Àlgebra lineal i geometria
1. Espais vectorials

Grau en Enginyeria Física
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Universitat Politècnica de Catalunya
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Outline

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The vector space $\mathbb{R}^n$

We consider the set of *n*-tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$$

and we call its elements **vectors**.

Notation: When we talk about $v \in \mathbb{R}^n$ we usually think of v as a column vector,

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

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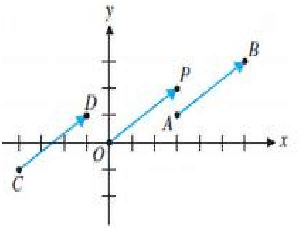
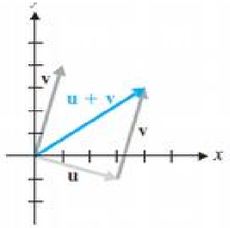
$\mathbb{R}^2$: Physical interpretation

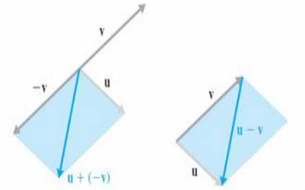
- View $(x, y) \in \mathbb{R}^2$ as a directed line segment between two points A and B, $(x, y) =$ "vector" $\overrightarrow{AB}$.
- $\overrightarrow{AB}$: the displacement needed to get from A to B: x units along the x-axis and y along the y-axis.
- Two vectors are equal if they represent the same displacement ($\Leftrightarrow$ they have the same length, direction, and sense).
- We can always think (x, y) as a vector of initial point $(0, 0)$ and end point (x, y).



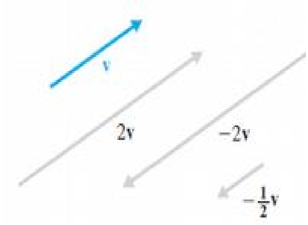
Operations in $\mathbb{R}^2$

We can sum or substract vectors





and multiply a vector by a constant (*scalar*)



$\mathbb{R}^3$

- Vectors in $\mathbb{R}^3$ have a similar physical interpretation
- We can also sum two vectors and multiply a vector by a scalar. These operations can be done in coordinates: if $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

$$c \cdot u = (cx_1, cx_2, cx_3) \text{ for any } c \in \mathbb{R}.$$

Operations in $\mathbb{R}^n$

In $\mathbb{R}^n$ we define the following operations:

sum: if $u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n.$$

scalar multiplication: if $u = (x_1, x_2, \ldots, x_n), c \in \mathbb{R}$, then

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Proposition

These operations in $\mathbb{R}^n$ satisfy the following properties:

1. $u + v = v + u$. *Commutativity*
2. $(u + v) + w = u + (v + w)$. *Associativity*
3. $\exists$ *an element* $\mathbf{0} \in \mathbb{R}^n$, *called the zero vector, such that* $u + \mathbf{0} = u$.
4. *For each* $u \in \mathbb{R}^n$, $\exists$ *an element* $-u \in \mathbb{R}^n$ *such that* $u + (-u) = \mathbf{0}$.
5. $c \cdot (u + v) = c \cdot u + c \cdot v$. *Distributivity*
6. $(c + d) \cdot u = c \cdot u + d \cdot u$. *Distributivity*
7. $c \cdot (d \cdot u) = (cd) \cdot u$.
8. $1 \cdot u = u$.

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Vector space over $\mathbb{K}$

Let $\mathbb{K}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or any (commutative) field (*"cos"*).

A **vector space over $\mathbb{K}$** (or **$\mathbb{K}$-e.v.**) is a set E with two operations $+$ and $\cdot$,

$+$ given $u, v \in E$, it assigns another element $u + v$ of E.

$\cdot$ given $u \in E$ and a scalar $c \in \mathbb{K}$, it assigns an element $c \cdot u \in E$

that satisfy the previous properties, i.e,

- $+$ is commutative, associative, has a neutral element (denoted $\mathbf{0}$ or $\vec{0}$) and every $u \in E$ has an opposite with respect to $+$ (denoted $-u$),
- $\cdot$ and $+$ satisfy:

$$c\cdot(u+v) = c\cdot u+c\cdot v,\ \ (c+d)\cdot u = c\cdot u+d\cdot u,\ \ c\cdot(d\cdot u) = (cd)\cdot u,\ \ 1\cdot u = u$$

for any $u, v \in E$ and $c, d \in \mathbb{K}$.

The elements of a $\mathbb{K}$-e.v. are called **vectors**.

Examples of vector spaces

- $\mathbb{K}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{K}\}$ is a $\mathbb{K}$-e.v. with the natural sum and product inherited by $\mathbb{K}$.
- $\mathcal{M}_{m\times n}(\mathbb{R})$=$m \times n$ matrices with entries in $\mathbb{R}$ and the natural operations of sum of matrices and multiplication by scalars is an $\mathbb{R}$-e.v.
- The set of polynomials of degree $\leq d$, $\mathbb{R}_d[x] = \{p(x) = a_0 + a_1 x + \ldots + a_d x^d \mid a_i \in \mathbb{R}\}$, is an $\mathbb{R}$-e.v. with the usual sum of polynomials and multplication by a scalar.
- $\mathbb{R}[x] = \{\text{polynomials in one variable x and coefficients in } \mathbb{R}\}$ is an $\mathbb{R}$-e.v.
- The set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-e.v. with the usual sum of functions ($f + g$ is the function $(f+g)(x) = f(x) + g(x)$) and product by a scalar ($c \cdot f$ is the function $(c \cdot f)(x) = cf(x)$).

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Properties

If E is a $\mathbb{K}$-e.v., then, $\forall u \in E$, $c \in \mathbb{K}$,

(a) $0 \cdot u = \mathbf{0} = c \cdot \mathbf{0}$,

(b) $(-1) \cdot u = -u$,

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Note: Usually we omit the dot "$\cdot$"

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Linear combinations

Definition

A vector u is a **linear combination** of vectors $u_1, \ldots, u_k$ if there are scalars $c_1, \ldots, c_k$ such that $u = c_1\, u_1 + \ldots + c_k\, u_k$ (the scalars c_i are the **coefficients** of the linear combination).

Finding out whether a vector in $\mathbb{K}^n$ is a linear combination of a collection of given vectors is equivalent to solving a linear system of equations:

Proposition

A system $Ax = b$ is consistent if and only if b is a linear combination of the columns of A.

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Vector subspaces

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Let E be a $\mathbb{K}$-e.v. Then a subset $V \neq \emptyset$ of E is a **vector subspace** if V is itself a vector space (with $+$ and $\cdot$ of E). This is equivalent to:

1. If u and v are in V, then $u + v$ is in V.
2. If u is in V and c is a scalar, then $c \cdot u$ is in V.

Ex:

- $V = \mathbb{K}^n$ is a vector subspace of $\mathbb{K}^n$.
- $V = \{\mathbf{0}\}$ is a vector subspace (of any E).
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0, 3z = 0\}$ is a vector subspace of $\mathbb{R}^3$.
- $F = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$.

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Remarks

- Every subspace contains the zero vector.
- Properties 1 and 2 can be combined:
  $V \neq \emptyset$ is a subspace $\Leftrightarrow$ for any $u_1, \ldots, u_k$ in V and $c_1, \ldots, c_k$ in $\mathbb{K}$, the linear combination
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  That is, vector subspaces are closed under linear combinations.

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Let $Ax = 0$ be a linear system, where $A \in \mathcal{M}_{m,n}(\mathbb{K})$. Then, the set of solutions $V = \{v \in \mathbb{K}^n \mid Av = 0\}$ is a vector subspace of $\mathbb{K}^n$.

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Definition

The set of all linear combinations of $v_1, v_2, \ldots, v_k$,

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is the called the **span of** $v_1, v_2, \ldots, v_k$ and is denoted as $[v_1, v_2, \ldots, v_k]$.

Proposition

$V = [v_1, v_2, \ldots, v_k]$ is a vector subspace and is the smallest subspace containing $\{v_1, \ldots, v_k\}$.

We say that $\{v_1, v_2, \ldots, v_k\}$ is a **system of generators of** V, and also that V is **spanned by** $v_1, v_2, \ldots, v_k$.

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Examples:

- $\mathbb{R}^n = [(1,0,\ldots,0),\ldots,(0,\ldots,0,1)]$.
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A vector space E is **finitely generated (f.g.)** if it is the span of a finite collection of vectors.

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Outline

Linear dependency

Definition

$v_1, v_2, \ldots, v_k \in E$ are **linearly dependent (l.d.)** if there are scalars $c_1, c_2, \ldots, c_k$, at least one $\neq 0$, such that $c_1\, v_1 + \ldots + c_k\, v_k = \mathbf{0}$. Otherwise, we say that $v_1, v_2, \ldots, v_k$ are **linearly independent (l.i.)**.

$v_1, v_2, \ldots, v_k$ are l.i. $\Leftrightarrow$ any linear combination $c_1\, v_1 + \ldots + c_k\, v_k = \mathbf{0}$ implies $c_1 = c_2 = \ldots = c_k = 0$.

Remarks:

1. Any set of vectors containing $\mathbf{0}$ is linearly dependent.
2. Two vectors v_1, v_2 are l.d. $\Leftrightarrow$ one is multiple of the other.
3. $v_1, v_2, \ldots, v_k$ in E are l.d. $\Leftrightarrow$ at least one of the vectors can be expressed as a linear combination of the others.
4. If $v_1, \ldots, v_k$ are l.i. and $u \notin [v_1, \ldots, v_k] \Rightarrow v_1, \ldots, v_k, u$ are l.i.

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Basis of a vector subspace

Definition

Let $V \subset E$ be a vector subspace. A collection of vectors $\{v_1, \ldots, v_k\}$ is a **basis of V** if

1. $V = [v_1, \ldots, v_k]$ and
2. $\{v_1, \ldots, v_k\}$ is linearly independent.

Ex: $(1, 1, 0), (0, 0, 1)$ is a basis of $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\}$.

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Standard basis

There are some **standard** (or *natural*, *canonical*) bases of certain vector spaces:

- $\mathbb{K}^n$: $\{e_1, e_2, \ldots, e_n\}$ where $e_i = (0, \ldots, \overset{i)}{1}, \ldots, 0)$ for $i = 1, 2, \ldots, n$.
- $\mathcal{M}_{m\times n}$: $\{E_{i,j}\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ where $E_{i,j} \in \mathcal{M}_{m\times n}$ has 1 at entry (i, j) and 0's elsewhere.
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Coordinates

Theorem

Let $B = \{v_1, \ldots, v_n\}$ be a basis of a $\mathbb{K}$-e.v. E. Then, for every vector $v \in E$, there is exactly one way to write v as a linear combination of the vectors in B, that is, there exist $c_1, \ldots, c_n \in \mathbb{K}$ such that $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ and moreover, these $c_1, \ldots, c_n$ are unique.

Definition

The $c_1, c_2, \ldots, c_n$ are called the **coordinates of v with respect to B**.

We will use the notation

$$v_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

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Coordinates: from E to $\mathbb{K}^n$

Taking coordinates of a vectors in a given basis preserves linear combinations:
If $B = \{v_1, \ldots, v_n\}$ is a basis of E and $u_1, \ldots, u_k$ are in E, then

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Basis and dimension

Proposition

Every (f.g.) $\mathbb{K}$-e.v. $E \neq \vec{0}$ has a basis.

Theorem (Steinitz substitution lemma)

Let E be an f.g. $\mathbb{K}$-e.v. Let $v_1, \ldots, v_m$ be generators of E and $u_1, \ldots, u_n \in E$ be l.i. Then, $n \leq m$ and one can substitute n vectors of $\{v_1, \ldots, v_m\}$ by $u_1, \ldots, u_n$ such that the new collection of vectors is still a system of generators for E.

Corollary (The Basis Theorem)

Any two bases of a f.g. vector space have the same number of elements.

Dimension of the vector space $\dim(E)$ = cardinal of any basis. By convention, $\dim(\{\vec{0}\}) = 0$

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Let E be a vector space of dimension n, $n \geq 1$. Then:

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2. *Any linearly independent set in E contains $\leq n$ vectors. Moreover, it can be extended to a basis of E (by choosing vectors of a given basis of E conveniently).*
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2. *Any linearly independent set in E contains $\leq n$ vectors. Moreover, it can be extended to a basis of E (by choosing vectors of a given basis of E conveniently).*
3. *Any linearly independent set of n vectors in E is a basis for E.*
4. *Any system of generators for E consisting of n vectors is a basis for E.*

Hence,

$$\begin{aligned} n &= \text{minimum number of elements in a system of generators of } E \\ &= \text{maximum number of l.i. vectors in } E. \end{aligned}$$

Theorem

Let $V_1 \subset V_2$ *be subspaces of* E *and* $\dim(E) = n$. *Then:*

1. $dim V_1 \leq dim V_2 \leq n$.
2. $dim V_1 = dim V_2$ *if and only if* $V_1 = V_2$.

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Rank (revisited)

Theorem

Given $v_1, v_2, \ldots, v_k \in \mathbb{K}^n$*, if* $A = (v_1, \ldots, v_k) \in \mathcal{M}_{n,k}(\mathbb{K})$*, then*

a) $v_1, v_2, \ldots, v_k$ *are l.d.* $\Leftrightarrow$ *the homogeneous system* $Ax = 0$ *has a nontrivial solution (indeterminate system).*

b) $v_1, v_2, \ldots, v_k$ *are l.i.* $\Leftrightarrow$ $rank(A) = k$*.*

c) $v_1, v_2, \ldots, v_k$ *are a system of generators of* $\mathbb{K}^n$ $\Leftrightarrow$ $rank(A) = n$*.*

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$\mathbb{K}^n$: Finding a basis from generators

If $V = [v_1, v_2, \ldots, v_k] \subset \mathbb{K}^n$, then a basis of V can be obtained by applying one the following methods:

1. Write the vectors $v_1, \ldots, v_k$ as the rows of a matrix A, and reduce A to row echelon form $\bar{A}$ (Gaussian elimination). The nonzero rows of $\bar{A}$ are a basis of V.
2. Write the vectors $v_1, \ldots, v_k$ as the columns of a matrix B. Then, reduce B to row echelon form $\bar{B}$ (Gaussian elimination). The columns of $\bar{B}$ with pivots indicate which vectors $v_1, \ldots, v_k$ to choose to obtain a basis of V.

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If $u_1, \ldots, u_k$ are linearly independent vectors of $\mathbb{K}^n$, then they can be extended to a basis of $\mathbb{K}^n$:

- Write the vectors $u_1, \ldots, u_k$ as the columns of a matrix B, and take $M = (B \mid I_n)$.
- Then, reduce M to row echelon form $\bar{M} = (\bar{B} \mid \bar{I}_n)$ (Gaussian elimination).
- Collect the columns of $\bar{I}_n$ with a pivot and choose the corresponding vectors of the standard basis (columns of I_n) of $\mathbb{K}^n$.
- $u_1, \ldots, u_k$ together with these last vectors form a basis of $\mathbb{K}^n$.

The same can be done if $u_1, \ldots, u_k$ are linearly independent vectors of a vector subspace V:
instead of I_n, take a matrix formed by a basis $v_1, \ldots, v_d$ of V and do the same process as above for $M = (u_1, \ldots, u_k | v_1, \ldots, v_d)$.

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Subspaces of $\mathbb{K}^n$: Generators $\leftrightarrow$ Equations

From "generators" to "equations":
If $V = [v_1, \ldots, v_k] \subset \mathbb{K}^n$:
Write $M = (v_1, \ldots, v_k)$, and form an augmented matrix $(M|x)$ with $x =$ column with entries $x_1, x_2, \ldots, x_n$.
Then $x \in [v_1, \ldots, v_k]$ if and only if $rank(M|x) = rank(M)$.
There are 2 options:

- Reduce M to echelon form $(\bar{M}|\bar{x})$ by Gaussian elimination $\Rightarrow$ a linear system of equations for V is obtained by writing the equations that correspond to zero rows of $\bar{M}$.
- If $rank(M) = k$, the equations are formed by the vanishing of the $(k+1) \times (k+1)$ minors of $(M|x)$ that contain a chosen non-zero $k \times k$ minor of M.

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From "equations" to "generators":

If $V = \{u \in \mathbb{K}^n \mid Au = 0\}$ (solutions to a homogeneous system):

- It is enough to solve the system to obtain a system of generators of V.
- Moreover, if we give values 0's and 1's to the free variables, these generators form a basis and $dim(V) = n - rank(A)$.

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Outline

Change of basis

Let $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be bases of E. Denote by $A_{B\to C}$ the $n \times n$ matrix whose columns are the coordinate vectors $(u_1)_C, \ldots, (u_n)_C$ of B with respect to C. This is the **change-of-basis matrix from B to C**:

$$A_{B\to C} = \begin{pmatrix} (u_1)_C & \ldots & (u_n)_C \end{pmatrix}.$$

Proposition

1. *$A_{B\to C} w_B = w_C$ for all $w \in E$.*
2. *$A_{B\to C}$ is invertible, and $(A_{B\to C})^{-1} = A_{C\to B}$.*
3. *If D is another basis of E, then $A_{C\to D} A_{B\to C} = A_{B\to D}$.*

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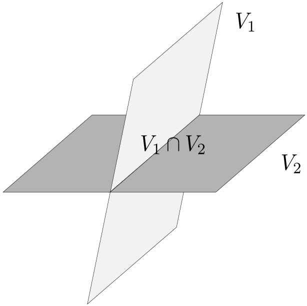
Outline

Intersection & sum of subspaces

Given V_1, V_2 vector subspaces of E, define

1. **Intersection of V_1 and V_2** is
   $V_1 \cap V_2 = \{v \in E \mid v \in V_1, v \in V_2\}$.
2. **Sum of V_1 and V_2** is
   $V_1 + V_2 = \{v_1 + v_2 \in E \mid v_1 \in V_1, v_2 \in V_2\}$.
   Computation: If $V_1 = [u_1, \ldots, u_r]$ and $V_2 = [v_1, \ldots, v_s]$, then
   $V_1 + V_2 = [u_1, \ldots, u_r, v_1, \ldots, v_s]$.



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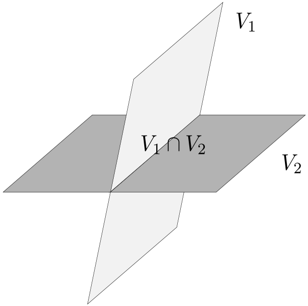


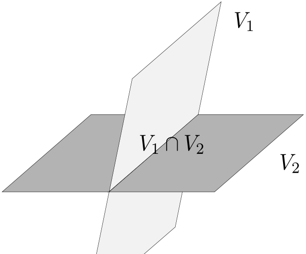
Theorem

1. *$V_1 \cap V_2$ and $V_1 + V_2$ are vector subspaces of E.*
2. *Grassmann formula: if $\dim(E) < \infty$, then*

$$dim(V_1 \cap V_2) + dim(V_1 + V_2) = dim(V_1) + dim(V_2).$$

Ex:

$$V_1 = [(1,0,1),(0,2,3)] \qquad V_1 \cap V_2 = [(1,0,1)]$$
$$V_2 = [(0,1,0),(1,1,1)] \qquad V_1 + V_2 = \mathbb{R}^3$$



Direct sum

Definition

E is the **direct sum** of subspaces F_1 and F_2 if any $w \in E$ can be written in a **unique way** as $w = v_1 + v_2$ with $v_1 \in F_1$, $v_2 \in F_2$. In this case we use the notation $E = F_1 \oplus F_2$.

Proposition

Let F_1, F_2 be two subspaces of E. Then $E = F_1 \oplus F_2$ if and only if the following two conditions hold:

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