Àlgebra lineal i geometria
4. Ortogonalitat

Grau en Enginyeria Física

2023-24

Universitat Politècnica de Catalunya
Departament de Matemàtiques

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BARCELONATECH

Outline

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The Euclidean scalar product

Definition

The **Euclidean scalar product** (or **dot product**) $< u, v >$ of two vectors $u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ is

$$< u, v > := u^t v = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

Properties:

1. $< u, u > \geq 0 \; \forall u$ and $< u, u >= 0 \Leftrightarrow u = 0$ (*positive definite*)
2. $< u, v >=< v, u >$ (*symmetric*).
3. *bilineal*:
   - $< a_1 u_1 + a_2 u_2, v >= a_1 < u_1, v > + a_2 < u_2, v >$,
   - $< u, a_1 v_1 + a_2 v_2 >= a_1 < u, v_1 > + a_2 < u, v_2 >$.

Any function $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that satisfies these properties is called a *scalar product*.

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Any function $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that satisfies these properties is called a *scalar product*.

Bilinear forms

Let E be an $\mathbb{R}$-e.v. A **bilinear form** on E is a map $\varphi : E \times E \longrightarrow \mathbb{R}$ such that, $\forall\ u, v, w \in E$ and $\lambda \in \mathbb{R}$:

(a) $\varphi(u+v, w) = \varphi(u, w) + \varphi(v, w)\ \varphi(\lambda u, w) = \lambda\varphi(u, w)$,

(b) $\varphi(w, u+v) = \varphi(w, u) + \varphi(w, v)\ \varphi(w, \lambda u) = \lambda\varphi(w, u)$.

If $\mathbf{u} = \{u_1, \dots, u_n\}$ is a basis of E, then the **matrix of φ in the basis u** is defined as

$$M_{\mathbf{u}}(\varphi) = \begin{pmatrix} \varphi(u_1, u_1) & \cdots & \varphi(u_1, u_n) \\ \vdots & & \vdots \\ \varphi(u_n, u_1) & \cdots & \varphi(u_n, u_n) \end{pmatrix}.$$

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Matrix of a bilinear form

Properties:

1. If $v_{\mathbf{u}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, w_{\mathbf{u}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \Rightarrow$

   $\varphi(v, w) = (x_1 \dots x_n) M_{\mathbf{u}}(\varphi) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and $M_{\mathbf{u}}(\varphi)$ is the unique matrix that satisfies this.

2. If $\mathbf{v}$ is another basis, then

$$M_{\mathbf{v}}(\varphi) = A^t_{\mathbf{v} \to \mathbf{u}} M_{\mathbf{u}}(\varphi) A_{\mathbf{v} \to \mathbf{u}}$$

A bilinear form φ is **symmetric** if $\varphi(u, v) = \varphi(v, u)$ for all u, v.
A bilinear form is symmetric $\Leftrightarrow M_{\mathbf{u}}(\varphi)$ is a symmetric matrix for any basis $\mathbf{u}$.

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Scalar products

Let E be an $\mathbb{R}$-e.v. and φ a bilinear form on E. One says that φ is **positive definite** if $\varphi(u, u) \geq 0$ with equality only when $u = 0$.

Definition

A **scalar product** on E is a symmetric, positive definite bilinear form $<,>: E \times E \longrightarrow \mathbb{R}$. An $\mathbb{R}$-e.v together with a scalar product is called a **Euclidean** vector space.

Examples:

- The Euclidean scalar product
- $E = \mathcal{F}([a, b], \mathbb{R}) = \{$ continuous real functions defined on $[a, b]\}$, then the following defines a scalar product on E:

$$< f, g >:= \int_a^b f(x)g(x)dx.$$

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Norm and distance

Let E be an $\mathbb{R}$-e.v. with scalar product $<,>$. The **norm** of $u \in E$ is $\|u\| = \sqrt{<u,u>}$.
If $<,>$ is the Euclidean product, the norm is called the *standard, Euclidean, or 2-norm* and is also denoted as $||u||_2$.
Properties: for any $u, v \in E$ and $c \in \mathbb{R}$

1. $\|u\| \geq 0 \ \forall u$ and $\|u\| = 0 \Leftrightarrow u = 0$;
2. $\|cu\| = |c|\|u\| \ c \in \mathbb{R}$;
3. $| < u, v > | \leq \|u\|\|v\|$ (Cauchy-Schwarz inequality)
4. $\|u + v\| \leq \|u\| + \|v\|$ (triangular inequality);

Any function $f : E \longrightarrow \mathbb{R}$ that satisfies properties 1,2,4 is called a *norm* (and is not necessarily defined through a scalar product).

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Other norms

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, one defines:

1. The **1-norm** (also called taxicab or Manhattan norm):

$$||x||_1 = |x_1| + \cdots + |x_n|.$$

2. The **maximum norm** (also called infinite norm):

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Distances and Angles

Let E be an $\mathbb{R}$-e.v. with scalar product $<,>$.

- A vector u is called a **unit** vector if $\|u\| = 1$. Given a vector $v \neq 0$, we can always find a unit vector in its direction: $v/\|v\|$ (we say that we have **normalized** v).
- The **distance** between two vectors $u, v \in E$, is $d(u,v) = \|u - v\|$.
- The (unoriented) **angle** between two vectors $u \neq 0, v \neq 0 \in E$ is the unique $\alpha \in [0, \pi]$ such that $\cos(\alpha) = \frac{<u,v>}{\|u\| \cdot \|v\|}$ (the sign of $\widehat{uv}$ depends on the orientation we choose).
- Two vectors u, v are **orthogonal** (also denoted $u \perp v$) if $< u, v >= 0$.
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- If $u \perp v$ and $u, v \neq 0 \Rightarrow u, v$ are l.i.

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Orthonormal basis

Definition

A basis $\{v_1, \ldots, v_d\}$ of a subspace $F \subseteq E$ is an **orthonormal basis** (b.o.n) if its vectors are

- pairwise orthogonal: $< v_i, v_j >= 0$ if $i \neq j$
- and normalized: $\|v_i\| = 1$ for $i = 1, 2, \ldots, d$.

- called **orthogonal** if pairwise orthogonal but not normalized.
- Ex: the standard basis is a b.o.n of $\mathbb{R}^n$ for Euclidean product.
- If $\mathbf{v} = \{v_1, \ldots, v_n\}$ is b.o.n. of $E \Rightarrow$ the coordinates of v in basis $\mathbf{v}$ are

$$(< v, v_1 >, \ldots, < v, v_n >).$$

- If $\mathbf{v}$ is b.o.n. of E and $\mathbf{u}$ basis of E then,

$$\mathbf{u} \text{ is b.o.n} \quad \Leftrightarrow \quad A^t_{\mathbf{u}\to\mathbf{v}} A_{\mathbf{u}\to\mathbf{v}} = I.$$

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$$\mathbf{u} \text{ is b.o.n} \quad \Leftrightarrow \quad A^t_{\mathbf{u}\to\mathbf{v}} A_{\mathbf{u}\to\mathbf{v}} = I.$$
- For any basis $\mathbf{u}$ of E, $\exists$ scalar product such that $\mathbf{u}$ is b.o.n for it.

Orthonormal basis

Definition

A basis $\{v_1, \dots, v_d\}$ of a subspace $F \subseteq E$ is an **orthonormal basis** (b.o.n) if its vectors are

- pairwise orthogonal: $< v_i, v_j >= 0$ if $i \neq j$
- and normalized: $\|v_i\| = 1$ for $i = 1, 2, \dots, d$.

- called **orthogonal** if pairwise orthogonal but not normalized.
- Ex: the standard basis is a b.o.n of $\mathbb{R}^n$ for Euclidean product.
- If $\mathbf{v} = \{v_1, \dots, v_n\}$ is b.o.n. of $E \Rightarrow$ the coordinates of v in basis $\mathbf{v}$ are

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Orthogonal matrices

An $n \times n$ matrix that satisfies $A^t A = I$ is called an **orthogonal** matrix.

- If we call the columns $u_1, \dots, u_n$, $A = \left(u_1 \dots u_n \right)$, then,
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- A is orthogonal $\Leftrightarrow A^{-1} = A^t$.
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- A is orthogonal $\Rightarrow \det A = \pm 1$.
- If A is orthogonal, then the corresponding endomorphism preserves the Euclidean scalar product:

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Examples of 2×2 orthogonal matrices

The following maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are linear and preserve norms:

- $f=$ **symmetry** with respect to a line l passing through the origin, $l = [v]$. E.g. $f(x, y) = (x, -y)$.
- $f=$**rotation** counterclockwise of angle α with respect to the origin; then

$$M_e(f) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

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Given a subspace F of a euclidean space E, the following algorithm produces a b.o.n. of F:

1. Take any basis of F: $u_1, \dots, u_d$ and define:
2. $v_1 := u_1$
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Outline

Spectral theorem

Theorem (Spectral Theorem)

Let A be a symmetric $n \times n$ matrix. Then A has real eigenvalues, diagonalizes, and there exists an orthonormal basis $\{v_1, \dots, v_n\}$ of eigenvectors (in the Euclidean product); if V has columns $v_1, \dots, v_n$, and D is the diagonal matrix of eigenvalues (in the corresponding order) then A decomposes as

$$A = VDV^t.$$

The orthonormal basis of eigenvectors is not difficult to find:

- If u, v are eigenvectors of A of eigenvalues $\lambda \neq \mu$, then $u \perp v$.
- If the eigenvalues are all distinct, then normalizing the eigenvectors we obtain an orthonormal basis of eigenvectors.
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Characterization of scalar products

Let A be a symmetric matrix.

- A is the matrix of a scalar product if and only if all eigenvalues of A are positive.
- Sylvester criterion: if δ_1, $\delta_2, \ldots, \delta_n$ are the principal minors of A, A is the matrix of a scalar product if and only if $\delta_i > 0 \; \forall i$.

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Cross-product in $\mathbb{R}^3$

The **cross-product** between two vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ of $\mathbb{R}^3$ is the following vector (in standard basis)

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\begin{aligned}
(u_1, u_2, u_3) \times (v_1, v_2, v_3) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).
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Main properties:

- bilineal
- $v \times u = -u \times v$ (anti-commutative)
- $u \times v$ is orthogonal to both u and v
- $\langle u \times v, w \rangle = \det(u, v, w)$
- $\|u \times v\| = \|u\| \cdot \|v\| \cdot |\sin(\widehat{uv})|$
- $u \times v = 0 \Leftrightarrow u, v$ linearly dependent
- If u, v are orthogonal and normalized $\Rightarrow u, v, u \times v$ is b.o.n.

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Cross-product in $\mathbb{R}^3$

The **cross-product** between two vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ of $\mathbb{R}^3$ is the following vector (in standard basis)

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Main properties:

- bilineal
- $v \times u = -u \times v$ (anti-commutative)
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Outline

Orthogonal complement

The **orthogonal complement** to a given subspace F of a Euclidean space E is the subspace

$$F^{\perp} = \{u \in E \,|\, u \perp v \text{ for all } v \in F\}.$$

Properties when E has finite dimension:

- If $F = [v_1, \ldots, v_d] \Rightarrow F^{\perp} = \left\{ u \in E \;\middle|\; \begin{matrix} < u, v_1 >= 0 \\ \vdots \\ < u, v_d >= 0 \end{matrix} \right\}$
- $(F^{\perp})^{\perp} = F$, $\quad F \subseteq G \Leftrightarrow G^{\perp} \subseteq F^{\perp}$,
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Orthogonal projection

Let E be a Euclidean space of dimension n.

Theorem (Orthogonal Decomposition)

$E = F \oplus F^{\perp}$ for any subspace F. This is, any $v \in E$ can be written in a unique way as $v = w + w'$ where $w \in F$ and $w' \in F^{\perp}$.

- w is called the *orthogonal projection* of v on F and is denoted as $proj_F(v)$,
- w' is called the *orthogonal projection* of v on $F^{\perp}$ and is denoted as $proj_{F^{\perp}}(v)$.
- Thus, $v = proj_F(v) + proj_{F^{\perp}}(v)$ and $proj_F(v)$ is the unique vector of F such that $v - proj_F(v)$ belongs to $F^{\perp}$.
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Geometric interpretation

Proposition

The orthogonal projection of v on F is the vector of F that is closest to v; this is,

$$\|v - proj_F(v)\| = \min_{w \in F}\{\|v - w\|\}$$

(and this equals $\|proj_{F^\perp}(v)\|$). The orthogonal projection $proj_F(v)$ is the best approximation to v in F.

Computation of the orthogonal projection

Proposition

$proj_F(v)$ is the unique vector w that satisfies $w \in F$ and $v - w \in F^{\perp}$. If F has basis $u_1, \ldots, u_d$, then $proj_F(v)$ is the unique vector w such that

$$w = c_1 u_1 + \ldots c_d u_d \in F \quad and \left\{ \begin{array}{c} < u_1, w > = < u_1, v > \\ \vdots \\ < u_d, w > = < u_d, v > \end{array} \right.$$

Thus, $proj_F(v)$ is the vector $c_1 u_1 + \cdots + c_d u_d$ such that $c_1, \ldots, c_d$ are solution to the system

$$\begin{pmatrix} < u_1, u_1 > & \ldots & < u_1, u_d > \\ \vdots & \vdots & \vdots \\ < u_d, u_1 > & \ldots & < u_d, u_d > \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} < u_1, v > \\ \vdots \\ < u_d, v > \end{pmatrix}$$

Orthogonal projection with orthogonal basis

Corollary

If $\dim F = 1$, $F = [u]$, *then* $proj_F(v) = \frac{<v,u>}{<u,u>}\ u$.

Proposition

If $u_1, \ldots, u_d$ *is an orthogonal basis of* F *and* $v \in \mathbb{R}^n$, *then*

$$proj_F(v) = \frac{< v, u_1 >}{< u_1, u_1 >} u_1 + \cdots + \frac{< v, u_d >}{< u_d, u_d >} u_d.$$

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If $u_1, \ldots, u_d$ *is an b.o.n. of* F *and* $v \in \mathbb{R}^n$, *then*

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That is, the coordinates of $proj_F(v)$ *in the basis* $u_1, \ldots, u_d$ *are* $< v, u_1 >, \ldots, < v, u_d >$.

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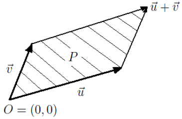
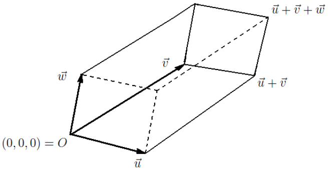
Determinants and volumes

From orthogonal projection and properties of cross product we can prove:

- In $\mathbb{R}^2$, the parallelogram determined by two vectors u, v has area equal to $|\det(u, v)|$.



- In $\mathbb{R}^3$, the parallelepiped determined by three vectors u, v, w has volume equal to $|\det(u, v, w)|$.



Outline

Singular value decomposition (SVD)

Theorem (Singular value decomposition)

Let A be a real $m \times n$ matrix. There there exists a decomposition $A = U \cdot D \cdot V^t$, where U is $m \times m$, V is $n \times n$, U, V are orthogonal and D is the following $m \times n$ matrix

$$D = \begin{pmatrix} \sigma_1 & & 0 & \cdots & 0 \\ & \ddots & & & \vdots \\ 0 & & \sigma_r & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

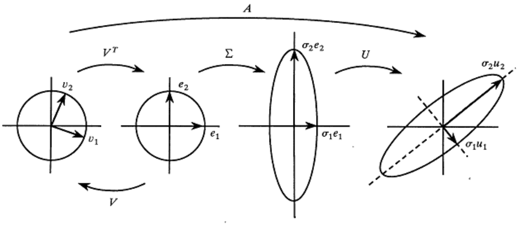
with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ and $r = \text{rank } A$.
$\sigma_1, \ldots, \sigma_r$ are called **singular values** *of A and are uniquely determined by A.*

Geometric interpretation of the SVD

If A is the standard matrix of a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, and we call $u_1, \ldots, u_m$, $v_1, \ldots, v_n$, the columns of U and V respectively, then D the matrix associated to f in orthonormal basis $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$:

$$A = M_e(f) = \underbrace{U}_{A_{u\to e}} * \underbrace{D}_{M_{v,u}(f)} * \underbrace{V^t}_{A_{e\to v}}$$

(note that $V^t = V^{-1} = A_{e\to v}$).



[T. Hern, C. Long]

How to get the SVD?

The singular values are determined by the matrix A:

$$A = UDV^t \Rightarrow A^tA = VD^tU^tUDV^t = VD^tDV^t$$

but U and V are not (almost determined in most cases). How do we compute the SVD?

(1) Diagonalize the symmetric matrix $S = A^t \cdot A$

(2) If $\lambda_1 \geq \cdots \geq \lambda_r$ are the non-zero eigenvalues of $S \Rightarrow$ the **singular values** are $\sigma_1 = \sqrt{\lambda_1}, \ldots \sigma_r = \sqrt{\lambda_r}$ (fact: A^tA always has non-negative eigenvalues).

(3) The columns of V are an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of S.

(4) $u_1 = \frac{1}{\sigma_1}Av_1, \ldots, u_r = \frac{1}{\sigma_r}Av_r$ are orthonormal vectors in $\mathbb{R}^m$ (which can be completed to an orthonormal basis of $\mathbb{R}^m$ if necessary) and they form the columns of U.

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The fundamental theorem of linear algebra

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let A be its standard matrix. Then $\mathbb{R}^n = \text{Nuc}(A) \oplus \text{Im}(A^t)$ ($\text{Im}(A^t)$=row space of A), $\mathbb{R}^m = \text{Im}(A) \oplus \text{Nuc}(A^t)$, these decompositions give orthogonal complements and there exist b.o.n.'s $v_1, \dots, v_n$ (of $\mathbb{R}^n$) and $u_1, \dots, u_m$ (of $\mathbb{R}^m$) such that

1. $\text{Im}(A) = [u_1, \dots, u_r]$
2. $\text{Nuc}(A) = [v_{r+1}, \dots, v_n]$
3. $\text{Im}(A^t) = [v_1, \dots, v_r]$
4. $\text{Nuc}(A^t) = [u_{r+1}, \dots, u_m]$

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The fundamental theorem of linear algebra

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let A be its standard matrix. Then $\mathbb{R}^n = \text{Nuc}(A) \oplus \text{Im}(A^t)$ ($\text{Im}(A^t)$=row space of A), $\mathbb{R}^m = \text{Im}(A) \oplus \text{Nuc}(A^t)$, these decompositions give orthogonal complements and there exist b.o.n.'s $v_1, \dots, v_n$ (of $\mathbb{R}^n$) and $u_1, \dots, u_m$ (of $\mathbb{R}^m$) such that

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2-norm of a matrix

To "measure" a linear map we measure how big the image of the unit sphere is under this map:

Definition

The **2-norm** of an $m \times n$ matrix A is

$$\|A\|_2 = \max_{\|x\|=1} \|Ax\|.$$

- This is a **matrix norm**: $\|A\|_2 \geq 0$, $\|A\|_2 = 0 \Leftrightarrow A = 0$, $\|cA\|_2 = |c|\|A\|_2$, $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$
- $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$
- $\|Av\| \leq \|A\|_2 \|v\| \ \forall v$.
- $\|AX\|_2 = \|A\|_2$ if X is an orthogonal matrix.
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Geometric consequence of the *SVD*:

Proposition

- $\|A\|_2 = \sigma_1$
- *The maximum is attained at* $\pm v_1$*:* $\max_{||x||=1} \|Ax\| = \|Av_1\|$.
- $\min_{||x||=1} \|Ax\| = \begin{cases} \sigma_n & \textit{if A has rank n, and is attained at } \pm v_n \\ 0 & \textit{if A has rank } < n \end{cases}$
- *If A is invertible,* $\|A^{-1}\|_2 = \frac{1}{\sigma_r}$.

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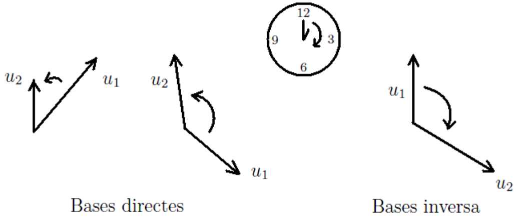
Outline

Orientation of $\mathbb{R}^2$

A basis u_1, u_2 of $\mathbb{R}^2$ has

- direct/positive orientation if the shortest rotation from u_1 to u_2 is counter-clockwise.
- inverse/negative if the shortest rotation from u_1 to u_2 is clockwise.



Orientations

In $\mathbb{R}^n$ we say that the standard basis has direct/positive orientation. For the other bases:

Definition

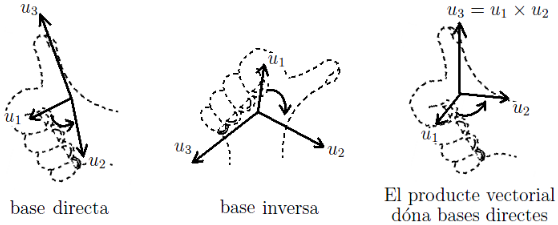
A basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ has direct/positive orientation, if

$$\det(u_1, u_2, \ldots, u_n) > 0$$

(computed in standard coordinates); otherwise, the basis is said to have inverse/negative orientation.

Geometric intuition in $\mathbb{R}^3$

In $\mathbb{R}^3$, to see if a basis u_1, u_2, u_3 has direct orientation we use the *right-hand rule*: put your thumb pointing to u_3 and if the sense of closing your hand is the same as the shortest from u_1 and u_2, then it has direct orientation.



- If $u, v \in \mathbb{R}^3$ are l.i. $\Rightarrow u, v, u \times v$ is a direct basis,

$$\det(u, v, u \times v) > 0.$$

Isometries

Definition

An endomorphism $f \in End(E)$ is an isometry if it preserves the scalar product,

$$\langle f(u), f(v)\rangle = \langle u, v\rangle \quad \forall u, v.$$

Ex: If A is an orthogonal matrix, then $x \overset{f}{\mapsto} Ax$ is an isometry.

Proposition

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map, the following are equivalent

- *f is an isometry*
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Direct/inverse isometries

Properties: if f is an isometry, then

- $\|f(u)\| = \|u\|$, for all $u \in E$.
- $d(f(x), f(y)) = d(x, y)$ for all x, y.
- angle between $f(u)$ and $f(v)$ = angle between u and v

Remark: if f is an isometry of $\mathbb{R}^n \Rightarrow \det(f) = \pm 1$ and if λ is an eigenvalue of f, then $|\lambda| = 1$.

- If $\det f = +1$ we say that it is a direct isometry (preserves orientation).
- If $\det f = +1$ we say that it is an inverse isometry (changes orientation).

Examples of isometries in $\mathbb{R}^2$

The following maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are isometries:

- $f=$ reflection/symmetry with along a line l passing through the origin, $l = [v]$. Then

$$f(x) = 2\frac{< v, x >}{< v, v >}v - x, \qquad M_e(f) = \frac{2}{< v, v >}vv^t - Id,$$

and taking u in $[v]$ this can be written as:

$$M_e(f) = Id - \frac{2}{< u, u >}u \cdot u^t.$$

- $f=$**rotation** counterclockwise of angle α with respect to the origin; then

$$M_e(f) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

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Classification of isometries in $\mathbb{R}^2$

Theorem

If f is an isometry of $\mathbb{R}^2$, then either

- $\det f = 1$ *and f is a counterclockwise rotation of angle α with respect to $(0,0)$ and in any direct b.o.n $\mathbf{u}$,*

$$M_{\mathbf{u}}(f) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

or

- $\det f = -1$ *and f is a reflection/symmetry along a line $[v] \ni (0,0)$; if $u \in [v]^\perp \Rightarrow$*

$$M_e(f) = Id - \frac{2}{< u, u >} u \cdot u^t \qquad M_{v,u}(f) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Example of isometry classification

Tell if the map $f(x,y) = (\frac{x-\sqrt{3}y}{2}, \frac{\sqrt{3}x+y}{2})$ is an isomtry and describe it.

- The standard matrix of f is $M = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$.
- M is orthogonal $\Rightarrow$ f is an isometry
- $det(M) = 1 \Rightarrow f$ is a rotation (by Theorem of Classification).
- To find angle α: according to the Theorem M must be of the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \Rightarrow \cos\alpha = 1/2, \sin\alpha = \sqrt{3}/2 \Rightarrow \alpha = \pi/3.$$

- Or also: $\alpha = \widehat{v, f(v)}\ \forall v \Rightarrow$ take any $v \in \mathbb{R}^2$, compute $\cos\alpha = \frac{<v,f(v)>}{\|v\|\|f(v)\|}$ and if $det(v, f(v)) > 0$ (resp. $det(v, f(v)) < 0$) take $\alpha \in [0, \pi]$ (resp. $\alpha \in [\pi, 2\pi]$).

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Examples of isometries in $\mathbb{R}^3$

$f =$ *rotation* of a certain angle with respect to a line $r \ni O = (0,0,0)$ (r is called *rotation axis*).

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Example of rotation:

f = rotation with $r = [e_3]$, oriented by e_3, and angle $\pi/3$. Then

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Matrix of a rotation

f = rotation with respect to $r = [u]$ (oriented by u) and angle θ. Take a positive b.o.n. $\mathbf{u} = u_1, u_2, u_3$ with $u_3 = \frac{u}{\|u\|}$ ("adapted b.o.n."), then

$$M_{\mathbf{u}}(f) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example: Axial symmetry

f = axial symmetry with respect to a line $r \ni O$.

- f = rotation of angle π with axis r (so $\det(f) = 1$).
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f = *specular reflexion/symmetry* along a plane $H \ni O$.

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- Example: if $H = \{z = 0\}$, then $f(x, y, z) = (x, y, -z)$
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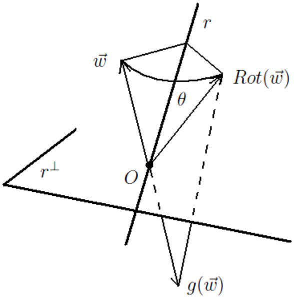
Example: Rotation followed by specular reflection

g = rotation R with axis $r = [u]$ and angle θ followed by a specular reflection along the orthogonal plane to r, $[u]^{\perp}$.



Classification of isometries in $\mathbb{R}^3$

Theorem

If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry, then either:

- **Case 1.** $\boxed{\det f = +1}$ *(direct isometry): f is a rotation with axis $r = [u] =$ VEPs of VAP 1.*
- **Case 2.** $\boxed{\det f = -1}$ *(indirect isometry): $f =$ rotation R of angle θ and axis $r = [u]$ (=VEPs of VAP -1), followed by a specular reflection S along plane $[u]^\perp$, $f = S \circ R$.*

Any isometry in $\mathbb{R}^3$ can be written as one of these 2 (case 1 if $\det f = 1$, case 2 if $\det f = -1$). Important: in case 2, the plane of reflection is orthogonal to rotation axis.

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Orient the axis by u and let θ be the angle of rotation. Take u of norm 1. Then if $\mathbf{u} = \{u_1, u_2, u_3 = u\}$ is a direct b.o.n (called basis adapted to f),

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$f = S \circ R$, R= rotation angle θ of axis $r = [u] \ni O$ (oriented by u), S specular reflection with respect to plane $H = [u]^{\perp}$.

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  - $[u]$: VEPs of VAP -1
  - Image of a vector v:

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$$M_e(f) = \begin{pmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}$$

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- Axis: VEPs of eigenvalue 1, $r = [u = (1,0,1)]$.
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SVD and rank approximation

Theorem (Eckhart-Young)

Let A be any matrix. If $A = UDV^t$ and the singular values of A are $\sigma_1, \ldots, \sigma_r$ then for any $k \leq r$,

$$M = U \begin{pmatrix} \sigma_1 & & 0 & \cdots & 0 \\ & \ddots & & & \vdots \\ 0 & & \sigma_k & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} V^t$$

is the matrix of rank k closest to A (in the sense that $\|A - M\|_2$ is minimal among matrices M of rank k). Note that $||A - M||_2 = \sigma_{k+1}$.

This is used in image compression, for example. Note that $A = \sigma_1 u_1 v_1^t + \sigma_2 u_2 v_2^t + \ldots + \sigma_r u_r v_r^t$.

Outline

Linear least squares approximation

Problem: $Ax = b$ might be incompatible due to measure errors in b, but we would still like to have an approximated solution:

$$\begin{array}{c}\text{Incompatible}\\ Ax = b \\ \text{system}\end{array} \quad \Leftrightarrow \quad b \notin \operatorname{Im}(A)$$

Want: $\tilde{x}$ such that $A\tilde{x}$ is as close to b as possible.

Definition

A **least squares solution of $Ax = b$** is a vector $\tilde{x}$ that minimizes $\|Ax - b\|$, that is

$$\|A\tilde{x} - b\| \leq \|Ax - b\| \text{ for all } x$$

Solution to the least squares problem

Solution given by Gauss (1801)

- Change b by the vector of $\mathrm{Im}(A)$ that is closest to b: the *orthogonal projection* of b in $\mathrm{Im}(A)$, $proj_{\mathrm{Im}(A)}(b)$.
- Then $\tilde{x}$ is a least squares solution $\Leftrightarrow$ $\tilde{x}$ is a solution of $Ax = proj_{\mathrm{Im}(A)}(b)$.
- If x is a least squares solution then it does not satisfy $Ax - b = \vec{0}$, but minimizes the norm $\|Ax - b\|$
- The **residual** measures how far $\tilde{x}$ is from a solution to the system:

  $$\text{residual} = A\tilde{x} - b \text{ (which is } = proj_{\mathrm{Im}(A)}(b) - b).$$

  norm of the residual: $\|A\tilde{x} - b\|$
- Important point: we do not need to compute $proj_{\mathrm{Im}(A)}(b)$ (see next slide).

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Theorem

- *$\tilde{x}$ is a least squares solution of $Ax = b$ if and only if it is a solution of the* **normal equations***:*

$$A^t Ax = A^t b.$$

- *If the rank of A equals the number of columns, then $A^t A$ is invertible and the least squares solution is unique and given by*

$$\tilde{x} = (A^t A)^{-1} A^t b$$

(although computing the inverse is not efficient)

- *If the original system is compatible, $\tilde{x}$ is a solution to the original system as well.*

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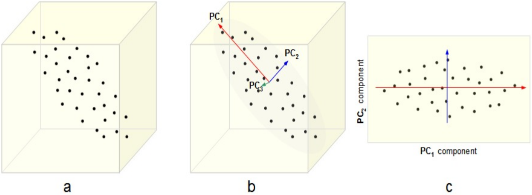
Outline

Principal component analysis

Goal: Given N data points in $\mathbb{R}^3$, $p_i = (x_i, y_i, z_i)$, $i = 1, \dots, N$ highly correlated, one wants to find $v_1 = (a, b, c)$ of norm 1 such that the set $\{t_i = ax_i + by_i + cz_i\}_i$ has maximum variance:



- Note that $proj_{[v_1]}(p_i) = t_i v_1$
- $v_1 = (a, b, c)$ is called the first principal component.
- Then one can look for $v_2 \in [v_1]^{\perp}$ (2nd principal component) maximizing variance of $proj_{[v_1]^{\perp}}(p_i)$.
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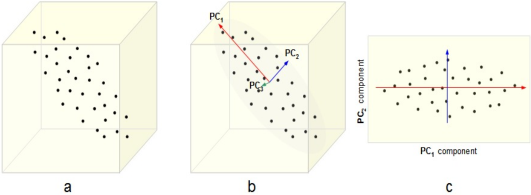

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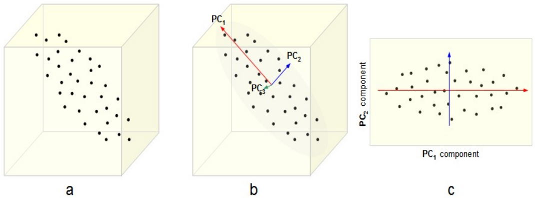

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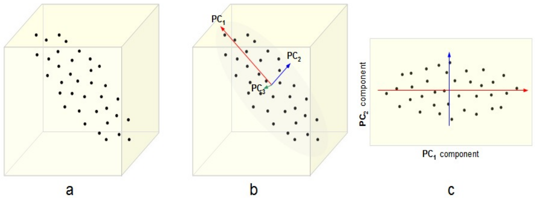

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Procedure

Assume that set $\{p_i\}$ is centered at the origin. Let

$$M = \begin{pmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{pmatrix} \text{ so that } \sum_i x_i = \sum_i y_i = \sum_i z_i = 0.$$

- Want $v_1 = (a, b, c)$ of norm 1 such that $\sum_i t_i^2 = \sum_i (ax_i + by_i + cz_i)^2 = ||Mv_1||$ is maximum.
- v_1 is the first column vector of V in the SVD: $M = UDV^t$.
- Then the matrix $M_2 = M - Mv_1 v_1^t$ has $proj_{[v_1]^\perp}(p_i)$ in its rows.
- $M_2 = \sigma_2 u_2 v_2^t + \ldots + \sigma_r u_r v_r^t$.
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Remarks:

- If the set $\{p_i\}$ is not centered at the origin we center it: let $(\bar{x}, \bar{y}, \bar{z}) = \sum_i (x_i, y_i, z_i)/N$, and consider

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  Proceed as before with this M and then sum $(\bar{x}, \bar{y}, \bar{z})$ to the final result.
- The matrix $M^t M$ is the *empirical covariance* matrix and the principal component v_1 is the dominant eigenvector of this matrix.
- The same can be done for clouds of points in $\mathbb{R}^n$.

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Scalar product in $\mathbb{C}^n$

Definition

In $\mathbb{C}^n$ the analogous to the dot product is the **standard hermitian product** $< u, v >$ of two vectors $u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$ is

$$< u, v > := u^t \overline{v} = x_1 \overline{y_1} + x_2 \overline{y_2} + \ldots + x_n \overline{y_n}.$$

Example:

$$u = \begin{pmatrix} 1 \\ i \\ 1-2i \end{pmatrix}, v = \begin{pmatrix} i \\ 0 \\ 3 \end{pmatrix} \Rightarrow < u, v > = (1 \; i \; 1-2i) \begin{pmatrix} -i \\ 0 \\ 3 \end{pmatrix} = 3 - 7i.$$

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1. $< u, u > \geq 0 \; \forall u$ (positive)
2. $< u, u >= 0 \Leftrightarrow u = 0$ (non-degenerate).
3. $< u, v >= \overline{< v, u >}$ ("hermitian").
4. "sesquilineal":
   - $< a_1 u_1 + a_2 u_2, v >= a_1 < u_1, v > + a_2 < u_2, v >$
   - $< u, a_1 v_1 + a_2 v_2 >= \overline{a_1} < u, v_1 > + \overline{a_2} < u, v_2 >$
5. In this case, the norm of a vector $u \in \mathbb{C}^n$ is $\|u\| = \sqrt{u^t \overline{u}} = \sqrt{|u_1|^2 + \ldots + |u_n|^2} \in \mathbb{R}$.
6. Orthonormal basis for this scalar product: same definition as before.
7. If we write the columns of a matrix $A = (v_1 \ldots v_d)$ then,

$$A^t \overline{A} = Id \quad \text{if and ony if } \{v_1, \ldots, v_d\} \text{ is an orthonormal basis.}$$

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   - $< a_1 u_1 + a_2 u_2, v >= a_1 < u_1, v > + a_2 < u_2, v >$;
   - $< u, a_1 v_1 + a_2 v_2 >= \overline{a_1} < u, v_1 > + \overline{a_2} < u, v_2 >$.
5. In this case, the norm of a vector $u \in \mathbb{C}^n$ is $\|u\| = \sqrt{u^t \overline{u}} = \sqrt{|u_1|^2 + \ldots + |u_n|^2} \in \mathbb{R}$.
6. Orthonormal basis for this scalar product: same definition as before.
7. If we write the columns of a matrix $A = (v_1 \ldots v_d)$ then,

   $$A^t \overline{A} = Id \quad \text{if and ony if } \{v_1, \ldots, v_d\} \text{ is an orthonormal basis.}$$

Unitary matrices

An $n \times n$ matrix that satisfies $\overline{A}^t A = Id$ is called a **unitary** matrix.

- If we call the columns $v_1, \dots, v_n$, $A = (v_1 \dots v_n)$, then,

$$\overline{A}^t A = Id \quad \text{if and ony if } \{v_1, \dots, v_n\} \text{ is an orthonormal basis.}$$

- A is unitary if and only if $A^{-1} = \overline{A}^t$.
- A is unitary $\Rightarrow |\det A| = 1$.
- If A is unitary, then the corresponding endomorphism preserves norms (preserves the measure of vectors):

$$||Ax|| = ||x|| \text{ for all } x$$

- A also preserves dot products and angles (and hence preserves orthogonality) and so it is a transformation that does not deform objects.

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Outline

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