.5 *[6] For every positive integer $n \equiv 0 \ (mod\ 4)$,*

$$\mu_s(4C_n) = 1.$$

In [6] the following conjecture is introduced.

Conjecture 3.6 *Let $m \geq 1$ and $n \geq 3$ be integers. If $mn \equiv 0 \pmod 4$ then*

$$\mu_s(mC_n) = 1.$$

The next conjecture is probably one of the most popular in the topic of super edge-magic labelings.

Conjecture 3.7 *[3] Every tree is super edge-magic.*

In terms of deficiency, the conjecture is equivalent to say that if T is a tree then $\mu_s(T) = 0$. But, what can we say when we deal with acyclic graphs that are not necessarily connected? Do we have any reason to think that given any forest F, then $\mu_s(F) = 0$? The answer to this last question is no. The following result provides us with a counterexample.

Theorem 3.8 *[8] Let nK_2 be a matching with n components. Then,*

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

However the following result found in [6] claims that the super edge-magic deficiency of any forest is always finite.

Theorem 3.9 *Let F be a forest. Then*

$$\mu_s(F) < +\infty.$$

In [6], Figueroa-Centeno et al. conjectured the following:

Conjecture 3.10 *Let F be a forest with exactly two connected components. Then*

$$\mu_s(F) \leq 1.$$

Up to now, this conjecture is still open and only a few particular examples known seem to confirm it.
In order to conclude this section we give the following result, stated by Muntaner and Rius in [16].

Theorem 3.11 *For every non-bipartite graph G there exists a natural number $N(G)$ such that if $\mu_s(G) \geq N(G)$ then $\mu_s(G) = +\infty$.*

4 Super edge-magic models

Generalizations of super edge-magic labelings can be found in the literature, for instance see [11]. In this section we introduce a new one, based on the interpretation of super edge-magic labelings as arithmetic progressions with difference 1.

Lemma 4.1 *[5] A graph $G = (V, E)$ of order p and size q is super edge-magic if and only if, there exists a bijective function $f : V \to \{i\}_{i=1}^{p}$ such that the set $S = \{f(x) + f(y) : xy \in E\}$ is a set of exactly q consecutive integers.*

Recall that a proper coloring of a graph G with n colors is an assignment of the n colors to the edges, in such a way that the edges incident with a common vertex receive distinct colors.
Consider a representation of K_n in such a way that its vertices coincide with vertices of an n-sided polygon. Each bijective assignment of the numbers from 1 up to n to the vertices induces a labeling of the edges, given by the sum of the adjacent vertex labels. Therefore, we have a proper coloring of the edges that admits a natural order given by the value of the colors. A graph of order n is super edge-magic if, when we have it as a subgraph of K_n, then the color of its edges respects the natural order. That is to say the values of the colors (edge labels) form an arithmetic progression of difference 1.
Given two natural numbers m, n, a *super edge-magic m-model of order n* is a proper coloring of the edges of K_n that uses m colors, with an m ordered set that determines a prestablised order of the m colors. We say that a graph G of order n is *super edge-magic with respect to the m-model* if and only if G can be found as a subgraph of K_n in a representation that uses consecutive colors in the ordered set, without any color repetition. Figure 4 shows the m-model of order n commented previously, for $m = 7$ and $n = 5$. The interior edges present a proper coloring of the cycle of order 5, that respects the order of the colors. In particular, we say that C_5 is super edge-magic with respect to this 7-model.
We point out that the number of edges of a super edge-magic graph with respect to an m-model is always less than or equal to m and that a graph that uses the m colors is always super edge-magic with respect to the m-model.
It is well known that even cycles do not admit any super edge magic labeling [3]. That is to say, they are not super edge-magic with respect to the first model described. In fact, in [15] an m-model with m colors so that no 2-regular graph of order m is super edge-magic with respect to it is provided. This model can be described taking as vertices the vertices of a regular

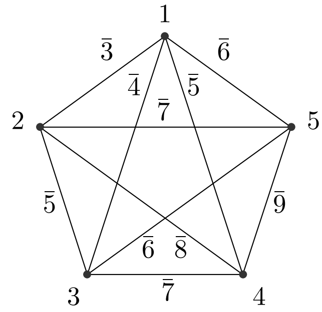

Figure 4: A 7-model of order 5.

n-gon, as edges its sides and diagonals and considering a new coloring that assigns to two different edges the same color if and only if they are parallel. The concept of super edge-magic models suggests new research lines, some of which can be found in [15]. We discuss the following ones:

Question 4.1 *Consider an even number m. Find an m-model of order m, namely ν_m, such that at least a 2-regular graph of order m is super edge-magic with respect to ν_m, or show that this model does not exist.*

Question 4.2 *Consider an even number m. Find an m-model of order m that maximizes the number of 2-regular graphs of order m that contains the model. Find such maximum.*

Question 4.3 *Given a (p, q)-graph G find, when possible, a q-model of order p that does not contain any subgraph isomorphic to G with a proper coloring.*

We conclude this section with an observation with respect to the first question. For $m = 4$, the only 2-regular graph of order 4 is C_4. A simple case by case proof allows us to conclude that there is not a proper coloring of K_4 with 4 colors that allows us to find C_4 using all colors. Nevertheless, the result in general is not true. Figure 5 shows a proper coloring of K_6 where there is contained a C_6 using all six colors. In relation with the second question, it is easy to check that there is no proper coloring of K_6 that admits a realization of $2C_3$ using all six colors. Therefore, in this case this maximum is exactly 1.

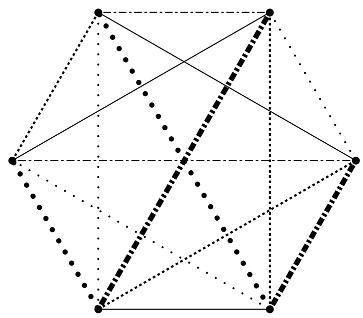

Figure 5: A proper coloring of K_6.

5 Queen labelings

In 1848 the German chess player Max Bezzel proposed the 8-queens problem. This problem consists of placing eight queens on a chessboard in such a way that not two queens attack each other. That is to say, no two queens can occupy the same row, column or diagonal.

It is in 1859 when Frank Nanck provides the first solution for this problem and extends the problem to what is today known as the n-queens problem. This problem has been studied by many well known mathematicians, as for instance Gauss and Cantor, and to count the number of solutions for a given n, has proven to be an extremely difficult problem. Some solutions have been found for a few values of n but it seems that the general problem is far away from being solved.

For instance it is known that there exist 92 solutions for the original 8 queens problem, 12 of which are essentially different.

Motivated by this problem, it was introduced in [2] the concept of *queen labeling* as follows: let $G = (V, E)$ be a digraph, where loops are allowed. A queen labeling of G is a bijective function $l : V \to [1, |V|]$ such that for every pair of arcs $(u_1, v_1), (u_2, v_2) \in E$, we have that $l(u_1) + l(v_1) \neq l(u_2) + l(v_2)$ and $l(u_1) - l(v_1) \neq l(u_2) - l(v_2)$.

If a digraph G admits a queen labeling, then we say that G is a *queen digraph*. The name of queen labeling comes from the correspondence established in [2], between the solutions of the n-queen problem and the queen labelings of 1-regular digraphs. That is to say, digraphs for which their underlying graph are 2-regular graphs, possible with loops, where each component has been oriented cyclically.

We want to remark the following open problem about queen labelings:

Question 5.1 *Characterize the set of* 1*-regular digraphs of order* n *with loops allowed that admit queen labelings.*

So far, not much is known about this problem. In [2] it is shown that if 3 divides $n(n-1)$, then the union of $n(n-1)/3$ cyclically oriented triangles, admits a queen labeling. We note that among all 1-regular digraphs of order 8, that are candidates to admit queen labelings, the only one that does not admit such labeling is the one with underlying graph $C_3 \cup C_5$.

6 Product labelings

Figueroa-Centeno et al. introduced similar concepts to magic and antimagic labelings, using products, instead of sums. The following definitions were first introduced in [4].
A graph $G = (V, E)$ of order p and size q is called *product-magic* if there is a bijective function from E onto $[1, q]$, such that the product of the labels of the edges incident with the same vertex is constant. If this product is distinct for each vertex, then the graph G is called *product antimagic.*
The graph G is called *product edge-magic* if there is a bijective function $f : V \cup E \to [1, p+q]$, such that the product $f(u) \cdot f(uv) \cdot f(v) = k$ for every $uv \in E$.
When the product $f(u) \cdot f(uv) \cdot f(v)$ is distinct for each edge $uv \in E$, then the graph is called *product edge-antimagic.* The following results and conjectures can be found in the original paper by Figueroa-Centeno et al. [4].

Theorem 6.1 *A graph* G *of size* q *is product-magic if and only if* $q \leq 1$*.*

Conjecture 6.2 *A connected graph* G *of order* q *is product antimagic if and only if* $q \geq 3$*.*

Theorem 6.3 *A graph* G *of size* q *without isolated vertices is product edge-magic if and only if* $q \leq 1$*.*

Theorem 6.4 *Every graph which is non-isomorphic to* K_2 *nor* $K_2 \cup N_n$ *is product edge-antimagic, where* N_n *is the graph formed by* n *isolated vertices.*

About Conjecture 6.2, Kaplan, Lev and Roditty proved in [13] that the following graphs are product antimagic:

- The disjoint union of cycles and paths, where each path has size at least 3.
- Connected graphs of order p and size q, where $q \geq 4p \ln p$.

- Graphs where each component has size at least 2, and the minimum degree is at least $8\sqrt{\ln q \ln(\ln q)}$, where q is the size of the graph.
- Every K-partite complet graph except for K_2 and $K_{1,2}$.
- The corona product of two graphs G and H, $G \bigodot H$, where G has no isolated vertices and H is regular.

Oleg Pikhurko obtained an excellent result characterizing the set of all graphs of large order that are product antimagic.

Theorem 6.5 *[17] Every connected graph of order larger than $n_0 = 10^{10^{20}}$ is product antimagic.*

This result suggest the following problem:

Question 6.1 *Improve the value for n_0.*

7 Conclusions

This paper is a journey through super edge-magic labelings and about other concepts that in one way or another keep some relation with them.
Graph labelings is a very active area of research and many challenging questions have emerge. Probably the most well known are the two conjectures that claim that every tree is graceful [19] and that every tree is harmonic [10], although lately, also the conjecture that claims that every tree is super edge-magic [3] has become very popular.
The survey by Gallian [9], which is periodically actualized, constitutes a very good source for the interested reader.

Acknowledgements

The research conducted in this document by first and third author has been supported by the Catalan Research Council under grant 2009SGR1387 and by the Spanish Research Council under project MTM2008-06620-C03-01.

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