Vector spaces

Bioinformatics Degree
Algebra

Departament de Matemàtiques

UNIVERSITAT POLITÈCNICA DE CATALUNYA
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Outline

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The vector space $\mathbb{R}^n$

We consider the set of *n*-tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

and we call its elements **vectors**.

Notation: When we talk about $v \in \mathbb{R}^n$ we usually think of v as a column vector,

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

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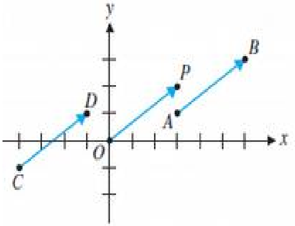
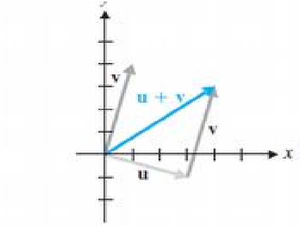
$\mathbb{R}^2$: Physical interpretation

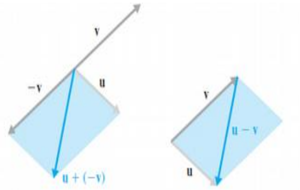
- View $(x, y) \in \mathbb{R}^2$ as a directed line segment between two points A and B, $(x, y) =$ "vector" $\overrightarrow{AB}$.
- $\overrightarrow{AB}$: the displacement needed to get from A to B: x units along the x-axis and y along the y-axis.
- Two vectors are equal if they represent the same displacement ($\Leftrightarrow$ they have the same length, direction, and sense).
- We can always think (x, y) as a vector of initial point $(0, 0)$ and end point (x, y).



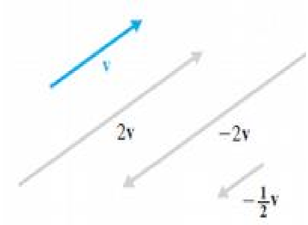
Operations in $\mathbb{R}^2$

We can sum or substract vectors





and multiply a vector by a constant (*scalar*)



$\mathbb{R}^3$

- Vectors in $\mathbb{R}^3$ have a similar physical interpretation
- We can also sum two vectors and multiply a vector by a scalar. These operations can be done in coordinates: if $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

$$c \cdot u = (cx_1, cx_2, cx_3) \text{ for any } c \in \mathbb{R}.$$

Operations in $\mathbb{R}^n$

In $\mathbb{R}^n$ we define the following operations:

sum: if $u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n.$$

scalar multiplication: if $u = (x_1, x_2, \ldots, x_n), c \in \mathbb{R}$, then

$$c \cdot u = (c\, x_1, c\, x_2, \ldots, c\, x_n) \in \mathbb{R}^n.$$

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Proposition

These operations in $\mathbb{R}^n$ satisfy the following properties:

1. $u + v = v + u$. *Commutativity*
2. $(u + v) + w = u + (v + w)$. *Associativity*
3. $\exists$ *an element* $\mathbf{0} \in \mathbb{R}^n$, *called the zero vector, such that* $u + \mathbf{0} = u$.
4. *For each* $u \in \mathbb{R}^n$, $\exists$ *an element* $-u \in \mathbb{R}^n$ *such that* $u + (-u) = \mathbf{0}$.
5. $c \cdot (u + v) = c \cdot u + c \cdot v$. *Distributivity*
6. $(c + d) \cdot u = c \cdot u + d \cdot u$. *Distributivity*
7. $c \cdot (d \cdot u) = (cd) \cdot u$.
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Examples

Any set that has two operations $+$ and $\cdot$ satisfying the previous property is called a vector space

Some other examples of vector spaces are:

- Solutions of a homogeneous linear system of equations.
- $m \times n$ matrices
- Polynomials of degree $\leq k$, $k \geq 1$
- Real functions

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Vector subspaces

F is a vector subspace of the vector space E if
$F \subseteq E$ and F a vector space itself.

Definition

Let F be a nonempty subset of $\mathbb{R}^n$. Then F is a vector subspace of $\mathbb{R}^n$ if the following conditions hold:

1. If u and v are in F, then $u + v$ is in F.
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Vector subspaces

The following are examples of vector subspaces:

- $V = \{\vec{0}\}$
- $V = \mathbb{R}^n$
- $F_1 = \{(\alpha, -2\alpha) \mid \alpha \in \mathbb{R}\}$
- $F_2 = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\}$
- $G_1 = \{(x, y) \in \mathbb{R}^2 \mid 2x - 5y = 0\}$
- $G_2 = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x - 5y + 3z = 0, x - y + z + 2t = 0\}$

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Linear Combination

Definition

We say that $u \in \mathbb{R}^n$ is a linear combination of $v_1, \ldots, v_k \in \mathbb{R}^n$ if there are $c_1, \ldots, c_k \in \mathbb{R}$ such that $u = c_1 v_1 + \ldots + c_k v_k$

Finding out if a given vector is a linear combination of a collection of vectors is equivalent to check whether a linear system of equations is consistent.

$$\begin{pmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} | \\ u \\ | \end{pmatrix}$$

Generators

Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$.

Definition

The span of $v_1, v_2, \ldots, v_k$ is the set of all linear combinations of $v_1, v_2, \ldots, v_k$:

$$[v_1, v_2, \ldots, v_k] = \{c_1 v_1 + \ldots + c_k v_k \mid c_1, \ldots, c_n \in \mathbb{R}\}.$$

If $[v_1, \ldots, v_k] = F$, we say that $\{v_1, v_2, \ldots, v_k\}$ is a system of generators for F, and also that F is spanned by $v_1, v_2, \ldots, v_k$.

Linear independence

Definition

The vectors $v_1, v_2, \ldots, v_k$ are linearly dependent if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that $c_1\, v_1 + \ldots + c_k\, v_k = \vec{0}$.
Otherwise, we say that $v_1, v_2, \ldots, v_k$ are linearly independent.

Theorem

The vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

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Basis

Definition

Let $F \subseteq \mathbb{R}^n$ be a vector subspace. An **ordered** collection of vectors $\{v_1, \ldots, v_k\}$ is a basis of F if

1. $F = [v_1, \ldots, v_k]$ (that is, $\{v_1, ...v_k\}$ is a system of generators of F) and
2. $v_1, \ldots, v_k$ are linearly independent.

Example-Definition

If $e_i = (0, \ldots, 1, \ldots, 0)$ for $i = 1, 2, \ldots, n$, then $e = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. This basis is called the standard basis for $\mathbb{R}^n$.

Notation: $[v_1, \ldots, v_k]$ is the generated set (the vector space),
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The importance of rank

Theorem

Given $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, *write* $A = (v_1, \ldots, v_k) \in M_{n,k}(\mathbb{R})$. *Then,*

- i) $v_1, v_2, \ldots, v_k$ *are linearly independent if and only if* $rank(A) = k$.
- ii) $v_1, v_2, \ldots, v_k$ *are a system of generators of* $\mathbb{R}^n$ *if and only if* $rank(A) = n$.
- i) + ii) $v_1, v_2, \ldots, v_k$ *are a basis for* $\mathbb{R}^n$ *if and only if* $k = rank(A) = n$.

Proposition

Given vectors $S = \{v_1, \ldots, v_k\}$ *in* $\mathbb{R}^n$*, the following are equivalent:*

1. S *is a basis of* $\mathbb{R}^n$;
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Given $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, *write* $A = (v_1, \ldots, v_k) \in M_{n,k}(\mathbb{R})$. *Then,*

i) $v_1, v_2, \ldots, v_k$ *are linearly independent if and only if* $rank(A) = k$.

ii) $v_1, v_2, \ldots, v_k$ *are a system of generators of* $\mathbb{R}^n$ *if and only if* $rank(A) = n$.

i) + ii) $v_1, v_2, \ldots, v_k$ *are a basis for* $\mathbb{R}^n$ *if and only if* $k = rank(A) = n$.

Proposition

Given vectors $S = \{v_1, \ldots, v_k\}$ *in* $\mathbb{R}^n$, *the following are equivalent:*

1. S *is a basis of* $\mathbb{R}^n$;
2. S *are linearly independent and* $k = n$;
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Dimension

Theorem (The Basis Theorem)

1. *Each basis of the space $\mathbb{R}^n$ has n vectors.*
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The cardinal of a basis of F is called the dimension of F.

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Outline

Vector subspaces

Proposition

- *$V = [v_1, v_2, \ldots, v_k]$ is a vector subspace of $\mathbb{R}^n$.*
- *Let $Ax = 0$ be a linear system, where $A \in M_{m,n}(\mathbb{R})$. Then, the set of solutions $V = \{v \in \mathbb{R}^n \mid Av = 0\}$ is a vector subspace of $\mathbb{R}^n$.*

In general, there are two ways to describe a vector subspace $F \subset \mathbb{R}^n$:

- through a system of generators: $F = [v_1, \ldots, v_k]$
- through an homogeneous linear system of equations: $F = \{v \in \mathbb{R}^n \mid Av = 0\}$

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Computing a basis of a vector subspace

If $F = [v_1, v_2, \ldots, v_k]$, a basis of F can be obtained by applying any of the following methods:

- Write the vectors $v_1, \ldots, v_k$ as the **rows** of a matrix A, and reduce A to row echelon form $\bar{A}$ (Gaussian elimination). The nonzero rows of $\bar{A}$ are a basis of F.
- Write the vectors $v_1, \ldots, v_k$ as the **columns** of a matrix B and reduce B to row echelon form $\bar{B}$ (Gaussian elimination). The columns of $\bar{B}$ with pivots indicate the vectors among $v_1, \ldots, v_k$ to choose to obtain a basis of F.

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Computing the dimension of a subspace

Proposition

- *If $F = [v_1, \ldots, v_k]$, then $\dim(F) = \mathrm{rank}\,(v_1, \ldots, v_k)$*
- *If $F = \{u \in \mathbb{R}^n \mid Au = 0\}$, then $\dim(F) = n - \mathrm{rank}(A)$.*

Theorem

Let $F \subseteq G$ be subspaces of $\mathbb{R}^n$. Then:

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Theorem

Let $F \subseteq G$ *be subspaces of* $\mathbb{R}^n$*. Then:*

- F, G *are finite-dimensional and* $dimF \leq dimG \leq n$*.*
- $dimF = dimG$ *if and only if* $F = G$*.*

Extending to a basis of $\mathbb{R}^n$

If $u_1, \ldots, u_k$ are linearly independent vectors, then they can be extended to a basis of $\mathbb{R}^n$:

- Write the vectors $u_1, \ldots, u_k$ as the columns of a matrix B, and take $M = (B \mid I_n)$.
- Then, reduce M to row echelon form $\bar{M} = (\bar{B} \mid \bar{I}_n)$ (Gaussian elimination).
- Collect the columns of $\bar{I}_n$ with a pivot and choose the corresponding vectors of the standard basis (columns of I_n) of $\mathbb{R}^n$.
- $u_1, \ldots, u_k$ together with these last vectors form a basis of $\mathbb{R}^n$.

The same can be done if $u_1, \ldots, u_k$ are linearly independent vectors of a vector subspace V:
instead of I_n, take a matrix formed by a basis $v_1, \ldots, v_d$ of V and do the same process as above for $M = (u_1, \ldots, u_k | v_1, \ldots, v_d)$.

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Subspaces: Equations $\leftrightarrow$ Generators

From "generators" to "equations":
If $V = [v_1, \ldots, v_k] \subset \mathbb{R}^n$:
Write $M = (v_1, \ldots, v_k)$, and form an augmented matrix $(M|x)$ with $x =$ column with entries $x_1, x_2, \ldots, x_n$.
Then $x \in [v_1, \ldots, v_k]$ if and only if $rank(M|x) = rank(M)$.
There are 2 options:

- Reduce M to echelon form $(\bar{M}|\bar{x})$ by Gaussian elimination $\Rightarrow$ a linear system of equations for V is obtained by writing the equations that correspond to zero rows of $\bar{M}$.
- If $rank(M) = k$, the equations are formed by the vanishing of the $(k+1) \times (k+1)$ minors of $(M|x)$ (only need those that a chosen non-zero $k \times k$ minor of M).

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Subspaces of $\mathbb{R}^n$: Generators $\leftrightarrow$ Equations

From "equations" to "generators":

If $V = \{u \in \mathbb{R}^n \mid Au = 0\}$ (solutions to a homogeneous system):

- It is enough to solve the system to obtain a system of generators of V.
- Moreover, if we give values 0's and 1's to the free variables, these generators form a basis and $dim(V) = n - rank(A)$.

We have proved:

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Outline

Coordinates

Theorem

Any element of a vector space can be written as a unique linear combination of the vectors of any basis of that space.

Given $u \in \mathbb{R}^n$ and $B = \{v_1, \ldots, v_n\}$ a basis for $\mathbb{R}^n$, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $u = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ and these $c_1, \ldots, c_n$ are unique.

Definition

The $c_1, c_2, \ldots, c_n$ are called the coordinates of v with respect to B. We will use the notation

$$v_B = \begin{pmatrix} c_1 \\ \ldots \\ c_n \end{pmatrix}.$$

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Examples

1 In the standard basis B of $\mathbb{R}^3$, the coordinates of $v = (-1, 2, -1)$ are $v_B = (-1, 2, -1)$, because

$$(-1, 2, -1) = (-1) \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (-1) \cdot (0, 0, 1).$$

2 In the basis $B' = \{(1, 0, 1), (0, 1, 1), (2, -1, 3)\}$, the coordinates of v relative to B' are $v_{B'} = (1, 1, -1)$, because

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Change of basis

Let $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be two bases of $\mathbb{R}^n$. Denote by $A_{B\to C}$ the $n \times n$ matrix whose columns are the coordinate vectors of the basis B with respect to C:

$$A_{B\to C} = ((u_1)_C, \ldots, (u_n)_C)\,.$$

This is the change-of-basis matrix from B to C.

Proposition

1. $A_{B\to C} \cdot w_B = w_C$ *for all* $w \in \mathbb{R}^n$.
2. $A_{B\to C}$ *is invertible, and* $(A_{B\to C})^{-1} = A_{C\to B}$.
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In $\mathbb{R}^3$, take the standard basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B' = \{(1,0,1),(0,1,1),(2,-1,3)\}$.
Then,

$$A_{B' \to B} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$A_{B \to B'} = A^{-1}_{B' \to B} = \frac{1}{2} \begin{pmatrix} 4 & 2 & -2 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Then,

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Outline

Intersection and sum of subspaces

Let F, G be vector subspaces in $\mathbb{R}^n$ then:
The intersection of F and G is $F \cap G = \{v \in \mathbb{R}^n \mid v \in F, v \in G\}$.
The sum of F and G is $F + G = \{v + w \in \mathbb{R}^n \mid v \in F, w \in G\}$.

Computation:
If $F = \{x \in \mathbb{R}^n \mid A_F x = 0\}$ and $G = \{x \in \mathbb{R}^n \mid A_G x = 0\}$, then

$$F \cap G = \{x \in \mathbb{R}^n \mid Ax = 0\}, \text{ where } A = \begin{pmatrix} A_F \\ A_G \end{pmatrix}.$$

If $F = [v_1, \ldots, v_r]$ and $G = [w_1, \ldots, w_s]$, then

$$F + G = [v_1, \ldots, v_r, w_1, \ldots, w_s].$$

Grassmann Formula

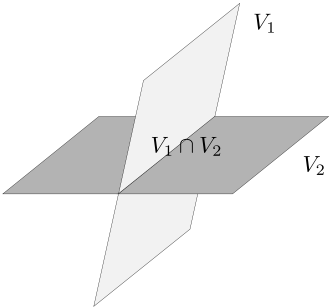
Theorem

- $F \cap G$ and $F + G$ are vector subspaces of $\mathbb{R}^n$.
- $dim(F+G) = dim(F) + dim(G) - dim(F \cap G)$.

Example

$F = [(1,0,1),(0,2,3)]$
$G = [(0,1,0),(1,1,1)]$

$F \cap G = [(1,0,1)]$
$F + G = \mathbb{R}^3$



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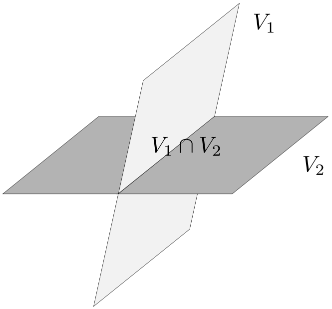

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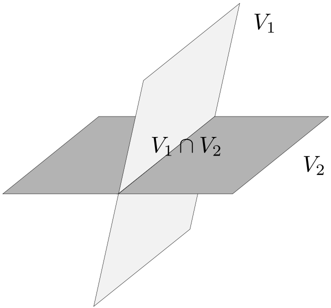

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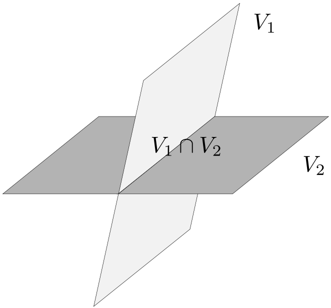

Direct sum

Definition

E is the **direct sum** of subspaces F_1 and F_2 if any $w \in E$ can be written in a **unique way** as $w = v_1 + v_2$ with $v_1 \in F_1$, $v_2 \in F_2$. In this case we use the notation $E = F_1 \oplus F_2$.

Proposition

Let F_1, F_2 be two subspaces of E. Then $E = F_1 \oplus F_2$ if and only if the following two conditions hold:

$$E = F_1 + F_2,$$

$$F_1 \cap F_2 = \{0\}.$$

If $E = F_1 \oplus F_2$, we say that F_2 is a **complementary subspace** to F_1 (and vice-versa).

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Outline

Python: vectors and operations

Vectors are introduced in numpy as $n \times 1$ matrices:

$\texttt{u} = \texttt{np.array}([1, 2, 0, -3])$ or $\texttt{v} = \texttt{np.array}([0, 5, -2, 7])$.

The sum of vectors in the same space is introduced with $+$ and the scalar multiplication with $*$:

$$\begin{aligned} \texttt{u} + \texttt{v} &= \texttt{np.array}([1, 7, -2, 4]) \\ (-3) * \texttt{u} &= \texttt{np.array}([-3, -6, 0, 9]) \end{aligned}$$

Python: Subspaces

If $F = [v_1, \ldots, v_k]$ we can compute $dim(F)$ with Python:

```
M = np.array([[v1], ..., [vk]]);
matrix_rank(M)
```

If $F = \{u \in \mathbb{R}^n \mid Au = 0\}$ we can compute $dim(F)$ with Python:

```
n-matrix_rank(A)
```