Diagonalization

Bioinformatics Degree
Algebra

Departament de Matemàtiques

UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

Outline

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Example: Population growth (Leslie model)

The Vollmar-Wasserman beetle lives for at most 3 years. We divide the female VW beetles into three age classes and call:

- x_1 = number of youths (beetles 0 to 1 years old)
- x_2 = number of juveniles (beetles 1 to 2 year old)
- x_3 = number of adults (beetles 2 to 3 year old)

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Example: Population growth (Leslie model)

We know that:

- Youths do not lay eggs. Female juveniles have an average of 4 youth females per year and female adults have an average of 3 youth females per year.
- The survival rate for youths is 50% (that is, the probability of a youths surviving to become a juvenile is 0.5), and the survival rate for juveniles is 25%.

Therefore we have:

$$\begin{aligned} \text{youths next year} &= 4x_2 + 3x_3 \\ \text{juveniles next year} &= 0.5x_1 \\ \text{adults next year} &= 0.25x_2 \end{aligned}$$

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If we write the number of youths, juveniles and adults as acolumn vector, next year we'll have

$$\begin{pmatrix} \sharp\text{youths} \\ \sharp\text{juveniles} \\ \sharp\text{adults} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, next year the population will bw $A\mathtt{x}$ and in k years $A^k\mathtt{x}$.

Goal: compute powers of matrices

- How do we compute A^k easily?
- If A is a diagonal matrix it is easy:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}$$

- If A is not diagonal but we can find a change of basis matrix P such that $P^{-1}AP$ is diagonal, then $A = PDP^{-1}$ and

$$A^k = PD(P)^{-1}PD(P)^{-1}\dots PD(P)^{-1}PD(P)^{-1} = PD^K P^{-1}$$

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Definitions and properties

Definition

An endomorphism of a vector space $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$.

Endomorphisms $\leftrightarrow$ square matrices.

Note: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ (d_i=ith value in the diagonal).

Definition

A vector $u \neq 0$ is an eigenvector of $f : \mathbb{R}^n \to \mathbb{R}^n$ with eigenvalue $\lambda \in \mathbb{R}$ if $f(u) = \lambda\, u$. In this case, we say that λ is an eigenvalue of f.

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Examples

1. Consider $f(x, y) = (x, 2y)$. Then, the standard matrix of f is

$$M(f) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- $e_1 = (1, 0)$ is an eigenvector of f with eigenvalue 1, and
- $e_2 = (0, 1)$ is an eigenvector of f with eigenvalue 2.

Examples

2. Consider $f(x, y) = (x + 5y, 5x + y)$. Then,
   - $u_1 = (1, 1)$ is eigenvector with eigenvalue 6;
   - $u_2 = (1, -1)$ is eigenvector with eigenvalue -4.

   The standard matrix of f is

$$M(f) = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}.$$

   In the basis $\mathbf{u} = \{u_1, u_2\}$, the matrix of f is diagonal and equal to

$$M_{\mathbf{u},\mathbf{u}}(f)(f) = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}.$$

The aim of this topic is to study the endomorphisms for which we can obtain a basis so that the matrix is diagonal.

Properties

Let f be an endomorphism of $\mathbb{R}^n$ and let $A = M_e(f)$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \text{Null}(A - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(A - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\text{Null}(A - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \text{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

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Characteristic polynomial

Definition

Given an endomorphism f of $\mathbb{R}^n$, let $A = M(f)$ be its standard matrix. The characteristic polynomial of f is computed as

$$P_f(x) = det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}$$

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- The roots of $P_f(x)$ are the eigenvalues of f, that is, λ is eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0 \Leftrightarrow \det(A - \lambda I) = 0$.
- $P_f(x)$ is a polynomial of degree n.
- $P_f(x)$ can be computed from the matrix of f on any basis $\mathbf{u}$ of $\mathbb{R}^n$: $P_f(x) = det(M_{\mathbf{u},\mathbf{u}}(f) - x\,Id)$.

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Characteristic polynomial

Example. Consider $f(x,y) = (2x+4y, x+5y)$. It has standard matrix

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$$

Its characteristic polynomial is

$$P_f(x) = \begin{vmatrix} 2-x & 4 \\ 1 & 5-x \end{vmatrix} = (2-x)(5-x) - 4\times 1 = x^2 - 7x + 6.$$

The roots of this polynomial are $\frac{7\pm\sqrt{(-7)^2-4\cdot 6}}{2}$, so 6 and 1. We have $P_f(x) = (x-6)(x-1)$.

Roots of polynomials

A root or zero of a polynomial $p(x)$ is a number a such that $p(a) = 0$.
Properties:

- a is a root of $p(x) \Leftrightarrow p(x)$ is a multiple of $(x - a)$, $p(x) = (x - a)q(x)$.
- a is a root of multiplicity m if m is the largest exponent such that $p(x) = (x - a)^m q(x)$ for some $q(x)$.
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Complex numbers

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- We can sum and multiply complex numbers
- The modulus (or absolute value, or norm) of $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

- The conjugate of $z = a + bi$ is $\bar{z} = a - bi$. It satisfies:

$$|\bar{z}| = |z|, \quad |z|^2 = z \cdot \bar{z}.$$

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Fundamental theorem of algebra

All roots of a polynomial are either real or complex numbers:

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

If $p(x)$ has coefficients in $\mathbb{R}$, then its (non-real) complex roots go in pairs z, $\bar{z}$

Outline

Multiplicities of eigenvalues

If $A = M(f)$,

Definition

- The algebraic multiplicity of λ, denoted by a_λ, is the multiplicity as a root of $P_f(x)$ (the number of times λ appears as a root of $P_f(x)$.
- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the subspace $\mathrm{Null}(A - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

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- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the subspace $\mathrm{Null}(A - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Multiplicities of eigenvalues

If $A = M(f)$,

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Diagonalization theorem

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An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
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Diagonalization (in terms of matrices)

Definition

A matrix $A \in M_n(\mathbb{R})$ is diagonalizable if there is an invertible matrix $P \in M_n(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

That is, A is diagonalizable if it is the standard matrix of a diagonalizable endomorphism f, $A = M(f)$; in this case P is the change of basis matrix from a basis $\mathbf{u}$ to the standard basis $\mathbf{e}$, $P = A_{\mathbf{u}\to\mathbf{e}}$.

$$A = A_{\mathbf{u}\to\mathbf{e}} D A_{\mathbf{u}\to\mathbf{e}}^{-1} = PDP^{-1},$$

$$D = P^{-1}AP = A_{\mathbf{e}\to\mathbf{u}} M(f) A_{\mathbf{u}\to\mathbf{e}} = M_{\mathbf{u}}(f)$$

Procedure to diagonalize endomorphisms (1)

Given an endomorphism f of $\mathbb{R}^n$, let $A = M_e(f)$ be its standard matrix.

1. Compute $P_f(x) = det(A - x\, Id)$.
2. Compute the roots $\lambda_1, \dots, \lambda_k$ of $P_f(x)$, these are the eigenvalues of f. For each eigenvalue λ_i, its algebraic multiplicity a_{λ_i} is the multiplicity it has as a root of $P_f(x)$. Note that $a_{\lambda_1} + \dots + a_{\lambda_k} = n$.
3. For each λ_i, compute $\text{Null}(A - \lambda_i Id)$, the subspace of all eigenvectors with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for some eigenvalue λ_i, then f does not diagonalize (and we are done).

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Procedure (2)

5. If $g_{\lambda_i} = a_{\lambda_i}$ for all i, for each eigenvalue λ_i take a basis of $\text{Null}(A - \lambda_i Id)$
6. Let $\mathbf{u}$ be the collection of all these vectors

Then,

- $\mathbf{u}$ is a basis of $\mathbb{R}^n$.
- $M_{\mathbf{u}}(f)$ is a diagonal matrix whose entries are the eigenvalues:

$$M_{\mathbf{u}}(f) = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

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Prodecure (3)

Recall that $D = M_{\mathbf{u}}(f)$ can be computed by doing a change of basis: if e is the standard basis of $\mathbb{R}^n$, then

$$A_{e\to\mathbf{u}}\, M(f)\, A_{\mathbf{u}\to e} = D.$$

(Equivalently, $A_{\mathbf{u}\to e}\, D\, A_{e\to\mathbf{u}} = M(f)$).

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A)$ = product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A)$ = sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

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Diagonalization in $\mathbb{C}$

If we allow ourselves to work with eigenvectors with entries in $\mathbb{C}$, we have the analogous result (taking into account that $P_f(x)$ already has all its roots in $\mathbb{C}$):

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Outline

Python

```
>>> import numpy as np
>>> from numpy.linalg import *
```

>>> A = np.array([[$a_{11}, \ldots, a_{1n}], \ldots, [a_{n1}, \ldots, a_{nn}$]])

To get eigenvalues and eigenvectors of A we do:

```
>>> eigenval, eigenvec = eig(A)
```

Then we call both outputs to see the list of eigenvalues

```
>>> eigenval
```

gives the collection of eigenvalues of A; we can call each: `eigenval[0]` is the first eigenvalue, `eigenval[1]` the second...

```
>>> eigenvec
```

gives a matrix P whose columns are eigenvectors of A; we can call each: `eigenvec[:,0]` gives the first eigenvector (column) (corresponding to `eigenval[0]`), `eigenvec[:,1]` gives the second...

Python

If we want to create the diagonal matrix with eigenvalues:

```
>>> D = np.zeros((n,n),dtype='complex128')
>>> for i in range(n):
      D[i,i] = eigenval[i]
```

(where it says n we need to put the size of the matrix). Then we can check if PDP^{-1} is indeed A:

```
>>> eigenvec@ D @inv(eigenvec)
```

To compute the powers of a matrix, A^k:

```
>>> matrix_power(A,k)
```

Note: In Python the complex number i is denoted as j. Modulus of a complex number z:

```
>>> abs(z)
```