# Linear maps

*Bioinformatics Degree*
*Algebra*

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# Outline

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# Definition

A linear map (or linear transformation) between $\mathbb{R}^n$ and $\mathbb{R}^m$ is a map that preserves linear combinations. More precisely,

## Definition

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map if

1. $f(u + v) = f(u) + f(v)$ for all $u, v \in \mathbb{R}^n$, and
2. $f(cv) = cf(v)$ for any $c \in \mathbb{R}$ and any $v \in \mathbb{R}^n$.

Examples

- $f(x) = 5x$
- $f(x, y) = (x + 2y, 3x, y - x)$
- $f(x, y, z) = x - 3y + z$
- $f(x, y, z) = (x, 2y, z)$
- $f(x_1, \dots, x_n) = (0, \dots, 0)$ is the *zero* map.
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# Properties of linear maps

If $f$ is a linear map, then:

- $f(0) = 0$
- $f(c_1 v_1 + \cdots + c_k v_k) = c_1 f(v_1) + \cdots + c_k f(v_k)$. This is equivalent to properties 1 and 2.
- $f$ is determined by the image of a basis.

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- $f$ is determined by the image of a basis.

## Standard matrix of a linear map

When we use coordinates in the standard bases, then linear maps

$$\begin{array}{rcl} f:\mathbb{R}^n & \to & \mathbb{R}^m \\ (x_1,\ldots,x_n) & \mapsto & (a_{11}x_1+\ldots+a_{1n}x_n,\cdots,a_{m1}x_1+\ldots+a_{mn}x_n) \end{array}$$

can be written in matrix notation as follows:

$$u \mapsto M(f)u, \text{ where } M(f) = (a_{i,j}).$$

The matrix $M(f)$ is called the standard matrix of the linear map $f$. It is a $m \times n$ matrix, and its columns are the vectors $f(e_i)$, $i = 1,\ldots,n$:

$$M(f) = \big(f(e_1),\cdots,f(e_n)\big)$$

# Outline

# Definitions

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a map.

- $f$ is injective if different vectors always have different images ($u \neq v$ implies $f(u) \neq f(v)$).
- $f$ is surjective if every vector $v$ in $\mathbb{R}^m$ is the image of some vector $u \in \mathbb{R}^n$, $v = f(u)$.
- $f$ is bijective if it is at the same time injective and surjective. A bijective linear map is called an *isomorphism*.

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## Null space of a linear map

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let $A$ be its standard matrix.

### Definition

The null space (or kernel) of a $f$ is the subspace

$$\mathrm{Null}(f) = \{v \in \mathbb{R}^n \mid f(v) = 0\} = \{x \in \mathbb{R}^n \mid Ax = 0\} = f^{-1}(0).$$

The null space of $f$ is the solution space of the homogeneous linear system $M(f)\,x = 0$. Hence, $\dim \mathrm{Null}(f) = n - \mathrm{rank}(M(f))$.

### Proposition

*The following are equivalent:*

1. *$f$ is injective;*
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# Range of a linear map

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let $A$ be its standard matrix.

## Definition

The range (or image) of $f$ is the vector subspace given by all the images of vectors, that is,
$\mathrm{R}(f) = \{v \in \mathbb{R}^m \mid v = f(u) \text{ for some u } \in \mathbb{R}^n\}$.

The range of $f$ is the vector space generated by the columns of $M(f)$, and $\dim \mathrm{R}(f) = \mathrm{rank}(M(f))$.

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# Dimensions of Nullspace and Range

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let $A$ be its standard matrix.

## Theorem (The rank theorem)

$$\dim \mathrm{Null}(f) + \dim \mathrm{R}(f) = n$$

Consequences:

- $f$ is injective $\Leftrightarrow$ $\mathrm{Null}(f) = 0 \Leftrightarrow \dim \mathrm{R}(f) = n$.
- $f$ is surjective $\Leftrightarrow \dim \mathrm{R}(f) = m \Leftrightarrow \dim \mathrm{Null}(f) = n - m$.
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# Composition of linear maps

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ be linear maps, the composition of $g$ with $f$ is the linear map $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ defined as:

$$\begin{array}{rcccc} g \circ f : \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^p \\ v & \mapsto & f(v) & \mapsto & g(f(v)) \end{array}.$$

If $M(f)$ and $M(g)$ are the standard matrix of $f$ and $g$ respectively, then the standard matrix of $g \circ f$ is

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# Inverse linear maps

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If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be linear maps, we say that $g$ is the inverse of $f$ (denoted as $g = f^{-1}$) if $g \circ f = f \circ g = Id$.

## Proposition

a) *If $f$ has standard matrix $M(f)$ and $g$ is the inverse of $f$, then the standard matrix of $g$ is $M(g) = M(f^{-1}) = (M(f))^{-1}$.*

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If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be linear maps, we say that $g$ is the inverse of $f$ (denoted as $g = f^{-1}$) if $g \circ f = f \circ g = Id$.

## Proposition

a) *If $f$ has standard matrix $M(f)$ and $g$ is the inverse of $f$, then the standard matrix of $g$ is $M(g) = M(f^{-1}) = (M(f))^{-1}$.*

b) *The following are equivalent:*
   1. *$f$ is invertible ($f$ is bijective)*
   2. $\det M(f) \neq 0$
   3. $\text{rank}(M(f)) = n$
   4. *$f$ is injective*
   5. *$f$ is surjective*

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# Outline

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map, let $B = \{u_1, \ldots, u_n\}$ be a basis of $\mathbb{R}^n$ and $C = \{v_1, \ldots, v_m\}$ be a basis of $\mathbb{R}^m$.

## Definition

The matrix of $f$ in bases $B, C$ has as columns the coordinates of $f(u_1), \ldots, f(u_n)$ in the basis $C$:

$$M_{B,C}(f) = \left(f(u_1)_{\mathbf{C}} \cdots f(u_n)_{\mathbf{C}}\right)$$

Properties:

- $M_{B,C}(f)(w_B) = (f(w))_C$ .
- If $B$ and $C$ are the standard bases, $M_{B,C}(f) = M(f)$.
- If we compute $\text{Null}(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of $\text{Null}(f)$ expressed in the basis $B$.
- If we compute $\text{R}(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of $\text{R}(f)$ expressed in the basis $C$.

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- If we compute Null$(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of Null$(f)$ expressed in the basis $B$.
- If we compute R$(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of R$(f)$ expressed in the basis $C$.

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- If $B$ and $C$ are the standard bases, $M_{B,C}(f) = M(f)$.
- If we compute Null$(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of Null$(f)$ expressed in the basis $B$.
- If we compute R$(f)$ using $M_{B,C}(f)$ instead of $M(f)$, we obtain the vectors of R$(f)$ expressed in the basis $C$.

# Change of basis

If $A_{C\to B}$ is the change-of-basis matrix from $C$ to $B$ (the standard basis of $\mathbb{R}^n$), and $A_{C'\to B'}$ is the change-of-basis matrix from $C'$ to $B'$ (the standard basis of $\mathbb{R}^m$), then:

$$M_{B,B'}(f) = A_{C'\to B'} M_{C,C'}(f) A_{C\to B}^{-1},$$
$$M_{C,C'}(f) = A_{C'\to B'}^{-1} M_{B,B'}(f) A_{C\to B}.$$