Àlgebra lineal i geometria
2. Aplicacions lineals

Grau en Enginyeria Física
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Universitat Politècnica de Catalunya
Departament de Matemàtiques

Marta Casanellas
Universitat Politècnica de Catalunya

Outline

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Definition

A **linear map** (or linear transformation) between two $\mathbb{K}-$e.v E and F is a map that preserves linear combinations. More precisely,

Definition

$f : E \longrightarrow F$ is a **linear map** if

1. $f(u+v) = f(u) + f(v)$ for all $u, v \in E$, and
2. $f(cv) = cf(v)$ for any $c \in \mathbb{K}$ and any $v \in E$.

Examples:

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $f(x, y) = (x + 2y, 3x, y - x)$
- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(x, y) = (-y, x)$ (*rotation* of $\pi/2$ centered at $(0, 0)$)
- $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $f(v) = \lambda.v$ for some $\lambda \in \mathbb{K}$ (*homothety*).
- $f : E \longrightarrow F$, $f(v) = \mathbf{0}$ $\forall v \in E$ is called *zero* map.
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Let $f : E \longrightarrow F$ be a map between $\mathbb{K}$-e.v. Then:

- f linear $\Leftrightarrow f(c_1 v_1 + \cdots + c_k v_k) = c_1 f(v_1) + \cdots + c_k f(v_k)$ $\forall v_1, \ldots, v_k \in E$ and $c_1, \ldots, c_k \in \mathbb{K}$.
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A linear map f is determined by the image of a basis (any basis):

Proposition

Given a basis $\{u_1, \ldots, u_n\}$ of E and any set of vectors $v_1, \ldots, v_n \in F$, there exists a unique linear map $f : E \longrightarrow F$ such that $f(u_i) = v_i \; \forall i$.

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Linear maps $\mathbb{K}^n \longrightarrow \mathbb{K}^m$ and matrices

- Basic example of linear map: If $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, the map $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ defined by

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto f(v) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

- All linear maps $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ are of this type: in standard coordinates they are defined as degree 1 homogeneous polynomials:

$$(x_1, \ldots, x_n) \mapsto (a_{1,1}x_1 + \ldots + a_{1,n}x_n, \cdots, a_{m,1}x_1 + \ldots + a_{m,n}x_n)$$

  and f corresponds to $v \mapsto Av$ where $A = (a_{i,j})$; the ith column of A is $f(e_i)$.
- The **standard matrix** $M(f)$ of a linear map $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is the $m \times n$ matrix whose columns are the vectors $f(e_i)$:

$$M(f) = (f(e_1) \cdots f(e_n))$$

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Let $f : E \longrightarrow F$ be a map between $\mathbb{K}$-e.v.

- f is **injective** if different vectors always have different images ($f(u) = f(v)$ implies $u = v$).
- f is **surjective** if every vector v in F is the image of a certain vector $u \in E$, $v = f(u)$.
- The set of all images of vectors is called the **image or range** of f,

$$\text{Im}(f) = \{v \in F \mid v = f(u) \text{ for some u } \in E\} \subseteq F$$

- f is surjective if and only if $\text{Im}(f) = F$.
- f is **bijective** if it is at the same time injective and surjective. A bijective linear map is called an **isomorphism**.

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Null space

Let $f : E \longrightarrow F$ be a linear map.

Definition

The **kernel** (*nucli*) of f is the subspace

$$\mathrm{Nuc}(f) = \{v \in E \mid f(v) = \mathbf{0}\} = f^{-1}(\{\mathbf{0}\}) \subset E.$$

Theorem

A linear map f is injective if and only if $\mathrm{Nuc}(f) = \{\mathbf{0}\}$.

If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is a linear map and A is its standard matrix, then

- $\mathrm{Nuc}(f) = \{v \in \mathbb{K}^n \mid f(v) = 0\} = \{x \in \mathbb{K}^n \mid Ax = 0\}$.
- $\dim \mathrm{Nuc}(f) = n - \mathrm{rank}(A)$.
- f is injective $\Leftrightarrow \mathrm{rank}(A) = n$ (=number of columns).
- f injective $\Rightarrow n \leq m$.

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Image and preimage of a subspace

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The **image of** $V \subseteq E$ is the set

$$f(V) := \{w \in F \mid w = f(u) \text{ for some } u \in V\}.$$

- If V is a subspace $\Rightarrow f(V)$ is also a subspace.
- If $V = [u_1, \ldots, u_d] \subset E \Rightarrow f(V) = [f(u_1), \ldots, f(u_d)] \subset F$.
- If $u_1, \ldots, u_d$ are linearly independent, $f(u_1), \ldots, f(u_d)$ do NOT need to be l.i.
- $\text{Im}(f) = f(E) = [f(u_1), \ldots, f(u_n)]$ if $\{u_1, \ldots, u_n\}$ is a basis of E.
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Image and preimage of a subspace

Let $f : E \longrightarrow F$ be a linear map.

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The **image of** $V \subseteq E$ is the set

$$f(V) := \{w \in F \,|\, w = f(u) \text{ for some } u \in V\}.$$

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Image for $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$

Let $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ be a linear map and let A be its standard matrix. Then,

- $\mathrm{Im}(f) = [\text{columns of A}]$.
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The **preimage** of $W \subseteq F$ is $f^{-1}(W) := \{u \in E \mid f(u) \in W\} \subseteq E$.

Lemma

1. *If $u \in E$ and $v \in F$ satisfy $f(u) = v$, then*

$$f^{-1}(\{v\}) = \{u + w \mid w \in \mathrm{Nuc}(f)\}.$$

2. *If W is a subspace, so is $f^{-1}(W)$.*

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Outline

Composition of linear maps

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be linear maps, the **composition** of g with f is the linear map $g \circ f : E \longrightarrow G$ defined as:

$$\begin{array}{rcccl} g \circ f : E & \xrightarrow{f} & F & \xrightarrow{g} & G \\ v & \mapsto & f(v) & \mapsto & (g \circ f)(v) := g(f(v)) \end{array}.$$

If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ has standard matrix A and $g : \mathbb{K}^m \longrightarrow \mathbb{K}^p$ has standard matrix $B \Rightarrow$ the standard matrix of $g \circ f$ is

$$M(g \circ f) = BA.$$

Inverse of linear maps

If $f : E \longrightarrow F$ is a linear map, we say that $g : F \longrightarrow E$ is the **inverse** of f (denoted as $g = f^{-1}$) if

$$g \circ f = f \circ g = Id.$$

Note: f is invertible $\Leftrightarrow$ f is bijective.

Invertible linear maps are called **isomorphisms**. Two $\mathbb{K}$-ev. are **isomorphic** if there exists an isomorphism $f : E \longrightarrow F$; in this case we use the notation $E \cong F$.

Properties:

- If f is iso. $\Rightarrow f^{-1}$ is a linear map.
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Theorem (The Rank theorem)

Let $f : E \longrightarrow F$ *be a linear map and assume that* E *has finite dimension. Then,* $\mathrm{Nuc}(f)$ *and* $\mathrm{Im}(f)$ *have finite dimension and*

$$\dim \mathrm{Nuc}(f) + \dim \mathrm{Im}(f) = \dim E$$

Characterizations of inj./surj. maps

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

- f is injective $\Leftrightarrow \text{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \text{Im}(f) = \dim E$.
- f is surjective $\Leftrightarrow \dim \text{Im}(f) = \dim F \Leftrightarrow \dim \text{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\text{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \text{Im}(f) = \dim F$.
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Isomorphism of e.v. of finite dimension

Proposition

If $\dim(E) = n$ *and* $B = \{v_1, \ldots, v_n\}$ *is a basis of* E*, then*

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{K}^n \\ v & \mapsto & v_B \end{array}$$

is an isomorphism.

Theorem

If E *and* F *are two* $\mathbb{K}$*-e.v. of finite dimension, then*

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Outline

Consider now linear maps $f : E \longrightarrow F$ between $\mathbb{K}$-e.v. of finite dimension, $n = \dim E$, $m = \dim F$. Let $\mathbf{u} = \{u_1, \ldots, u_n\}$ and $\mathbf{v} = \{v_1, \ldots, v_m\}$ be bases of E and F (resp.).

Definition

The **matrix of f in bases u,v** is the $m \times n$ matrix whose columns are the coordinates of $f(u_1), \ldots, f(u_n)$ in the basis $\mathbf{v}$:

$$M_{\mathbf{u},\mathbf{v}}(f) = \Big(f(u_1)_{\mathbf{v}} \cdots f(u_n)_{\mathbf{v}} \Big).$$

Properties:

- If $E = \mathbb{K}^n$, $F = \mathbb{K}^m$ and $\mathbf{u}$, $\mathbf{v}$ are the standard bases $\Rightarrow$ this matrix is the *standard matrix* $M(f)$.
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Change of basis as matrices of linear maps

If $A_{\mathbf{u}\to\mathbf{e}}$ is the change-of-basis matrix from **u** to **e**, then this matrix can be thought as the **matrix of the Identity map** in certain basis:

$$A_{\mathbf{u}\to\mathbf{e}} = M_{\mathbf{u},\mathbf{e}}(Id).$$

Note: The matrix of the identity map is the Identity matrix if we put the same basis at both sides.

If $A_{\mathbf{u}\to\mathbf{u'}}$ is the change-of-basis matrix from **u** to **u'**, and $A_{\mathbf{v}\to\mathbf{v'}}$ is the change-of-basis matrix from **v** to **v'**, then:

$$M_{\mathbf{u'},\mathbf{v'}}(f) = A_{\mathbf{v}\to\mathbf{v'}}\, M_{\mathbf{u},\mathbf{v}}(f)\, A^{-1}_{\mathbf{u}\to\mathbf{u'}}\,,$$

$$M_{\mathbf{u},\mathbf{v}}(f) = A^{-1}_{\mathbf{v}\to\mathbf{v'}}\, M_{\mathbf{u'},\mathbf{v'}}(f)\, A_{\mathbf{u}\to\mathbf{u'}}\,.$$

The vector space of linear maps

The set of linear maps between $\mathbb{K}$-e.v, E, F is denoted as $L(E, F)$. This is a $\mathbb{K}$-e.v with the usual sum and product by scalars of maps: if $f, g \in L(E, F)$ and $c \in \mathbb{K}$,

+ $f + g$ is the map $(f + g)(v) := f(v) + g(v)$, $v \in E$.
· $c \cdot f$ is the map $(c \cdot f)(v) := cf(v)$, $v \in E$.

Theorem

Let $\mathbf{u} = \{u_1, \ldots, u_n\}$ and $\mathbf{v} = \{v_1, \ldots, v_m\}$ be bases of E and F, respectively. Then the map

$$\begin{array}{rcl} \varphi : L(E,F) & \longrightarrow & \mathcal{M}_{m\times n}(\mathbb{K}) \\ f & \mapsto & M_{\mathbf{u},\mathbf{v}}(f) \end{array}$$

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Endomorphisms

An **endomorphism** is a linear map from E to itself.

Notation

- $End(E) = \{f : E \longrightarrow E \mid f \text{ linear map } \}$.
- If $f \in End(E)$ and $\mathbf{u} = \{u_1, \ldots, u_n\}$ is a basis of E, we denote by $M_{\mathbf{u}}(f)$ the matrix $M_{\mathbf{u},\mathbf{u}}(f)$.
- Using composition we can define f^m for any $m \in \mathbb{N}$:

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Endomorphisms

An **endomorphism** is a linear map from E to itself.

Notation

- $End(E) = \{f : E \longrightarrow E \mid f \text{ linear map } \}$.
- If $f \in End(E)$ and $\mathbf{u} = \{u_1, \dots, u_n\}$ is a basis of E, we denote by $M_{\mathbf{u}}(f)$ the matrix $M_{\mathbf{u},\mathbf{u}}(f)$.
- Using composition we can define f^m for any $m \in \mathbb{N}$:

$$f^m = f \circ \overset{m)}{\dots} \circ f.$$

- $M_{\mathbf{u}}(f^m) = M_{\mathbf{u}}(f)^m$, for any basis $\mathbf{u}$

Determinant of an endomorphism

Definition

The **determinant** of an endomorphism $f \in End(E)$ (E of finite dimension) is the determinant of its matrix in *any* basis $\mathbf{u}$,

$$\det(f) = \det(M_{\mathbf{u}}(f)).$$

This does not depend on the basis and

$$\det(g \circ f) = \det g \det f.$$

Trace

- The trace does not depend on the basis either: if **u** and **v** are two basis of E ($\dim E < \infty$), then

$$\mathrm{tr}(M_{\mathbf{u}}(f)) = \mathrm{tr}(M_{\mathbf{v}}(f)).$$

- This is known as **the trace of the endomorphism** and denoted as $\mathrm{tr}(f)$.

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Invariant subspaces

Let $f \in End(E)$ and $F \subseteq E$ be a subspace.

Definition

F is **f-invariant** (or invariant by f) if $f(F) \subseteq F$.
In this case we define the **restriction** of f to F, as the endomorphism $f_{|F} \in End(F)$ defined by $f_{|F}(v) := f(v)$.

Proposition

Let $\mathbf{u} = \{u_1 \dots u_n\}$ be a basis of E obtained by extension of a basis $B = \{u_1, \dots, u_d\}$ of a subspace $F \subset E$. Then F is f-invariant if and only if

$$M_{\mathbf{u}}(f) = \left(\begin{array}{c|c} A & * \\ \hline \mathbf{0} & * \end{array}\right),$$

where $A \in \mathcal{M}_d(\mathbb{K})$. In this case, $A = M_B(f_{|F})$.

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Outline

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