Linear systems

Bioinformatics Degree
Algebra

Departament de Matemàtiques

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Outline

Linear systems

Solving linear systems

Python

Outline

Outline

Outline

Linear systems

Definition

A system of m linear equations with n variables is a collection of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where the coefficients a_{ij}, the constant terms $b_1, b_2, \ldots, b_m$ and the values that the unknowns $x_1, x_2, \ldots, x_n$ are real numbers.

A system is homogenous if $b_i = 0$ for $i = 1, \ldots, m$.

Linear systems

A particular solution is a list of values for the unknowns $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ that is a solution to all the equations.
The general solution is the set of all the solutions to the system.

Geometric interpretation
From a geometric point of view, the general solution to a linear system describes a linear variety (a point, a line, a plane, etc.).
Each particular solution is a point of the linear variety.

Matrix expression of a linear system

Any linear system can be put as a matrix equation $Ax = b$ by taking

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

The matrix A is called the matrix of the system.
The augmented matrix is $(A \mid b)$.

Number of solutions

Theorem

Any linear system has either (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions.

A linear system is consistent if it has one or more solutions. If it does not have solutions, it is inconsistent.

Example

(i) $\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$ *(ii)* $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$ *(iii)* $\begin{cases} x_1 - x_2 = 0 \\ x_2 - x_2 = 0 \end{cases}$

Number of solutions

Theorem

Any linear system has either (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions.

A linear system is consistent if it has one or more solutions. If it does not have solutions, it is inconsistent.

Example

(i) $\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$ *(ii)* $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$ *(iii)* $\begin{cases} x_1 - x_2 = 0 \\ x_2 - x_2 = 0 \end{cases}$

Rouché-Frobenius Theorem

The matrix expression of linear systems of equations allow us to know how many solutions the system has:

Theorem (Rouché-Frobenius)

- *$Ax = b$ is consistent* **if and only if** *$rank(A) = rank(A|b)$.*

  In this case, its set of solutions depends on $n - rank(A)$ free variables. This value is known as the degrees of freedom of the system.

- *In particular, if $n = rank(A)$ the solution is unique.*

Rouché-Frobenius Theorem

The matrix expression of linear systems of equations allow us to know how many solutions the system has:

Theorem (Rouché-Frobenius)

- *$Ax = b$ is consistent* **if and only if** *$rank(A) = rank(A|b)$.*

  In this case, its set of solutions depends on $n - rank(A)$ free variables. This value is known as the degrees of freedom of the system.

- *In particular, if $n = rank(A)$ the solution is unique.*

Rouché-Frobenius Theorem

The matrix expression of linear systems of equations allow us to know how many solutions the system has:

Theorem (Rouché-Frobenius)

- *$Ax = b$ is consistent* **if and only if** *$rank(A) = rank(A|b)$.*

  In this case, its set of solutions depends on $n - rank(A)$ free variables. This value is known as the degrees of freedom of the system.

- *In particular, if $n = rank(A)$ the solution is unique.*

Outline

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.
Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.
Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.
Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.
Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.

Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Solving systems: Gaussian elimination

Goal: convert the system $Ax = b$ to a simpler system using elementary transformations.

Consider the augmented matrix $(A \mid b)$ and

1st step Reduce $(A \mid b)$ to **row echelon form**.

2nd step Solve the system by **back substitution** if it is consistent.

- The number of pivots (rank) of the row echelon form of A and $(A|b)$ tells us whether the system is consistent or not.
- If the system is consistent, then the **leading variables** corresponding to pivots can be written in terms of the other variables (called **free variables**).
- The number of free variables is the **degrees of freedom** of the system.

Back substitution and Gauss-Jordan elimination

The back substitution step can also be performed by elementary row operations on the row echelon form of $(A|b)$ by **Gauss-Jordan elimination**:

Once we have a matrix in *row echelon form*, do:

1. start with the rightmost pivot and use an operation of type E_2 to convert it to 1.
2. from bottom to top: make all the entries above the pivot equal to zero using type E_3.
3. Repeat the previous steps the next column to the left (so, from right to left).

Back substitution and Gauss-Jordan elimination

The back substitution step can also be performed by elementary row operations on the row echelon form of $(A|b)$ by **Gauss-Jordan elimination**:

Once we have a matrix in *row echelon form*, do:

1. start with the rightmost pivot and use an operation of type E_2 to convert it to 1.
2. from bottom to top: make all the entries above the pivot equal to zero using type E_3.
3. Repeat the previous steps the next column to the left (so, from right to left).

Back substitution and Gauss-Jordan elimination

The back substitution step can also be performed by elementary row operations on the row echelon form of $(A|b)$ by **Gauss-Jordan elimination**:

Once we have a matrix in *row echelon form*, do:

1. start with the rightmost pivot and use an operation of type E_2 to convert it to 1.
2. from bottom to top: make all the entries above the pivot equal to zero using type E_3.
3. Repeat the previous steps the next column to the left (so, from right to left).

Reduced row echelon form

In this way we obtain a matrix in **row reduced echelon form**, that is a matrix of the following form:

$$A = \begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition

A matrix is in **row reduced echelon form** if it is in row echelon form and

- all pivots are 1
- the pivots are the only non-zero entries in its column.

Reduced row echelon form

In this way we obtain a matrix in **row reduced echelon form**, that is a matrix of the following form:

$$A = \begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition

A matrix is in **row reduced echelon form** if it is in row echelon form and

- all pivots are 1
- the pivots are the only non-zero entries in its column.

Reduced row echelon form

In this way we obtain a matrix in **row reduced echelon form**, that is a matrix of the following form:

$$A = \begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition

A matrix is in **row reduced echelon form** if it is in row echelon form and

- all pivots are 1
- the pivots are the only non-zero entries in its column.

Row reduced echelon form

- If A square and the row reduced echelon form is Id_n, then $Ax = b$ can be trivially solved: the solution is the new independent term

$$(A \,|\, b) \sim \cdots \sim (Id_n \,|\, b') \quad \text{so} \quad Ax = b \Leftrightarrow Id_n x = b' \Leftrightarrow x = b'$$

- Whereas the row echelon form of A is not unique, the row *reduced* echelon form is unique

Row reduced echelon form

- If A square and the row reduced echelon form is Id_n, then $Ax = b$ can be trivially solved: the solution is the new independent term

$$(A \mid b) \sim \cdots \sim (Id_n \mid b') \quad \text{so} \quad Ax = b \Leftrightarrow Id_n x = b' \Leftrightarrow x = b'$$

- Whereas the row echelon form of A is not unique, the row *reduced* echelon form is unique

Solving simultaneous systems

Goal: solve systems with the same $m \times n$ matrix A but different independent terms,

$$Ax^{(1)} = b^{(1)}, Ax^{(2)} = b^{(2)}, \ldots, Ax^{(r)} = b^{(r)}.$$

Equivalently: find X $m \times r$ matrix such that

$$AX = \underbrace{\begin{pmatrix} b^{(1)} \, b^{(2)} \, \ldots \, b^{(r)} \end{pmatrix}}_{B}.$$

matrix equation $AX = B$

Efficient solution: Gauss-Jordan elimination to the following augmented matrix

$$\left(A \,|\, b^{(1)} \, b^{(2)} \, \ldots \, b^{(r)} \right)$$

Solving simultaneous systems

Goal: solve systems with the same $m \times n$ matrix A but different independent terms,

$$Ax^{(1)} = b^{(1)}, Ax^{(2)} = b^{(2)}, \ldots, Ax^{(r)} = b^{(r)}.$$

Equivalently: find X $m \times r$ matrix such that

$$AX = \underbrace{\begin{pmatrix} b^{(1)} \, b^{(2)} \, \ldots \, b^{(r)} \end{pmatrix}}_{B}.$$

matrix equation $AX = B$

Efficient solution: Gauss-Jordan elimination to the following augmented matrix

$$\left(A \,|\, b^{(1)} \, b^{(2)} \, \ldots \, b^{(r)} \right)$$

Application: finding the inverse of a matrix

The previous algorithm is useful to find the inverse of a matrix.
Input: a square matrix A.
Output: the inverse of A if A is nonsingular, or that the inverse does not exist (if A is singular).

1. Form the $n \times 2n$ matrix $M = (A \mid Id_n)$
2. Reduce M to row echelon form (*Gaussian elimination*). This process generates a zero row in the left half of M if and only if A has no inverse.
3. Reduce the matrix to its row reduced echelon form (*Gauss-Jordan*). In the end, we obtain $M \sim (Id_n \mid B)$, where the identity matrix Id_n has replaced A in the left half.
4. Then $A^{-1} = B$, the matrix that is now in the right half.

Application: finding the inverse of a matrix

The previous algorithm is useful to find the inverse of a matrix.
Input: a square matrix A.
Output: the inverse of A if A is nonsingular, or that the inverse does not exist (if A is singular).

1. Form the $n \times 2n$ matrix $M = (A \mid Id_n)$
2. Reduce M to row echelon form (*Gaussian elimination*). This process generates a zero row in the left half of M if and only if A has no inverse.
3. Reduce the matrix to its row reduced echelon form (*Gauss-Jordan*). In the end, we obtain $M \sim (Id_n \mid B)$, where the identity matrix Id_n has replaced A in the left half.
4. Then $A^{-1} = B$, the matrix that is now in the right half.

Application: finding the inverse of a matrix

The previous algorithm is useful to find the inverse of a matrix.
Input: a square matrix A.
Output: the inverse of A if A is nonsingular, or that the inverse does not exist (if A is singular).

1. Form the $n \times 2n$ matrix $M = (A \mid Id_n)$
2. Reduce M to row echelon form (*Gaussian elimination*). This process generates a zero row in the left half of M if and only if A has no inverse.
3. Reduce the matrix to its row reduced echelon form (*Gauss-Jordan*). In the end, we obtain $M \sim (Id_n \mid B)$, where the identity matrix Id_n has replaced A in the left half.
4. Then $A^{-1} = B$, the matrix that is now in the right half.

Application: finding the inverse of a matrix

The previous algorithm is useful to find the inverse of a matrix.
Input: a square matrix A.
Output: the inverse of A if A is nonsingular, or that the inverse does not exist (if A is singular).

1. Form the $n \times 2n$ matrix $M = (A \mid Id_n)$
2. Reduce M to row echelon form (*Gaussian elimination*). This process generates a zero row in the left half of M if and only if A has no inverse.
3. Reduce the matrix to its row reduced echelon form (*Gauss-Jordan*). In the end, we obtain $M \sim (Id_n \mid B)$, where the identity matrix Id_n has replaced A in the left half.
4. Then $A^{-1} = B$, the matrix that is now in the right half.

Outline

Python

```
import numpy as np
from numpy.linalg import *
```

A $=$ np.array([[$a_{11}, \ldots, a_{1n}$], [$a_{21}, \ldots, a_{2n}$], $\ldots$, [$a_{n1}, \ldots, a_{nn}$]])

b $=$ np.array([$b_1, b_2, \ldots, b_m$])

If A is an invertible square matrix, we can solve the system by using:

```
solve(A,b)
```

Python

```
import numpy as np
from numpy.linalg import *
```

A $= \texttt{np.array}([[\mathtt{a}_{11}, \ldots, \mathtt{a}_{1n}], [\mathtt{a}_{21}, \ldots, \mathtt{a}_{2n}], \ldots, [\mathtt{a}_{n1}, \ldots, \mathtt{a}_{nn}]])$

b $= \texttt{np.array}([\mathtt{b}_1, \mathtt{b}_2, \ldots, \mathtt{b}_m])$

If A is an invertible square matrix, we can solve the system by using:

```
solve(A,b)
```