# SUPPLEMENTARY MATERIALS: A NOVEL ALGEBRAIC APPROACH TO TIME-REVERSIBLE EVOLUTIONARY MODELS*

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## SM1. Gluing parameters.

*Proof of Lemma* 4.5. By rooting the tree $T'$ at $s$ we have

$$p^T_{\boldsymbol{i}_\alpha,k,\boldsymbol{i}_\beta} = \sum_{\boldsymbol{j}\in ext(\boldsymbol{i}_\alpha,k,\boldsymbol{i}_\beta)} \pi_k \prod_{e\in E(T')} M^e_{j_{p(e)},j_{c(e)}} =$$

$$= \frac{1}{\pi_k} \sum_{\substack{v\,\in\, Int(T_1)\,\cup\, Int(T_2)\\ j_v\,\in\,\Sigma}} \left(\pi_k \prod_{e\in E(T_1)} M^e_{j_{p(e)},j_{c(e)}}\right)\left(\pi_k \prod_{e\in E(T_2)} M^e_{j_{p(e)},j_{c(e)}}\right) =$$

$$= \frac{1}{\pi_k}\left(\sum_{\substack{v\,\in\, Int(T_1)\\ j_v\,\in\,\Sigma}} \pi_k \prod_{e\in E(T_1)} M^e_{j_{p(e)},j_{c(e)}}\right)\left(\sum_{\substack{v\,\in\, Int(T_2)\\ j_v\,\in\,\Sigma}} \pi_k \prod_{e\in E(T_2)} M^e_{j_{p(e)},j_{c(e)}}\right)$$

and the claim follows. ∎

## SM2. Computations.

**SM2.1. Computations for the tripod.** The list of 9 quadric binomials, 29 cubic binomials, and 3 quintic binomials generating $I_T$ for a tripod $T$ mentioned in Proposition 5.3 is:

### Generators of degree 2

**(1)** $\bar{p}_{222}\bar{p}_{441} - \frac{\pi_{34}-\pi_{12}}{\pi_{34}}\bar{p}_{221}\bar{p}_{442}$

**(2)** $\bar{p}_{332}\bar{p}_{441} + \frac{\pi_{12}}{\pi_{34}}\bar{p}_{331}\bar{p}_{442}$

**(3)** $\bar{p}_{222}\bar{p}_{414} - \frac{\pi_{34}-\pi_{12}}{\pi_{34}}\bar{p}_{212}\bar{p}_{424}$

**(4)** $\bar{p}_{323}\bar{p}_{414} + \frac{\pi_{12}}{\pi_{34}}\bar{p}_{313}p_{424}$

**(5)** $\bar{p}_{144}\bar{p}_{222} - \frac{\pi_{34}-\pi_{12}}{\pi_{34}}\bar{p}_{122}\bar{p}_{244}$

**(6)** $\bar{p}_{332}\bar{p}_{221} - \frac{\pi_{12}}{\pi_{12}-\pi_{34}}\bar{p}_{331}\bar{p}_{222}$

**(7)** $\bar{p}_{144}\bar{p}_{233} + \frac{\pi_{12}}{\pi_{34}}\bar{p}_{133}\bar{p}_{244}$

**(8)** $\bar{p}_{122}\bar{p}_{233} - \frac{\pi_{12}}{\pi_{12}-\pi_{34}}\bar{p}_{133}\bar{p}_{222}$

**(9)** $\bar{p}_{323}\bar{p}_{212} - \frac{\pi_{12}}{\pi_{12}-\pi_{34}}\bar{p}_{313}\bar{p}_{222}$

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*Supplementary material for SIAP MS#M160530.
https://doi.org/10.1137/23M1605302

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**Generators of degree 3**

**(10)** $\bar p_{144}\bar p_{424}\bar p_{442} - \frac{\pi_1\pi_2\pi_{34}}{(\pi_1-\pi_2)^2}\bar p_{122}\bar p_{444}^2$

**(11)** $\bar p_{244}\bar p_{414}\bar p_{442} - \frac{\pi_1\pi_2\pi_{34}}{(\pi_1-\pi_2)^2}\bar p_{212}\bar p_{444}^2$

**(12)** $\bar p_{244}\bar p_{424}\bar p_{441} - \frac{\pi_1\pi_2\pi_{34}}{(\pi_1-\pi_2)^2}\bar p_{221}\bar p_{444}^2$

**(13)** $\bar p_{144}\bar p_{414}\bar p_{441} - \frac{\pi_1\pi_2\pi_{12}}{(\pi_1-\pi_2)^2}\bar p_{111}\bar p_{444}^2$

**(14)** $\bar p_{122}\bar p_{414}\bar p_{441} - \frac{\pi_{12}}{\pi_{34}}\bar p_{111}\bar p_{424}\bar p_{442}$

**(15)** $\bar p_{144}\bar p_{212}\bar p_{441} - \frac{\pi_{12}}{\pi_{34}}\bar p_{111}\bar p_{244}\bar p_{442}$

**(16)** $\bar p_{122}\bar p_{212}\bar p_{441} - \frac{\pi_{12}}{\pi_{34}-\pi_{12}}\bar p_{111}\bar p_{222}\bar p_{442}$

**(17)** $\bar p_{133}\bar p_{212}\bar p_{441} + \bar p_{111}\bar p_{233}\bar p_{442}$

**(18)** $\bar p_{122}\bar p_{313}\bar p_{441} + \bar p_{111}\bar p_{323}\bar p_{442}$

**(19)** $\bar p_{122}\bar p_{313}\bar p_{441} + \bar p_{111}\bar p_{323}\bar p_{442}$

**(20)** $\bar p_{144}\bar p_{221}\bar p_{414} - \frac{\pi_{12}}{\pi_{34}}\bar p_{111}\bar p_{244}\bar p_{424}$

**(21)** $\bar p_{122}\bar p_{221}\bar p_{414} - \frac{\pi_{12}}{\pi_{34}-\pi_{12}}\bar p_{111}\bar p_{222}\bar p_{424}$

**(22)** $\bar p_{133}\bar p_{221}\bar p_{414} + \bar p_{111}\bar p_{233}\bar p_{424}$

**(23)** $\bar p_{122}\bar p_{331}\bar p_{414} + \bar p_{111}\bar p_{332}\bar p_{424}$

**(24)** $\bar p_{144}\bar p_{212}\bar p_{221} - \frac{\pi_{12}}{\pi_{34}-\pi_{12}}\bar p_{111}\bar p_{222}\bar p_{244}$

**(25)** $\bar p_{122}\bar p_{212}\bar p_{221} - \frac{\pi_{12}\pi_{34}}{(\pi_{12}-\pi_{34})^2}\bar p_{111}\bar p_{222}^2$

**(26)** $\bar p_{133}\bar p_{212}\bar p_{221} - \frac{\pi_{34}}{\pi_{12}-\pi_{34}}\bar p_{111}\bar p_{233}\bar p_{222}$

**(27)** $\bar p_{144}\bar p_{313}\bar p_{221} + \bar p_{111}\bar p_{323}\bar p_{244}$

**(28)** $\bar p_{122}\bar p_{313}\bar p_{221} - \frac{\pi_{34}}{\pi_{12}-\pi_{34}}\bar p_{111}\bar p_{323}\bar p_{222}$

**(29)** $\bar p_{331}\bar p_{323}\bar p_{233} - \frac{\pi_3\pi_4\pi_{12}}{(\pi_3-\pi_4)^2}\bar p_{333}^2\bar p_{221}$

**(30)** $\bar p_{111}\bar p_{323}\bar p_{233} - \frac{\pi_{12}}{\pi_{34}}\bar p_{133}\bar p_{313}\bar p_{221}$

**(31)** $\bar p_{144}\bar p_{331}\bar p_{212} + \bar p_{111}\bar p_{332}\bar p_{244}$

**(32)** $\bar p_{122}\bar p_{331}\bar p_{212} - \frac{\pi_{34}}{\pi_{12}-\pi_{34}}\bar p_{111}\bar p_{332}\bar p_{222}$

**(33)** $\bar p_{133}\bar p_{331}\bar p_{212} - \frac{\pi_{34}}{\pi_{12}}\bar p_{111}\bar p_{332}\bar p_{233}$

**(34)** $\bar p_{122}\bar p_{333}^2 - \frac{(\pi_3-\pi_4)^2}{\pi_3\pi_4\pi_{12}}\bar p_{133}\bar p_{332}\bar p_{323}$

**(35)** $\bar p_{122}\bar p_{313}\bar p_{331} - \frac{\pi_{34}}{\pi_{12}}\bar p_{111}\bar p_{332}\bar p_{323}$

**(36)** $\bar p_{133}\bar p_{313}\bar p_{331} - \frac{\pi_3\pi_4\pi_{34}}{(\pi_3-\pi_4)^2}\bar p_{111}\bar p_{333}^2$

**(37)** $\bar p_{244}\bar p_{424}\bar p_{442} - \frac{\pi_1\pi_2(\pi_3+\pi_4)^2}{(\pi_1-\pi_2)^2(\pi_{34}-\pi_{12})}\bar p_{222}\bar p_{444}^2$

**(38)** $\bar p_{332}\bar p_{323}\bar p_{233} - \frac{\pi_3\pi_4\pi_{12}^2}{(\pi_3-\pi_4)^2(\pi_{12}^2-\pi_{34}^4)}\bar p_{333}^2\bar p_{222}$

**(39)** $\bar p_{333}^2\bar p_{212} - \frac{(\pi_{12}-\pi_{34})(\pi_3-\pi_4)^2}{\pi_3\pi_4\pi_{12}}\bar p_{313}\bar p_{332}\bar p_{233}$

**Generators of degree 5**

**(40)** $\bar p_{144}\bar p_{333}^2\bar p_{414}\bar p_{442} + \frac{\pi_1\pi_2(\pi_3-\pi_4)^2}{\pi_3\pi_4(\pi_1-\pi_2)^2}\bar p_{133}\bar p_{313}\bar p_{332}\bar p_{444}^2$

**(41)** $\bar p_{144}\bar p_{333}^2\bar p_{424}\bar p_{441} + \frac{\pi_1\pi_2(\pi_3-\pi_4)^2}{\pi_3\pi_4(\pi_1-\pi_2)^2}\bar p_{133}\bar p_{331}\bar p_{323}\bar p_{444}^2$

**(42)** $\bar p_{333}^2\bar p_{244}\bar p_{414}\bar p_{441} + \frac{\pi_1\pi_2(\pi_3-\pi_4)^2}{\pi_3\pi_4(\pi_1-\pi_2)^2}\bar p_{313}\bar p_{331}\bar p_{233}\bar p_{444}^2$

**SM2.2. Computations for quartets.** In the following lemma we compute the coordinates in $B_n$ for any point in evolving on $T = 12|34$ under the TN93 model. The coordinates for points on the variety of $T = 13|24$ or $14|23$ can be obtained by correspondingly permuting the subindices of the coordinates. This result is used in Proposition 5.11.

LEMMA SM2.1. *Consider the tree $T = 12|34$ evolving under the evolutionary model $\mathcal{M} = TN93$. In the basis $B_4$, any tensor $p = \varphi_T(\Lambda^1, \ldots, \Lambda^5)$ has monomial coordinates except for $\bar p_{iiii}$ and $\bar p_{iijj}$ for $i, j \in \{2, 3, 4\}$. These coordinates can be obtained by* (4.4) *from the following expressions of the coordinates of $q = \varphi_T(Id, Id, Id, Id, \Lambda)$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (non listed coordinates are zero):*

**(1)** $\bar q_{1111} = \lambda_1$

**(2)** $\bar q_{1122} = \bar q_{2211} = \frac{1}{\pi_{12}\pi_{34}}\lambda_1$

**(3)** $\bar q_{1133} = \bar q_{3311} = \frac{\pi_{34}}{\pi_3\pi_4}\lambda_1$

**(4)** $\bar{q}_{1144} = \bar{q}_{4411} = \frac{\pi_{12}}{\pi_1\pi_2}\lambda_1$

**(5)** $\bar{q}_{1212} = \bar{q}_{1221} = \bar{q}_{2112} = \bar{q}_{2121} = \frac{1}{\pi_{12}\pi_{34}}\lambda_2$

**(6)** $\bar{q}_{1222} = \bar{q}_{2122} = \bar{q}_{2212} = \bar{q}_{2221} = \frac{\pi_{34}-\pi_{12}}{\pi_{34}^2\pi_{12}^2}\lambda_2$

**(7)** $\bar{q}_{1233} = \bar{q}_{2133} = \bar{q}_{3312} = \bar{q}_{3321} = -\frac{1}{\pi_3\pi_4}\lambda_2$

**(8)** $\bar{q}_{1244} = \bar{q}_{2144} = \bar{q}_{4412} = \bar{q}_{4421} = \frac{1}{\pi_1\pi_2}\lambda_2$

**(9)** $\bar{q}_{1313} = \bar{q}_{1331} = \bar{q}_{3113} = \bar{q}_{3131} = \frac{\pi_{34}}{\pi_3\pi_4}\lambda_3$

**(10)** $\bar{q}_{1323} = \bar{q}_{1332} = \bar{q}_{2313} = \bar{q}_{2331} = \bar{q}_{3123} = \bar{q}_{3132} = \bar{q}_{3213} = \bar{q}_{3231} = -\frac{1}{\pi_3\pi_4}\lambda_3$

**(11)** $\bar{q}_{1333} = \bar{q}_{3133} = \bar{q}_{3313} = \bar{q}_{3331} = \frac{\pi_{34}(\pi_4-\pi_3)}{\pi_3^2\pi_4^2}\lambda_3$

**(12)** $\bar{q}_{1414} = \bar{q}_{1441} = \bar{q}_{4114} = \bar{q}_{4141} = \frac{\pi_{12}}{\pi_1\pi_2}\lambda_4$

**(13)** $\bar{q}_{1424} = \bar{q}_{1442} = \bar{q}_{2414} = \bar{q}_{2441} = \bar{q}_{4124} = \bar{q}_{4142} = \bar{q}_{4214} = \bar{q}_{4241} = \frac{1}{\pi_1\pi_2}\lambda_4$

**(14)** $\bar{q}_{1444} = \bar{q}_{4144} = \bar{q}_{4414} = \bar{q}_{4441} = \frac{\pi_{12}(\pi_2-\pi_1)}{\pi_1^2\pi_2^2}\lambda_4$

**(15)** $\bar{q}_{2222} = \frac{1}{\pi_{12}^2\pi_{34}^2}\lambda_1 + \frac{(\pi_{12}-\pi_{34})^2}{\pi_{12}^3\pi_{34}^3}\lambda_2$

**(16)** $\bar{q}_{2233} = \bar{q}_{3322} = \frac{1}{\pi_{12}\pi_3\pi_4}\lambda_1 + \frac{\pi_{12}-\pi_{34}}{\pi_3\pi_4\pi_{12}\pi_{34}}\lambda_2$

**(17)** $\bar{q}_{2244} = \bar{q}_{4422} = \frac{1}{\pi_1\pi_2\pi_{34}}\lambda_1 + \frac{\pi_{34}-\pi_{12}}{\pi_1\pi_2\pi_{12}\pi_{34}}\lambda_2$

**(18)** $\bar{q}_{2323} = \bar{q}_{2332} = \bar{q}_{3223} = \bar{q}_{3232} = \frac{1}{\pi_3\pi_4\pi_{34}}\lambda_3$

**(19)** $\bar{q}_{2333} = \bar{q}_{3233} = \bar{q}_{3323} = \bar{q}_{3332} = \frac{\pi_3-\pi_4}{\pi_3^2\pi_4^2}\lambda_3$

**(20)** $\bar{q}_{2424} = \bar{q}_{2442} = \bar{q}_{4224} = \bar{q}_{4242} = \frac{1}{\pi_1\pi_2\pi_{12}}\lambda_4$

**(21)** $\bar{q}_{2444} = \bar{q}_{4244} = \bar{q}_{4424} = \bar{q}_{4442} = \frac{\pi_2-\pi_1}{\pi_1^2\pi_2^2}\lambda_4$

**(22)** $\bar{q}_{3333} = \frac{\pi_{34}^2}{\pi_3^2\pi_4^2}\lambda_1 + \frac{\pi_{12}\pi_{34}}{\pi_3^2\pi_4^2}\lambda_2 + \frac{\pi_{34}(\pi_3-\pi_4)^2}{\pi_3^3\pi_4^3}\lambda_3$

**(23)** $\bar{q}_{3344} = \bar{q}_{4433} = \frac{\pi_{12}\pi_{34}}{\pi_1\pi_2\pi_3\pi_4}\lambda_1 - \frac{\pi_{12}\pi_{34}}{\pi_1\pi_2\pi_3\pi_4}\lambda_2$

**(24)** $\bar{q}_{4444} = \frac{\pi_{12}^2}{\pi_1^2\pi_2^2}\lambda_1 + \frac{\pi_{12}\pi_{34}}{\pi_1^2\pi_2^2}\lambda_2 + \frac{\pi_{12}(\pi_1-\pi_2)^2}{\pi_1^3\pi_2^3}\lambda_4$

*Proof.* Let $T_1$ be the tripod with leaves $l_1, l_2, l_s$ and $T_2$ be the tripod with leaves $l_s, l_3, l_4$ so that $T$ is the gluing $T_1 * T_2$. Define tensors $q^{T_1} = \varphi_{T_1}(Id, Id, \Lambda)$ and $\bar{p}^{T_2} = \varphi_{T_2}(Id, Id, Id)$ evolving on $T_1$ and $T_2$ respectively; hence $q = q_1 * q_2$.

By Theorem 4.6 the coordinates of $q$ in the basis $B_n$ can be obtained by the matrix product

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 11 | * | 0 | 0 | 0 |
| 22 | * | * | 0 | 0 |
| 33 | * | * | * | 0 |
| 44 | * | * | 0 | * |
| 12 | 0 | * | 0 | 0 |
| 21 | 0 | * | 0 | 0 |
| 13 | 0 | 0 | * | 0 |
| 31 | 0 | 0 | * | 0 |
| 23 | 0 | 0 | * | 0 |
| 32 | 0 | 0 | * | 0 |
| 14 | 0 | 0 | 0 | * |
| 41 | 0 | 0 | 0 | * |
| 24 | 0 | 0 | 0 | * |
| 42 | 0 | 0 | 0 | * |
| 34 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 |

| | 11 | 22 | 33 | 44 | 12 | 21 | 13 | 31 | 23 | 32 | 14 | 41 | 24 | 42 | 34 | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | * | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | * | * | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | * | 0 | 0 | 0 | * | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | * | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | 0 | 0 |

in the sense that $q_{i_1 i_2 i_3 i_4}$ is the product of row $(i_1, i_2)$ of the first matrix and column $(i_3, i_4)$ of the second matrix up to multiplication by a scalar product. As $*$ entries are monomials in (actually they are linear entries in some $\lambda_k$ in the first matrix and scalar entries in the second), non-monomial entries only appear when multiplying rows and columns with indices $(2, 2), (3, 3), (4, 4)$. From this matrix product we get the entries that appear in the list. ∎

## SM3. Flattening matrices.

TABLE SM1
*Flattening matrix of $\bar{q}$ for a quartet $T = \{12 \mid 34\}$.*

| | $(1,1)$ | $(1,4)$ | $(4,1)$ | $(2,4)$ | $(4,2)$ | $(4,4)$ | $(2,2)$ | $(1,2)$ | $(2,1)$ | $(3,3)$ | $(1,3)$ | $(3,1)$ | $(2,3)$ | $(3,2)$ | $(3,4)$ | $(4,3)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $\bar{q}_{1111}$ | 0 | 0 | 0 | 0 | $\bar{q}_{1144}$ | $\bar{q}_{1122}$ | 0 | 0 | $\bar{q}_{1133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,4)$ | 0 | $\bar{q}_{1414}$ | $\bar{q}_{1414}$ | $\bar{q}_{1424}$ | $\bar{q}_{1424}$ | $\bar{q}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,1)$ | 0 | $\bar{q}_{1414}$ | $\bar{q}_{1414}$ | $\bar{q}_{1424}$ | $\bar{q}_{1424}$ | $\bar{q}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,4)$ | 0 | $\bar{q}_{1424}$ | $\bar{q}_{1424}$ | $\bar{q}_{2424}$ | $\bar{q}_{2424}$ | $\bar{q}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,2)$ | 0 | $\bar{q}_{1424}$ | $\bar{q}_{1424}$ | $\bar{q}_{2424}$ | $\bar{q}_{2424}$ | $\bar{q}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,4)$ | $\bar{q}_{1144}$ | $\bar{q}_{1444}$ | $\bar{q}_{1444}$ | $\bar{q}_{2444}$ | $\bar{q}_{2444}$ | $\bar{q}_{4444}$ | $\bar{q}_{2244}$ | $\bar{q}_{1244}$ | $\bar{q}_{1244}$ | $\bar{q}_{3344}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,2)$ | $\bar{q}_{1122}$ | 0 | 0 | 0 | 0 | $\bar{q}_{2244}$ | $\bar{q}_{2222}$ | $\bar{q}_{1222}$ | $\bar{q}_{1222}$ | $\bar{q}_{3322}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,2)$ | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{1244}$ | $\bar{q}_{1222}$ | $\bar{q}_{1212}$ | $\bar{q}_{1212}$ | $\bar{q}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,1)$ | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{1244}$ | $\bar{q}_{1222}$ | $\bar{q}_{1212}$ | $\bar{q}_{1212}$ | $\bar{q}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(3,3)$ | $\bar{q}_{1133}$ | 0 | 0 | 0 | 0 | $\bar{q}_{3344}$ | $\bar{q}_{3322}$ | $\bar{q}_{1233}$ | $\bar{q}_{1233}$ | $\bar{q}_{3333}$ | $\bar{q}_{1333}$ | $\bar{q}_{1333}$ | $\bar{q}_{3233}$ | $\bar{q}_{3233}$ | 0 | 0 |
| $(1,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{1333}$ | $\bar{q}_{1313}$ | $\bar{q}_{1313}$ | $\bar{q}_{1323}$ | $\bar{q}_{1323}$ | 0 | 0 |
| $(3,1)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{1333}$ | $\bar{q}_{1313}$ | $\bar{q}_{1313}$ | $\bar{q}_{1323}$ | $\bar{q}_{1323}$ | 0 | 0 |
| $(2,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{3233}$ | $\bar{q}_{1323}$ | $\bar{q}_{1323}$ | $\bar{q}_{3232}$ | $\bar{q}_{3232}$ | 0 | 0 |
| $(3,2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{q}_{3233}$ | $\bar{q}_{1323}$ | $\bar{q}_{1323}$ | $\bar{q}_{3232}$ | $\bar{q}_{3232}$ | 0 | |
| $(3,4)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE SM2

*Flattening matrix of $\bar{p}$ for a quartet $T = \{12 \mid 34\}$. The highlighted submatrices have rank 1 as stated in Lemma 5.13. The entries highlighted in dark gray are generically non-zero. Therefore, the 18 quadratic edge invariants in Theorem 5.14 arise from considering all 2-minors of the grey submatrices containing the dark grey entry.*

| | (1,1) | (1,4) | (4,1) | (2,4) | (4,2) | (4,4) | (2,2) | (1,2) | (2,1) | (3,3) | (1,3) | (3,1) | (2,3) | (3,2) | (3,4) | (4,3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1,1) | $\bar{p}_{1111}$ | 0 | 0 | 0 | 0 | $\bar{p}_{1144}$ | $\bar{p}_{1122}$ | 0 | 0 | $\bar{p}_{1133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,4) | 0 | $\bar{p}_{1414}$ | $\bar{p}_{1441}$ | $\bar{p}_{1424}$ | $\bar{p}_{1442}$ | $\bar{p}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,1) | 0 | $\bar{p}_{4114}$ | $\bar{p}_{4141}$ | $\bar{p}_{4124}$ | $\bar{p}_{4142}$ | $\bar{p}_{4144}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,4) | 0 | $\bar{p}_{2414}$ | $\bar{p}_{2441}$ | $\bar{p}_{2424}$ | $\bar{p}_{2442}$ | $\bar{p}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,2) | 0 | $\bar{p}_{4214}$ | $\bar{p}_{4241}$ | $\bar{p}_{4224}$ | $\bar{p}_{4242}$ | $\bar{p}_{4244}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,4) | $\bar{p}_{4411}$ | $\bar{p}_{4414}$ | $\bar{p}_{4441}$ | $\bar{p}_{4424}$ | $\bar{p}_{4442}$ | $\bar{p}_{4444}$ | $\bar{p}_{4422}$ | $\bar{p}_{4412}$ | $\bar{p}_{4421}$ | $\bar{p}_{4433}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,2) | $\bar{p}_{2211}$ | 0 | 0 | 0 | 0 | $\bar{p}_{2244}$ | $\bar{p}_{2222}$ | $\bar{p}_{2212}$ | $\bar{p}_{2221}$ | $\bar{p}_{2233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,2) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1244}$ | $\bar{p}_{1222}$ | $\bar{p}_{1212}$ | $\bar{p}_{1221}$ | $\bar{p}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,1) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2144}$ | $\bar{p}_{2122}$ | $\bar{p}_{2112}$ | $\bar{p}_{2121}$ | $\bar{p}_{2133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (3,3) | $\bar{p}_{3311}$ | 0 | 0 | 0 | 0 | $\bar{p}_{3344}$ | $\bar{p}_{3322}$ | $\bar{p}_{3312}$ | $\bar{p}_{3321}$ | $\bar{p}_{3333}$ | $\bar{p}_{3313}$ | $\bar{p}_{3331}$ | $\bar{p}_{3323}$ | $\bar{p}_{3332}$ | 0 | 0 |
| (1,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1333}$ | $\bar{p}_{1313}$ | $\bar{p}_{1331}$ | $\bar{p}_{1323}$ | $\bar{p}_{1332}$ | 0 | 0 |
| (3,1) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3133}$ | $\bar{p}_{3113}$ | $\bar{p}_{3131}$ | $\bar{p}_{3123}$ | $\bar{p}_{3132}$ | 0 | 0 |
| (2,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2333}$ | $\bar{p}_{2313}$ | $\bar{p}_{2331}$ | $\bar{p}_{2323}$ | $\bar{p}_{2332}$ | 0 | 0 |
| (3,2) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3233}$ | $\bar{p}_{3213}$ | $\bar{p}_{3231}$ | $\bar{p}_{3223}$ | $\bar{p}_{3232}$ | 0 | |
| (3,4) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table SM3

*Flattening matrix of $\bar{p}$ for a quartet $T = 12 \mid 34$. The entries highlighted in dark gray form a (generically) non-vanishing 2-minor and the submatrix formed by the entries highlighted in (dark and light) gray has rank 2. Therefore, the 4 cubic edge invariants in Theorem 5.14 arise from considering all 3-minors of the grey submatrix containing the dark grey minor.*

| | $(1,1)$ | $(1,4)$ | $(4,1)$ | $(2,4)$ | $(4,2)$ | $(4,4)$ | $(2,2)$ | $(1,2)$ | $(2,1)$ | $(3,3)$ | $(1,3)$ | $(3,1)$ | $(2,3)$ | $(3,2)$ | $(3,4)$ | $(4,3)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $\bar{p}_{1111}$ | 0 | 0 | 0 | 0 | $\bar{p}_{1144}$ | $\bar{p}_{1122}$ | 0 | 0 | $\bar{p}_{1133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,4)$ | 0 | $\bar{p}_{1414}$ | $\bar{p}_{1441}$ | $\bar{p}_{1424}$ | $\bar{p}_{1442}$ | $\bar{p}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,1)$ | 0 | $\bar{p}_{4114}$ | $\bar{p}_{4141}$ | $\bar{p}_{4124}$ | $\bar{p}_{4142}$ | $\bar{p}_{4144}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,4)$ | 0 | $\bar{p}_{2414}$ | $\bar{p}_{2441}$ | $\bar{p}_{2424}$ | $\bar{p}_{2442}$ | $\bar{p}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,2)$ | 0 | $\bar{p}_{4214}$ | $\bar{p}_{4241}$ | $\bar{p}_{4224}$ | $\bar{p}_{4242}$ | $\bar{p}_{4244}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,4)$ | $\bar{p}_{4411}$ | $\bar{p}_{4414}$ | $\bar{p}_{4441}$ | $\bar{p}_{4424}$ | $\bar{p}_{4442}$ | $\bar{p}_{4444}$ | $\bar{p}_{4422}$ | $\bar{p}_{4412}$ | $\bar{p}_{4421}$ | $\bar{p}_{4433}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,2)$ | $\bar{p}_{2211}$ | 0 | 0 | 0 | 0 | $\bar{p}_{2244}$ | $\bar{p}_{2222}$ | $\bar{p}_{2212}$ | $\bar{p}_{2221}$ | $\bar{p}_{2233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,2)$ | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1244}$ | $\bar{p}_{1222}$ | $\bar{p}_{1212}$ | $\bar{p}_{1221}$ | $\bar{p}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,1)$ | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2144}$ | $\bar{p}_{2122}$ | $\bar{p}_{2112}$ | $\bar{p}_{2121}$ | $\bar{p}_{2133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(3,3)$ | $\bar{p}_{3311}$ | 0 | 0 | 0 | 0 | $\bar{p}_{3344}$ | $\bar{p}_{3322}$ | $\bar{p}_{3312}$ | $\bar{p}_{3321}$ | $\bar{p}_{3333}$ | $\bar{p}_{3313}$ | $\bar{p}_{3331}$ | $\bar{p}_{3323}$ | $\bar{p}_{3332}$ | 0 | 0 |
| $(1,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1333}$ | $\bar{p}_{1313}$ | $\bar{p}_{1331}$ | $\bar{p}_{1323}$ | $\bar{p}_{1332}$ | 0 | 0 |
| $(3,1)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3133}$ | $\bar{p}_{3113}$ | $\bar{p}_{3131}$ | $\bar{p}_{3123}$ | $\bar{p}_{3132}$ | 0 | 0 |
| $(2,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2333}$ | $\bar{p}_{2313}$ | $\bar{p}_{2331}$ | $\bar{p}_{2323}$ | $\bar{p}_{2332}$ | 0 | 0 |
| $(3,2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3233}$ | $\bar{p}_{3213}$ | $\bar{p}_{3231}$ | $\bar{p}_{3223}$ | $\bar{p}_{3232}$ | 0 | |
| $(3,4)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(4,3)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE SM4

*Flattening matrix of $\bar{p}$ for a quartet $T = 12 \mid 34$. The entries highlighted in dark gray form a (generically) non-vanishing 3-minor and the submatrix formed by the entries highlighted in (dark and light) gray has rank 3. Therefore, 7 of the quartic edge invariants in Theorem 5.14 arise from considering all 4-minors of the grey submatrix containing the dark grey minor.*

| | (1,1) | (1,4) | (4,1) | (2,4) | (4,2) | (4,4) | (2,2) | (1,2) | (2,1) | (3,3) | (1,3) | (3,1) | (2,3) | (3,2) | (3,4) | (4,3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1,1) | $\bar{p}_{1111}$ | 0 | 0 | 0 | 0 | $\bar{p}_{1144}$ | $\bar{p}_{1122}$ | 0 | 0 | $\bar{p}_{1133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,4) | 0 | $\bar{p}_{1414}$ | $\bar{p}_{1441}$ | $\bar{p}_{1424}$ | $\bar{p}_{1442}$ | $\bar{p}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,1) | 0 | $\bar{p}_{4114}$ | $\bar{p}_{4141}$ | $\bar{p}_{4124}$ | $\bar{p}_{4142}$ | $\bar{p}_{4144}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,4) | 0 | $\bar{p}_{2414}$ | $\bar{p}_{2441}$ | $\bar{p}_{2424}$ | $\bar{p}_{2442}$ | $\bar{p}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,2) | 0 | $\bar{p}_{4214}$ | $\bar{p}_{4241}$ | $\bar{p}_{4224}$ | $\bar{p}_{4242}$ | $\bar{p}_{4244}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,4) | $\bar{p}_{4411}$ | $\bar{p}_{4414}$ | $\bar{p}_{4441}$ | $\bar{p}_{4424}$ | $\bar{p}_{4442}$ | $\bar{p}_{4444}$ | $\bar{p}_{4422}$ | $\bar{p}_{4412}$ | $\bar{p}_{4421}$ | $\bar{p}_{4433}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,2) | $\bar{p}_{2211}$ | 0 | 0 | 0 | 0 | $\bar{p}_{2244}$ | $\bar{p}_{2222}$ | $\bar{p}_{2212}$ | $\bar{p}_{2221}$ | $\bar{p}_{2233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,2) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1244}$ | $\bar{p}_{1222}$ | $\bar{p}_{1212}$ | $\bar{p}_{1221}$ | $\bar{p}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,1) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2144}$ | $\bar{p}_{2122}$ | $\bar{p}_{2112}$ | $\bar{p}_{2121}$ | $\bar{p}_{2133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (3,3) | $\bar{p}_{3311}$ | 0 | 0 | 0 | 0 | $\bar{p}_{3344}$ | $\bar{p}_{3322}$ | $\bar{p}_{3312}$ | $\bar{p}_{3321}$ | $\bar{p}_{3333}$ | $\bar{p}_{3313}$ | $\bar{p}_{3331}$ | $\bar{p}_{3323}$ | $\bar{p}_{3332}$ | 0 | 0 |
| (1,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1333}$ | $\bar{p}_{1313}$ | $\bar{p}_{1331}$ | $\bar{p}_{1323}$ | $\bar{p}_{1332}$ | 0 | 0 |
| (3,1) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3133}$ | $\bar{p}_{3113}$ | $\bar{p}_{3131}$ | $\bar{p}_{3123}$ | $\bar{p}_{3132}$ | 0 | 0 |
| (2,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2333}$ | $\bar{p}_{2313}$ | $\bar{p}_{2331}$ | $\bar{p}_{2323}$ | $\bar{p}_{2332}$ | 0 | 0 |
| (3,2) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3233}$ | $\bar{p}_{3213}$ | $\bar{p}_{3231}$ | $\bar{p}_{3223}$ | $\bar{p}_{3232}$ | 0 | |
| (3,4) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE SM5

*Flattening matrix of $\bar{p}$ for a quartet $T = 12 \mid 34$. The entries highlighted in dark gray form a (generically) non-vanishing 3-minor and the submatrix formed by the entries highlighted in (dark and light) gray has rank 3. Therefore, 7 of the quartic edge invariants in Theorem 5.14 arise from considering all 4-minors of the grey submatrix containing the dark grey minor.*

| | (1,1) | (1,4) | (4,1) | (2,4) | (4,2) | (4,4) | (2,2) | (1,2) | (2,1) | (3,3) | (1,3) | (3,1) | (2,3) | (3,2) | (3,4) | (4,3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1,1) | $\bar{p}_{1111}$ | 0 | 0 | 0 | 0 | $\bar{p}_{1144}$ | $\bar{p}_{1122}$ | 0 | 0 | $\bar{p}_{1133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,4) | 0 | $\bar{p}_{1414}$ | $\bar{p}_{1441}$ | $\bar{p}_{1424}$ | $\bar{p}_{1442}$ | $\bar{p}_{1444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,1) | 0 | $\bar{p}_{4114}$ | $\bar{p}_{4141}$ | $\bar{p}_{4124}$ | $\bar{p}_{4142}$ | $\bar{p}_{4144}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,4) | 0 | $\bar{p}_{2414}$ | $\bar{p}_{2441}$ | $\bar{p}_{2424}$ | $\bar{p}_{2442}$ | $\bar{p}_{2444}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,2) | 0 | $\bar{p}_{4214}$ | $\bar{p}_{4241}$ | $\bar{p}_{4224}$ | $\bar{p}_{4242}$ | $\bar{p}_{4244}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,4) | $\bar{p}_{4411}$ | $\bar{p}_{4414}$ | $\bar{p}_{4441}$ | $\bar{p}_{4424}$ | $\bar{p}_{4442}$ | $\bar{p}_{4444}$ | $\bar{p}_{4422}$ | $\bar{p}_{4412}$ | $\bar{p}_{4421}$ | $\bar{p}_{4433}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,2) | $\bar{p}_{2211}$ | 0 | 0 | 0 | 0 | $\bar{p}_{2244}$ | $\bar{p}_{2222}$ | $\bar{p}_{2212}$ | $\bar{p}_{2221}$ | $\bar{p}_{2233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (1,2) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1244}$ | $\bar{p}_{1222}$ | $\bar{p}_{1212}$ | $\bar{p}_{1221}$ | $\bar{p}_{1233}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (2,1) | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2144}$ | $\bar{p}_{2122}$ | $\bar{p}_{2112}$ | $\bar{p}_{2121}$ | $\bar{p}_{2133}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| (3,3) | $\bar{p}_{3311}$ | 0 | 0 | 0 | 0 | $\bar{p}_{3344}$ | $\bar{p}_{3322}$ | $\bar{p}_{3312}$ | $\bar{p}_{3321}$ | $\bar{p}_{3333}$ | $\bar{p}_{3313}$ | $\bar{p}_{3331}$ | $\bar{p}_{3323}$ | $\bar{p}_{3332}$ | 0 | 0 |
| (1,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{1333}$ | $\bar{p}_{1313}$ | $\bar{p}_{1331}$ | $\bar{p}_{1323}$ | $\bar{p}_{1332}$ | 0 | 0 |
| (3,1) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3133}$ | $\bar{p}_{3113}$ | $\bar{p}_{3131}$ | $\bar{p}_{3123}$ | $\bar{p}_{3132}$ | 0 | 0 |
| (2,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{2333}$ | $\bar{p}_{2313}$ | $\bar{p}_{2331}$ | $\bar{p}_{2323}$ | $\bar{p}_{2332}$ | 0 | 0 |
| (3,2) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{p}_{3233}$ | $\bar{p}_{3213}$ | $\bar{p}_{3231}$ | $\bar{p}_{3223}$ | $\bar{p}_{3232}$ | 0 | |
| (3,4) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (4,3) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |