Àlgebra lineal i geometria
3. Diagonalització

Grau en Enginyeria Física
2023-24

Universitat Politècnica de Catalunya
Departament de Matemàtiques

Marta Casanellas
Universitat Politècnica de Catalunya

Outline

Eigenvalues and Eigenvectors

Diagonalization theorem

Applications

Bibliography

Outline

Eigenvalues and Eigenvectors

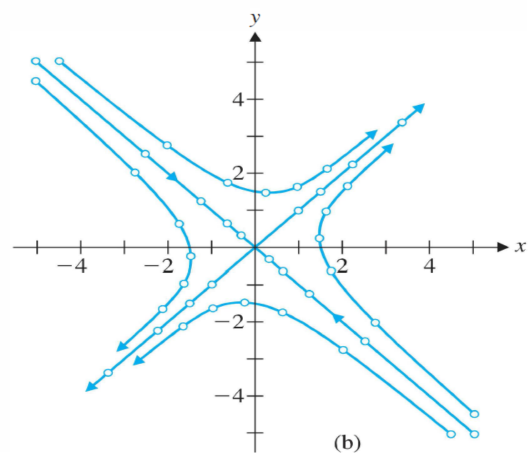
Diagonalization theorem

Applications

Bibliography

Motivation

$$A^m x$$



$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},$$

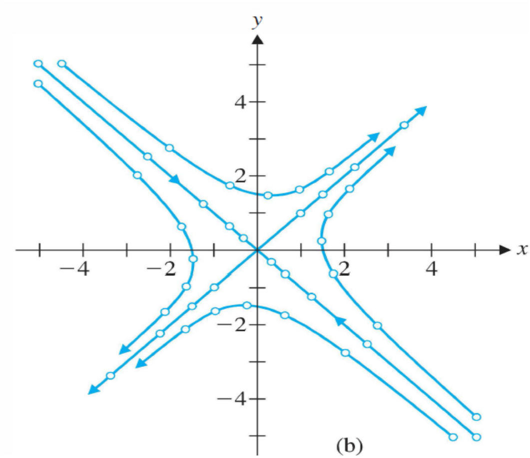
$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \dots$$

[D.Poole]

Motivation

$$A^m x$$



$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},$$

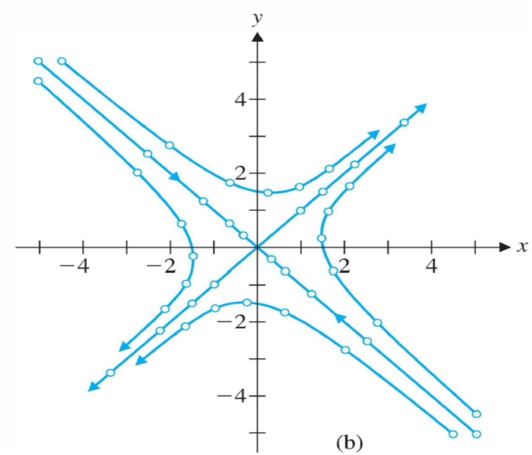
$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots$$

[D.Poole]

Motivation

$$A^m x$$



$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots$$

[D.Poole]

Motivation

From now on, E is a $\mathbb{K}$-e.v. of finite dimension n and $f \in End(E)$.

- Goal: compute powers of matrices.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is **diagonalizable** in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathbf{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

Motivation

From now on, E is a $\mathbb{K}$-e.v. of finite dimension n and $f \in End(E)$.

- Goal: compute powers of matrices.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is **diagonalizable** in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathbf{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

Motivation

From now on, E is a $\mathbb{K}$-e.v. of finite dimension n and $f \in End(E)$.

- Goal: compute powers of matrices.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is **diagonalizable** in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathbf{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

Motivation

From now on, E is a $\mathbb{K}$-e.v. of finite dimension n and $f \in End(E)$.

- Goal: compute powers of matrices.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is **diagonalizable** in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathrm{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

Motivation

From now on, E is a $\mathbb{K}$-e.v. of finite dimension n and $f \in End(E)$.

- Goal: compute powers of matrices.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is **diagonalizable** in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathbf{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

- An $n \times n$ matrix **diagonalizes** if there exists an invertible matrix P such that $P^{-1}MP$ is a diagonal matrix.
- If M diagonalizes, then $M = PDP^{-1}$ for a certain diagonal matrix D. Hence, M^m can be easily computed:

$$M^m = PDP^{-1}PDP^{-1} \dots PDP^{-1}PDP^{-1} = PD^mP^{-1}.$$

- An $n \times n$ matrix **diagonalizes** if there exists an invertible matrix P such that $P^{-1}MP$ is a diagonal matrix.
- If M diagonalizes, then $M = PDP^{-1}$ for a certain diagonal matrix D. Hence, M^m can be easily computed:

$$M^m = PDP^{-1}PDP^{-1} \ldots PDP^{-1}PDP^{-1} = PD^mP^{-1}.$$

Eigenvalues and eigenvectors

Remark: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ ($d_i = i$th value in the diagonal).

Definition

Let $f \in End(E)$. A vector $u \neq 0 \in E$ is an **eigenvector** (**VEP**) of f if $f(u) = \lambda\, u$ for some $\lambda \in \mathbb{K}$. In this case, we say that λ is an **eigenvalue** (**VAP**) of f and that u is an eigenvector with eigenvalue λ.

Example

Consider the endomorphism of $\mathbb{R}^2$ given by $f(x, y) = (5x, 2y)$. Then, $e_1 = (1, 0)$ is an eigenvector of f with eigenvalue 1, and $e_2 = (0, 1)$ is an eigenvector of f with eigenvalue 2.
The standard matrix of f is

$$M_e(f) = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Eigenvalues and eigenvectors

Remark: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ ($d_i = i$th value in the diagonal).

Definition

Let $f \in End(E)$. A vector $u \neq 0 \in E$ is an **eigenvector** (**VEP**) of f if $f(u) = \lambda\, u$ for some $\lambda \in \mathbb{K}$. In this case, we say that λ is an **eigenvalue** (**VAP**) of f and that u is an eigenvector with eigenvalue λ.

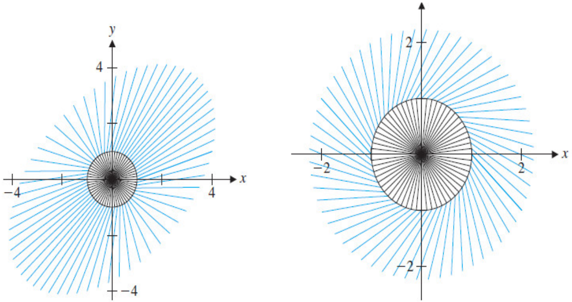
Example

Consider the endomorphism of $\mathbb{R}^2$ given by $f(x, y) = (5x, 2y)$.
Then, $e_1 = (1, 0)$ is an eigenvector of f with eigenvalue 1, and $e_2 = (0, 1)$ is an eigenvector of f with eigenvalue 2.
The standard matrix of f is

$$M_e(f) = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Geometric interpretation

In black v; in blue $f(v)$.



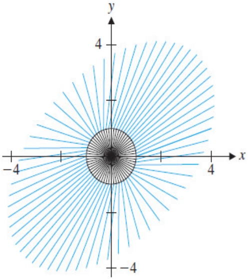
[D.Poole]

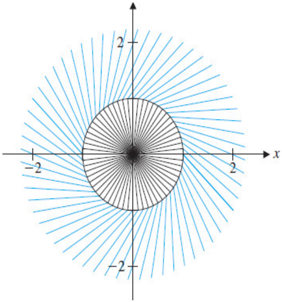
A map with eigenvectors $[(1,1)]$ and $[(1,-1)]$

A map with no eigenvectors in $\mathbb{R}$.

Geometric interpretation

In black v; in blue $f(v)$.





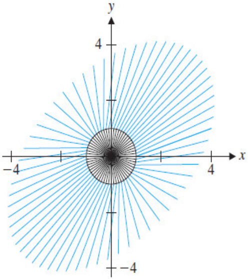
[D.Poole]

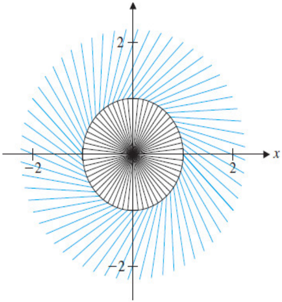
A map with eigenvectors $[(1,1)]$ and $[(1,-1)]$

A map with no eigenvectors in $\mathbb{R}$.

Geometric interpretation

In black v; in blue $f(v)$.





[D.Poole]

A map with eigenvectors $[(1,1)]$ and $[(1,-1)]$

A map with no eigenvectors in $\mathbb{R}$.

Eigenvectors and eigenvalues

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is a VAP of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP 0 $\Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvectors and eigenvalues

Lemma

- $u \in E$ *is a VEP of VAP* $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ *and* $u \neq 0$.
- $\lambda \in \mathbb{K}$ *is a VAP of* $f \Leftrightarrow \det(f - \lambda Id) = 0$.

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP 0 $\Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvectors and eigenvalues

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is a VAP of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP $0 \Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvectors and eigenvalues

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is a VAP of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP $0 \Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvectors and eigenvalues

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is a VAP of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP 0 $\Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvectors and eigenvalues

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is a VAP of $f \Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

For each λ VAP of f,

$$E_\lambda := \mathrm{Nuc}(f - \lambda Id) \subseteq E$$

is called the **eigenspace** of λ (**subespai propi**). This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

- $E_0 = \mathrm{Nuc}(f)$; $u \neq 0$ is a VEP with VAP 0 $\Leftrightarrow u \in \mathrm{Nuc}(f)$.
- The **spectrum** of f is the set of all its eigenvalues in $\mathbb{K}$ and is denoted by $\sigma(f)$.

Eigenvalues

Definition

The **characteristic polynomial of $A \in \mathcal{M}_n(\mathbb{K})$** is

$$P_A(x) := det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}.$$

If $f \in End(E)$, the **characteristic polynomial of f** is $p_A(x)$ where $A = M_{\mathbf{u}}(f)$ for some basis $\mathbf{u}$.

Properties

Proposition

1. $P_f(x)$ *does not depend on the basis* $\mathbf{u}$ *chosen.*
2. $P_f(x)$ *is a polynomial of degree* n, $P_f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$. *Moreover, if* $M_{\mathbf{u}}(f) = (a_{i,j})$, $\Rightarrow$
$$c_n = (-1)^n,$$
$$c_{n-1} = (-1)^{n-1}\mathrm{tr}(f) = (-1)^{n-1}(a_{1,1} + \ldots + a_{n,n}),$$
$$c_0 = \det(f).$$
3. *The roots of* $P_f(x)$ *are the eigenvalues of* f*, that is,* $\lambda \in \mathbb{K}$ *is an eigenvalue of* $f \Leftrightarrow P_f(\lambda) = 0$.

Properties

Proposition

1. $P_f(x)$ *does not depend on the basis* $\mathbf{u}$ *chosen.*
2. $P_f(x)$ *is a polynomial of degree* n*,* $P_f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$*. Moreover, if* $M_{\mathbf{u}}(f) = (a_{i,j})$, $\Rightarrow$
$$c_n = (-1)^n,$$
$$c_{n-1} = (-1)^{n-1}\mathrm{tr}(f) = (-1)^{n-1}(a_{1,1} + \ldots + a_{n,n}),$$
$$c_0 = \det(f).$$
3. *The roots of* $P_f(x)$ *are the eigenvalues of* f*, that is,* $\lambda \in \mathbb{K}$ *is an eigenvalue of* $f \Leftrightarrow P_f(\lambda) = 0$.

Properties

Proposition

1. *$P_f(x)$ does not depend on the basis $\mathbf{u}$ chosen.*
2. *$P_f(x)$ is a polynomial of degree n,*
   $P_f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$. Moreover, if
   $M_{\mathbf{u}}(f) = (a_{i,j})$, $\Rightarrow$

$$c_n = (-1)^n,$$

$$c_{n-1} = (-1)^{n-1}\mathrm{tr}(f) = (-1)^{n-1}(a_{1,1} + \ldots + a_{n,n}),$$

$$c_0 = \det(f).$$

3. *The roots of $P_f(x)$ are the eigenvalues of f, that is,*
   $\lambda \in \mathbb{K}$ is an eigenvalue of $f \Leftrightarrow P_f(\lambda) = 0$.

Properties

Proposition

1. *$P_f(x)$ does not depend on the basis $\mathbf{u}$ chosen.*
2. *$P_f(x)$ is a polynomial of degree n,*
   $P_f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$. *Moreover, if* $M_{\mathbf{u}}(f) = (a_{i,j})$, $\Rightarrow$
   $$c_n = (-1)^n,$$
   $$c_{n-1} = (-1)^{n-1}\mathrm{tr}(f) = (-1)^{n-1}(a_{1,1} + \ldots + a_{n,n}),$$
   $$c_0 = \det(f).$$
3. *The roots of $P_f(x)$ are the eigenvalues of f, that is,* $\lambda \in \mathbb{K}$ *is an eigenvalue of f* $\Leftrightarrow P_f(\lambda) = 0$.

Algebraic and geometric multiplicity of an eigenvalue

Definition

If λ is an eigenvalue of f, the **algebraic multiplicity of λ** , denoted by a_λ, is the multiplicity as a root of $P_f(x)$.

Definition

The **geometric multiplicity of λ**, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Nuc}(f - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Algebraic and geometric multiplicity of an eigenvalue

Definition

If λ is an eigenvalue of f, the **algebraic multiplicity of λ** , denoted by a_λ, is the multiplicity as a root of $P_f(x)$.

Definition

The **geometric multiplicity of λ**, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Nuc}(f - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Algebraic and geometric multiplicity of an eigenvalue

Definition

If λ is an eigenvalue of f, the **algebraic multiplicity of λ** , denoted by a_λ, is the multiplicity as a root of $P_f(x)$.

Definition

The **geometric multiplicity of λ**, denoted by g_λ, is the dimension of the vector subspace $\text{Nuc}(f - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Outline

Eigenvalues and Eigenvectors

Diagonalization theorem

Applications

Bibliography

Linear independency of VEPs

Let $f \in End(E)$. Then,

Lemma

- *If u, v are VEP's of different VAP's $\Rightarrow$ u, v are l.i.*
- *If $\lambda_1, \ldots, \lambda_r$ are different VAP's $\Rightarrow$ the sum $E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum,*

$$E_{\lambda_1} + \ldots + E_{\lambda_r} = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$$

Corollary

If $\lambda_1, \ldots \lambda_r$ are different eigenvalues of f and $B_i = \{v_1^i, \ldots v_{d_i}^i\}$ is a basis of E_{λ_i} for $i = 1, \ldots, r$, then $B_1 \cup \ldots \cup B_r$ is a collection of linearly independent vectors.

Linear independency of VEPs

Let $f \in End(E)$. Then,

Lemma

- *If u, v are VEP's of different VAP's $\Rightarrow$ u, v are l.i.*
- *If $\lambda_1, \ldots, \lambda_r$ are different VAP's $\Rightarrow$ the sum $E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum,*

$$E_{\lambda_1} + \ldots + E_{\lambda_r} = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$$

Corollary

If $\lambda_1, \ldots \lambda_r$ are different eigenvalues of f and $B_i = \{v_1^i, \ldots v_{d_i}^i\}$ is a basis of E_{λ_i} for $i = 1, \ldots, r$, then $B_1 \cup \ldots \cup B_r$ is a collection of linearly independent vectors.

Linear independency of VEPs

Let $f \in End(E)$. Then,

Lemma

- *If u, v are VEP's of different VAP's $\Rightarrow$ u, v are l.i.*
- *If $\lambda_1, \ldots, \lambda_r$ are different VAP's $\Rightarrow$ the sum $E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum,*

$$E_{\lambda_1} + \ldots + E_{\lambda_r} = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$$

Corollary

If $\lambda_1, \ldots \lambda_r$ are different eigenvalues of f and $B_i = \{v^i_1, \ldots v^i_{d_i}\}$ is a basis of E_{λ_i} for $i = 1, \ldots, r$, then $B_1 \cup \ldots \cup B_r$ is a collection of linearly independent vectors.

Linear independency of VEPs

Let $f \in End(E)$. Then,

Lemma

- *If u, v are VEP's of different VAP's $\Rightarrow$ u, v are l.i.*
- *If $\lambda_1, \ldots, \lambda_r$ are different VAP's $\Rightarrow$ the sum $E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum,*

$$E_{\lambda_1} + \ldots + E_{\lambda_r} = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$$

Corollary

If $\lambda_1, \ldots \lambda_r$ are different eigenvalues of f and $B_i = \{v_1^i, \ldots v_{d_i}^i\}$ is a basis of E_{λ_i} for $i = 1, \ldots, r$, then $B_1 \cup \ldots \cup B_r$ is a collection of linearly independent vectors.

Diagonalization theorem

Theorem (First Diagonalization Thm)

An endomorphism f of E is diagonalizable in $\mathbb{K}$ if and only if

1. *$P_f(x)$ has all its roots $\lambda_1, \ldots, \lambda_k$ in $\mathbb{K}$ (P_f fully decomposes in $\mathbb{K}$)*
   and
2. *for every VAP λ_i, the algebraic multiplicity and geometric multiplicity are equal: $g_{\lambda_i} = a_{\lambda_i}$.*

If it diagonalizes, it does so in a basis of VEP's.

Corollary

If all the roots of $P_f(x)$ are in $\mathbb{K}$ and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Theorem (First Diagonalization Thm)

An endomorphism f of E is diagonalizable in $\mathbb{K}$ if and only if

1. *$P_f(x)$ has all its roots $\lambda_1, \dots, \lambda_k$ in $\mathbb{K}$ (P_f fully decomposes in $\mathbb{K}$)*
   and
2. *for every VAP λ_i, the algebraic multiplicity and geometric multiplicity are equal: $g_{\lambda_i} = a_{\lambda_i}$.*

If it diagonalizes, it does so in a basis of VEP's.

Corollary

If all the roots of $P_f(x)$ are in $\mathbb{K}$ and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute $\mathrm{Nuc}(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute $\mathrm{Nuc}(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute $\mathrm{Nuc}(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute $\mathrm{Nuc}(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

Procedure to diagonalize an endomorphism

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute $\text{Nuc}(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

Procedure to diagonalize an endomorphism

In this case, for each eigenvalue λ_i, let $\{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ be a basis for $\mathrm{Nuc}(A - \lambda_i\, Id)$. Then,

1. $\mathbf{v} = \bigcup_{i=1}^k \{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ is a basis of $\mathbb{R}^n$.
2. $M_{\mathbf{v}}(f)$ is a diagonal matrix:

$$M_{\mathbf{v}}(f) = D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

Recall that $M_{\mathbf{v}}(f)$ can be computed by doing a change of basis: if $\mathbf{e}$ is the standard basis of $\mathbb{R}^n$ and , then

$$A_{\mathbf{e}\to\mathbf{v}}\, A\, A_{\mathbf{v}\to\mathbf{e}} = D.$$

(Equivalently, $A_{\mathbf{v}\to\mathbf{e}}\, D\, A_{\mathbf{e}\to\mathbf{v}} = A$).

Procedure to diagonalize an endomorphism

In this case, for each eigenvalue λ_i, let $\{v_1^i, \ldots, v_{a_{\lambda_i}}^i\}$ be a basis for $\mathrm{Nuc}(A - \lambda_i\, Id)$. Then,

1. $\mathbf{v} = \bigcup_{i=1}^k \{v_1^i, \ldots, v_{a_{\lambda_i}}^i\}$ is a basis of $\mathbb{R}^n$.
2. $M_{\mathbf{v}}(f)$ is a diagonal matrix:

$$M_{\mathbf{v}}(f) = D = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_k \end{pmatrix}.$$

Recall that $M_{\mathbf{v}}(f)$ can be computed by doing a change of basis: if $\mathbf{e}$ is the standard basis of $\mathbb{R}^n$ and , then

$$A_{\mathbf{e}\to\mathbf{v}}\, A\, A_{\mathbf{v}\to\mathbf{e}} = D.$$

(Equivalently, $A_{\mathbf{v}\to\mathbf{e}}\, D\, A_{\mathbf{e}\to\mathbf{v}} = A$).

Procedure to diagonalize an endomorphism

In this case, for each eigenvalue λ_i, let $\{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ be a basis for $\mathrm{Nuc}(A - \lambda_i\, Id)$. Then,

1. $\mathbf{v} = \bigcup_{i=1}^k \{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ is a basis of $\mathbb{R}^n$.
2. $M_{\mathbf{v}}(f)$ is a diagonal matrix:

$$M_{\mathbf{v}}(f) = D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

Recall that $M_{\mathbf{v}}(f)$ can be computed by doing a change of basis: if $\mathbf{e}$ is the standard basis of $\mathbb{R}^n$ and , then

$$A_{\mathbf{e}\to\mathbf{v}}\, A\, A_{\mathbf{v}\to\mathbf{e}} = D.$$

$$(\text{Equivalently, } A_{\mathbf{v}\to\mathbf{e}}\, D\, A_{\mathbf{e}\to\mathbf{v}} = A).$$

Triangularization of endomorphisms

Lemma

For any $f \in End(\mathbb{R}^n)$ there exists a basis $\mathbf{u}$ of $\mathbb{C}^n$ in which $M_{\mathbf{u}}(f)$ is triangular, has VAP's $\lambda_1, \ldots, \lambda_n$ (repeated if necessary) of f in the diagonal and

$$\det(A) = \lambda_1 \ldots \lambda_n,$$

$$tr(A) = \lambda_1 + \ldots + \lambda_n.$$

Better than triangular: we could get a Jordan canonical form, i.e. a block-diagonal matrix formed by blocks of type:

$$\begin{pmatrix} \lambda & 0 & \ldots & 0 \\ 1 & \lambda & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & \\ 0 & \ldots & 1 & \lambda \end{pmatrix}.$$

Outline

Eigenvalues and Eigenvectors

Diagonalization theorem

Applications

Bibliography

Study of $A^k x$ when $k \to \infty$

Let $A \in \mathcal{M}_n(\mathbb{R})$, with $A = PDP^{-1}$, $D = diag(\lambda_1, \ldots, \lambda_n)$, $P =$ change-of-basis $= A_{\mathbf{v} \to \mathbf{e}}$, $\mathbf{v} = \{v_1, \ldots, v_n\}$. Then,

- $A^k = PD^k P^{-1}$.
- If $x = c_1 v_1 + \cdots + c_n v_n \Rightarrow A^k x = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i|$, then λ_1^k grows faster than λ_i^k so, if $c_1 \neq 0$,

$$A^k x \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big, and}$$

- This is the "power method": the technical basis to efficiently compute VAPs (and VEPs).

Study of $A^k x$ when $k \to \infty$

Let $A \in \mathcal{M}_n(\mathbb{R})$, with $A = PDP^{-1}$, $D = diag(\lambda_1, \dots, \lambda_n)$, $P =$ change-of-basis $= A_{\mathbf{v}\to\mathbf{e}}$, $\mathbf{v} = \{v_1, \dots, v_n\}$. Then,

- $A^k = PD^kP^{-1}$.
- If $x = c_1 v_1 + \cdots + c_n v_n \Rightarrow A^k x = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i|$, then λ_1^k grows faster than λ_i^k so, if $c_1 \neq 0$,

$$A^k x \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big, and}$$

- This is the "power method": the technical basis to efficiently compute VAPs (and VEPs).

Study of $A^k x$ when $k \to \infty$

Let $A \in \mathcal{M}_n(\mathbb{R})$, with $A = PDP^{-1}$, $D = diag(\lambda_1, \dots, \lambda_n)$, $P =$ change-of-basis $= A_{\mathbf{v}\to\mathbf{e}}$, $\mathbf{v} = \{v_1, \dots, v_n\}$. Then,

- $A^k = PD^kP^{-1}$.
- If $x = c_1v_1 + \cdots + c_nv_n \Rightarrow A^kx = c_1\lambda_1^k v_1 + \cdots + c_n\lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i|$, then λ_1^k grows faster than λ_i^k so, if $c_1 \neq 0$,

$$A^kx \sim c_1\lambda_1^k v_1 \quad \text{for } k \text{ big, and}$$

- This is the "power method": the technical basis to efficiently compute VAPs (and VEPs).

Study of $A^k x$ when $k \to \infty$

Let $A \in \mathcal{M}_n(\mathbb{R})$, with $A = PDP^{-1}$, $D = diag(\lambda_1, \dots, \lambda_n)$, $P =$ change-of-basis $= A_{\mathbf{v}\to\mathbf{e}}$, $\mathbf{v} = \{v_1, \dots, v_n\}$. Then,

- $A^k = PD^kP^{-1}$.
- If $x = c_1 v_1 + \cdots + c_n v_n \Rightarrow A^k x = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i|$, then λ_1^k grows faster than λ_i^k so, if $c_1 \neq 0$,

$$A^k x \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big, and}$$

- This is the "power method": the technical basis to efficiently compute VAPs (and VEPs).

Discrete dynamical systems

Definition

A **homogeneous linear discrete dynamical system** is a matrix equation of the form

$$\mathtt{x}(k+1) = A\,\mathtt{x}(k), \quad k \in \mathbb{N},$$

where A is an $n \times n$ square matrix , and

$$\mathtt{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix} \in \mathbb{R}^n.$$

The vector $\mathtt{x}(0)$ is called an **initial condition**.
A **solution** (or trajectory) is a collection of vectors $\{\mathtt{x}(k)\}_{k\geq 0}$ such that each $\mathtt{x}(k)$ satisfies the equation above.

Lemma

The solutions to $\mathtt{x}(k+1) = A\,\mathtt{x}(k)$ are $\{\mathtt{x}(k) = A^k\mathtt{x}(0)\}_k$.

Discrete dynamical systems

Definition

A **homogeneous linear discrete dynamical system** is a matrix equation of the form

$$\mathtt{x}(k+1) = A\,\mathtt{x}(k), \quad k \in \mathbb{N},$$

where A is an $n \times n$ square matrix , and

$$\mathtt{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix} \in \mathbb{R}^n.$$

The vector $\mathtt{x}(0)$ is called an **initial condition**.
A **solution** (or trajectory) is a collection of vectors $\{\mathtt{x}(k)\}_{k\geq 0}$ such that each $\mathtt{x}(k)$ satisfies the equation above.

Lemma

The solutions to $\mathtt{x}(k+1) = A\,\mathtt{x}(k)$ are $\{\mathtt{x}(k) = A^k\mathtt{x}(0)\}_k$.

Discrete linear systems (cont.)

If A diagonalizes with eigenvalues $\lambda_1, \dots, \lambda_n$, and $\mathbf{v} = \{v_1, \dots, v_n\}$ is a basis of eigenvectors, the solutions $x(k) = A^k x(0)$ satisfy

- If $x(0) = c_1 v_1 + \cdots + c_n v_n \Rightarrow$
  $x(k) = A^k x(0) = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i| \Rightarrow \lambda_1^k$ grows faster than λ_i^k and if $c_1 \neq 0$,

$$x(k) \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big.}$$

Definition

If $|\lambda_1| > |\lambda_i|$, λ_1 is called the dominant eigenvalue.

Discrete linear systems (cont.)

If A diagonalizes with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\mathbf{v} = \{v_1, \ldots, v_n\}$ is a basis of eigenvectors, the solutions $x(k) = A^k x(0)$ satisfy

- If $x(0) = c_1 v_1 + \cdots + c_n v_n \Rightarrow$ $x(k) = A^k x(0) = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i| \Rightarrow \lambda_1^k$ grows faster than λ_i^k and if $c_1 \neq 0$,

$$x(k) \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big.}$$

Definition

If $|\lambda_1| > |\lambda_i|$, λ_1 is called the dominant eigenvalue.

Discrete linear systems (cont.)

If A diagonalizes with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\mathbf{v} = \{v_1, \ldots, v_n\}$ is a basis of eigenvectors, the solutions $x(k) = A^k x(0)$ satisfy

- If $x(0) = c_1 v_1 + \cdots + c_n v_n \Rightarrow$
  $x(k) = A^k x(0) = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i| \Rightarrow \lambda_1^k$ grows faster than λ_i^k and if $c_1 \neq 0$,

$$x(k) \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big.}$$

Definition

If $|\lambda_1| > |\lambda_i|$, λ_1 is called the dominant eigenvalue.

Stochastic matrices

Definition

A (column) **stochastic matrix** is a non-negative $n \times n$ matrix whose columns sum to 1.

A similar definition can be made for rows.

If A is a stochastic matrix we have:

- 1 is an eigenvalue of A.
- If x sums to 1, then Ax still sums to 1.

Stochastic matrices

Definition

A (column) **stochastic matrix** is a non-negative $n \times n$ matrix whose columns sum to 1.

A similar definition can be made for rows.
If A is a stochastic matrix we have:

- 1 is an eigenvalue of A.
- If x sums to 1, then Ax still sums to 1.

Stochastic matrices

Definition

A (column) **stochastic matrix** is a non-negative $n \times n$ matrix whose columns sum to 1.

A similar definition can be made for rows.
If A is a stochastic matrix we have:

- 1 is an eigenvalue of A.
- If x sums to 1, then Ax still sums to 1.

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

- *$1 > |\lambda|$ for any other VAP λ*
- $g_1 = 1$
- *1 has a nonegative VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to sum to 1, then we have*

$$\lim A^k = (v\, v \dots v)$$

$$and \qquad \lim A^k \mathrm{x} = v$$

for any positive vector x that sums to 1.

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

- *$1 > |\lambda|$ for any other VAP λ*
- $g_1 = 1$
- *1 has a nonegative VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to sum to 1, then we have*

$$\lim A^k = (v\, v \ldots v)$$

$$and \qquad \lim A^k \mathrm{x} = v$$

for any positive vector x that sums to 1.

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

- *$1 > |\lambda|$ for any other VAP λ*
- $g_1 = 1$
- *1 has a nonegative VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to sum to 1, then we have*

$$\lim A^k = (v\, v \dots v)$$

$$\text{and} \qquad \lim A^k \mathrm{x} = v$$

for any positive vector x that sums to 1.

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

- *$1 > |\lambda|$ for any other VAP λ*
- $g_1 = 1$
- *1 has a nonegative VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to sum to 1, then we have*

$$\lim A^k = (v\, v \ldots v)$$

$$\text{and} \qquad \lim A^k \mathrm{x} = v$$

for any positive vector x that sums to 1.

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

- $1 > |\lambda|$ *for any other VAP* λ
- $g_1 = 1$
- *1 has a nonegative VEP* v.
- *no other VAP has positive eigenvectors.*
- *If we take* v *to sum to 1, then we have*

$$\lim A^k = (v\, v \dots v)$$

$$and \qquad \lim A^k \mathrm{x} = v$$

for any positive vector x *that sums to 1.*

Perron-Frobenius Theorem for stochastic matrices

Theorem

If A is a positive stochastic matrix, then 1 is VAP and

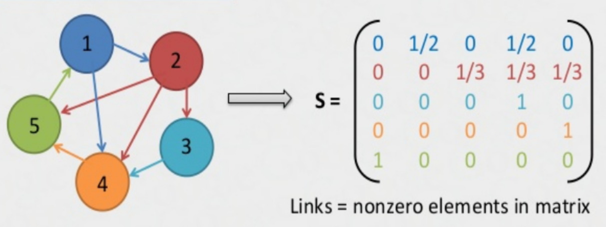
- *$1 > |\lambda|$ for any other VAP λ*
- $g_1 = 1$
- *1 has a nonegative VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to sum to 1, then we have*

$$\lim A^k = (v\, v \dots v)$$

$$and \qquad \lim A^k \mathtt{x} = v$$

for any positive vector $\mathtt{x}$ that sums to 1.

Ranking de pàgines web (Google pagerank)



Want v positive such that $Av = av$ $(A = S^t)$ for some $a \Rightarrow v=$ dominant eigenvector, $a = 1$.
Computation of v: $\lim A^k \mathtt{x}$ for any positive $\mathtt{x}$.

Cayley-Hamilton theorem

Theorem (Cayley-Hamilton)

If $f \in End(\mathbb{R}^n)$ has characteristic polynomial $P_f(x) = a_0 + a_1 x + \ldots + a_n x^n$ and A is its standard matrix of, then

$$a_0 Id + a_1 A + \cdots + a_n A^n = 0.$$

Consequences:

- A^n can be computed as a linear combination of $Id, A, A^2, \ldots, A^{n-1}$
- If A is invertible $\Rightarrow$ A^{-1} can be computed as a linear combination of $Id, A, \ldots, A^{n-1}$
- Useful to compute $exp(A)$ (next slide).

Cayley-Hamilton theorem

Theorem (Cayley-Hamilton)

If $f \in End(\mathbb{R}^n)$ has characteristic polynomial $P_f(x) = a_0 + a_1 x + \ldots + a_n x^n$ and A is its standard matrix of, then

$$a_0 Id + a_1 A + \cdots + a_n A^n = 0.$$

Consequences:

- A^n can be computed as a linear combination of $Id, A, A^2, \ldots, A^{n-1}$
- If A is invertible $\Rightarrow$ A^{-1} can be computed as a linear combination of $Id, A, \ldots, A^{n-1}$
- Useful to compute $exp(A)$ (next slide).

Cayley-Hamilton theorem

Theorem (Cayley-Hamilton)

If $f \in End(\mathbb{R}^n)$ *has characteristic polynomial* $P_f(x) = a_0 + a_1 x + \ldots + a_n x^n$ *and* A *is its standard matrix of, then*

$$a_0 Id + a_1 A + \cdots + a_n A^n = 0.$$

Consequences:

- A^n can be computed as a linear combination of $Id, A, A^2, \ldots, A^{n-1}$
- If A is invertible $\Rightarrow$ A^{-1} can be computed as a linear combination of $Id, A, \ldots, A^{n-1}$
- Useful to compute $exp(A)$ (next slide).

Cayley-Hamilton theorem

Theorem (Cayley-Hamilton)

If $f \in End(\mathbb{R}^n)$ *has characteristic polynomial* $P_f(x) = a_0 + a_1 x + \ldots + a_n x^n$ *and* A *is its standard matrix of, then*

$$a_0 Id + a_1 A + \cdots + a_n A^n = 0.$$

Consequences:

- A^n can be computed as a linear combination of $Id, A, A^2, \ldots, A^{n-1}$
- If A is invertible $\Rightarrow$ A^{-1} can be computed as a linear combination of $Id, A, \ldots, A^{n-1}$
- Useful to compute $exp(A)$ (next slide).

Exponential of a matrix

If f is a diagonalizable endomorphism with standard matrix $A \in M_n(\mathbb{R})$, so that $A = S\ D\ S^{-1}$, where D diagonal: $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $S = A_{\mathbf{v}\to\mathbf{e}}$, and $\mathbf{v} = \{v_1, \ldots, v_n\}$ is the corresponding basis of eigenvectors. Then, we define the **exponential** of the matrix A:

$$e^A = S\ e^D\ S^{-1} = S \begin{pmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n} \end{pmatrix} S^{-1}$$

and this coincides with $\sum_{n\geq 0} \frac{A^n}{n!}$.

Real matrices with complex eigenvalues

If A is a real matrix and we allow diagonalization in $\mathbb{K} = \mathbb{C}$, then VAP's and VEP's go "in pairs":

- $p_A(x) \in \mathbb{R}[x] \Rightarrow \lambda$ is a VAP of A if and only if $\bar{\lambda}$ is also a VAP.
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $w \in \mathbb{C}^n$ is a VEP with VAP λ, then we can write $w = u + iv$ where $u, v \in \mathbb{R}^n$.
- Then, $\bar{w} := u - iv$ is an eigenvector with eigenvalue $\bar{\lambda}$.

If we want to work only in $\mathbb{R}$, the one can rearrange complex VAP's and VEP's in conjugate pairs to obtain a "diagonalization" of A in 2×2 blocks: use $[u, v]$ instead of $w, \bar{w}$.

Real matrices with complex eigenvalues

If A is a real matrix and we allow diagonalization in $\mathbb{K} = \mathbb{C}$, then VAP's and VEP's go "in pairs":

- $p_A(x) \in \mathbb{R}[x] \Rightarrow \lambda$ is a VAP of A if and only if $\bar{\lambda}$ is also a VAP.
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $w \in \mathbb{C}^n$ is a VEP with VAP λ, then we can write $w = u + iv$ where $u, v \in \mathbb{R}^n$.
- Then, $\bar{w} := u - iv$ is an eigenvector with eigenvalue $\bar{\lambda}$.

If we want to work only in $\mathbb{R}$, the one can rearrange complex VAP's and VEP's in conjugate pairs to obtain a "diagonalization" of A in 2×2 blocks: use $[u, v]$ instead of $w, \bar{w}$.

Real matrices with complex eigenvalues

If A is a real matrix and we allow diagonalization in $\mathbb{K} = \mathbb{C}$, then VAP's and VEP's go "in pairs":

- $p_A(x) \in \mathbb{R}[x] \Rightarrow \lambda$ is a VAP of A if and only if $\bar{\lambda}$ is also a VAP.
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $w \in \mathbb{C}^n$ is a VEP with VAP λ, then we can write $w = u + iv$ where $u, v \in \mathbb{R}^n$.
- Then, $\bar{w} := u - iv$ is an eigenvector with eigenvalue $\bar{\lambda}$.

If we want to work only in $\mathbb{R}$, the one can rearrange complex VAP's and VEP's in conjugate pairs to obtain a "diagonalization" of A in 2×2 blocks: use $[u, v]$ instead of $w, \bar{w}$.

Real matrices with complex eigenvalues

If A is a real matrix and we allow diagonalization in $\mathbb{K} = \mathbb{C}$, then VAP's and VEP's go "in pairs":

- $p_A(x) \in \mathbb{R}[x] \Rightarrow \lambda$ is a VAP of A if and only if $\bar{\lambda}$ is also a VAP.
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $w \in \mathbb{C}^n$ is a VEP with VAP λ, then we can write $w = u + iv$ where $u, v \in \mathbb{R}^n$.
- Then, $\bar{w} := u - iv$ is an eigenvector with eigenvalue $\bar{\lambda}$.

If we want to work only in $\mathbb{R}$, the one can rearrange complex VAP's and VEP's in conjugate pairs to obtain a "diagonalization" of A in 2×2 blocks: use $[u, v]$ instead of $w, \bar{w}$.

Outline

Eigenvalues and Eigenvectors

Diagonalization theorem

Applications

Bibliography

Bibliography

Basic:

- D. Poole, Linear Algebra, A modern introduction (3rd edition), Brooks/Cole, 2011. Chapter 6.

Additional

- Hernández Rodríguez, E.; Vàzquez Gallo, M.J.; Zurro Moro, M.A. Álgebra lineal y geometría [en línia]