Bull. Aust. Math. Soc. **84** (2011), 137–152
doi:10.1017/S0004972711002280

BI-MAGIC AND OTHER GENERALIZATIONS OF SUPER EDGE-MAGIC LABELINGS

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(Received 27 November 2010)

Dedicated to the memory of Professor Gary Bloom

Abstract

In this paper, we use the product $\otimes_h$ in order to study super edge-magic labelings, bi-magic labelings and optimal k-equitable labelings. We establish, with the help of the product $\otimes_h$, new relations between super edge-magic labelings and optimal k-equitable labelings and between super edge-magic labelings and edge bi-magic labelings. We also introduce new families of graphs that are inspired by the family of generalized Petersen graphs. The concepts of super bi-magic and r-magic labelings are also introduced and discussed, and open problems are proposed for future research.

2010 *Mathematics subject classification*: primary 05C78.
Keywords and phrases: super edge-magic, bi-magic, k-equitable.

1. Introduction

For most of the graph-theory terminology and notation utilized in this paper we follow either [5] or [14], unless otherwise specified. In particular we may allow graphs to have loops; however no multiple edges will be allowed unless we are in Section 4. Let $G = (V, E)$ be a graph. We say that a graph G is a (p, q)-graph if $|V| = p$ and $|E| = q$. Kotzig and Rosa introduced in [10] the concept of edge-magic labeling. A bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ is an *edge-magic labeling* of G if there exists an integer k such that the sum $f(x) + f(xy) + f(y) = k$ for all $xy \in E$. In 1998, Enomoto *et al.* [6] defined the concepts of super edge-magic graphs and super edge-magic labelings. A *super edge-magic labeling* is an edge-magic labeling with the extra condition that $f(V) = \{i\}_{i=1}^{p}$. It is worthwhile mentioning that an equivalent labeling had already appeared in the literature in 1991 under the name of *strongly indexable labeling* [1]. A graph that admits a (super) edge-magic labeling is called a *(super) edge-magic graph.*

The research conducted in this document by first and third author has been supported by the Spanish Research Council under project MTM2008-06620-C03-01 and by the Catalan Research Council under grant 2009SGR1387.

In 2000, Figueroa *et al.* [7] provided a very useful characterization of super edge-magic graphs that we state in the next lemma.

LEMMA 1.1. *A (p, q)-graph G is super edge-magic if and only if there is a bijective function $\bar{f} : V \longrightarrow \{i\}_{i=1}^{p}$ such that the set $S_E = \{\bar{f}(u) + \bar{f}(v) : uv \in E\}$ is a set of q consecutive integers.*

In [8], Figueroa *et al.* introduced the concept of *super edge-magic digraph* as follows: a digraph $D = (V, E)$ is super edge-magic if its underlying graph is super edge-magic. In general, we say that a digraph D admits a labeling f if its underlying graph admits the labeling f. In this paper we will use super edge-magic digraphs in order to achieve our goals. In [4] Bloom and Ruiz introduced a generalization of *graceful labelings* (see [9] for a formal definition of graceful labeling), which they called *k-equitable labelings*. Let $G = (V, E)$ be a (p, q)-graph and let $g : V \longrightarrow \mathbb{Z}$ be an injective function with the property that the new function $h : E \longrightarrow \mathbb{N}$ defined by the rule $h(uv) = |g(u) - g(v)|$ for all $uv \in E$ assigns the same integer to exactly k edges. Then g is said to be a k-equitable labeling and G a *k-equitable graph*. In [4] the authors called a k-equitable labeling, *optimal*, when g assigns all the elements of the set $\{i\}_{i=1}^{p}$ to the elements of V. Both Bloom and Wojciechowski [15, 16], and independently Barrientos [2], proved that C_n is optimal k-equitable if and only if k is a proper divisor of n ($k \neq n$).

From now on, we will use the notation $\mathrm{und}(D)$ in order to denote the underlying graph of a digraph D. At this point let $D = (V, E)$ with $V \subset \mathbb{N}$ be any digraph. We define the adjacency matrix of D, denoted by $A(D)$, to be the matrix such that the rows and columns are named after the vertices of D in increasing order, and an entry (i, j) of the matrix is 1 if and only if $(i, j) \in E$. Otherwise, the entry (i, j) is 0.

In [8], Figueroa *et al.* defined the following product: let $D = (V, E)$ be a digraph with adjacency matrix $A(D) = (a_{i,j})$ and let $\Gamma = \{F_i\}_{i=1}^{m}$ be a family of m digraphs with the same set of vertices V'. Assume that $h : E \longrightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D. Then the digraph $D \otimes_h \Gamma$ is defined by the following:

(1) $V(D \otimes_h \Gamma) = V \times V'$;
(2) $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \Longleftrightarrow [(a_1, a_2) \in E(D) \wedge (b_1, b_2) \in E(h(a_1, a_2))]$.

An alternative way of defining the same product is through adjacency matrices, since we can obtain the adjacency matrix of $D \otimes_h \Gamma$ as follows.

(1) If $a_{i,j} = 0$ then $a_{i,j}$ is multiplied by the $p' \times p'$ 0-square matrix.
(2) If $a_{i,j} = 1$ then $a_{i,j}$ is multiplied by $A(h(i, j))$ where $A(h(i, j))$ is the adjacency matrix of the digraph $h(i, j)$.

Note that when h is constant, $D \otimes_h \Gamma$ is the Kronecker product. From now on, let S_n denote the set of all super edge-magic 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. We also denote by Σ_n the set of all 1-regular digraphs of order n.

The following results were introduced in [8].

THEOREM 1.2. *Let D be a (super) edge-magic digraph and let $h : E(D) \longrightarrow S_n$ be any function. Then* $\mathrm{und}(D \otimes_h S_n)$ *is (super) edge-magic.*

THEOREM 1.3. *Let $\overrightarrow{C_m}$ be a strong orientation of C_m and let $h : E(\overrightarrow{C_m}) \longrightarrow S_n$ be any constant function. Then* $\mathrm{und}(\overrightarrow{C_m} \otimes_h S_n) = \gcd(m, n) C_{\mathrm{lcm}[m,n]}$.

THEOREM 1.4. *Let F be an acyclic graph. Consider any function $h : E(\overrightarrow{F}) \longrightarrow \Sigma_n$. Then* $\mathrm{und}(\overrightarrow{F} \otimes_h \Sigma_n) = nF$.

Using this product, in the original paper, Figueroa *et al.* were able to find exponential lower bounds for the number of nonisomorphic labelings of different types, and different families of graphs.

2. Generalizations of generalized Petersen graphs and the $\otimes_h$-product

The *generalized Petersen graph* $P(n; k)$, $n \geq 3$ and $1 \leq k \leq \lceil (n-1)/2 \rceil$, consists of an outer n-cycle $x_0 x_1 \cdots x_{n-1} x_0$, a set of n-spokes $x_i y_i$, $0 \leq i \leq n-1$, and n inner edges of the form $y_i y_{i+_n k}$, where $+_n$ denotes the sum of two elements in the group $\mathbb{Z}_n$. In this section we propose two possible generalizations of this family, one replacing the k step of the inner edges by a permutation and another one, increasing the number of levels. We denote by $\mathfrak{S}_n$ the set of permutations of $\{0, 1, \ldots, n-1\}$.

Let $n \geq 3$ and let $\pi \in \mathfrak{S}_n$. The *first generalization of a generalized Petersen graph* considered in this paper $GGP(n; \pi)$, consists of an outer n-cycle $x_0 x_1 \cdots x_{n-1} x_0$, a set of n-spokes $x_i y_i$, $0 \leq i \leq n-1$ and n inner edges defined by $y_i y_{\pi(i)}$, $i = 0, \ldots, n-1$. Notice that, if we consider the permutation π defined by $\pi(i) = i +_n k$ then $GGP(n; \pi) = P(n; k)$.

Let $m \geq 2, n \geq 3$ and $\pi_2, \ldots, \pi_m \in \mathfrak{S}_n$. The *second generalization of a generalized Petersen graph* considered in this paper $GGP(n; \pi_2, \ldots, \pi_m)$ is a graph with vertex set $\bigcup_{j=1}^{m} \{x_i^j : i = 0, \ldots, n-1\}$, an outer n-cycle $x_0^1 x_1^1 \cdots x_{n-1}^1 x_0^1$, and inner edges $x_i^{j-1} x_i^j$ and $x_i^j x_{\pi_j(i)}^j$, for $j = 2, \ldots, m$, and $i = 0, \ldots, n-1$. Notice that, $GGP(n; \pi_2, \ldots, \pi_m) = P_m \times C_n$, when $\pi_j(i) = i +_n 1$ for every $j = 2, \ldots, m$.

The graphs $GGP(9; \pi)$ and $GGP(5; \pi_2, \pi_3)$ are shown in Figure 1, where $\pi \in \mathfrak{S}_9$, $\pi_2, \pi_3 \in \mathfrak{S}_5$ and $\pi = (0, 1, 8, 3, 4, 2, 6, 7, 5)$, $\pi_2 = (0, 2, 4, 1, 3)$ and $\pi_3(i) = i +_5 1$.

Let $\overrightarrow{LP_m}$ be the digraph obtained from a path of m-vertices, in such a way that we can travel from one leaf to the other following the directions of the arrows, with a loop attached at each vertex.

PROPOSITION 2.1. *Let $\overrightarrow{C_n}$ be a strong connected digraph obtained from a cycle of order n where n is odd. Then*

$$\mathrm{und}(\overrightarrow{LP_m} \otimes \overrightarrow{C_n}) = P_m \times C_n.$$

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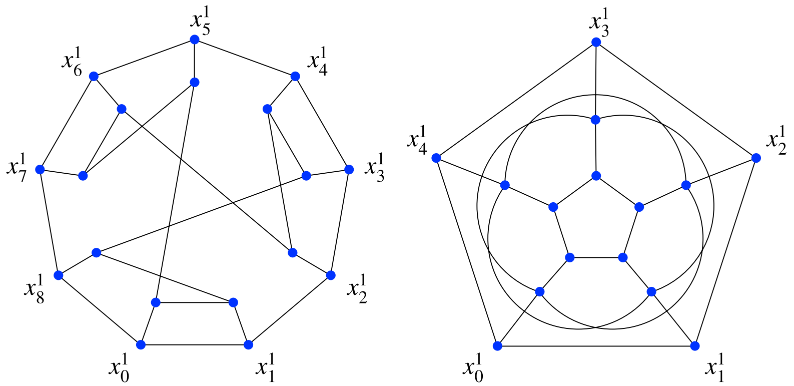

FIGURE 1. The graphs $GGP(9; \pi)$ and $GGP(5; \pi_2, \pi_3)$.

PROOF. By definition,

$$V(\overrightarrow{LP_m} \otimes \overrightarrow{C_n}) = V(P_m \times C_n).$$

Let $a_0 a_1 \cdots a_{m-1}$ and $b_0 b_1 \cdots b_{n-1}$ be directed paths respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{C_n}$. Then $((a_i, b_j), (a_{i'}, b_{j'}))$ is an arc in $\overrightarrow{LP_m} \otimes \overrightarrow{C_n}$ if and only if $(a_i, a_{i'}) \in E(\overrightarrow{LP_m})$ and $j' = j +_n 1$. That is, all arcs are of the form either $((a_i, b_j), (a_i, b_{j+_n 1}))$ or $((a_i, b_j), (a_{i+_m 1}, b_{j+_n 1}))$. □

From now on, let us denote by $\sigma_k \in \mathfrak{S}_n$ the permutation defined by $\sigma_k(i) = i +_n k$.

PROPOSITION 2.2. *Let n be an odd integer and let $\pi \in \mathfrak{S}_n$. Assume that, for some $h : E(\overrightarrow{LP_2}) \longrightarrow S_n$, we obtain*

$$\operatorname{und}(\overrightarrow{LP_2} \otimes_h S_n) = GGP(n; \pi).$$

Then there exists $h' : E(\overrightarrow{LP_m}) \longrightarrow S_n$ such that

$$\operatorname{und}(\overrightarrow{LP_m} \otimes_{h'} S_n) = GGP\Bigg(n; \overbrace{\sigma_1, \ldots, \sigma_1}^{(m-2)\text{ times}}, \pi\Bigg).$$

PROOF. Let $a_0 a_1 \cdots a_{m-1}$ and $b_0 b_1$ be the directed paths induced respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{LP_2}$. Let $h' : E(\overrightarrow{LP_m}) \longrightarrow S_n$ be the function defined by

$$h'(e) = \begin{cases} h(b_1 b_1) & \text{if } e = a_{m-1} a_{m-1}, \\ h(b_0 b_1) & \text{if } e = a_{m-2} a_{m-1}, \\ h(b_0 b_0) & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{und}(\overrightarrow{LP_m} \otimes_{h'} S_n) = GGP\Bigg(n; \overbrace{\sigma_1, \ldots, \sigma_1}^{(m-2)\text{ times}}, \pi\Bigg).$$

This concludes the proof. □

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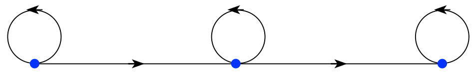

FIGURE 2. The digraph $\overrightarrow{LP_3}$.

We can introduce a slight modification in h' in order to construct, for each $l < m$, $GGP(n; \pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$.

PROPOSITION 2.3. *Let n be an odd integer. Assume that, for some $h : E(\overrightarrow{LP_2}) \longrightarrow S_n$, we obtain*

$$\mathrm{und}(\overrightarrow{LP_2} \otimes_h S_n) = GGP(n; \pi).$$

Then for each l, $1 < l \leq m$, there exists $h'_l : E(\overrightarrow{LP_m}) \longrightarrow S_n$ such that

$$\mathrm{und}(\overrightarrow{LP_m} \otimes_{h'_l} S_n) = GGP(n; \pi_2, \ldots, \pi_m),$$

where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$.

PROOF. The result follows from Proposition 2.2 when $l = m$. Hence, we only need to consider the case when $l < m$. Let $a_0 a_1 \cdots a_{m-1}$ and $b_0 b_1$ be the directed paths induced respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{LP_2}$. Assume that $\Gamma \in S_n$ and denote by $\overleftarrow{\Gamma}$ the oriented digraph obtained from Γ by reversing all its arcs. Let $h'_l : E(\overrightarrow{LP_m}) \longrightarrow S_n$ be the function defined by

$$h'_l(e) = \begin{cases} h(b_1 b_1) & \text{if } e = a_{l-1} a_{l-1}, \\ h(b_0 b_1) & \text{if } e = a_{l-2} a_{l-1}, \\ h(b_0 b_0) & \text{if } e = a_{l-2} a_{l-2}, \\ \overleftarrow{h(b_0 b_1)} & \text{if } e = a_{l-1} a_l, \\ h(b_0 b_0) & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{und}(\overrightarrow{LP_m} \otimes_{h'_l} S_n) = GGP(n; \pi_2, \ldots, \pi_m),$$

where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$. □

Let $x_0 x_1 \cdots x_{m-1} x_0$ be the outer cycle of $P(m; k)$ with spokes $x_i y_i$, $0 \leq i \leq m-1$, and inner edges $y_i y_{i+_m k}$. We denote by $\overrightarrow{P(m; k)}$ the oriented graph obtained from $P(m; k)$ by orienting the edges of the outer cycle from x_i to $x_{i+_m 1}$, the inner edges from y_i to $y_{i+_m k}$ and the spokes from the outer cycle to the inner one.

PROPOSITION 2.4. *Let m, n be two positive integers such that $\gcd(m, n) = 1$ with n odd. Then*

$$\mathrm{und}(\overrightarrow{P(m; k)} \otimes \overrightarrow{C_n}) = P(mn; k + mr),$$

where r is the smallest positive integer such that $k +_n mr = 1$.

PROOF. Let $v_0 v_1 \cdots v_{n-1} v_0$ be the cycle $\overrightarrow{C_n}$, where each vertex is identified with the corresponding label of a super edge-magic labeling of $\overrightarrow{C_n}$. Then

$$V(\overrightarrow{P(m;k)} \otimes \overrightarrow{C_n}) = \{(x_i, v_j), (y_i, v_j)\}_{i=0,\dots,m-1}^{j=0,\dots,n-1}$$

and

$$E(\overrightarrow{P(m;k)} \otimes \overrightarrow{C_n}) = \{((x_i, v_j), (x_{i+_m 1}, v_{j+_n 1})), ((y_i, v_j), (y_{i+_m k}, v_{j+_n 1})), \\ ((x_i, v_j), (y_i, v_{j+_n 1}))\}_{i=0,\dots,m-1}^{j=0,\dots,n-1}.$$

By Theorem 1.3, the digraph induced by the vertices of the form (x_i, v_j) is a cycle of length mn with a strong orientation. By the definition of the Kronecker product, we have mn spokes of the form $((x_i, v_j), (y_i, v_{j+_n 1}))$ and inner edges of the form $((y_i, v_j), (y_{i+_m k}, v_{j+_n 1}))$. Let us see now that $d((x_i, v_{j-_n 1}), (x_{i+_m k}, v_j)) = k + mr$, where r is the smallest positive integer such that $k +_n mr = 1$. By the definition of $\overrightarrow{P(m;k)}$ there is a directed path of length k from x_i to $x_{i+k} = k$. Thus $d((x_i, v_j), (x_i, v_{j+_n m})) = m$ and hence

$$\begin{aligned} d((x_i, v_{j-1}), (x_{i+k}, v_j)) &= d((x_i, v_{j-1}), (x_{i+k}, v_{j-1+k})) \\ &\quad + d((x_{i+k}, v_{j-1+k}), (x_{i+k}, v_j)) \\ &= k + d((x_{i+k}, v_{j-mr}), (x_{i+k}, v_j)) = k + mr. \end{aligned}$$

This completes the proof. □

2.1. (Super) edge-magic *GGP*. Since every digraph $\overrightarrow{LP_m}$ admits a super edge-magic labeling (just label the vertices of the path following the arrows in increasing order) we can apply Theorem 1.2 to extend the class of graphs that are super edge-magic, by adding every *GGP* that can be obtained from the $\otimes_h$-product of the $\overrightarrow{LP_m}$ with S_n. For instance, next we propose an alternative proof for the following theorem found in [6, 7].

THEOREM 2.5 [6, 7]. *Let m, n be two integers, n odd. Then $P_m \times C_n$ is super edge-magic.*

PROOF. Since, by Theorem 1.2 $\overrightarrow{LP_m} \otimes \overrightarrow{C_n}$ is super edge-magic and by Proposition 2.1 $\operatorname{und}(\overrightarrow{LP_m} \otimes \overrightarrow{C_n}) = P_m \times C_n$, the result follows. □

THEOREM 2.6. *The Petersen graph is super edge-magic. Moreover we have the following results.*

(i) *For each $m \geq 2$, $1 < l \leq m$ and $1 \leq k \leq 2$, the graph $GGP(5; \pi_2, \dots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \sigma_k$, is super edge-magic.*
(ii) *For each $1 \leq k \leq 2$, the graph $P(5n; k + 5r)$ is super edge-magic, where r is the smallest positive integer such that $k +_n 5r = 1$.*

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PROOF. Let a_0a_1 be a directed path in $\overrightarrow{LP_2}$. Let $\overrightarrow{C_5}$ be the directed cycle defined by $1 \to 4 \to 2 \to 5 \to 3 \to 1$ and $\overrightarrow{C_1} \cup \overrightarrow{C_4}$ the digraph $1 \to 5 \to 3 \to 4 \to 1$ with a loop labeled 2. We can obtain the Petersen graph from $\overrightarrow{LP_2} \otimes_h \{\overrightarrow{C_5}, \overrightarrow{C_1} \cup \overrightarrow{C_4}\}$, where h is defined by $h(a_0a_0) = h(a_1a_1) = \overrightarrow{C_5}$ and $h(a_0a_1) = \overrightarrow{C_1} \cup \overrightarrow{C_4}$. By Theorem 2.5 $P(5; 1)$ is super edge-magic. Thus, applying Proposition 2.3 together with Theorem 1.2, we obtain (i). Similarly, by Proposition 2.4 and Theorem 1.2 we obtain (ii). □

3. Edge bi-magic

A (p, q)-graph $G = (V, E)$ is said to have an *edge bi-magic labeling* if there exists a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that for each edge $uv \in E$, $f(u) + f(uv) + f(v) \in \{k_1, k_2\}$, where k_1, k_2 are two distinct constants. In this case, the graph is said to be *edge bi-magic*. If we add the extra condition that $f(V) = \{i\}_{i=1}^{p}$ then we say that f is a *super edge bi-magic labeling* and G a *super edge bi-magic graph*. In this section, we study the complete graphs that are edge bi-magic and we introduce new classes of (super) edge bi-magic graphs. In particular, we generalize the class of edge bi-magic graphs that was given by Rajan *et al.* in [11]. We also prove that the product introduced in [8] is useful for providing new families of edge bi-magic graphs.

The next theorem gives necessary conditions for a complete graph to be edge bi-magic, provided that the magic constants are of the same parity. It is similar to [13, Theorem 2.11]. See also [12].

THEOREM 3.1. *Suppose that K_p has an edge bi-magic labeling with magic constants k_1, k_2 such that $k_1 + k_2$ is an even integer. The number ν of vertices that receive even labels satisfies the following condition.*

(i) *If $p \equiv 0$ or $3 \pmod 4$ and k_1 is even then $\nu = \frac{1}{2}(p - 1 \pm \sqrt{p+1})$.*
(ii) *If $p \equiv 1$ or $2 \pmod 4$ and k_1 is even then $\nu = \frac{1}{2}(p - 1 \pm \sqrt{p-1})$.*
(iii) *If $p \equiv 0$ or $3 \pmod 4$ and k_1 is odd then $\nu = \frac{1}{2}(p + 1 \pm \sqrt{p+1})$.*
(iv) *If $p \equiv 1$ or $2 \pmod 4$ and k_1 is odd then $\nu = \frac{1}{2}(p + 1 \pm \sqrt{p+1})$.*

PROOF. The proof is similar to the one given [13, Theorem 2.11]. It is only relevant to note that k_1 and k_2 are of the same parity. □

LEMMA 3.2. *Let G be a super edge bi-magic graph of order $p > 4$ without loops. Then its size is at most $4p - 10$.*

PROOF. Let G be a super edge bi-magic graph of order $p > 4$ without loops and let f be a super edge bi-magic labeling of G. Consider the set $S_E = \{f(u) + f(v) : uv \in E(G)\}$. Then if we allow repetitions in S_E we have that

$$S_E \subset \{3, 4, \ldots, 2p - 1\} \cup \{5, \ldots, 2p - 3\}.$$

Therefore, the size of a super edge bi-magic graph without loops is at most $4p - 10$. □

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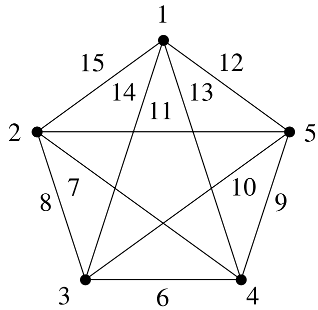

FIGURE 3. A super edge bi-magic labeling of K_5.

REMARK 3.3. This upper bound is tight. Figure 3 shows an edge bi-magic labeling of K_5. Using Lemma 3.2 we obtain that the graph K_n is not super edge bi-magic for $n > 5$.

The next lemma gives a characterization of super edge bi-magic graphs in terms of arithmetic progressions. In some sense, it is a similar result to Lemma 1.1 for the case of super edge-magic labelings given by Figueroa *et al.* in [7].

LEMMA 3.4. *A graph labeling of G is super edge bi-magic if and only if the set of sum labels of adjacent vertices (including repetitions) can be partitioned into two sets S and S' and there exists an integer r such that $S \cup (S' - r)$ is a set of consecutive integers.*

PROOF. In order to prove the necessity assume that there exists a super edge bi-magic labeling of G. Let k and k' be the two magic constants and let S (respectively S') be the sums of the labels of adjacent vertices with magic sum k (respectively k'). Thus $(k - S) \cup (k' - S')$ forms a set of consecutive integers (the labels of the edges). Hence, so do the sets $(S - k) \cup (S' - k')$ and $S \cup (S' - (k' - k))$. Let us prove the converse. Let $S \cup (S' - r) = \{a_1 < \cdots < a_q\}$ and assume first that $a_1 \in S$. We have that $a_i + p + q - i + 1 = k$ is constant. For each $1 \leq i \leq q$ we assign to the corresponding edge the label $p + q - i + 1$. Thus, for each $a_i \in S$ we have $a_i + p + q - i + 1 = k$, whereas if $a_i \in S' - r$ we obtain that $a_i + r + p + q - i + 1 = k + r = k'$. We proceed similarly in case $a_1 + r \in S'$. □

3.1. Some constructions of (super) edge bi-magic graphs. Let $G = (V, E)$ be a graph and let $S \subset V$. We denote by $G *_S u$ the graph obtained from G by adding a new vertex u and the edge set $\{uv : v \in S\}$ and by $G \wedge_S \{u_i\}_{i=1}^{|S|}$ the graph obtained from G by adding a leaf $v_i u_i$ to each vertex of $v_i \in S$. Furthermore, in general we write $G \wedge_S \{u_i^j\}_{i=1,\ldots|S|}^{j=1,\ldots,n_i}$ to denote the graph obtained from G by adding leaves $v_i u_i^j$, $j = 1, \ldots, n_i$ to each vertex of $v_i \in S$.

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PROPOSITION 3.5. *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let $S \subset V$ be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers. Then the graph $G *_S u$ is (super) edge bi-magic.*

PROOF. Let $G *_S u = (V', E')$ and assume that $s = \max\{f(x) \mid x \in S\}$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+|S|+1}$ such that

$$f'(x) = \begin{cases} f(x) + 1 & \text{if } x \in V \cup E, \\ 1 & \text{if } x = u, \\ p + q + 2 + i & \text{if } x = uv,\ v \in S, \text{ and } f(v) = s - i. \end{cases}$$

Then f' is a (super) edge bi-magic labeling of $G *_S u = (V', E')$ with magic constants $k_1 = k + 3$ and $k_2 = p + q + s + 4$, where k is the magic sum for f. □

The graph $PY(n)$ is the graph obtained from the cylinder $C_3 \times P_n$ by adding a new vertex and joining it to the three vertices of the cycle on the top.

COROLLARY 3.6 [11, Theorem 1]. *The graph $PY(n)$ is edge bi-magic.*

PROOF. Recall that $\text{und}(\overrightarrow{LP_n} \otimes \overrightarrow{C_3}) = C_3 \times P_n$. In particular, it admits a (super) edge-magic labeling, with the vertices of the cycle on the top labeled with $\{1, 2, 3\}$. Thus, the construction of Proposition 3.5 produces an edge bi-magic labeling of $PY(n)$. □

PROPOSITION 3.7. *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let S be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers and $|S|$ is odd. Then the graph $G \wedge_S \{u_i\}_{i=1}^{|S|}$ is (super) edge bi-magic.*

PROOF. Let $G \wedge_S \{u_i\}_{i=1}^{|S|} = (V', E')$ and assume that $s = \max\{f(x) \mid x \in S\}$ and that the new edges are $v_i u_i$ where $f(v_i) = s - i + 1$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+|S|+1}$ such that

$$f'(x) = \begin{cases} f(x) + |S| & \text{if } x \in V \cup E, \\ \dfrac{|S| - 1}{2} + \dfrac{i + 1}{2} & \text{if } x = u_i \text{ and } i \text{ is odd}, \\ \dfrac{i}{2} & \text{if } x = u_i \text{ and } i \text{ is even}, \\ p + q + |S| + l & \text{if } x = v_i u_i \text{ and } i = 2l - 1, \\ p + q + |S| + \dfrac{|S| + 1}{2} + l & \text{if } x = v_i u_i \text{ and } i = 2l. \end{cases}$$

Then f' is a (super) edge bi-magic labeling of $G \wedge_S \{u_i\}_{i=1}^{|S|}$ with magic constants $k_1 = k + 3|S|$ and $k_2 = p + q + s + (5|S| + 3l)/2$, where k is the magic sum of f. □

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PROPOSITION 3.8. *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let S be a subset of vertices such that $f(v_i) = s - d(i-1)$ for each $v_i \in S$ with $d > 1$. Then the graph $G \wedge_S \{u_i^j\}_{i=1,\ldots|S|}^{j=1,\ldots,n_i}$, where $n_{2l-1} = d - 1$ and $n_{2l} = 1$, is (super) edge bi-magic.*

PROOF. Let $G \wedge_S \{u_i^j\}_{i=1,\ldots,|S|}^{j=1,\ldots,n_i} = (V', E')$. Let $r = (d-1)\lceil |S|/2 \rceil + \lfloor |S|/2 \rfloor$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+2r}$ such that

$$f'(x) = \begin{cases} f(x) + r & \text{if } x \in V \cup E, \\ (l-1)d + j & \text{if } x = u_{2l-1}^j, \\ ld & \text{if } x = u_{2l}^1, \\ p + q + r + ld - j & \text{if } x = v_{2l-1}u_{2l-1}^j, \\ p + q + r + ld & \text{if } x = v_{2l}u_{2l}^1. \end{cases}$$

Then f' is a (super) edge bi-magic labeling of $G \wedge_S \{u_i^j\}_{i=1,\ldots,|S|}^{j=1,\ldots,n_i}$ with magic constants $k_1 = k + 3r$ and $k_2 = p + q + d + 2r + s$, where k is the magic sum of f. □

3.2. (Super) edge bi-magic graphs obtained using $\otimes_h$-product. We present a simplified proof of the main result found in [8]. Recall that S_n denotes the set of all super edge-magic 1-regular labeled digraphs of odd order n.

THEOREM 3.9. *Let D be a (super) edge-magic digraph and let $h : E(D) \to S_n$ be any function. Then the graph* und$(D \otimes_h S_n)$ *is (super) edge-magic.*

PROOF. As in the original paper, we rename the vertices of D and each element of S_n after the labels of their corresponding edge-magic and super edge-magic labelings respectively. We also define the labels as in [8, Theorem 3.1].

(1) If $(i, j) \in V(D \otimes_h S_n)$ we assign to the vertex the label: $n(i-1) + j$.
(2) If $((i, j), (i', j')) \in E(D \otimes_h S_n)$ we assign to the arc the label: $n(e-1) + (3n+3)/2 - (j + j')$, where e is the label of (i, i') in D.

Notice that, since each element Γ of S_n is labeled with a super edge-magic labeling, by [8, Corollary 1.1] we have

$$\{(3n+3)/2 - (j + j') : (j, j') \in E(\Gamma)\} = \{1, 2, \ldots, n\}.$$

Thus, the set of labels in $D \otimes_h S_n$ covers all elements in $\{1, 2, \ldots, n(|V(D)| + |E(D)|)\}$. Moreover, for each arc $((i, j)(i', j')) \in E(D \otimes_h S_n)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the sum of labels is constant and equal to

$$n(i + i' + e - 3) + (3n+3)/2. \tag{3.1}$$

That is, $n(\mathrm{val}_f - 3) + (3n+3)/2$. We also notice that if the digraph D is super edge-magic then the vertices of $D \otimes_h S_n$ receive the smallest labels. □

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Using this proof we can extend the previous result to the case of edge bi-magic digraphs.

THEOREM 3.10. *Let D be a (super) edge bi-magic digraph and let $h : E(D) \to S_n$ be any function. Then the graph* $\mathrm{und}(D \otimes_h S_n)$ *is (super) edge bi-magic.*

PROOF. Let k_1 and k_2 be the valences for a (super) edge bi-magic labeling of D. From the proof of Theorem 1.2, it is clear that for each arc $((i, j), (i', j')) \in E(D \otimes_h S_n)$, coming from an arc (i, i') in D labeled with e, the induced sum (3.1) belongs to $\{n(k_1 - 3) + (3n + 3)/2, n(k_2 - 3) + (3n + 3)/2\}$. □

4. k-equitable

In this section, we use the $\otimes_h$-product in order to construct k-equitable labelings of new families of graphs. In this case, the input elements are k-equitable digraphs and 1-regular super edge-magic digraphs. However, instead of applying the product directly, we have to use what we call the rotation of a super edge-magic digraph.

4.1. Rotations of super edge-magic digraphs. Let $M = (a_{i,j})$ be a square matrix of order n and let $M^R = (a^R_{i,j})$ be the matrix obtained from M where $a^R_{i,j} = a_{n+1-j,i}$. Graphically this corresponds to a rotation of the matrix by $\pi/2$ radians clockwise (see Example 4.1). We say that M^R is the *rotation of the matrix M*. Note that the digraph corresponding to M^R may contain loops and double arcs. Therefore, in this section we may work with digraphs for which their underlying graphs contain multiple edges. Recall that if we write S_n then n is odd.

EXAMPLE 4.1. A matrix M and its rotation M^R

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to M^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

LEMMA 4.2. *Let $D \in S_n$, and assume that each vertex is named after the label of a super edge-magic labeling. Let $A = (a_{i,j})$ be its adjacency matrix. If $a^R_{i,j} = 1$ then*

$$|i - j| \le \frac{n-1}{2}.$$

PROOF. By [8, Corollary 1.1], if $A = (a_{i,j})$ is the adjacency matrix of $D \in S_n$ and $a_{i,j} = 1$ then $(n + 3)/2 \le i + j \le (3n + 1)/2$. Hence, since $a^R_{i,j} = a_{n+1-j,i}$, if $a^R_{i,j} = 1$ it follows that $(n + 3)/2 \le n + 1 - j + i \le (3n + 1)/2$. Therefore, $-(n - 1)/2 \le i - j \le (n - 1)/2$ and we obtain the result. □

A digraph S is said to be a *rotation super edge-magic of order n* if its adjacency matrix is the rotation matrix of the adjacency matrix of a super edge-magic 1-regular digraph of order n. We denote by RS_n the set of all digraphs that are rotation super edge-magic of order n. The following corollaries are easy observations.

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COROLLARY 4.3. *Let S be a digraph in RS_n and let k be an integer. If $|k| \leq (n-1)/2$ then there exists a unique arc $(i, j) \in E(S)$ such that $i - j = k$.*

PROOF. Let $D \in S_n$ be the digraph where S is coming from. Let $A = (a_{i,j})$ be the adjacency matrix of D, where every vertex takes the label of a super edge-magic labeling of D. Note that, since A comes from a super edge-magic labeling of a 1-regular digraph, every secondary diagonal ($\nearrow$) contains at most a 1, and the diagonals that contain the 1s are consecutive. Moreover, in each main diagonal ($\searrow$) of A^R there appears at most a 1 and the diagonals that contain the 1s are consecutive. $\square$

COROLLARY 4.4. *For each digraph D and each constant function $h : E(D) \longrightarrow RS_n$ one of the weakly connected components of $D \otimes_h RS_n$ is isomorphic to D.*

PROOF. Let S be a digraph in RS_n. By Corollary 4.3 we know that S contains a loop. Let (j, j) be a loop in S. Then the subdigraph of $D \otimes_h RS_n$ induced by the vertices of the form (i, j) for $i \in V(D)$ is isomorphic to D. $\square$

REMARK 4.5. Inheriting the notation used in this section, let A be the adjacency matrix of a super edge-magic digraph D of order n. We have that, $A^R = A^t P$, where A^t is the transpose of A, and $P = (p_{i,j})$ where $p_{i,j} = 1$ if $i + j = n + 1$ but $p_{i,j} = 0$ otherwise. Clearly, $(A^R)^t$ is the adjacency matrix of some digraph in RS_n. That is, there exists a (possibly) different super edge-magic labeling of D, such that if B is its induced adjacency matrix then $B^t P = (A^R)^t$. Thus, $B = PA^t P$.

EXAMPLE 4.6. Let D be the super edge-magic digraph $1 \to 5 \to 3 \to 4 \to 1$ and a loop in 2. Its adjacency matrix A is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which has rotation matrix

$$A^R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $(A^R)^t = B^t P$ where $B = PA^t P$. That is, B is the adjacency matrix of a super edge-magic digraph obtained by reversing the arcs of D and by interchanging the labels by σ, where σ is the permutation on $\{1, \ldots, n\}$ defined by $\sigma(i) = n + 1 - i$. In our example, the super edge-magic digraph defined by B is $1 \to 5 \to 2 \to 3 \to 1$ and the loop in 4.

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REMARK 4.7. Let M^{3R} be the matrix obtained from M by rotating $3\pi/2$ radians in the clockwise sense the columns of M. That is, $M^{3R} = PA^t$. Note that this different rotation of the adjacency matrix of a super edge-magic labeled digraph has the same properties of M^R.

4.2. Main theorem. Let D be a k-equitable digraph where the vertices are identified by the labels of a k-equitable labeling of D. Let us consider the induced labeling on $V(G \otimes_h RS_n)$ that assigns the label $n(i-1)+j$ to the vertex (i, j). One can easily see that all labels are distinct and that, in case the labeling of D is optimal, all elements in $\{1, \ldots, n \cdot |V(D)|\}$ are used. Moreover, by the product definition of $\otimes_h$, $|n(i-i') + (j-j')|$ is an induced arc label if and only if $(i, i') \in E(D)$ and $(j, j') \in E(h(i, i'))$.

LEMMA 4.8. *Let D be a k-equitable digraph, and let $((i, j), (i', j'))$, $((r, s), (r', s'))$ be two arcs of $D \otimes_h RS_n$. If $|n(i-i') + (j-j')| = |n(r-r') + (s-s')|$ then $|i-i'| = |r-r'|$ and $|s-s'| = |j-j'|$.*

PROOF. Note that the equality $n(i-i') + (j-j') = \pm(n(r-r') + (s-s'))$ implies that there exists $\alpha \in \mathbb{Z}$ such that $|\alpha n| = |\pm(s-s') - (j-j')|$. Thus, by Lemma 4.2, $|\alpha n| \le n-1$. Hence, $\alpha = 0$ and therefore $|j-j'| = |s-s'|$ and $|i-i'| = |r-r'|$. □

THEOREM 4.9. *Let D be an (optimal) k-equitable digraph and let $h : E(D) \to RS_n$ be any function. Then $D \otimes_h RS_n$ is (optimal) k-equitable.*

PROOF. Assume that $|n(i-i') + (j-j')|$ is an arc label induced by a k-equitable labeling of D. There exist exactly k arcs in D, (i_l, i'_l) , $1 \le l \le k$ such that $|i_l - i'_l| = |i-i'|$. Thus $|n(i_l - i'_l)| = |n(i-i')|$ and by Lemma 4.2 we have that

$$|n(i_l - i'_l)| - \frac{n-1}{2} \le |n(i-i') + (j-j')| \le |n(i_l - i'_l)| + \frac{n-1}{2}.$$

Hence, we obtain that

$$||n(i-i') + (j-j')| - |n(i_l - i'_l)|| \le \frac{n-1}{2}$$

and by Corollary 4.3 there exist two different arcs (r, r'), $(s, s') \in E(h(i_l, i'_l))$ such that

$$||n(i-i') + (j-j')| - |n(i_l - i'_l)|| = |r-r'| = |s-s'|$$

with $r - r' \le 0 \le s - s'$.

Therefore, either $|n(i-i') + (j-j')| = |n(i_l - i'_l) + r - r'|$ or $|n(i-i') + (j-j')| = |n(i_l - i'_l) + s - s'|$. In the first case $((i_l, r), (i'_l, r'))$ is labeled with $|n(i-i') + (j-j')|$, whereas in the second case $((i_l, s), (i'_l, s'))$ is labeled with $|n(i-i') + (j-j')|$.

Moreover, assume that

$$|n(i-i') + (j-j')| = |n(r-r') + (s-s')|.$$

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By Lemma 4.8, $|i - i'| = |r - r'|$ and $|s - s'| = |j - j'|$. That is, $|n(i - i')| = |n(r - r')|$ and we only have k-possible arcs with the same label.

In particular, if the k-equitable labeling of D is optimal, then the induced labeling on $D \otimes_h RS_n$ is also optimal. □

Recall that cycles are k-equitable for each proper divisor k of their size. By giving a nonoptimal labeling, it was stated in [3] that the union of vertex-disjoint k-equitable graphs is k-equitable. Using Theorem 4.9, we can provide optimal k-equitable labelings of n copies of trees, for n odd.

THEOREM 4.10. *Let n be an odd integer and let F be an optimal k-equitable forest for each proper divisor k of $|E(F)|$. Then nF is optimal k-equitable for each proper divisor k of $|E(F)|$.*

PROOF. Clearly, each rotation of a super edge-magic 1-regular digraph gives a 1-regular digraph. In particular, by Theorem 1.4 we have that $\mathrm{und}(\overrightarrow{F} \otimes_h \Sigma_n) = nF$. Thus, since F is optimal k-equitable for each proper divisor k of $|E(F)|$, Theorem 4.9 implies that nF is optimal k-equitable for each proper divisor k of $|E(F)|$. □

THEOREM 4.11. *Let $m - 1$, n be odd integers. Then nC_m is optimal k-equitable for all proper divisors k of m.*

PROOF. Let $\overrightarrow{C_n}$ be a strong orientation of C_n and assume that M is the adjacency matrix of $\overrightarrow{C_n}$ where each vertex is identified with the label of a super edge-magic labeling of $\overrightarrow{C_n}$. The matrix M^R obtained by rotating $\pi/2$ radians clockwise is the adjacency matrix of a digraph $\overrightarrow{RC_n} = \overrightarrow{C_1} \cup \overrightarrow{C_{n_1}} \cup \cdots \cup \overrightarrow{C_{n_k}}$. Let $\overleftarrow{RC_n}$ be the digraph obtained from $\overrightarrow{RC_n}$ by reversing all its arcs. Consider a function $h : E(\overrightarrow{C_m}) \to \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\}$ such that two consecutive arcs in $\overrightarrow{C_m}$, namely (x, y), (y, z) have $h(x, y) \neq h(y, z)$. Assume that $a_1 a_2 \cdots a_m$ is a directed path in $\overrightarrow{C_m}$. Then for each $(i, j) \in E(h(a_1, a_2))$ we obtain that $(a_1, i)(a_2, j)(a_3, i) \cdots (a_m, j)(a_1, i)$ is a cycle of length m in $\overrightarrow{C_m} \otimes_h \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\}$. That is,

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\} \simeq n\overrightarrow{C_m}.$$

Thus, since every cycle is optimal k-equitable for each proper divisor k of the size, the result follows by Theorem 4.9. □

5. (Super) edge r-magic graphs. Open problems

A (p, q)-graph $G = (V, E)$ admits an *edge r-magic labeling* if there exists a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that for each edge $uv \in E$, $f(u) + f(uv) + f(v) \in \{k_1, k_2, \ldots, k_r\}$ where $\{k_1, \ldots, k_r\}$ are r distinct constants. In this case, the graph is said to be edge r-magic. If we add the extra condition

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that $f(V) = \{i\}_{i=1}^{p}$ then we say that f is a *super edge r-magic labeling* and G a *super edge r-magic graph*.

The next lemma is an extension of Lemma 3.4 for the case of super edge r-magic graphs. The proof works similarly.

LEMMA 5.1. *A graph labeling of a graph G is super edge r-magic if and only if the set of sum labels of adjacent vertices (including repetitions) can be partitioned into r sets $S_0, S_1, \ldots, S_{r-1}$ and there exist $r-1$ integers $c_1, c_2, \ldots, c_{r-1}$ such that $S_0 \cup (S_1 - c_1) \cup \cdots \cup (S_{r-1} - c_{r-1})$ is a set of consecutive integers.*

With a similar proof as in Section 3.2 we can state the following result.

THEOREM 5.2. *Let D be a (super) edge r-magic digraph and let $h : E(D) \to S_n$. Then the graph* $\mathrm{und}(D \otimes_h S_n)$ *is (super) edge r-magic.*

Clearly, each graph is edge r-magic for some r. Thus a natural question appears.

QUESTION 5.3. Given a graph G, find the minimum r such that G is edge r-magic.

Similarly, we can study the following aspect.

QUESTION 5.4. Let G be an edge r-magic graph. Find an edge r-magic labeling f of G that minimizes the difference $k_r - k_1$, where k_1 and k_r are, respectively, the minimum and the maximum magic constants of f.

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