Diagonalization

Bioinformatics Degree
Algebra

Departament de Matemàtiques

UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Example: Population growth (Leslie model)

The Vollmar-Wasserman beetle lives for at most 3 years. We divide the female VW beetles into three age classes and call:

- x_1 = number of youths (beetles 0 to 1 years old)
- x_2 = number of juveniles (beetles 1 to 2 year old)
- x_3 = number of adults (beetles 2 to 3 year old)

We want to study the number of youths, juveniles and adults after k years.

Example: Population growth (Leslie model)

The Vollmar-Wasserman beetle lives for at most 3 years. We divide the female VW beetles into three age classes and call:

- x_1 = number of youths (beetles 0 to 1 years old)
- x_2 = number of juveniles (beetles 1 to 2 year old)
- x_3 = number of adults (beetles 2 to 3 year old)

We want to study the number of youths, juveniles and adults after k years.

Example: Population growth (Leslie model)

The Vollmar-Wasserman beetle lives for at most 3 years. We divide the female VW beetles into three age classes and call:

- x_1 = number of youths (beetles 0 to 1 years old)
- x_2 = number of juveniles (beetles 1 to 2 year old)
- x_3 = number of adults (beetles 2 to 3 year old)

We want to study the number of youths, juveniles and adults after k years.

Example: Population growth (Leslie model)

The Vollmar-Wasserman beetle lives for at most 3 years. We divide the female VW beetles into three age classes and call:

- x_1 = number of youths (beetles 0 to 1 years old)
- x_2 = number of juveniles (beetles 1 to 2 year old)
- x_3 = number of adults (beetles 2 to 3 year old)

We want to study the number of youths, juveniles and adults after k years.

Example: Population growth (Leslie model)

We know that:

- Youths do not lay eggs. Female juveniles have an average of 4 youth females per year and female adults have an average of 3 youth females per year.
- The survival rate for youths is 50% (that is, the probability of a youths surviving to become a juvenile is 0.5), and the survival rate for juveniles is 25%.

Therefore we have:

$$\begin{aligned} \text{youths next year} &= 4x_2 + 3x_3 \\ \text{juveniles next year} &= 0.5x_1 \\ \text{adults next year} &= 0.25x_2 \end{aligned}$$

Example: Population growth (Leslie model)

We know that:

- Youths do not lay eggs. Female juveniles have an average of 4 youth females per year and female adults have an average of 3 youth females per year.
- The survival rate for youths is 50% (that is, the probability of a youths surviving to become a juvenile is 0.5), and the survival rate for juveniles is 25%.

Therefore we have:

$$\begin{aligned} \text{youths next year} &= 4x_2 + 3x_3 \\ \text{juveniles next year} &= 0.5x_1 \\ \text{adults next year} &= 0.25x_2 \end{aligned}$$

Example: Population growth (Leslie model)

If we write the number of youths, juveniles and adults as acolumn vector, next year we'll have

$$\begin{pmatrix} \sharp\text{youths} \\ \sharp\text{juveniles} \\ \sharp\text{adults} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, next year the population will bw $A\mathtt{x}$ and in k years $A^k\mathtt{x}$.

Goal: compute powers of matrices

- How do we compute A^k easily?
- If A is a diagonal matrix it is easy:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \Rightarrow \quad D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}$$

- If A is not diagonal but we can find a change of basis matrix P such that $P^{-1}AP$ is diagonal, then $A = PDP^{-1}$ and

$$A^k = PD(P)^{-1}PD(P)^{-1} \dots PD(P)^{-1}PD(P)^{-1} = PD^K P^{-1}$$

$$A^k = P \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix} P^{-1}$$

Goal: compute powers of matrices

- How do we compute A^k easily?
- If A is a diagonal matrix it is easy:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}$$

- If A is not diagonal but we can find a change of basis matrix P such that $P^{-1}AP$ is diagonal, then $A = PDP^{-1}$ and

$$A^k = PD(P)^{-1}PD(P)^{-1} \dots PD(P)^{-1}PD(P)^{-1} = PD^KP^{-1}$$

$$A^k = P \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix} P^{-1}$$

Goal: compute powers of matrices

- How do we compute A^k easily?
- If A is a diagonal matrix it is easy:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}$$

- If A is not diagonal but we can find a change of basis matrix P such that $P^{-1}AP$ is diagonal, then $A = PDP^{-1}$ and

$$A^k = PD(P)^{-1}PD(P)^{-1}\dots PD(P)^{-1}PD(P)^{-1} = PD^K P^{-1}$$

$$A^k = P \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix} P^{-1}$$

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Definitions and properties

Definition

An endomorphism of a vector space $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$.

Endomorphisms $\leftrightarrow$ square matrices.

Note: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ (d_i=ith value in the diagonal).

Definition

A vector $u \in \mathbb{R}^n$ is said to be an eigenvector of $f : \mathbb{R}^n \to \mathbb{R}^n$ with eigenvalue $\lambda \in \mathbb{R}$ if $f(u) = \lambda\, u$. In this case, we say that λ is an eigenvalue of f.

Definitions and properties

Definition

An endomorphism of a vector space $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$.

Endomorphisms $\leftrightarrow$ square matrices.
Note: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ (d_i=ith value in the diagonal).

Definition

A vector $u \in \mathbb{R}^n$ is said to be an eigenvector of $f : \mathbb{R}^n \to \mathbb{R}^n$ with eigenvalue $\lambda \in \mathbb{R}$ if $f(u) = \lambda\, u$. In this case, we say that λ is an eigenvalue of f.

Examples

1. Consider $f(x, y) = (x, 2y)$. Then, the standard matrix of f is

$$M_e(f) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- $e_1 = (1, 0)$ is an eigenvector of f with eigenvalue 1, and
- $e_2 = (0, 1)$ is an eigenvector of f with eigenvalue 2.

Examples

2. Consider $f(x, y) = (x + 5y, 5x + y)$. Then,
   - $u_1 = (1, 1)$ is eigenvector with eigenvalue 6;
   - $u_2 = (1, -1)$ is eigenvector with eigenvalue -4.

   The standard matrix of f is

   $$M_e(f) = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}.$$

   In the basis $\mathbf{u} = \{u_1, u_2\}$, the matrix of f is diagonal and equal to

   $$M_{\mathbf{u},\mathbf{u}}(f)(f) = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}.$$

The aim of this topic is to study the endomorphisms for which we can obtain a basis so that the matrix is diagonal.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \text{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\text{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \text{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \text{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\text{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \text{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \mathrm{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\mathrm{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \mathrm{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \text{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\text{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \text{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \text{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\text{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \text{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Properties

Let f be an endomorphism of $\mathbb{R}^n$.

Lemma

- *$u \in \mathbb{R}^n$ is an eigenvector of eigenvalue $\lambda \Leftrightarrow u \in \mathrm{Null}(f - \lambda Id)$ and $u \neq 0$.*
- *λ is an eigenvalue of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each eigenvalue λ of f, $\mathrm{Null}(f - \lambda Id)$ is called the **eigenspace** of λ and contains all eigenvectors of eigenvalue λ (plus $\mathbf{0}$).

- 0 is an eigenvalue of $f \Leftrightarrow \mathrm{Null}(f) \neq \{\mathbf{0}\}$.
- The **spectrum** of f is the set of all its eigenvalues; it is denoted by $\sigma(f)$.

Characteristic polynomial

Definition

Given an endomorphism f of $\mathbb{R}^n$, let $A = M(f)$ be its standard matrix. The characteristic polynomial of f is computed as

$$P_f(x) = det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}$$

.

- The roots of $P_f(x)$ are the eigenvalues of f, that is, λ is eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0 \Leftrightarrow \det(A - \lambda I) = 0$.
- $P_f(x)$ is a polynomial of degree n.
- $P_f(x)$ can be computed from the matrix of f on any basis $\mathbf{u}$ of $\mathbb{R}^n$: $P_f(x) = det(M_{\mathbf{u},\mathbf{u}}(f) - x\,Id)$.

Characteristic polynomial

Definition

Given an endomorphism f of $\mathbb{R}^n$, let $A = M(f)$ be its standard matrix. The characteristic polynomial of f is computed as

$$P_f(x) = det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}$$

.

- The roots of $P_f(x)$ are the eigenvalues of f, that is, λ is eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0 \Leftrightarrow \det(A - \lambda I) = 0$.
- $P_f(x)$ is a polynomial of degree n.
- $P_f(x)$ can be computed from the matrix of f on any basis $\mathbf{u}$ of $\mathbb{R}^n$: $P_f(x) = det(M_{\mathbf{u},\mathbf{u}}(f) - x\,Id)$.

Characteristic polynomial

Definition

Given an endomorphism f of $\mathbb{R}^n$, let $A = M(f)$ be its standard matrix. The characteristic polynomial of f is computed as

$$P_f(x) = det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}$$

.

- The roots of $P_f(x)$ are the eigenvalues of f, that is, λ is eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0 \Leftrightarrow \det(A - \lambda I) = 0$.
- $P_f(x)$ is a polynomial of degree n.
- $P_f(x)$ can be computed from the matrix of f on any basis $\mathbf{u}$ of $\mathbb{R}^n$: $P_f(x) = det(M_{\mathbf{u},\mathbf{u}}(f) - x\,Id)$.

Characteristic polynomial

Definition

Given an endomorphism f of $\mathbb{R}^n$, let $A = M(f)$ be its standard matrix. The characteristic polynomial of f is computed as

$$P_f(x) = det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}$$

.

- The roots of $P_f(x)$ are the eigenvalues of f, that is, λ is eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0 \Leftrightarrow \det(A - \lambda I) = 0$.
- $P_f(x)$ is a polynomial of degree n.
- $P_f(x)$ can be computed from the matrix of f on any basis $\mathbf{u}$ of $\mathbb{R}^n$: $P_f(x) = det(M_{\mathbf{u},\mathbf{u}}(f) - x\, Id)$.

Characteristic polynomial

Example. Consider $f(x,y) = (2x+4y, x+5y)$. It has standard matrix

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$$

Its characteristic polynomial is

$$P_f(x) = \begin{vmatrix} 2-x & 4 \\ 1 & 5-x \end{vmatrix} = (2-x)(5-x) - 4 \times 1 = x^2 - 7x + 6.$$

The roots of this polynomial are $\frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 6}}{2}$, so 6 and 1. We have $P_f(x) = (x-6)(x-1)$.

Roots of polynomials

A root or zero of a polynomial $p(x)$ is a number a such that $p(a) = 0$.
Properties:

- a is a root of $p(x) \Leftrightarrow p(x)$ is a multiple of $(x - a)$ $(p(x) = (x - a)q(x))$.
- a is a root of multiplicity m if $p(x) = (x - a)^m q(x)$ for some $q(x)$.
- Any real polynomial factorizes as a product of degree 1 and degree 2 polynomials with real coefficients.

Roots of polynomials

A root or zero of a polynomial $p(x)$ is a number a such that $p(a) = 0$.

Properties:

- a is a root of $p(x) \Leftrightarrow p(x)$ is a multiple of $(x - a)$ $(p(x) = (x - a)q(x))$.
- a is a root of multiplicity m if $p(x) = (x - a)^m q(x)$ for some $q(x)$.
- Any real polynomial factorizes as a product of degree 1 and degree 2 polynomials with real coefficients.

Roots of polynomials

A root or zero of a polynomial $p(x)$ is a number a such that $p(a) = 0$.

Properties:

- a is a root of $p(x) \Leftrightarrow p(x)$ is a multiple of $(x - a)$ $(p(x) = (x - a)q(x))$.
- a is a root of multiplicity m if $p(x) = (x - a)^m q(x)$ for some $q(x)$.
- Any real polynomial factorizes as a product of degree 1 and degree 2 polynomials with real coefficients.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Fundamental theorem of algebra

- A polynomial might NOT have real roots:

$$x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1}$$

- We denote by i the square root of -1, $i = \sqrt{-1}$
- A complex number is an expression $a + bi$ with $a, b \in \mathbb{R}$. Ex: $3 + 2i$.
- The set of complex numbers is denoted as $\mathbb{C}$ and contains $\mathbb{R}$.
- All roots of a polynomial are either real or complex numbers.

Theorem

If $p(x)$ is a polynomial of degree n, then it has n complex roots counted with multiplicity.

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Multiplicities of eigenvalues

Definition

- The algebraic multiplicity of λ, denoted by a_λ, is the multiplicity as a root of $P_f(x)$ (the number of times λ appears as a root of $P_f(x)$.
- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Null}(f - \lambda Id)$, that is, $n - rk(M(f) - \lambda I) > 0$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Multiplicities of eigenvalues

Definition

- The algebraic multiplicity of λ, denoted by a_λ, is the multiplicity as a root of $P_f(x)$ (the number of times λ appears as a root of $P_f(x)$.
- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Null}(f - \lambda Id)$, that is, $n - rk(M(f) - \lambda I) > 0$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Multiplicities of eigenvalues

Definition

- The algebraic multiplicity of λ, denoted by a_λ, is the multiplicity as a root of $P_f(x)$ (the number of times λ appears as a root of $P_f(x)$.
- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Null}(f - \lambda Id)$, that is, $n - rk(M(f) - \lambda I) > 0$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Multiplicities of eigenvalues

Definition

- The algebraic multiplicity of λ, denoted by a_λ, is the multiplicity as a root of $P_f(x)$ (the number of times λ appears as a root of $P_f(x)$.
- The geometric multiplicity of λ, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Null}(f - \lambda Id)$, that is, $n - rk(M(f) - \lambda I) > 0$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Diagonalization theorem

Definition

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
2. *for every eigenvalue λ_i, the algebraic multiplicity and geometric multiplicity are equal: $g_{\lambda_i} = a_{\lambda_i}$.*

In particular, if all the roots of $P_f(x)$ are real and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Definition

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
2. *for every eigenvalue λ_i, the algebraic multiplicity and geometric multiplicity are equal: : $g_{\lambda_i} = a_{\lambda_i}$.*

In particular, if all the roots of $P_f(x)$ are real and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Definition

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
2. *for every eigenvalue λ_i, the algebraic multiplicity and geometric multiplicity are equal: : $g_{\lambda_i} = a_{\lambda_i}$.*

In particular, if all the roots of $P_f(x)$ are real and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Definition

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
2. *for every eigenvalue λ_i, the algebraic multiplicity and geometric multiplicity are equal: : $g_{\lambda_i} = a_{\lambda_i}$.*

In particular, if all the roots of $P_f(x)$ are real and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Definition

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if there is a basis $\mathbf{u}$ of $\mathbb{R}^n$ such that $M_{\mathbf{u},\mathbf{u}}(f)$ is diagonal.

Theorem (Diagonalization)

An endomorphism f of $\mathbb{R}^n$ is diagonalizable in $\mathbb{R}$ if and only if

1. *all the roots $\lambda_1, \dots, \lambda_k$ of $P_f(x)$ (the eigenvalues of f) are real;*
2. *for every eigenvalue λ_i, the algebraic multiplicity and geometric multiplicity are equal: : $g_{\lambda_i} = a_{\lambda_i}$.*

In particular, if all the roots of $P_f(x)$ are real and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization for matrices

Definition

A matrix $A \in M_n(\mathbb{R})$ is diagonalizable if there is an invertible matrix $P \in M_n(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

That is, A is diagonalizable if it is the standard matrix of a diagonalizable endomorphism f, $A = M(f)$; in this case P is the change of basis matrix from a basis $\mathbf{u}$ to the standard basis $\mathbf{e}$, $P = A_{\mathbf{u}\to\mathbf{e}}$.

$$A = A_{\mathbf{u}\to\mathbf{e}} D A_{\mathbf{u}\to\mathbf{e}}^{-1} = PDP^{-1},$$

$$D = P^{-1}AP = A_{\mathbf{e}\to\mathbf{u}} M(f) A_{\mathbf{u}\to\mathbf{e}} = M_{\mathbf{u}}(f)$$

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let $A = M_e(f)$ be its standard matrix.

1. Compute $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$, these are the eigenvalues of f. For each eigenvalue λ_i, its algebraic multiplicity a_{λ_i} is the multiplicity it has as a root of $P_f(x)$. Note that $a_{\lambda_1} + \ldots + a_{\lambda_k} = n$.
3. For each λ_i, compute Null$(A - \lambda_i Id)$, the subspace of all eigenvectors with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for some eigenvalue λ_i, then f does not diagonalize (and we are done).

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let $A = M_e(f)$ be its standard matrix.

1. Compute $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$, these are the eigenvalues of f. For each eigenvalue λ_i, its algebraic multiplicity a_{λ_i} is the multiplicity it has as a root of $P_f(x)$. Note that $a_{\lambda_1} + \ldots + a_{\lambda_k} = n$.
3. For each λ_i, compute Null$(A - \lambda_i Id)$, the subspace of all eigenvectors with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for some eigenvalue λ_i, then f does not diagonalize (and we are done).

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let $A = M_e(f)$ be its standard matrix.

1. Compute $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$, these are the eigenvalues of f. For each eigenvalue λ_i, its algebraic multiplicity a_{λ_i} is the multiplicity it has as a root of $P_f(x)$. Note that $a_{\lambda_1} + \ldots + a_{\lambda_k} = n$.
3. For each λ_i, compute Null$(A - \lambda_i Id)$, the subspace of all eigenvectors with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for some eigenvalue λ_i, then f does not diagonalize (and we are done).

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let $A = M_e(f)$ be its standard matrix.

1. Compute $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$, these are the eigenvalues of f. For each eigenvalue λ_i, its algebraic multiplicity a_{λ_i} is the multiplicity it has as a root of $P_f(x)$. Note that $a_{\lambda_1} + \ldots + a_{\lambda_k} = n$.
3. For each λ_i, compute $\text{Null}(A - \lambda_i Id)$, the subspace of all eigenvectors with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for some eigenvalue λ_i, then f does not diagonalize (and we are done).

Procedure to diagonalize an endomorphism

5. If $g_{\lambda_i} = a_{\lambda_i}$ for all i, for each eigenvalue λ_i take a basis of $\text{Null}(A - \lambda_i Id)$
6. Let $\mathbf{u}$ be the collection of all these vectors

Then,

- $\mathbf{u}$ is a basis of $\mathbb{R}^n$.
- $M_{\mathbf{u}}(f)$ is a diagonal matrix whose entries are the eigenvalues:

$$M_{\mathbf{u}}(f) = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

Procedure to diagonalize an endomorphism

5. If $g_{\lambda_i} = a_{\lambda_i}$ for all i, for each eigenvalue λ_i take a basis of $\text{Null}(A - \lambda_i \, Id)$
6. Let $\mathbf{u}$ be the collection of all these vectors

Then,

- $\mathbf{u}$ is a basis of $\mathbb{R}^n$.
- $M_{\mathbf{u}}(f)$ is a diagonal matrix whose entries are the eigenvalues:

$$M_{\mathbf{u}}(f) = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

Diagonalizing endomorphisms

Recall that $D = M_{\mathbf{u}}(f)$ can be computed by doing a change of basis: if e is the standard basis of $\mathbb{R}^n$, then

$$A_{e\to\mathbf{u}}\, M(f)\, A_{\mathbf{u}\to e} = D.$$

(Equivalently, $A_{\mathbf{u}\to e}\, D\, A_{e\to\mathbf{u}} = M(f)$).

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A)$ = product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A)$ = sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A)$ = product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A)$ = sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A)$ = product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A)$ = sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A) =$ product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A) =$ sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A) =$ product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A) =$ sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Properties of the characteristic polynomial

If $A = M(f)$, we have:

- The term of $P_f(x)$ of degree n is $(-1)^n$.
- The term of $P_f(x)$ of degree $n-1$ is $(-1)^{n-1}tr(A)$ (the trace tr of a matrix A is defined as the sum of its diagonal entries).
- The constant term of $P_f(x)$ is $\det(A)$.

Note:

- Determinant $\det(A)$ = product of all eigenvalues (repeated if multiplicity > 1)
- Trace $tr(A)$ = sum of all eigenvalues (repeated if multiplicity > 1)

This might help in getting the eigenvalues of 2×2 or 3×3 matrices.

Outline

Motivation

Eigenvalues and Eigenvectors

Diagonalization theorem

Python

Python

```
>>> import numpy as np
>>> from numpy.linalg import *
```

>>> $A = np.array([[a_{11}, \ldots, a_{1n}], \ldots, [a_{n1}, \ldots, a_{nn}]])$

To get eigenvalues and eigenvectors of A we do:

```
>>> eigenval, eigenvec = eig(A)
```

Then we call both outputs to see the list of eigenvalues

```
>>> eigenval
```

gives the collection of eigenvalues of A; we can call each: `eigenval[0]` is the first eigenvalue, `eigenval[1]` the second...

```
>>> eigenvec
```

gives a matrix P whose columns are eigenvectors of A; we can call each: `eigenvec[:,0]` gives the first eigenvector (column) (corresponding to `eigenval[0]`), `eigenvec[:,1]` gives the second...

Python

If we want to create the diagonal matrix with eigenvalues:

```
>>> D = np.zeros((n,n),dtype='complex128')
>>> for i in range(n):
      D[i,i] = eigenval[i]
```

(where it says n we need to put the size of the matrix). Then we can check if PDP^{-1} is indeed A:

```
>>> eigenvec@ D @inv(eigenvec)
```

Note: In Python the complex number i is denoted as j.

To compute the powers of a matrix, A^k:

```
>>> matrix_power(A,k)
```