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Analytic families of reducible linear quasi-periodic differential equations

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Abstract. In this paper we study the existence of analytic families of reducible linear quasi-periodic differential equations in matrix Lie algebras. Under suitable conditions we show, by means of a Kolmogorov–Arnold–Moser (KAM) scheme, that a real analytic quasi-periodic system close to a constant matrix can be modified by the addition of a time-free matrix that makes it reducible to constant coefficients. If the system depends analytically on external parameters, then this modifying term is also analytic.

As a major application, we prove the analyticity of resonance tongue boundaries in Hill's equation with a small quasi-periodic forcing. Several consequences for the spectrum of Schrödinger operators with quasi-periodic forcing are derived. In particular, we prove that, generically, the spectrum of Schrödinger operators with a small real analytic and quasi-periodic potential has all spectral gaps open and, therefore, it is a Cantor set. Some other applications are included for linear quasi-periodic systems on $so(3, \mathbb{R})$ and $sp(n, \mathbb{R})$.

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1. *Introduction and main result*

In this paper we deal with linear equations in a certain matrix Lie algebra with quasi-periodic coefficients and a dependence on certain external parameters. Linear equations with quasi-periodic coefficients are of the type

$$x'(t) = a(t)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and a is a matrix depending quasi-periodically on time (with frequency $\omega \in \mathbb{R}^d$) and belongs to g, a matrix Lie subalgebra of $gl(n, \mathbb{R})$. The fact that a is quasi-periodic means that there exists a lift (continuous at least), $A : \mathbb{T}^d \to g$, with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, such that

$$a(t) = A(\omega t)$$

for all $t \in \mathbb{R}$.

We can make this quasi-periodicity more evident by writing the corresponding lifted system of (1) for $(X, \theta) \in G \times \mathbb{T}^d$:

$$X' = A(\theta)X, \quad \theta' = \omega, \tag{2}$$

which defines a flow on $G \times \mathbb{T}^d$, because $A \in g$. Here G is a matrix Lie group of $GL(n, \mathbb{R})$ whose Lie algebra is g. Therefore, the map

$$X \in g \mapsto \exp(X) \in G$$

is a well-defined diffeomorphism between g and a neighbourhood of the identity in G.

We are interested in the reducibility of this kind of equation to constant coefficients. We will say that the linear quasi-periodic equation on g, (2), is reducible to constant coefficients whenever there exists a matrix function $Z : \mathbb{T}^d \to G$ (at least of class C^1) and a constant matrix $B \in g$ (called the *Floquet matrix*) such that the change of variables

$$X = Z(\theta)Y$$

transforms the system (2) into

$$Y' = BY, \quad \theta' = \omega. \tag{3}$$

Remark 1. For general quasi-periodic equations we cannot expect the transformation Z to be defined on $\mathbb{T}^d$, but on some finite covering of it. This is evident even in the periodic case, where, unless the system is complexified, one must choose the transformation to have a period that is double that of the original equation. However, in this paper we will not change the Floquet matrix and we will be able to show the reducibility of the system by means of a transformation, close to the identity, defined on $\mathbb{T}^d$ itself.

The reducibility of (2) to (3) is equivalent to the fulfillment of the following *homological equation* for the transformation $Z : \mathbb{T}^d \to G$:

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B \tag{4}$$

for all $\theta \in \mathbb{T}^d$, where $\partial_\omega \cdot = \langle \nabla_\theta \cdot, \omega \rangle$ is the derivative in the ω-direction.

We will not deal with general equations such as (2), but only with those which are perturbations of systems with constant coefficients and which depend on parameters. That is, we will consider systems of the form

$$x' = (A_0 + P(\theta, \mu))x, \quad \theta' = \omega, \tag{5}$$

where $A_0 \in g$ is a constant matrix, $P = P(\theta, \mu)$ belongs to g for real values of θ and μ and it is real analytic.

Equation (5) is a perturbation of a system with constant coefficients if $P(\theta, \mu)$ is small. For this kind of system and several different contexts, one has that (5) is reducible to constant coefficients for almost all values of μ provided some generic conditions are met (see [**Eli92**] for the case of $sl(2, \mathbb{R})$ and [**Kri99b**] for compact Lie algebras). This also seems to be the situation for general analytic quasi-periodic perturbations of systems with constant coefficients (see [**JS92, JS96**] for results in positive measure). We would like to stress that, even in the cases where almost-everywhere reducibility holds, there exist generic sets of μ for which reducibility does not hold (see [**Eli92, Eli02**]).

Even if a system such as (5) is reducible to constant coefficients, the Floquet matrix will not, in general, be A_0 again. One can try, however, to modify (9) in a way such that the perturbed system is reducible with Floquet matrix A_0. This is an old idea going back to [**Mos67**], see Remark 6. We will try to obtain a real analytic matrix function $\xi^* \in g$ such that

$$x' = (A_0 + P(\theta, \mu) - \xi^*(\mu))x, \quad \theta' = \omega, \tag{6}$$

is reducible to the constant-coefficients system

$$y' = A_0 y, \quad \theta' = \omega \tag{7}$$

by means of a transformation

$$x = \exp(X(\theta, \mu))y, \tag{8}$$

where $X \in g$ is real analytic in both θ and μ. If the transformation succeeds then the equation $\xi^*(\mu) = 0$ will determine an analytic family of systems of (5) that are reducible to (7). This allows us to study the problem of the persistence and analyticity of these families. To achieve our goal we will have to impose analyticity to the original system in both θ and μ, the smallness of P and some arithmetic properties on the eigenvalues of A_0 and the frequencies ω.

One of our main motivations is to be able to detect bifurcations of the Floquet matrices of reducible systems. We will introduce a scaling function $\chi = \chi(\mu)$, also real analytic, of a suitable order $k \in \mathbb{N}$ of scaling and consider the system

$$x' = \chi(\mu)^k (A_0 + \chi(\mu) P(\theta, \mu))x, \quad \theta' = \omega. \tag{9}$$

Under some additional hypothesis on A_0, we will show that a suitable modification of (9)

$$x' = \chi(\mu)^k (A_0 + \chi(\mu) P(\theta, \mu) - \chi(\mu)\xi^*(\mu))x, \quad \theta' = \omega$$

is reducible to constant coefficients with Floquet matrix $\chi(\mu)^k A_0$, where both ξ^* and the transformation depend analytically on μ. The treatment of this scaled case is postponed to §1.2.

Remark 2. The requirement that the Floquet matrix of (6) is A_0 (or $\chi(\mu)^k A_0$ in the scaled case) is again imposed so as to preserve the good arithmetic relations between the eigenvalues of A_0 and the frequency vector ω.

1.1. *Formulation of the main result.* Let us first formulate the main result without the scaling parameter χ. The reducibility of the modified system (6) to (7) by means of the transformation (8) requires that $Z = \exp(X)$ satisfies the homological equation

$$\partial_\omega Z(\theta, \mu) = (A_0 + P(\theta, \mu) - \xi^*(\mu))Z - Z A_0, \tag{10}$$

where the unknowns are X and ξ^*. If we try to solve the homological equation (10) by means of (a modified) Newton's quadratic method, we obtain the linear version (in X) of (10), namely

$$\partial_\omega X(\theta, \mu) = [A_0, X] + P(\theta, \mu) - \xi^*(\mu). \tag{11}$$

Without imposing extra conditions, this equation needs not to have a solution (even formally) and if there is such a solution it may not be unique. In addition, considering different choices of ξ^* may be interesting in different contexts. As we want the convergence issues to be separated from the formal (algebraic) aspects, we will assume that (11) is solvable in the following way. In our notation, $C^{\mathrm{a}}_\rho(\mathbb{T}^d, \mathbb{R})$ will stand for the space of real analytic functions from $P : \mathbb{T}^d \to g$ having an analytic extension to $|\mathrm{Im}\,\theta| < \rho$ and satisfying

$$|P|_\rho := \sup_{|\mathrm{Im}\,\theta| < \rho} |P(\theta)| < \infty.$$

Definition 3. Given a matrix Lie algebra, g, a quartet (A_0, C, S, ω) is said to be *admissible* if $A_0 \in g$, $C, S : g \to g$ are linear operators with $C^2 = C$ and there exist positive constants c, ν such that, for all real analytic $P \in C^{\mathrm{a}}_\rho(\mathbb{T}^d, g)$, the equations

$$\partial_\omega X(\theta) = [A_0, X(\theta)] + P(\theta) - C(\bar{P}), \quad \bar{X} = S(\bar{P}), \tag{12}$$

where the bar denotes the average of a quasi-periodic function, have a unique real analytic solution $X : \mathbb{T}^d \to g$ that satisfies the estimate

$$|X|_{\rho-\delta} \le c \frac{|P|_\rho}{\delta^\nu} \tag{13}$$

for all $0 < \delta < \rho$.

The main result now reads as follows.

THEOREM 4. *Let $g \subset gl(n, \mathbb{R})$ be a matrix Lie algebra, (A_0, C, S, ω) an admissible quartet, with positive constants c, ν, and ρ_0 a positive number. Then there exists a constant $\varepsilon = \varepsilon(\rho_0, c, \nu, |A_0|) > 0$ such that for any real analytic matrix-function $P : \mathbb{T}^d \to g$ such that*

$$|P|_{\rho_0} \leq \varepsilon$$

there exists a $\xi^* \in g$, with $|\xi^*| \leq 2\varepsilon$ and $\xi^* = C(\xi^*)$, such that the modified system

$$x' = (A_0 + P(\theta) - \xi^*)x, \quad \theta' = \omega \tag{14}$$

is reducible to the constant-coefficients system

$$y' = A_0 y, \quad \theta' = \omega \tag{15}$$

by means of a transformation $x = Z(\theta)y$, of the form $Z = \exp(X)$, where $X : \mathbb{T}^d \to g$ is real analytic and

$$|X|_{\rho_0/2} \leq \tilde{c}(\nu)\varepsilon\rho_0^{-\nu},$$

where $\tilde{c}(\nu)$ is a constant. Moreover, if P depends real analytically on $\mu \in \mathbb{R}^p$ in a certain ball around the origin then both X and ξ^* depend real analytically on μ in a narrower ball.

A more convenient version of the previous theorem for some applications will be stated in §1.2. The proof of both theorems will be given in §5.

Remark 5. The modifying term $\xi^*(\mu)$ will also be called the *counterterm*.

Remark 6. This result is a reformulation of Moser [**Mos67**] who introduced the counterterm. Some linear versions can be found in [**BMS76**] and [**Kat70**]. For a similar result in the discrete and smooth context, see [**Kri99a**].

1.2. *On admissible* (A_0, C, S, ω). Assume that $A_0 \in g$ and $\omega \in \mathbb{R}^d$ are rationally independent and fixed. One would like to have an effective method to determine operators $C, S : g \to g$ such that the quartet (A_0, C, S, ω) is admissible. A criterion of this kind requires two conditions: one algebraic (which allows to compute a formal solution of this problem) and another Diophantine (so that the previous formal solution is an actual solution).

Let us try to formally solve (12) in terms of the Fourier coefficients of $P : \mathbb{T}^d \to g$. Writing

$$X(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} X_{\mathbf{k}} e^{i\langle\mathbf{k},\theta\rangle}, \quad P(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} P_{\mathbf{k}} e^{i\langle\mathbf{k},\theta\rangle}$$

and equating the Fourier coefficients in (12) one obtains

$$i\langle\mathbf{k}, \omega\rangle X_{\mathbf{k}} = [A_0, X_{\mathbf{k}}] + P_{\mathbf{k}} \tag{16}$$

for $\mathbf{k} \neq 0$ and

$$0 = [A_0, X_{\mathbf{0}}] + P_{\mathbf{0}} - C(P_{\mathbf{0}}) \tag{17}$$

for $\mathbf{k} = 0$. Thus, the only condition on the operators S and C is that

$$0 = [A_0, S(P_\mathbf{0})] + P_\mathbf{0} - C(P_\mathbf{0}) \tag{18}$$

must hold for all $P_\mathbf{0} \in g$. This can be understood at a more geometrical level making use of the adjoint operator, which is the following linear operator on g:

$$\begin{array}{rccc} \mathrm{ad}_{A_0}: & g & \to & g \\ & X & \mapsto & [A_0, X]. \end{array}$$

In terms of this operator, (18) holds for all $P_\mathbf{0} \in g$ if and only if the operator $C - \mathrm{ad}_{A_0} \circ S$ is the identity on g. To solve the equations for the remaining Fourier coefficients (16) one needs that $i\langle \mathbf{k}, \omega\rangle I - \mathrm{ad}_{A_0}$ is an invertible operator for all $\mathbf{k} \in \mathbb{Z}^d - \{0\}$. Thus, we meet the required condition on rational independence for the formal solution

$$\lambda - i\langle \mathbf{k}, \omega\rangle \neq 0 \tag{19}$$

for all

$$\lambda \in \mathrm{Spec}(\mathrm{ad}_{A_0})$$

and $\mathbf{k} \neq 0$. Note that the eigenvalues of ad_{A_0} will be of the form $\lambda' - \lambda''$, for λ', λ'' in the spectrum of A_0.

If we want this formal solution to be an actual solution of the homological equation (12), we need to strengthen this non-resonance condition to have good control over the small divisors. This is summarized in the following lemma.

LEMMA 7. *Assume that $A_0 \in g$, $\omega \in \mathbb{R}^d$ and that there exist linear operators $C, S : g \to g$, with $C^2 = C$ such that $C - \mathrm{ad}_{A_0} \circ S$ is the identity on g. If there exist positive constants τ, K such that the following Diophantine condition*

$$\inf_{\lambda \in \mathrm{Spec}(\mathrm{ad}_{A_0})} |\lambda - i\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^\tau} \quad \textit{for all } \mathbf{k} \in \mathbb{Z}^d \setminus \{0\} \tag{20}$$

is satisfied, then the quartet (A_0, C, S, ω) is admissible.

To prove the lemma, it suffices to represent (16) and (17) in terms of a basis of the Lie algebra g and then use the standard Diophantine conditions.

Example 8. If $A_0 \in g$ has all eigenvalues equal and ω is Diophantine, that is, there exist positive constants K and τ such that the frequency vector ω satisfies

$$|\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^\tau} \quad \text{for all } \mathbf{k} \in \mathbb{Z}^d \setminus \{0\},$$

then, choosing C to be the identity and S to be zero, the quartet (A_0, C, S, ω) is admissible.

As the counterterm $\xi^*(\mu)$ in Theorem 4 satisfies that $C(\xi^*) = \xi^*$ and the persistence of a family of reducible quasi-periodic systems of (9) with Floquet matrix A_0 requires the condition $\xi^*(\mu) = 0$, it is important that the dimension of the image of C in g, which we denote as L_C, is as small as we can. Note that in Example 8, the dimension of this space is not necessarily minimal, as the special properties of A_0 are not used. The following lemma gives a condition of this kind.

LEMMA 9. *Let $A_0 \in g$ and ω satisfy the Diophantine conditions (20). Then the minimal dimension of L_C in g is the dimension of g as a subspace of $gl(n, \mathbb{R})$ minus the dimension of the image of* ad_{A_0} *in g.*

Proof. As

$$C - \mathrm{ad}_{A_0} \circ S = I,$$

it is clear that $\dim L_C \geq \dim \ker \mathrm{ad}_{A_0}$. If L_C is chosen to be exactly $\ker \mathrm{ad}_{A_0}$ and S so that $\mathrm{ad}_{A_0} \circ S = \mathrm{ad}_{A_0}$ the optimal bound is attained. □

Remark 10. Once L_C is chosen to be the kernel of ad_{A_0}, the operator S is any linear operator satisfying $\mathrm{ad}_{A_0} \circ S = \mathrm{ad}_{A_0}$.

In particular, if $\mathrm{Spec}(\mathrm{ad}_{A_0}) = \{0\}$ and ω is Diophantine, there exist choices of C and S such that the quartet $(\chi^k A_0, C, S, \omega)$ is admissible for all values of the parameter χ in $\mathbb{R}$. One can use this uniformity in χ to obtain the following theorem.

THEOREM 11. *Let $g \subset gl(n, \mathbb{R})$ be a matrix Lie algebra, let (A_0, C, S, ω) be admissible, with positive constants c, ν and such that* $\mathrm{Spec}(\mathrm{ad}_{A_0}) = \{0\}$. *Let ρ_0 be a positive number. Then there exists a positive constant $\varepsilon = \varepsilon(\rho_0, c, \nu, |A_0|)$ such that for any real analytic matrix-function $P : \mathbb{T}^d \to g$ such that*

$$|P|_{\rho_0} \leq \varepsilon$$

and for any $|\chi| \leq 1$, there exists a $\xi^* \in g$, with $\xi^* = C(\xi^*)$, such that the modified system

$$x' = \chi^k(A_0 + \chi P(\theta) - \chi \xi^*)x, \quad \theta' = \omega \tag{21}$$

is reducible to the constant-coefficients system

$$y' = \chi^k A_0 y, \quad \theta' = \omega \tag{22}$$

by means of a transformation $x = Z(\theta)y$, of the form $Z = \exp(\chi X)$, where $X : \mathbb{T}^d \to g$ is real analytic and

$$|X|_{\rho_0/2} \leq \tilde{c}(\nu)\varepsilon\rho_0^{-\nu},$$

where $\tilde{c}$ is a constant. Moreover, if P and χ depend real analytically on $\mu \in \mathbb{R}^p$ in a certain ball around the origin, then both X and ξ^* depend real analytically on μ in a narrower ball.

An example, which will be used in the following section, is the following.

Example 12. Let $g = sp(1, \mathbb{R}) = sl(2, \mathbb{R})$ and A_0 be the nilpotent matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the algebra $sl(2, \mathbb{R})$ we can consider the basis formed by the elements

$$X_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and let $(x_{11}, x_{12}, x_{21})^{\mathrm{T}}$ be the coordinates of an element $X \in g$ with respect to this basis. As

$$[A_0, X] = \begin{pmatrix} x_{21} & -2x_{11} \\ 0 & -x_{21} \end{pmatrix},$$

the spectrum of $\mathrm{ad}_{A_0} : g \to g$ reduces to the zero eigenvalue with multiplicity three and its kernel is the linear subspace of g spanned by X_{12}. We can choose C, in the above coordinates, as

$$c_{11}(\xi) = 0, \quad c_{12}(\xi) = 0, \quad c_{21}(\xi) = \xi_{21}$$

and, for example,

$$s_{11}(\xi) = -\tfrac{1}{2}\xi_{12}, \quad s_{12}(\xi) = 0, \quad s_{21}(\xi) = \xi_{11}.$$

With these definitions (A_0, C, S, ω) is admissible.

1.3. *Outline.* Before ending the introduction, let us outline the contents of the present paper. In §2 we include the main application of Theorems 4 and 11, which is the analyticity of resonance tongue boundaries in Hill's equation with quasi-periodic forcing,

$$x'' + (a + bq(t))x = 0,$$

where q is a quasi-periodic function with frequency ω, $q(t) = Q(\omega t)$ and a, b are real parameters. The resonance tongues, which determine the closure of the areas in the (a, b)-plane where the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a - bQ(\theta) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta' = \omega \tag{23}$$

is reducible to a hyperbolic matrix, are shown to have real analytic boundaries, $a = a(b)$, provided that ω is Diophantine, Q real analytic and $|b|$ sufficiently small. These tongue boundaries, for every $|b|$ small are characterized by the fact that system (23) is reducible to constant coefficients with Floquet matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

depending on whether the two boundaries of the tongue merge or not. This allows us to use Theorems 4 and 11 to study the persistence and analyticity of such families of quasi-periodic systems reducible to these Floquet matrices. Hill's equation with quasi-periodic forcing shows up as the eigenvalue equation of one-dimensional Schrödinger operators with quasi-periodic potential and the real analyticity of tongue boundaries has direct applications to the structure of their spectrum. In §2.3 we prove the genericity of 'having all gaps open' (and, in particular, Cantor spectrum) for Schrödinger operators with (small) real analytic potentials and Diophantine frequencies.

In §3 linear equations with quasi-periodic coefficients in $so(3, \mathbb{R})$ are considered. To end the applications we show how the results of §2, together with some arguments of Sacker–Sell spectral theory can be used to study hyperbolicity boundaries of Hamiltonian systems in higher dimensions. This is done in §4.

The proofs of Theorems 4 and 11 are postponed to §5, where a classical Kolmogorov–Arnold–Moser (KAM) scheme is presented. These proofs will be given in a unified way because they only differ in some details. Finally, we include three appendices. Appendices A and B deal with generalizations of Theorem 4 to the context of multiple resonances and the presence of a time-reversing symmetry, respectively. Appendix C illustrates some higher-dimensional phenomena numerically, to be compared with the results in §4.

2. *Analyticity of tongue boundaries in quasi-periodic Hill's equation and applications*

In this section we present the main application of Theorems 4 and 11, which is the analyticity of resonance tongue boundaries of Hill's equation with quasi-periodic forcing. This problem, together with the main application Theorem 13, are presented in §2.1. The proof of Theorem 13 is given in §2.2 and the applications to the spectrum of quasi-periodic Schrödinger operators, in §2.3.

2.1. *Set-up: analyticity of tongue boundaries.* In [**BPS03**] the stability and instability zones of Hill's equation with quasi-periodic forcing,

$$x'' + (a + bq(t))x = 0, \tag{24}$$

with $(a, b) \in \mathbb{R}^2$ and q quasi-periodic with frequency $\omega \in \mathbb{R}^d$, were studied for small values of $|b|$. Hill's equation with quasi-periodic forcing is a generalization and extension of the classical, periodic Hill equation. Both the periodic and the quasi-periodic case occur as a first variation equation in the stability analysis of periodic solutions and lower-dimensional tori in Hamiltonians with few degrees of freedom (see [**Eli88, JV97, Bou97**]).

As the function q is quasi-periodic (later on we also assume that it is real analytic) there exists a real analytic lift $Q : \mathbb{T}^d \to \mathbb{R}$ such that $q(t) = Q(\omega t)$ for all $t \in \mathbb{R}$. In particular, Hill's equation (24) can be made autonomous introducing some new angular variables $\theta \in \mathbb{T}^d$,

$$x'' + (a + bQ(\theta))x = 0, \quad \theta' = \omega \tag{25}$$

and as a first-order quasi-periodic linear system introducing $y = x'$ and

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a - bQ(\theta) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta' = \omega. \tag{26}$$

Note that any choice of the form $q_\phi(t) = Q(\omega t + \phi)$, with $\phi \in \mathbb{T}^d$, gives rise to a Hill equation of the form (24).

For fixed $b \in \mathbb{R}$, Hill's equation shows up as the eigenvalue equation of the one-dimensional quasi-periodic Schrödinger operator

$$(H_{-bQ,\omega,\phi}x)(t) = -x''(t) - bQ(t)x(t), \tag{27}$$

which is an essentially self-adjoint operator on $L^2(\mathbb{R})$. If ω is rationally independent, the spectrum of this operator is independent of the choice of $\phi \in \mathbb{T}^d$ and it will be denoted by $\sigma_{-bQ,\omega}$. In this setting, the parameter a is called the energy (or the spectral parameter). The study of such quasi-periodic Schrödinger operators is relevant in quantum physics, see [**Sok85**]. For an exposition on the relations between the quasi-periodic Schrödinger operator and its associated eigenvalue equation, see [**BPS03**] and the reviews [**Joh83, Eli98a, Eli98b, Eli99, Sim82**].

Let us now quickly consider the periodic case, in our notation $d = 1$. In this case the flow (26) is reducible to constant coefficients by means of a periodic transformation. The corresponding Floquet matrix depends on (a, b), although it is not uniquely determined. In the context of the study of the stability and instability regions of (26) in the periodic case, resonance tongues are defined as the connected components in the (a, b)-plane where the Floquet matrix of (26) is nilpotent.

In the quasi-periodic case, this definition makes no sense, as a system such as (26) with $d > 1$ and ω rationally independent does not need to be reducible to constant coefficients. Nevertheless, resonance tongues can be defined by means of the rotation number introduced in [**JM82**] (see also [**Her83, AS83**]). Let us now briefly review this object, following [**JM82**].

The rotation number of (24) is defined as

$$\mathrm{rot}(a, b) = \lim_{T\to+\infty} \frac{\arg(x'(T) + i x(T))}{2\pi T},$$

where x is any non-trivial solution of (24). This number exists and is independent of the particular solution. The map

$$(a, b) \in \mathbb{R}^2 \mapsto \mathrm{rot}(a, b)$$

is continuous and, for fixed b, is a non-decreasing function of a and vanishes if $a < a^*$ for some a^*. The introduction of resonance tongues is motivated by the 'Gap Labelling Theorem' [**JM82**], which states that in the open intervals where the rotation number is constant, it must be of the form

$$\alpha = \frac{\langle \mathbf{k}, \omega\rangle}{2},$$

where $\mathbf{k} \in \mathbb{Z}^d$ is a suitable multi-integer such that $\langle \mathbf{k}, \omega\rangle \geq 0$. This motivates the introduction of resonance tongues, for any $\mathbf{k} \in \mathbb{Z}^d$ such that $\langle \mathbf{k}, \omega\rangle \geq 0$, as the set

$$\mathcal{R}(\mathbf{k}) = \{(a, b) \in \mathbb{R}^2 \mid \mathrm{rot}(a, b) = \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle\}.$$

Resonance tongues can be linked to the spectrum of Schrödinger operators with quasi-periodic potential (27) as a function of b. Indeed, a value a belongs to the spectrum $\sigma_{-bQ,\omega}$ if and only if it is not a point of constancy of $a \mapsto \mathrm{rot}(a, b)$. Thus, the interior of a resonance tongue is the union of all the intervals of the resolvent set of $\sigma_{-bQ,\omega}$, called the *non-collapsed spectral gaps*. Whenever the closure of a certain spectral gap degenerates to a point, this point is called a *collapsed spectral gap*. When two collapsed gaps occur in a same tongue for two different values of b, we speak of an *instability pocket* in the (a, b)-plane, see Figure 1. Compare also with [**BL95, BS00**] for analogues in the periodic case.

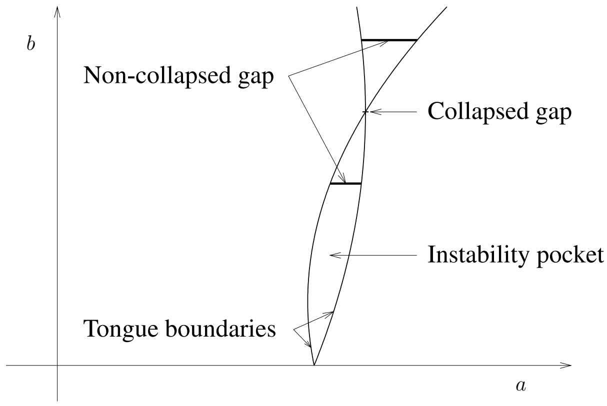

FIGURE 1. Resonance tongue with pocket in the (a, b)-plane giving rise to spectral gaps on each horizontal line with constant b. Note how collapses of gaps correspond to crossings of the tongue boundaries at the extremities of an instability pocket.

In the periodic case, it is well known (see [**Rel69**] and [**MW79**]) that the functions defining the tongue boundaries, also called *band functions*, are real analytic (in b). In [**BPS03**] the problem of the regularity of boundaries of resonance tongues was addressed in the quasi-periodic case. Under the assumption of analyticity for Q and Diophantine conditions on ω it was proved that the resonance tongue boundaries are C^∞ functions of b, for small values of $|b|$. That is, for any resonance tongue there exist two C^∞-functions a_1 and a_2 such that the tongue is the set of those (a, b) that satisfy

$$\min(a_1(b), a_2(b)) \le a \le \max(a_1(b), a_2(b)).$$

Note that the functions $\min(a_1(b), a_2(b))$ or $\max(a_1(b), a_2(b))$ need not be C^∞ (or even differentiable) at a crossing of tongue boundaries, see Figure 1. Beyond this perturbative result on the C^∞ regularity of tongue boundaries, these are Lipschitz functions with a constant that depends only on $\|Q\|_{C(\mathbb{T}^d)}$, see [**BPS03**].

Theorems 4 and 11 can be used to show that tongue boundaries are, in fact, real analytic for $|b|$ small enough. This is the contents of the following result.

THEOREM 13. *Consider Hill's equation with quasi-periodic forcing (24),*

$$x'' + (a + bQ(\omega t))x = 0,$$

with function $Q : \mathbb{T}^d \to \mathbb{R}$ being real analytic and the frequency vector ω, Diophantine. Assume that for $(a_0, b_0) \in \mathbb{R}^2$ the associated quasi-periodic flow on $\mathbb{R}^2 \times \mathbb{T}^d$, (26), is reducible to constant coefficients and it is at a tongue boundary. Then we have the following.

(i) *If a_0 is at the end of a non-collapsed gap of $\sigma_{-b_0 Q,\omega}$, the tongue boundary $a = a(b)$ such that $a(b_0) = a_0$ is real analytic in a neighbourhood of b_0 and for $(a, b) = (a(b), b)$, the skew-product is reducible to constant coefficients.*

(ii) *If a_0 is a collapsed gap of $\sigma_{-b_0 Q,\omega}$, the two tongue boundaries $a_i = a_i(b)$ for $i = 1, 2$, with $a_i(b_0) = a_0$, are real analytic functions in a neighbourhood of b_0. Moreover, for $(a, b) = (a_i(b), b)$, $i = 1, 2$, the skew-product is reducible to constant coefficients.*

In both cases the reducing transformations depend real analytically on both θ and b.

As for $b = 0$ the skew-product associated to Hill's equation is always reducible to constant coefficients (it is already in this form), one has the following consequence.

COROLLARY 14. *If the potential Q is analytic and ω Diophantine, every tongue boundary is an analytic function of b in a neighbourhood of $b = 0$.*

2.2. *Proof of Theorem 13.* To prove Theorem 13 we will have to distinguish between collapsed and non-collapsed gaps at some point. Nevertheless, both cases have the passage to a perturbative situation as a common starting point.

Fix (a_0, b_0) as in Theorem 13. By hypothesis the skew-product is reducible to constant coefficients, whose Floquet matrix we denote by A_0. This matrix belongs to $sl(2, \mathbb{R})$ (as our setting is Hamiltonian) and satisfies $A_0^2 = 0$, because a_0 is at the endpoint of a spectral gap. Moreover, the gap is collapsed if and only if $A_0 = 0$ (see [**BPS03**]).

Let $R : \mathbb{T}^d \to G$ be a real analytic reducing transformation for $(a, b) = (a_0, b_0)$ given by the hypothesis. After this transformation, the skew-product becomes

$$y' = \left(A_0 + (a - a_0 + (b - b_0)Q(\theta)) \begin{pmatrix} r_{11}r_{12} & r_{12}^2 \\ -r_{11}^2 & -r_{11}r_{12} \end{pmatrix} \right) y, \quad \theta' = \omega, \tag{28}$$

where the r_{ij} are the components of R. We introduce now $\mu = (\alpha, \beta)$, where $\alpha = a - a_0$ and $\beta = b - b_0$, as the new local perturbation parameters. We denote as $P(\theta, \mu)$ the time dependent part of (28), that is,

$$P(\theta, \mu) = (\alpha + \beta Q(\theta)) \begin{pmatrix} r_{11}r_{12} & r_{12}^2 \\ -r_{11}^2 & -r_{11}r_{12} \end{pmatrix}. \tag{29}$$

Let us now distinguish between collapsed and non-collapsed gaps.

Non-collapsed gap. In this case, A_0 satisfies $A_0^2 = 0$ but $A_0 \neq 0$. After performing a change of basis if necessary, we may assume that

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using Example 12 and denoting by p_{ij} the components of P, the choices

$$C \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & -p_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ p_{21} & 0 \end{pmatrix}$$

and

$$S \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & -p_{11} \end{pmatrix} = \begin{pmatrix} -p_{12}/2 & 0 \\ p_{11} & p_{12}/2 \end{pmatrix}$$

make (A_0, C, S, ω) admissible, as ω is Diophantine. Thus, by Theorem 4, there exist a real analytic function $\xi_{21}^*(\mu)$, defined in a neighbourhood of the origin, and a real analytic $X = X(\theta, \mu) \in g$ such that $Z(\theta, \mu) = \exp(X(\theta, \mu))$ satisfies

$$\partial_\omega Z = (A_0 + P(\theta, \mu) - \xi^*(\mu))Z - ZA_0, \tag{30}$$

where

$$\xi^*(\mu) = \begin{pmatrix} 0 & 0 \\ \xi^*_{21}(\mu) & 0 \end{pmatrix}.$$

Therefore, for the values of μ for which $\xi^*_{21}(\mu) = 0$, system (28) (and thus also the original skew-product for parameters $(a, b) = (a_0, b_0) + \mu$) is analytically reducible to a constant-coefficients system with Floquet matrix A_0.

Hence, to prove item (i), we only need to show that the equation

$$\xi^*_{21}(\alpha, \beta) = 0 \tag{31}$$

can be inverted to obtain an analytic function $\alpha = \alpha(\beta)$. Note that, averaging (30) with respect to θ and using (29), one obtains the relation

$$\xi^*_{21}(\alpha, \beta) = -\alpha \cdot \overline{r^2_{11}} - \beta \cdot \overline{Qr^2_{11}} + O_2(\mu),$$

where $O_2(\mu)$ collects terms of order greater than one in μ and we recall that the bar denotes averaging with respect to time. As $\overline{r^2_{11}} \neq 0$ (because r_{11} is a non-trivial quasi-periodic solution of Hill's equation), the Implicit Function Theorem yields a real analytic function $\alpha = \alpha(\beta)$ for which (31) holds.

Collapsed gap. As we have said before, in this case the Floquet matrix is $A_0 = 0$. To prove the analyticity of tongue boundaries we first use Theorem 4 to obtain a representation of the quasi-periodic system suitable to show that either the two tongue boundaries coincide (and are real analytic) or they have a finite order of contact. Then, using techniques from [**BPS03**] and Theorem 11 we prove that the tongue boundaries are real analytic in the latter case.

Using Example 8 and as ω is Diophantine, the choice $C = I$ and $S = 0$ makes (A_0, C, S, ω) admissible. Hence, we can apply Theorem 4 to obtain real analytic functions $\xi^*_{11}(\mu)$, $\xi^*_{12}(\mu)$ and $\xi^*_{21}(\mu)$, defined in a neighbourhood of the origin, and a real analytic $X = X(\theta, \mu) \in g$ such that $Z(\theta, \mu) = \exp(X(\theta, \mu))$ satisfies

$$\partial_\omega Z(\theta, \mu) = (P(\theta, \mu) - \xi^*(\mu))Z(\theta, \mu), \tag{32}$$

where P is defined as in (29) and

$$\xi^*(\mu) = \begin{pmatrix} \xi^*_{11}(\mu) & \xi^*_{12}(\mu) \\ \xi^*_{21}(\mu) & -\xi^*_{11}(\mu) \end{pmatrix}.$$

As will be seen in Appendix B, if q is even, $q(t) = q(-t)$ for all $t \in \mathbb{R}$, then $\xi_{11} \equiv 0$. Most of the considerations in the present section are simpler in this reversible setting.

We want to find two functions $\alpha_1(\beta)$ and $\alpha_2(\beta)$ such that system (28) is reducible to a Floquet matrix $B(\beta)$ satisfying $B^2 = 0$ if $\alpha = \alpha_{1,2}(\beta)$. In [**BPS03**] it was shown that these α_i were C^∞ functions. In principle, it could happen that these two boundaries have a C^∞-tangency, but that they are not equal. First of all we shall rule out this possibility.

Note that the reducing transformation Z in (32) also defines a conjugation from the original unmodified system

$$\partial_\omega Z(\theta, \mu) = P(\theta, \mu)Z(\theta, \mu) - Z(\theta, \mu)(Z^{-1}(\theta, \mu)\xi^*(\mu)Z(\theta, \mu)).$$

We will now study the analyticity of the boundaries of the resonance tongues of the system

$$x' = Z^{-1}(\theta, \mu)\xi^*(\mu)Z(\theta, \mu)x, \quad \theta' = \omega. \tag{33}$$

This system has the property that for every fixed value of θ, the matrix

$$S(\theta, \mu) = Z^{-1}(\theta, \mu)\xi^*(\mu)Z(\theta, \mu)$$

is similar to ξ^*, although the system is not necessarily conjugated to constant coefficients. In particular, the eigenvalues of S do not change with θ. For this system, one has the following lemma.

LEMMA 15. *If* $\det \xi^*(\mu) > 0$ *then the rotation number of the quasi-periodic system (33) is strictly non-zero.*

Proof. Converting to polar coordinates, $\varphi = \arg(z_2 + iz_1)$ the flow on $\mathbb{S}^1 \times \mathbb{T}^d$ is given by equations

$$\varphi' = -s_{21}(\theta)\sin^2\varphi + s_{12}(\theta)\cos^2\varphi + 2s_{11}(\theta)\cos\varphi\sin\varphi, \quad \theta' = \omega. \tag{34}$$

The right-hand side is a quadratic form given by the matrix $-JS$. This last quadratic form is definite if, and only if, $\det S > 0$, which is equivalent to $\det \xi^*(\mu) > 0$. □

Moreover, averaging (32) and keeping in mind the definition of P in (29), we can compute the first terms of $\xi^*(\mu)$:

$$\xi^*_{12}(\alpha, \beta) = \alpha \cdot \overline{r_{12}^2} + \beta \cdot \overline{Qr_{12}^2} + O_2(\mu),$$
$$\xi^*_{21}(\alpha, \beta) = -\alpha \cdot \overline{r_{11}^2} - \beta \cdot \overline{Qr_{11}^2} + O_2(\mu),$$
$$\xi^*_{11}(\alpha, \beta) = \alpha \cdot \overline{r_{11}r_{12}} + \beta \cdot \overline{Qr_{11}r_{12}} + O_2(\mu).$$

As in the case of a collapsed gap all solutions are quasi-periodic with frequency ω, by selecting suitable initial conditions the transformation R can always be chosen so that

$$\overline{r_{11}^2} = \overline{r_{12}^2} = 1 \quad \text{and} \quad \overline{r_{11}r_{12}} = 0,$$

the expression for the determinant of ξ^* becomes

$$\det \xi^*(\mu) = \alpha^2 + O(\alpha\beta, \beta^2, \alpha^3).$$

This, together with Weierstrass Preparation theorem, shows that we can write, in a neighbourhood of the origin,

$$\det \xi^*(\mu) = F(\alpha, \beta)(\alpha^2 + g_1(\beta)\alpha + g_2(\beta)),$$

where g_1, g_2 and F are real analytic functions with $F(0, 0) = 1$. For $\beta = 0$, using Lemma 15 and the non-decreasing character of the rotation number with respect to α, the rotation number goes from negative to positive values when α crosses zero. By continuity, for $\beta \neq 0$ and $|\beta|$ small, one should find some zero set in α. Therefore, the two roots of

$$\alpha^2 + g_1(\beta)\alpha + g_2(\beta) = 0$$

are real for all real values of β, so that

$$\det \xi^*(\alpha, \beta) = F(\alpha, \beta)(\alpha - \alpha_1^*(\beta))(\alpha - \alpha_2^*(\beta)),$$

where $\alpha_{1,2}^* = \alpha_{1,2}^*(\beta)$ are two real analytic functions, see [**Rel69**].

As α_1^* and α_2^* are real analytic functions both vanishing at zero, either they coincide or they have a tangency of some order. Let us assume first that they coincide, that is $\alpha_1^*(\beta) = \alpha_2^*(\beta)$ for all β. Using the continuity and the monotonicity in α of the rotation number, the rotation number of (33) is strictly positive if $\alpha > \alpha_1^*(\beta)$ and strictly negative if $\alpha < \alpha_1^*(\beta)$. Therefore, the two tongue boundaries coincide in a neighbourhood of zero and they are given by $\alpha_1^*(\beta)$.

If $\alpha_1^* \neq \alpha_2^*$, there exists an integer $p \geq 1$ and a constant $C \neq 0$ such that

$$\alpha_2^*(\beta) - \alpha_1^*(\beta) = C\beta^p + O_{p+1}(\beta).$$

We are going to see that this p is precisely the order of contact of the two tongue boundaries at $\beta = 0$ and that the latter are real analytic functions. Note that α_1^* and α_2^* need not be the parameterization of the tongue boundaries.

If the order of contact between α_1^* and α_2^* is p then, after some changes of variables in α (which are described in [**BPS03**]), the matrix ξ^* can be assumed to be of the form

$$\xi^*(\alpha, \beta) = \begin{pmatrix} S_3(\alpha, \beta) & S_2(\alpha, \beta) \\ -S_1(\alpha, \beta) & -S_3(\alpha, \beta) \end{pmatrix}$$

with

$$\begin{aligned} S_1(\alpha, \beta) &= \alpha + \sigma_1(\beta) + \alpha\rho_1(\alpha, \beta), \\ S_2(\alpha, \beta) &= \alpha + \sigma_2(\beta) + \alpha\rho_2(\alpha, \beta), \\ S_3(\alpha, \beta) &= \sigma_3(\beta) + \alpha\rho_3(\alpha, \beta), \end{aligned}$$

where

$$\sigma_j(\beta) = \sum_{k \geq p} m_{j,k}\beta^k, \quad j = 1, 2, 3, \quad (m_{1,p} - m_{2,p})^2 + m_{3,p}^2 > 0$$

and the possible terms in S_1 and S_2 of degree less than p in β, which must be equal, are included inside α with a suitable redefinition of α.

The equation

$$\gamma^2 + (m_{1,p} + m_{2,p})\gamma + m_{1,p}m_{2,p} - m_{3,p}^2 = 0$$

has two different roots which we denote as γ_1 and γ_2. Taking one of these, for instance γ_1, we perform the change of variables $\alpha = \gamma_1\beta^p + \delta\beta^p$, which means that we restrict our study to a wedge around a boundary of the 'unperturbed' problem of width $\delta\beta^p$, with δ small. In the new variables, δ and β, the matrix ξ^* becomes

$$\beta^p \left(\begin{pmatrix} m_{3,p} & (m_{1,p} + \gamma_1) + \delta \\ -(m_{2,p} + \gamma_1) - \delta & -m_{3,p} \end{pmatrix} + O(\beta) \right).$$

Therefore, the system (33) becomes

$$x' = \beta^p \left(\begin{pmatrix} m_{3,p} & (m_{1,p} + \gamma_1) + \delta \\ -(m_{2,p} + \gamma_1) - \delta & -m_{3,p} \end{pmatrix} + \beta P(\theta, \delta, \beta) \right) x, \quad \theta' = \omega, \quad (35)$$

where we have used that X is of the order of β. The terms of order β^p in this expression will be written as

$$\begin{pmatrix} a & b+\delta \\ -c-\delta & -a \end{pmatrix}.$$

Due to the definition of δ and γ_1, $bc - a^2 = 0$. Also, b and c cannot be zero at the same time because in this case the order of contact of α_1^* and α_2^* would be greater than p. Let us assume that $b > 0$ (the other cases are treated similarly). This assumption means that the determinant of (35) is

$$\beta^{2p}(\delta(b+c+\delta) + O(\beta)).$$

If we fix a $\delta < 0$ then an application of Coppel's criterion [**Cop78**] shows that (35) has an exponential dichotomy if $|\beta| > 0$ is small enough. As the same can be done for γ_2 this shows that the order of contact between the actual boundaries of the resonance tongue is exactly p. This allows us to put ourselves in the context of [**BPS03**] where it was the shown that the Taylor expansion of tongue boundaries up to any finite order is determined by the normal form up to this order. Moreover, as higher-order terms of the normal form do not change the first terms of the Taylor expansion, we may assume that, after suitable changes in the variable δ, system (35) is of the form

$$x' = \beta^p \left(\begin{pmatrix} a & b+\delta \\ -c-\delta & -a \end{pmatrix} + \beta^2 P_1(\theta, \delta, \beta) \right) x, \quad \theta' = \omega. \tag{36}$$

Comparing (36) with the format of Theorem 11 (see, for instance, (21)), the role of χ is now played by β. The extra factor β (in the term $\beta^2 P_1$) is needed to ensure that $|\beta P_1|$ is small enough. Additional β factors in the non-autonomous part are easily obtained by pushing the normal form some steps beyond p and modifying again the definition of α. To be able to apply Theorem 11, one should put the non-autonomous part of the matrix in (36) in the form of (21).

The analytical (in α, β) conjugation given by

$$T = \begin{pmatrix} \sqrt{\dfrac{b+\delta}{b+c+\delta}} & 0 \\ \dfrac{-a}{\sqrt{(b+\delta)(b+c+\delta)}} & \sqrt{\dfrac{b+c+\delta}{b+\delta}} \end{pmatrix}$$

transforms (36) into

$$x' = \beta^p \left(\begin{pmatrix} 0 & b+c+\delta \\ -\delta & 0 \end{pmatrix} + \beta^2 Q_1(\theta, \delta, \beta) \right) x, \quad \theta' = \omega,$$

where Q_1 is a new perturbation, and the change given by

$$\begin{pmatrix} b+c+\delta & 0 \\ 0 & (b+c+\delta)^{-1} \end{pmatrix}$$

transforms it into

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ -\delta(b+c+\delta) & 0 \end{pmatrix} + \beta^2 R_1(\theta, \delta, \beta) \right) x, \quad \theta' = \omega, \tag{37}$$

being R_1 a perturbation defined by the conjugation and Q_1.

Now we are in situation to apply Theorem 11 to the system

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta^2 R_1(\theta, \delta, \beta) \right) x, \quad \theta' = \omega. \tag{38}$$

This yields the existence of a real analytic function $\xi_{21} = \xi_{21}(\delta, \beta)$ such that

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ -\beta\xi_{21}(\delta, \beta) & 0 \end{pmatrix} + \beta^2 R_1(\theta, \delta, \beta) \right) x, \quad \theta' = \omega$$

is reducible to

$$x' = \beta^p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, \quad \theta' = \omega, \tag{39}$$

for $|\delta|, |\beta|$ small enough. Clearly, one also has that the counterterm

$$\begin{pmatrix} 0 & 0 \\ -\delta(b + c + \delta) + \beta\xi_{21}(\delta, \beta) & 0 \end{pmatrix}$$

makes (37) reducible to (39). Therefore, the equation

$$\delta(b + c + \delta) - \beta\xi_{21}(\delta, \beta) = 0$$

determines one of the components of the boundary of the resonance tongue (see [**BPS03**]). Note that, as $b + c > 0$, this can be written as $\delta_1 = \delta_1(\beta) = O(\beta)$, so that the expression for this part of the tongue boundary is

$$\alpha_1(\beta) = \gamma_1\beta^p + \delta_1(\beta)\beta^p = \gamma_1\beta^p + O(\beta^{p+1})$$

as we wanted to see (the case of γ_2 is treated similarly). This shows the analyticity of tongue boundaries around a collapsed gap and finishes the proof of Theorem 13. □

Remark 16. It is not possible to do the appropriate changes for a whole neighbourhood of $(\alpha, \beta) = (0, 0)$ in view of the format of the transformation T or other similar transformations.

2.3. *Applications to the spectrum of quasi-periodic Schrödinger operators.* Theorem 13 on the analyticity of tongue boundaries can be strengthened in conjunction with Eliasson's reducibility theorem [**Eli92**], which states reducibility at tongue boundaries under the hypothesis of analyticity of the potential and Diophantine character of the frequency ω.

THEOREM 17. [**Eli92**] *Let $\omega \in DC(c, \sigma, \mathbb{R}^d)$ be Diophantine and $Q : \mathbb{T}^d \to \mathbb{R}$ be real analytic in a strip of width $\rho > 0$. Then there is a constant $C = C(c, \sigma, \rho) > 0$ such that if we define*

$$\lambda_0(s) = \begin{cases} \left(\dfrac{s}{C}\right)^2 & s \geq C \\ -\infty & s < C \end{cases}$$

then the following hold for $a > \lambda_0(|Q|_\rho)$.

(i) *If the rotation number* $\operatorname{rot}(a, b)$ *of (24) is Diophantine or rational, with respect to ω, then the corresponding skew-product flow is reducible to constant coefficients (with frequency $\omega/2$).*

(ii) *If $a > \lambda_0$ is at the endpoint of a non-collapsed spectral gap then the Floquet matrix $B \in sl(2,\mathbb{R})$ satisfies $B^2 = 0$, being $B = 0$ if and only if the gap is collapsed.*

According to this theorem, under the hypothesis made on ω and Q, Hill's equation is reducible at the tongue boundaries for small values of $|b|$ and all the values of a, or a large enough once b has been fixed. The analyticity of tongue boundaries holds in this domain as a consequence of Theorem 13.

COROLLARY 18. *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be real analytic and $\omega \in DC(c, \tau, \mathbb{R}^d)$ be Diophantine. Then there is a constant $C > 0$, such that the tongue boundaries are real analytic if $|b| < C$.*

The analyticity of tongue boundaries in an open domain for (a, b) can be used to study the genericity of 'having all gaps open' for a certain value of b. That is, to study the opening of all spectral gaps for a certain value of b. The function space will be, for some $\rho > 0$, the space $C^{\mathrm{a}}_\rho(\mathbb{T}^d, \mathbb{R})$ of real analytic functions $Q : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|\mathrm{Im}\,\theta| < \rho$ and

$$|Q|_\rho < \infty.$$

THEOREM 19. *Let $\omega \in DC(c, \tau, \mathbb{R}^d)$ and $\rho > 0$ be fixed. Then, there exists a constant $C = C(c, \tau, \rho)$ such that for a generic potential (i.e. in a G_δ set) in*

$$\{Q \in C^{\mathrm{a}}_\rho(\mathbb{T}^d, \mathbb{R}) : |Q|_\rho < C\},$$

with respect to the $|\cdot|_\rho$-topology, the operator

$$(H_{Q,\omega,\phi}x)(t) = -x''(t) + Q(\omega t + \phi)x(t)$$

has all spectral gaps open and, thus, it is a Cantor set.

This result answers a problem raised by Moser and Pöschel [**MP84**] asking whether having all spectral gaps open is generic or, at least, having all spectral gaps open for energies a large enough. Under the same hypothesis of the theorem, [**Eli92**] already proves the genericity of Cantor spectrum. This has also been obtained for generic pairs $(\omega, Q) \in \mathbb{R}^d \times C^\delta(\mathbb{T}^d)$, with $0 \le \delta < 1$ in [**Joh91**] (see also [**FJP02**]). For discrete Schrödinger operators, see [**Pui04**] and references therein. The proof uses the following lemma from [**BPS03**].

LEMMA 20. [**BPS03**] *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be a real analytic potential and ω be Diophantine. Let $a_1(b)$ and $a_2(b)$ be the two (analytic) tongue boundaries in a neighbourhood of zero for some resonance* **k**. *Then*

$$a_1'(0) = Q_0 - |Q_{\mathbf{k}}| \quad \textit{and} \quad a_2'(0) = Q_0 + |Q_{\mathbf{k}}|,$$

where the $Q_{\mathbf{k}}$ are the Fourier coefficients of Q and $a_i' = da_i/db$.

The proof of Theorem 19 is then a consequence of the analyticity of the tongue boundaries when the quasi-periodic system is reducible to constant coefficients. Indeed, if the two tongue boundaries of a certain resonance have a transversality at the origin, then the set of values of $|b| \le C$ for which the two tongue boundaries merge is finite. Since there is a countable set of resonance tongues the result follows.

Remark 21. As it will be shown by Theorem 23, the condition that all the tongues are transversal at $b = 0$ is not necessary. The only requirement is that these tongues have some order of transversality at $b = 0$. If this is the case, the proof above shows that, if $Q = bV$ with $|V|_\rho = 1$, then the spectrum $\sigma(bV, \omega)$ has all gaps open for all $|b| < C$ except for a countable set.

Using Eliasson's result in the upper part of the spectrum, one can also conclude genericity of 'having all gaps open' for quasi-periodic Schrödinger operators at large energies.

COROLLARY 22. *Fix a frequency $\omega \in DC(c, \tau, \mathbb{R}^d)$. Then, the spectrum of the Schrödinger operator of a generic potential in $C^a_\rho(\mathbb{T}^d)$ has always a component in which all spectral gaps are open. That is, there is a constant $R > 0$, depending only on c, τ and the norm of Q, such that the spectrum of the operator restricted to the interval $[R, +\infty)$ has all gaps open.*

Let us now sketch the proof of this corollary. Let Q have all harmonics different from zero. By Corollary 18, the tongue boundaries of

$$x'' + (a - bQ(\omega t))x = 0$$

are analytic if $a \geq \lambda_0(|b||Q|_\rho)$ (see [**Eli92**, Eliasson's theorem 17]). Fix $b_0 > 0$ and let $R_1 > 0$ be such that $R_1 \geq \lambda_0(|b||Q|_\rho)$ for all $|b| \leq b_0$. This means that in the domain $[R_1, +\infty) \times [0, b_0]$ of the parameter plane, tongue boundaries are analytic. Assume that a tongue boundary lies in this domain for $|b| \leq b_0$. As it is analytic there, then their crossings form a finite set at most.

As tongue boundaries are globally Lipschitz functions of b with uniform Lipschitz constant [**BPS03**], there is a $R \geq R_1$ such that any tongue emanating from any $a_0 \geq R$ at $b = 0$ satisfies that $a \geq R_1$ for $0 \leq b \leq b_0$. In particular, the spectrum of a generic potential in $[R, +\infty)$ has all gaps open.

Finally, one can also study the question of the opening of all gaps for a particular potential.

THEOREM 23. *Let $d \geq 2$. Then, there is an exceptional set $\mathcal{A} \subset \mathbb{R}^d$, of zero measure, such that if $\omega = (\omega_1, \ldots, \omega_d) \notin \mathcal{A}$, then there is a constant C_ω such that for all values of b, except for a countable set, with $|b| \leq C_\omega$, the spectrum of the operator*

$$Hx = -x'' + b\sum_{j=1}^{d} c_j \cos(\omega_j t)x,$$

where the constants c_j are all different from zero and satisfy the normalization $c_1^2 + \cdots + c_d^2 = 1$, has all gaps open.

This follows immediately from the next result, which describes the order of contact of the 'quasi-Mathieu' equation,

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t)\right)\right)x = 0. \tag{40}$$

THEOREM 24. [**BPS03**] *Consider the reversible near-Mathieu equation with quasi-periodic forcing (40) as above. Then, the order of tangency at* $b = 0$ *of the* $\mathbf{k}^*$*th resonance tongue is greater than or equal to* $|\mathbf{k}^*|$ *and it is exactly* $|\mathbf{k}^*|$ *if and only if* ω *does not belong to* $\mathcal{A}(\mathbf{k}^*)$*, where* $\mathcal{A}(\mathbf{k}^*)$ *is an algebraic subset of the Diophantine frequency vectors.*

3. *Analytic families of reducible linear quasi-periodic systems in* $so(3, \mathbb{R})$

In this section we consider the existence of analytic families of reducible linear quasi-periodic systems in $so(3, \mathbb{R})$, the algebra of all real antisymmetric matrices (hence with zero trace). Quasi-periodic linear equations in $so(3, \mathbb{R})$ have been studied in [**Eli02, Kri99a, Mos98**]. Let us start reviewing some basic facts on the geometry of $so(3, \mathbb{R})$.

If we denote by J_1, J_2, J_3 the following Pauli matrices

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then (J_1, J_2, J_3) is a basis of $so(3, \mathbb{R})$, and for the Lie bracket the following relation holds

$$[J_1, J_2] = J_3, \tag{41}$$

together with the other two circular permutations of indices 1, 2, 3. Using this basis, we can express the Lie bracket $[X, Y]$ for any $X, Y \in so(3, \mathbb{R})$. Indeed, assume that $\mathbf{e}, \mathbf{f} \in \mathbb{R}^3$ are such that

$$X = e_1 J_1 + e_2 J_2 + e_3 J_3 \quad \text{and} \quad Y = f_1 J_1 + f_2 J_2 + f_3 J_3.$$

Then it follows that

$$[X, Y] = \begin{vmatrix} e_2 & f_2 \\ e_3 & f_3 \end{vmatrix} J_1 - \begin{vmatrix} e_1 & f_1 \\ e_3 & f_3 \end{vmatrix} J_2 + \begin{vmatrix} e_1 & f_1 \\ e_2 & f_2 \end{vmatrix} J_3.$$

This expression yields an identification

$$v : (so(3, \mathbb{R}), [\cdot, \cdot]) \to (\mathbb{R}^3, \wedge),$$

where $\wedge$ is the exterior product by sending

$$v([X, Y]) = v(X) \wedge v(Y).$$

If $A_0 \in so(3, \mathbb{R}^3)$, then its eigenvalues are 0 and $\pm i|v(A_0)|$, where the norm on $(\mathbb{R}^3, \wedge)$ is assumed to be Euclidean.

Consider now a linear equation with quasi-periodic coefficients in $so(3, \mathbb{R})$. This means that there is a map $A : \mathbb{T}^d \to so(3, \mathbb{R})$ and a frequency vector such that

$$x' = A(\theta)x, \quad \theta' = \omega, \tag{42}$$

where now $x \in \mathbb{R}^3$. Therefore, there exist $a_i : \mathbb{T}^d \to \mathbb{R}$ for $i = 1, 2, 3$, such that

$$A(\theta) = \begin{pmatrix} 0 & a_1(\theta) & a_3(\theta) \\ -a_1(\theta) & 0 & a_2(\theta) \\ -a_3(\theta) & -a_2(\theta) & 0 \end{pmatrix}.$$

Let us restrict our attention to systems that are perturbations of a constant matrix. That is, consider equations of the form

$$A(\theta, \mu) = A_0 + P(\theta, \mu) \tag{43}$$

with $A_0, P \in so(3, \mathbb{R})$ such that $P(\cdot; 0) = 0$. To study analytic families of the above equations that have a constant Floquet matrix it is necessary to have an expression for the adjoint operator $\mathrm{ad}_{A_0} : g \to g$. In the basis (J_1, J_2, J_3) one can assume, after a change of basis, that

$$A_0 = |v(A_0)| J_3$$

so that, if $X = x_1 J_1 + x_2 J_2 + x_3 J_3$, then

$$\mathrm{ad}_{A_0}(X) = \begin{pmatrix} 0 & |v(A_0)| & 0 \\ -|v(A_0)| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and the eigenvalues of ad_{A_0} are 0 and $\pm|v(A_0)|$. To check the non-resonance condition

$$|v(A_0)| - \langle \mathbf{k}, \omega \rangle \neq 0 \tag{44}$$

when $\mathbf{k} \in \mathbb{Z}^d$ we have to consider three possibilities: it is always different from zero (*irrational* case); it vanishes for $\mathbf{k} = 0$ (*degenerate* case); or it vanishes for some $\mathbf{k} \neq 0$ (*rational* case). Let us treat these three cases separately.

3.1. *Irrational case.* Assume that the non-resonance condition (44) is satisfied for all $\mathbf{k} \in \mathbb{Z}^d$. To be under the hypothesis of Lemma 7 one must impose the additional Diophantine hypothesis:

$$\big|\, |v(A_0)| - \langle \mathbf{k}, \omega \rangle \big| \geq \frac{K}{|\mathbf{k}|^\tau} \quad \text{for } \mathbf{k} \neq \mathbf{0}, \tag{45}$$

where K, τ are some fixed positive constants. If, for

$$P_{\mathbf{0}} = p_1 J_1 + p_2 J_2 + p_3 J_3 \in so(3, \mathbb{R})$$

we define

$$S(P) = \left(-\frac{p_2}{|v(A_0)|}, \frac{p_1}{|v(A_0)|}, 0 \right)$$

and

$$C(P) = (0, 0, p_3)$$

in the (J_1, J_2, J_3)-basis of $so(3, \mathbb{R})$, the quartet (A_0, C, S, ω) is admissible by Lemma 7. Therefore, there exists a real analytic function $p_3 = p_3(\mu)$ such that the system

$$x' = (A_0 + P(\theta, \mu) - p_3(\mu) J_3) x, \quad \theta' = \omega \tag{46}$$

is reducible to

$$x' = A_0.$$

In particular, the condition $p_3(\mu) = 0$, which is real analytic, determines an analytic family of reducible systems with Floquet matrix A_0.

3.2. *Degenerate case.* This case corresponds to $|v(A_0)| = 0$ so that $A_0 = 0$. According to Example 8, if ω is Diophantine, we can choose the counterterm C to be the identity and the operator S to be identically zero.

Applying Theorem 4, there exist real analytic functions ξ_1^*, ξ_2^* and ξ_3^* of μ and a real analytic matrix $X = X(\theta, \mu)$ in $so(3, \mathbb{R})$ such that the transformation $Z = \exp(X)$ satisfies

$$\partial_\omega Z(\theta, \mu) = \left(P(\theta, \mu) - \sum_{j=1}^{3} \xi_j^*(\mu) J_j \right).$$

In particular, the three conditions $\xi^*(\mu) = 0$ determine an analytic family of reducible subsystems of (42) with Floquet matrix A_0.

3.3. *Rational case.* This resonant case is characterized by the existence of some $\mathbf{k}_0 \neq 0$ such that

$$|v(A_0)| = \langle \mathbf{k}_0, \omega \rangle.$$

Note that, even if ω is Diophantine, the Diophantine condition (20) does not hold, although this can be overcome, see Appendix A.

Nevertheless this situation of rational dependence can be reduced to the previous degenerate case. Indeed, denote by $\alpha = |v(A_0)|$ the positive eigenvalue of ad_{A_0} and assume, as before, A_0 is of the form $A_0 = \alpha J_3$.

Let $y(t) = \exp(\alpha t J_3)$. As $\alpha = \langle \mathbf{k}_0, \omega \rangle$, y is quasi-periodic with $y(t) = Y(\omega t) \in SO(3, \mathbb{R})$ being

$$Y(\theta) = \exp(\langle \mathbf{k}_0, \theta \rangle J_3),$$

which is real analytic. After the change of variables

$$x = Y(\theta) y$$

the new unperturbed matrix is zero and we are in the degenerate case.

4. *Hyperbolicity boundaries in higher dimensions*

In this section we consider the problem of the generalization of Theorem 13 to higher-dimensional Hamiltonian systems. Such a system has the form

$$x' = H(\theta, \mu) x, \quad \theta' = \omega, \tag{47}$$

where $H \in sp(m, \mathbb{R})$ depends analytically on the angles $\theta \in \mathbb{T}^d$ and the external parameters $\mu \in \mathbb{R}^p$ in some neighbourhood of the origin. In what follows the frequency ω is also assumed to be Diophantine. The dimension of (47) is thus $n = 2m$.

We are interested in the regions in the parameter space $\mu \in \mathbb{R}^p$ for which system (47) has an *exponential dichotomy*. Let us now quickly review this property. For a proper exposition and more references see [**SS78, HdlL04**].

4.1. *Exponential dichotomy and Sacker–Sell spectral theory.* A system like (47),

$$x' = A(\theta)x, \quad \theta' = \omega, \tag{48}$$

but not necessarily Hamiltonian, has an exponential dichotomy if and only if the only solution that is bounded on $\mathbb{R}$ is the trivial one. Exponential dichotomy is equivalent to a certain splitting of the product space $\mathbb{R}^n \times \mathbb{T}^d$ into stable and unstable subbundles as we now explain.

If we denote by $M(t;\phi)$ a fundamental matrix for

$$x' = A(\omega t + \phi)x,$$

then the stable subbundle can be defined as

$$\mathcal{S} = \Big\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d \text{ such that } \lim_{t\to+\infty} |M(t;\theta)x| = 0\Big\}$$

and the unstable subbundle as

$$\mathcal{U} = \Big\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d \text{ such that } \lim_{t\to-\infty} |M(t;\theta)x| = 0\Big\}.$$

For a general system (48) these subbundles need not be continuous as a function of θ but, if (48) has an exponential dichotomy, then these subbundles are continuous (in fact as regular as the original system see [**Joh80**] and [**JS81**]) and, for all values of θ, the sections

$$\mathcal{S}(\theta) = \{(x,\theta) \in \mathcal{S}\} \quad \text{and} \quad \mathcal{U}(\theta) = \{(x,\theta) \in \mathcal{U}\}$$

satisfy $\mathbb{R}^n = \mathcal{S}(\theta) \oplus \mathcal{U}(\theta)$. In fact, if such a decomposition into continuous subbundles holds, then the system has an exponential dichotomy.

Sacker–Sell spectral theory relies on the concept of exponential dichotomy and it is defined as follows. A value $\lambda \in \mathbb{R}$ is in the *Sacker–Sell spectrum* of (50), $\Sigma(A,\omega)$, if the system

$$x' = (A(\theta) - \lambda I)x, \quad \theta' = \omega \tag{49}$$

does not have an exponential dichotomy. The spectral theorem [**SS78**] states that the Sacker–Sell spectrum is the union of, at most, n disjoint intervals, called the *spectral intervals*. Moreover, if $\lambda_1 < \lambda_2$ do not belong to the spectrum (that is, they lie in the resolvent set) and if $\mathcal{S}_{\lambda_1}$ is the stable subbundle of (49) when $\lambda = \lambda_1$ and $\mathcal{U}_{\lambda_2}$ the unstable subbundle when $\lambda = \lambda_2$, then $\mathcal{S}_{\lambda_1} \cap \mathcal{U}_{\lambda_2}$ is an invariant subbundle of (50) and the Sacker–Sell spectrum of the restriction of the flow to this subbundle is $\Sigma(A,\omega) \cap (\lambda_1, \lambda_2)$.

An important property of exponential dichotomy is that it is persistent under small quasi-periodic perturbations of the original system. This is the linear operator version of the normal hyperbolicity results for dynamical systems, see, for instance, [**HPS77**]. In particular, this means that a value of the resolvent set of (48) belongs to it for small perturbations of the system.

In [**HdlL04**], the structure of this spectrum for discrete-time quasi-periodic flows was considered. These results apply to the case of continuous flows like (50) by taking Poincaré maps. They derive several additional properties when the flow is Hamiltonian like (47).

First of all, if λ belongs to the Sacker–Sell spectrum of a quasi-periodic Hamiltonian skew-product flow, then $-\lambda$ also belongs to it. In particular, if there is a spectral interval including the zero, it is symmetric with respect to zero. Also, it has as a consequence that the restriction of the flow to any invariant subbundle whose restricted flow has a non-symmetric spectrum (like the stable or unstable subbundles) is not Hamiltonian.

There is one case when the restriction of a quasi-periodic Hamiltonian flow to an invariant subbundle is again Hamiltonian. Let $\mathcal{C}$ be an invariant subbundle of (47) whose spectrum (that is the spectrum of the restriction of the flow to it) is symmetric with respect the origin. Then, the restriction of the flow is Hamiltonian [**HdlL04**]. This kind of subbundle, and their restricted flows, are particularly important, as the exponential dichotomy of (47) is equivalent to the exponential dichotomy of this reduced flow. Such a continuous subbundle will be called a *central subbundle*. Note that the whole space $\mathbb{R}^{2m} \times \mathbb{T}^d$ is always a trivial central subbundle.

4.2. *An example with analytic boundaries.* Here we will focus only on the following quasi-periodic system of equations

$$\begin{aligned} x_0'' &= -ax_0 + b\sum_{j=0}^{m} Q_{0j}(\theta)x_j, \\ x_k'' &= \lambda_k^2 x_k + b\sum_{j=0}^{m} Q_{kj}(\theta)x_j, \quad k = 1, \dots, m, \\ \theta' &= \omega, \end{aligned} \tag{50}$$

where $a, b, \lambda_1, \dots, \lambda_m$ are real parameters such that

$$\lambda_1, \dots, \lambda_m > \lambda_0 > 0, \tag{51}$$

for some λ_0 and the Q_{kj} are real analytic functions. This can be written as a Hamiltonian system like (47) if we set

$$H(\theta) = H_0 + bH_1(\theta) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & I \\ \hline -a & 0 & \\ 0 & \Lambda & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline Q & 0 \end{array}\right),$$

where Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ and $Q = (Q_{kj})_{k,j}$ if $y = (x, x')^{\mathrm{T}}$ and

$$y' = (H_0 + bH_1(\theta))y, \quad \theta' = \omega. \tag{52}$$

When $b = 0$ this system is in constant coefficients and in this case the study of the Sacker–Sell spectrum and associated invariant subbundles is trivial. Assume furthermore that $a > -\lambda_0^2$. Then the Sacker–Sell spectrum is exactly the union

$$S_{a,0} \cup C_{a,0} \cup U_{a,0},$$

where

$$S_{a,0} = \{-\lambda_1\} \cup \cdots \cup \{-\lambda_k\}, \quad U_{a,0} = \{\lambda_k\} \cup \cdots \cup \{\lambda_1\}$$

and $C_{a,0}$ is $\{0\}$ if $a \geq 0$ or $\{-\sqrt{-a}\} \cup \{\sqrt{-a}\}$ if $a < 0$. Therefore system (52) has an exponential dichotomy if, and only if, $a < 0$.

Let $\mathcal{C}_{a,0}$ be the invariant subbundle corresponding to the $C_{a,0}$. If b is small enough, then the Sacker–Sell spectrum of (52) has three components $S_{a,b}$, $C_{a,b}$ and $U_{a,b}$ separated by λ_0 and $-\lambda_0$. The corresponding spectral subbundles $\mathcal{S}_{a,b}$, $\mathcal{C}_{a,b}$ and $\mathcal{U}_{a,b}$, which are real analytic, depend real analytically on a, b for $a > -\lambda_0$ and b small enough [**Joh80**].

The flow on the central subbundle is again Hamiltonian and two-dimensional, as for $b = 0$ the central subbundle $\mathcal{C}_{a,0}$ is given by $x_1 = \cdots = x_m = x'_1 = \cdots = x'_m = 0$ and the reduced flow (in the coordinates (x_0, x'_0, θ)) is precisely given by

$$x''_0 = -ax'_0, \quad \theta' = \omega. \tag{53}$$

By the analytic dependence of the subbundles, the reduced flow on the central subbundle $\mathcal{C}_{a,b}$ is given, in some new coordinates $(\xi_1, \xi_2)^{\mathrm{T}}$, by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}' = \left(\begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} + bP(\theta, a, b) \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \theta' = \omega, \tag{54}$$

where P is a Hamiltonian matrix function depending real analytically on θ and (a, b) for $a > -\lambda_0$ and small values of $|b|$. Note that, after this reduction to the central subbundle, (52) has an exponential dichotomy if and only if (54) has an exponential dichotomy.

As for $b = 0$, (54) reduces to (53), the resonances of our problem are the values of a of the form

$$\left(\frac{\langle \mathbf{k}, \omega \rangle}{2} \right)^2 \tag{55}$$

for some $\mathbf{k} \in \mathbb{Z}^d$ not identically zero, because for these values all of the solutions of (53) are quasi-periodic with frequency $\omega/2$. These resonant values of a lie precisely at the boundaries of hyperbolic regions.

THEOREM 25. *Assume that a_0 is of the form (55) for some non-zero $\mathbf{k} \in \mathbb{Z}^d$ and that $\lambda_1, \ldots, \lambda_m$ satisfy (51) for some $\lambda_0 > 0$. Then there exists a $\beta_0 > 0$ and two real-analytic functions a_1 and a_2, such that for $|b| < \beta_0$:*

(i) *if (a, b) is such that $(a - a_1(b))(a - a_2(b)) < 0$ system (52) has an exponential dichotomy;*

(ii) *if $(a - a_1(b))(a - a_2(b)) = 0$ system (52) does not have an exponential dichotomy;*

(iii) *for all $\varepsilon > 0$, there exists an a, with $\varepsilon > (a - a_1(b))(a - a_2(b)) > 0$, such that (52) does not have an exponential dichotomy.*

Proof. Theorem 13 also holds for systems of the form (54) so that, for any resonant value of a_0 there exist two real-analytic functions a_1 and a_2 such that $a_1(0) = a_2(0) = a_0$, which parameterize the boundaries of the region in the (a, b)-plane with zero rotation number for $|b| < \beta_0$. As the regions of constancy of the rotation number of (54) correspond to the regions of exponential dichotomy, the result follows. □

After this example it is natural to ask whether similar results are true when in an equation such as (50) the dimension of the central subbundle is greater than two. Appendix C presents a numerical exploration of the case $\dim C_{a,0} = 4$. There is a strong evidence that the boundaries of the regions of exponential dichotomy are not analytic.

5. *Proofs of Theorems 4 and 11*

In this section we will prove Theorems 4 and 11. The proofs of both results follow the same guidelines and we will prove them at the same time. We give the proof of Theorem 11 because there is an additional element, the scaling factor, which has to be taken into account. The reader interested in Theorem 4 only can replace χ by 1 whenever it appears. Recall that Theorem 11 requires some additional properties on the eigenvalues of the adjoint of A_0.

We will first prove Theorem 11 disregarding the dependence on the external parameters μ and then we will explain what needs to be done in order to prove the analyticity with respect to these parameters.

To prove Theorem 4 we must show the existence of a constant element $\xi^* \in g$, with $C(\xi^*) = \xi^*$, and $Z : \mathbb{T}^d \to G$, of the form $Z = \exp(\chi X)$ with $X \in g$ small (therefore Z is close to the identity), such that

$$\partial_\omega Z(\theta) = \chi^k (A_0 + \chi P(\theta) - \chi \xi^*) Z - Z \chi^k A_0, \quad \theta \in \mathbb{T}^d. \tag{56}$$

This is a nonlinear homological equation that we will try to solve by Newton's quadratic method (following [**Mos67, BMS76**]). It is an iterative process in which the final transformation Z will be given as the infinite composition of the transformations that will be defined at each step. Note that ξ^* is not yet known and it will have to be determined along the iterative process. To make this more evident we write this equation as

$$\partial_\omega Z(\theta) = \chi^k (A_0 + \chi P(\theta) - \chi \eta) Z - Z \chi^k A_0, \quad \theta \in \mathbb{T}^d, \tag{57}$$

where $\eta \in g$ is now a variable. At each step of the iterative process we will define new transformations Z^r in G and $\xi^r : g \to g$, which will reduce the system to constant coefficients up to a certain perturbation that will become smaller and smaller. The domains in θ of the composition of the transformations $Z^0, \dots, Z^r$ will shrink to a narrower, but non-empty, complex strip of $\mathbb{T}^d$. The domains for which the composition $\xi^0 \circ \cdots \circ \xi^r$ is defined will quickly shrink to zero and the image of zero under this composition will define the sought ξ^*. To see this more clearly, we proceed a bit further in this iterative process before giving the inductive lemma. Writing $Z = \exp(\chi X)$, the linear version of (57), with respect to the size of the perturbation, becomes

$$\partial_\omega X(\theta) = \chi^k ([A_0, X] + P^0(\theta) - \eta^0), \quad \theta \in \mathbb{T}^d, \tag{58}$$

where we have written $P^0 = P$ and $\eta^0 = \eta$ to stress that this is the first step of an iterative process. The admissibility of $(\chi^k A_0, C, S, \omega)$ implies that (58) can be uniquely solved in any strip of $\mathbb{T}^d$ narrower than ρ provided that η^0 is taken equal to $\hat{\eta}^0 = C(\overline{P})$ and we set $\overline{X} = S(\overline{P})$. Let $X^0(\theta)$ be the solution for this choice of η^0. Then $Z^0 = \exp(\chi X^0)$ satisfies

$$\begin{aligned} \partial_\omega Z^0(\theta) = \chi^k (A_0 + \chi P^0(\theta) - \chi \eta^0) Z^0 \\ - Z^0 \chi^k (A_0 + \chi P_1(\theta, \eta^0) - \chi \eta^0 + \chi C(\overline{P})), \quad \theta \in \mathbb{T}^d, \end{aligned} \tag{59}$$

where $P_1(\theta, \eta^0)$ is the new perturbation defined by the above equation. Up to now we have defined the transformation Z^0, but we have not yet defined the transformation for η^0

to render it closer to zero. In order to put the right-hand side of (59) in the form of the left-hand side we introduce a new variable η^1 satisfying

$$\eta^1 = \eta^0 - C(\overline{P}). \tag{60}$$

This trivially defines a diffeomorphism $\xi^0 : g \to g$

$$\eta^1 \mapsto \xi^0(\eta^1) = \eta^0 = \eta^1 + C(\overline{P}),$$

which allows us to express (59) in the new variable η^1 as follows:

$$\begin{aligned}\partial_\omega Z^0(\theta) &= \chi^k(A_0 + \chi P^0(\theta) - \chi\xi^0(\eta^1))Z^0(\theta)\\ &\quad - Z^0(\theta)\chi^k(A_0 + \chi P^1(\theta, \eta^1) - \chi\eta^1), \quad \theta \in \mathbb{T}^d\end{aligned} \tag{61}$$

if we set $P^1(\theta, \eta^1) = P_1(\theta, \xi^0(\eta^1))$.

The point in choosing these transformations Z^0 and ξ^0 is that the perturbations on the right-hand side are much smaller than those on the left. This will be shown in the following section. We would like to stress that each step of the transformation involves two changes of variables. First, using the admissibility of $(\chi^k A_0, C, S, \omega)$, we perform the change Z, which implies considering narrower strips around the torus $\mathbb{T}^d$. Second, inverting (60) we perform a change in the variable η so that the system in this new variable is closer to A_0. Of course, in this first step, the transformation ξ^0 defined by (60) is globally a diffeomorphism, but in the next steps the domains of definition of the transformation of η will rapidly shrink to zero.

5.1. *The inductive lemma.* To prove that at each step of the iterative process the transformed system belongs to the same Lie algebra g, we use the following proposition.

PROPOSITION 26. *Let g be a matrix Lie algebra and $Y : \mathbb{R} \to g$ any smooth function. Then, we have the following.*
(i) *For all $t \in \mathbb{R}$ the element $(\exp(Y(t)))' \exp(-Y(t))$ belongs to g.*
(ii) *If $a, b : \mathbb{R} \to gl(n, \mathbb{R})$ are continuous functions that satisfy the equation*

$$(\exp(Y(t)))' = a(t)\exp(Y(t)) - \exp(Y(t))\, b(t), \quad t \in \mathbb{R},$$

then $a(t) \in g$ for $t \in \mathbb{R}$ if and only if $b(t) \in g$ for $t \in \mathbb{R}$.

Proof. First of all, note that (ii) is a direct consequence of (i). Indeed, if $X(t) = \exp(Y(t))$, then one has the identities

$$b = -X^{-1}X' + X^{-1}aX = (X^{-1})'X + X^{-1}aX,$$

and also

$$a = X'X^{-1} + XbX^{-1}.$$

Then (ii) follows from (i), applying (i) to $-Y$ and Y, respectively, and using the invariance of g by conjugations by matrices in G. The proof of (i) makes use of the following property of Lie algebras (see [**Pos86**], for instance). Let $Y_0, Y_1 \in g$. Then

$$\exp(Y_0 + tY_1)\exp(-Y_0) = \exp\biggl(t\frac{\exp(\mathrm{ad}_{Y_0}) - I}{\mathrm{ad}\, Y_0}Y_1 + o(t)\biggr),$$

where $(\exp(\operatorname{ad} Y_0) - I)/\operatorname{ad} Y_0$ is the sum of an operator series

$$I + \frac{\mathrm{ad}_{Y_0}}{2!} + \cdots + \frac{(\mathrm{ad}_{Y_0})^k}{(k+1)!} + \cdots$$

and $\mathrm{ad}_{Y_0} : g \to g$ is the adjoint operator of Y_0. Let $t_0 \in \mathbb{R}$ and write $Y_0 = Y(t_0)$ and $Y_1 = Y'(t_0)$, both belonging to g. Note that

$$\begin{aligned}\frac{d}{dt}(\exp(Y(t)))_{t=t_0}\exp(-Y(t_0)) &= \frac{d}{dt}(\exp(Y_0 + (t-t_0)Y_1))_{t=t_0}\exp(-Y_0)\\ &= \frac{d}{dt}(\exp(Y_0 + (t-t_0)Y_1)\exp(-Y_0))_{t=t_0}\\ &= \frac{\exp(\mathrm{ad}_{Y_0}) - I}{\mathrm{ad}_{Y_0}} Y_1 \in g,\end{aligned}$$

and the proposition follows. □

Now we can state the inductive lemma.

LEMMA 27. (The inductive lemma) *Assume that (A_0, C, S, ω) is admissible with constants c and ν. Fix a complex domain*

$$\mathcal{D}^r: \quad |\operatorname{Im}\theta| < \rho_r, \quad |\eta^r| < \sigma_r$$

and a constant $0 < \delta_r < \rho_r$. Then there exists a constant $K = K(g)$ such that if P^r is analytic on $\mathcal{D}^r$, belongs to g for real values of (θ, η^r), and

$$|P^r|_{\mathcal{D}^r} = \sup_{(\theta,\eta^r)\in\mathcal{D}^r} |P^r(\theta, \eta^r)| \le \varepsilon_r < K\sigma_r \tag{62}$$

then, in the domain

$$\mathcal{D}_{r+1}: \quad |\operatorname{Im}\theta| < \rho_r - \delta_r, \quad |\eta^r| < \sigma_r/2$$

the transformation

$$Z^r(\theta, \eta^r) = \exp(\chi X^r(\theta, \eta^r)), \tag{63}$$

where $X^r(\theta, \eta^r)$ satisfies

$$\partial_\omega X^r = \chi^k([A_0, X^r] + P^r - C(\overline{P^r}(\eta^r))), \quad \overline{X^r} = S(\overline{P^r}(\eta^r)), \tag{64}$$

is real analytic and the equation

$$\eta^{r+1} = \eta^r - C(\overline{P^r}(\eta^r)) \tag{65}$$

defines an analytic diffeomorphism ξ^r

$$\eta^{r+1} \in D(0, \varepsilon_r) \mapsto \xi^r(\eta^{r+1}) \in D(0, 2\varepsilon_r)$$

such that the equation

$$\begin{aligned}\partial_\omega Z^r(\theta, \xi^r(\eta^{r+1})) &= \chi^k(A_0 + \chi P^r(\theta, \xi^r(\eta^{r+1})) - \chi\xi^r(\eta^{r+1}))Z^r\\ &\quad - Z^r\chi^k(A_0 + \chi P^{r+1}(\theta, \eta^{r+1}) - \chi\eta^{r+1})\end{aligned} \tag{66}$$

holds in the domain

$$\mathcal{D}^{r+1}: \quad |\mathrm{Im}\,\theta| < \rho_r - \delta_r, \quad |\eta^{r+1}| < \varepsilon_r$$

with the estimates

$$|X^r|_{\mathcal{D}_{r+1}} \leq M := c\frac{\varepsilon_r}{\delta_r^{\nu}}, \tag{67}$$

$$|P^{r+1}|_{\mathcal{D}^{r+1}} \leq (e^{|\chi|M} - 1)(\varepsilon_r + 5\varepsilon_r e^{|\chi|M} + 2|A_0|e^{|\chi|M}M) + |A_0|\,|\chi|M^2 e^{2|\chi|M} \tag{68}$$

and

$$|D_{\eta^{r+1}}\xi^r|_{\varepsilon_r} \leq 1 + c_1\frac{\varepsilon_r}{\sigma_r}, \tag{69}$$

where the constant c_1 depends only on g.

Remark 28. The estimate (67) comes from the admissibility of $(\chi^k A_0, C, S, \omega)$ and it is included in the statement of the lemma only for the sake of completeness.

Remark 29. The difference between the domains $\mathcal{D}_{r+1}$ and $\mathcal{D}^{r+1}$ is due to the η component. The restriction for the η component in $\mathcal{D}^{r+1}$ allows us to define the map

$$(\theta, \eta^{r+1}) \in \mathcal{D}^{r+1} \mapsto (\theta, \xi^r(\eta^{r+1})) \in \mathcal{D}_{r+1},$$

which inverts (65). Similarly to what we did for the first step, a perturbation P_{r+1} : $\mathcal{D}_{r+1} \to g$ is defined by

$$\begin{aligned}\partial_\omega Z^r(\theta, \eta^r) = {} & \chi^k(A_0 + \chi P^r(\theta, \eta^r) - \chi\eta^r)Z^r \\ & - Z^r\chi^k(A_0 + \chi P_{r+1}(\theta, \eta^r) - \chi\eta^r + \chi C(\overline{P^r}(\eta^r))),\end{aligned} \tag{70}$$

and later on we will define the perturbation $P^{r+1} : \mathcal{D}^{r+1} \to g$ as

$$P^{r+1}(\theta, \eta^{r+1}) = P_{r+1}(\theta, \xi^r(\eta^{r+1}))$$

so that (66) holds.

Proof. First of all we compute P_{r+1} in terms of Z^r, X^r, A_0, P^r and η^r. The identities (63) and (64) determine P_{r+1} when $\chi \neq 0$

$$\begin{aligned}P_{r+1}(\theta, \eta^r) = {} & (I - (Z^r)^{-1})(\eta^r - C(\overline{P^r}(\eta^r))) - (Z^r)^{-1} \\ & \times \left(\frac{1}{\chi}[A_0, \chi X^r - Z^r] + P^r(I - Z^r) + \eta^r(Z^r - I) + \frac{1}{\chi^{k+1}}\partial_\omega(Z^r - \chi X^r)\right)\end{aligned} \tag{71}$$

on $\mathcal{D}_{r+1}$. For the proof of Theorem 11, one also has to define the value for $\chi = 0$. This can be done taking the limit of the above expression when $\chi \to 0$ and obtain

$$P_{r+1}(\theta, \eta^r) = 0. \tag{72}$$

As A_0, X^r and P^r belong to g for real values of θ and $\eta^r \in g$, then necessarily $P_{r+1} \in g$ for these real values, due to Proposition 26. In order to be able to define

$$P^{r+1}(\theta, \eta^{r+1}) = P_{r+1}(\theta, \xi^r(\eta^{r+1})), \quad (\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$$

we first need to know that (65) can be inverted so that the map ξ^r can be defined.

5.1.1. *Inversion of (65).* Let $F^r(\eta^r) = C(\overline{P^r}(\eta^r))$. Then F^r is analytic on the ball $D(0, \sigma_r)$. By Cauchy estimates we have

$$|D_{\eta^r} F^r|_{\sigma_r/2} \leq c' \frac{|F^r|_{\sigma_r}}{\sigma_r - \sigma_r/2} \leq 2c' \frac{\varepsilon_r}{\sigma_r},$$

where c' is a constant depending only on g. Assume that

$$K < \min\left(\frac{1}{4}, \frac{1}{4c'}\right). \tag{73}$$

In this case, (65) is invertible when $|\eta^r| < \sigma_r/2$ and, as

$$|\xi^r(\eta^{r+1})| \leq |\eta^{r+1}| + |F^r(\xi^r(\eta^r))|,$$

then for $|\eta^{r+1}| < \varepsilon_r$ one has

$$|\eta^r| < 2\varepsilon_r.$$

As $\varepsilon_r > K\sigma_r$,

$$2\varepsilon_r \leq \frac{\sigma_r}{2},$$

and the map ξ^r,

$$\eta^{r+1} \in D(0, \varepsilon_r) \mapsto \xi^r(\eta^{r+1}) \in D\left(0, \frac{\sigma_r}{2}\right),$$

is well-defined. Moreover,

$$|D_{\eta^{r+1}} \xi^r|_{\varepsilon_r} \leq \frac{1}{1 - |D_{\eta^r} F^r|_{\sigma_r/2}} \leq \frac{1}{1 - 2c'(\varepsilon_r/\sigma_r)} \leq 1 + c_1 \frac{\varepsilon_r}{\sigma_r},$$

writing $c_1 = 4c'$, as we wanted to show.

5.1.2. *Bounds for* P^{r+1}. Once we have inverted (65) we can now estimate $P^{r+1}(\theta, \eta^{r+1}) = P_{r+1}(\theta, \xi^r(\eta^{r+1}))$ on $\mathcal{D}^{r+1}$, which, in virtue of (71), can be expressed as follows:

$$\begin{aligned} P^{r+1}(\theta, \eta^{r+1}) = {} & (I - Z^{-1})(\xi^r(\eta^{r+1}) - C(\overline{P}(\xi^r(\eta^{r+1})))) \\ & - Z^{-1}\left(\frac{1}{\chi}[A_0, \chi X - Z] + P^r(\theta, \xi^r(\eta^{r+1}))(I - Z)\right. \\ & \left. + \xi^r(\eta^{r+1})(Z - I) + \frac{1}{\chi^{k+1}} \partial_\omega (Z - \chi X)\right), \end{aligned} \tag{74}$$

where we write $Z = Z^r(\theta, \xi^r(\eta^{r+1}))$ and $X = X^r(\theta, \xi^r(\eta^{r+1}))$ only for simplicity. To bound this remainder, we will estimate all of the terms in the above expression. First of all note that, as $(\theta, \xi^r(\eta^{r+1})) \in \mathcal{D}_{r+1}$ for $(\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$, then

$$|X|_{\mathcal{D}^{r+1}} \leq |X^r|_{\mathcal{D}_{r+1}} \leq c \frac{\varepsilon_r}{\delta_r{}^\nu} =: M.$$

Now we are ready to bound the terms of (74)

$$|I - Z^{-1}|_{\mathcal{D}^{r+1}} = \left| \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} (\chi X)^j \right|_{\mathcal{D}^{r+1}} \leq e^{|\chi| M} - 1.$$

$$|\xi^r - C(\overline{P^r}(\xi^r))|_{\mathcal{D}^{r+1}} = |\eta^{r+1}|_{\mathcal{D}^{r+1}} \le \varepsilon_r.$$
$$|Z^{-1}|_{\mathcal{D}^{r+1}} \le e^{|\chi|M}.$$

$$\begin{aligned}\left|\frac{1}{\chi}[A_0, \chi X - Z]\right|_{\mathcal{D}^{r+1}} &= \frac{1}{|\chi|}|[A_0, I + \chi X - Z]|_{\mathcal{D}^{r+1}} \\ &\le \frac{2}{|\chi|}|A_0||I + \chi X - Z|_{\mathcal{D}^{r+1}} \\ &\le \frac{2}{|\chi|}|A_0|(e^{|\chi|M} - 1 - |\chi|M) \le |\chi|M^2|A_0|\exp(|\chi|M).\end{aligned}$$

$$\begin{aligned}|P^r|_{\mathcal{D}^{r+1}} &\le \varepsilon_r. \\ |I - Z|_{\mathcal{D}^{r+1}} &\le e^{|\chi|M} - 1. \\ |\xi^r|_{\mathcal{D}^{r+1}} &\le 2\varepsilon_r.\end{aligned}$$

$$\begin{aligned}\frac{1}{|\chi^{k+1}|}|\partial_\omega(Z - \chi X)|_{\mathcal{D}^{r+1}} &= \frac{1}{|\chi^{k+1}|}\left|\sum_{j=2}^{\infty}\frac{1}{j!}\partial_\omega((\chi X)^j)\right|_{\mathcal{D}^{r+1}} \\ &\le \frac{1}{|\chi^{k+1}|}(\exp(|\chi||X|_{\mathcal{D}^{r+1}}) - 1)|\chi|\,|\partial_\omega X|_{\mathcal{D}^{r+1}} \\ &\le (e^{|\chi|M} - 1)(2|A_0|M + 2\varepsilon_r).\end{aligned}$$

Collecting all of these estimates we have

$$\begin{aligned}|P^{r+1}|_{\mathcal{D}^{r+1}} &\le (e^{|\chi|M} - 1)\varepsilon_r + e^{|\chi|M}\{|\chi|M^2|A_0|e^{|\chi|M} + \varepsilon_r(e^{|\chi|M} - 1) \\ &\quad + 2\varepsilon_r(e^{|\chi|M} - 1) + (e^{|\chi|M} - 1)(2|A_0|M + 2\varepsilon_r)\} \\ &\le (e^{|\chi|M} - 1)(\varepsilon_r + 5\varepsilon_r e^{|\chi|M} + 2|A_0|e^{|\chi|M}M) \\ &\quad + |A_0|\,|\chi|M^2e^{2|\chi|M}, \qquad (75)\end{aligned}$$

which holds for all χ, even for $\chi = 0$, due to the choice of (72). This proves the last estimate (68). □

5.2. *The iterative construction.* To finish the proof we must show that the iterative process that was started at the beginning of the section can be continued up to any order (by suitably choosing the right domains) and that this process is convergent. As the first step of an iterative process define

$$P^0(\theta, \eta^0) = P(\theta),$$

which is analytic in the complex strip $|\mathrm{Im}\,\theta| < \rho_0$. Having fixed this constant, we will define sequences $(\rho_r)_r$, $(\delta_r)_r$, $(\varepsilon_r)_r$ and $(\sigma_r)_r$ such that the inductive lemma can be applied up to any finite order and which guarantee the existence of the constant ξ^* and the reducing transformation that we will call Z^*.

Take

$$\rho_r = \rho_0\left(\frac{1}{2} + \frac{1}{2^{r+1}}\right), \quad \rho_{r+1} = \rho_r - \delta_r, \quad \delta_r = \frac{\rho_0}{2^{r+2}}, \quad r \ge 0$$

as the sequences that will define the successive domains for the angles θ. In order to overcome the problems caused by the presence of small divisors, we define the sequences $(\varepsilon_r)_r$ and $(\sigma_r)_r$ as

$$\varepsilon_{r+1} = \varepsilon_r^{3/2}, \quad \varepsilon_r = \varepsilon_0^{(3/2)^r}, \quad \sigma_{r+1} = \varepsilon_r, \quad r \geq 0$$

and $\sigma_0 = \varepsilon_0^{2/3}$, which will be completely determined once we fix the initial ε_0. In order to do so, we first state the inequalities that we want the sequences $(\rho_r)_r$, $(\delta_r)_r$, $(\varepsilon_r)_r$ and $(\sigma_r)_r$ to satisfy. In the notation of the inductive lemma, the conditions we impose are

$$\varepsilon_r < K\sigma_r, \quad r \geq 0, \tag{76}$$

$$M = c\frac{\varepsilon_r}{\delta_r^\nu} < \varepsilon_r^{1/2}, \quad r \geq 0 \tag{77}$$

and

$$(e^{|\chi|M} - 1)(\varepsilon_r + 5\varepsilon_r e^{|\chi|M} + 2|A_0|e^{|\chi|M}M) + |A_0|\,|\chi|M^2e^{2|\chi|M} < \varepsilon_r^{3/2}, \quad r \geq 0. \tag{78}$$

Now we must choose ε_0 so that these conditions are satisfied.

Choice of ε_0. The choice of ε_0 will be very conservative. First of all, condition (76) is equivalent to

$$\frac{\varepsilon_r}{\varepsilon_{r-1}} = \varepsilon_0^{1/2(3/2)^{r-1}} < K, \quad r \geq 0,$$

provided that we set $\varepsilon_{-1} = \varepsilon_0^{2/3}$ for consistency. The conditions for $r \geq 0$ hold if

$$\varepsilon_0 < K^3, \tag{79}$$

because $K < 1/4$. Writing (77) in terms of ε_0 and ρ_0 we obtain

$$c\left(\frac{4}{\rho_0}\right)^\nu \cdot 2^{r\nu}\varepsilon_0^{1/2(3/2)^r} < 1,$$

which holds choosing

$$\varepsilon_0 < \min\left(\exp\left(\frac{\nu\log(1/4)}{\log(3/2)}\right), \frac{1}{c^2}\left(\frac{\rho_0}{4}\right)^{2\nu}\right). \tag{80}$$

Finally using that, by the above assumptions, $M < \varepsilon_r^{1/2}$ (which is less than one), we can estimate the left-hand side of (78) as follows

$$(e^{|\chi|M} - 1)(\varepsilon_r + 5\varepsilon_r e^{|\chi|M} + 2|A_0|e^{|\chi|M}M) + |A_0|\,|\chi|M^2e^{2|\chi|M} = C_1M\varepsilon_r + C_2M^2, \tag{81}$$

with

$$C_1 = e^{|\chi|}|\chi|(1 + 5e^{|\chi|})$$

and

$$C_2 = 3|A_0|\,|\chi|e^{2|\chi|},$$

where we have used that $M < 1$. Since we want the right-hand side of (81) to be smaller than $\varepsilon_r^{3/2}$, we need to impose extra conditions, in addition to (79) and (80). Indeed, writing the definition of M, condition (78) holds if

$$C_1 c \frac{\varepsilon_r^2}{\delta_r^\nu} + C_2 c^2 \frac{\varepsilon_r^2}{\delta_r^{2\nu}} < \varepsilon_r^{3/2},$$

which is equivalent to

$$C_1 c \frac{\varepsilon_r^{1/2}}{\delta_r^\nu} + C_2 c^2 \frac{\varepsilon_r^{1/2}}{\delta_r^{2\nu}} < 1.$$

The left-hand side of this expression is bounded by

$$C_1 c \frac{\varepsilon_r^{1/2}}{\delta_r^\nu} + C_2 c^2 \frac{\varepsilon_r^{1/2}}{\delta_r^{2\nu}} < c\varepsilon_r^{1/2}\, 4^{r\nu}\left(C_1\left(\frac{4}{\rho_0}\right)^\nu + C_2 c\left(\frac{4}{\rho_0}\right)^{2\nu}\right).$$

This expression is less than one (which implies condition (78)) if we take

$$\varepsilon_0 < \min\left(\exp\left(\frac{2\nu\log(1/4)}{\log(3/2)}\right), \frac{1}{C_3^2}\right), \tag{82}$$

where

$$C_3 = c\left(C_1\left(\frac{4}{\rho_0}\right)^\nu + C_2 c\left(\frac{4}{\rho_0}\right)^{2\nu}\right).$$

Therefore, taking ε_0 satisfying the bounds (79), (80) and (82) the estimates (76)–(78) follow for all $r \geq 0$.

The iterative process. Once we have chosen ε_0, the sequences $(\varepsilon_r)_r$ and $(\sigma_r)_r$ are defined and the inductive lemma can be applied up to any finite order to obtain analytic maps

$$X^r : \mathcal{D}_{r+1} \to g \quad \text{and} \quad Z^r = \exp(\chi X^r),$$
$$P^{r+1} : \mathcal{D}^{r+1} \to g$$

and

$$\xi^r : D(0, \sigma_{r+1}) \to D(0, 2\sigma_{r+1}) \subset D\left(0, \frac{\sigma_r}{2}\right),$$

which satisfy the homological equation (66) with the estimates

$$\begin{aligned} |P^{r+1}|_{\mathcal{D}^{r+1}} &< \varepsilon_r^{3/2} = \varepsilon_{r+1}, \\ |D_{\eta^{r+1}}\xi^r|_{\varepsilon_r} &< 1 + c_1\varepsilon_{r-1}^{1/2} = 1 + c_1\sigma_r^{1/2}, \\ |X^r|_{\mathcal{D}_{r+1}} &< \varepsilon_r^{1/2}, \end{aligned} \tag{83}$$

for $r \geq 0$. Writing

$$\xi_r = \xi_{r-1} \circ \xi^r = \xi^0 \circ \xi^1 \circ \cdots \circ \xi^r,$$

which is a real analytic map on $B(0, \sigma_{r+1})$, and

$$Z_r(\theta, \eta^{r+1}) = Z_{r-1}(\theta, \xi^r(\eta^{r+1})) \cdot Z^r(\theta, \xi^r(\eta^{r+1})), \quad (\theta, \eta^{r+1}) \in \mathcal{D}^{r+1},$$

which is also G-real analytic, we obtain, for all $r \geq 0$, the equation

$$\partial_\omega Z_r(\theta, \eta^{r+1}) = \chi^k(A_0 + \chi P^0(\theta) - \chi \xi_r(\eta^{r+1}))Z_r - Z_r \chi^k(A_0 + \chi P^{r+1}(\theta, \eta^{r+1}) - \chi \eta^{r+1})$$

for $(\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$. To prove the conjugation

$$\partial_\omega Z^*(\theta, 0) = \chi^k(A_0 + \chi P^0(\theta) - \chi \xi^*(0))Z^* - Z^* \chi^k A_0,$$

for $|\mathrm{Im}\,\theta| < \rho_0/2$, we only need to show that the sequences $(Z_r)_r$ and $(\xi_r)_r$ converge uniformly on $\mathcal{D}^* = \lim_{r\to\infty} \mathcal{D}^r$ to Z^* and ξ^* respectively, because from the estimates (83) the perturbations $(P^r)_r$ converge to zero on $\mathcal{D}^*$.

Existence of ξ^*. The desired ξ^* will be the limit of the sequence $(\xi_r(0))_r$. First of all note that if the limit exists, then it will belong to g and satisfy the identity $C(\xi^*) = \xi^*$ and the bound $|\xi^*| \leq 2\varepsilon_0$, as this holds for all $r \geq 0$. Now let us prove the convergence of the sequence.

Using the estimates (83)

$$|\xi_{r+1}(0) - \xi_r(0)| = |\xi_r(\xi^{r+1}(0)) - \xi_r(0)| \leq |D\xi_r|_{\sigma_{r+1}} |\xi^{r+1}(0)| < |D\xi_r|_{\sigma_{r+1}} \sigma_{r+1}.$$

As $\xi_r = \xi_{r-1} \circ \xi^r$, then

$$D\xi_r(\eta^{r+1}) = (D\xi_{r-1})(\xi^r(\eta^{r+1}))(D\xi^r)(\eta^{r+1}),$$

so, if $|\eta^{r+1}| < \sigma_{r+1}$,

$$\begin{aligned} |D\xi_r|_{\sigma_{r+1}} &\leq |D\xi^r|_{\sigma_{r+1}} \cdot |D\xi^{r-1}|_{\sigma_r} \cdot \dots \cdot |D\xi^1|_{\sigma_2} \cdot |D\xi^0|_{\sigma_1} \\ &\leq (1 + c_1\sigma_r^{1/2})(1 + c_1\sigma_{r-1}^{1/2}) \cdot \dots \cdot (1 + c_1\sigma_1^{1/2})(1 + c_1\sigma_0^{1/2}) \\ &\leq \prod_{j=0}^{\infty}(1 + c_1\sigma_j^{1/2}) \\ &\leq \exp\left(\sum_{j=0}^{\infty}(c_1\sigma_j^{1/2})\right) < \exp(2c_1\sigma_0^{1/2}) < \infty \end{aligned}$$

because, by (76), $\sigma_{r+1}/\sigma_r < K < 1/4$, so that

$$\sum_{j=0}^{\infty} \sigma_j^{1/2} < \sum_{j=0}^{\infty} \frac{\sigma_0^{1/2}}{2^j} = 2\sigma_0^{1/2}.$$

Therefore, $(\xi_r(0))_r$ is a Cauchy sequence and it converges to $\xi^* \in g$, with $C(\xi^*) = \xi^*$.

Existence of Z^*. We follow the same idea to prove the existence of ξ^*. As

$$Z_r(\theta, \eta^{r+1}) = Z_{r-1}(\theta, \xi^r(\eta^{r+1}))Z^r(\theta, \xi^r(\eta^{r+1})), \quad (\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$$

for $r \geq 1$ and

$$Z_0(\theta, \eta^1) = Z^0(\theta, \xi^0(\eta^1)),$$

then, for $|\mathrm{Im}\,\theta| < \rho_0/2$,

$$
\begin{aligned}
&|Z_{r+1}(\theta,0) - Z_r(\theta,0)| \\
&\quad = |Z_r(\theta,\xi^{r+1}(0))Z^{r+1}(\theta,\xi^{r+1}(0)) - Z_r(\theta,0)| \\
&\quad = |Z_r(\theta,\xi^{r+1}(0)) - Z_r(\theta,0) + Z_r(\theta,\xi^{r+1}(0))(Z^{r+1}(\theta,\xi^{r+1}(0)) - I)| \\
&\quad \le |Z_r(\theta,\xi^{r+1}(0)) - Z_r(\theta,0)| + |Z_r(\theta,\xi^{r+1}(0))| \cdot |Z^{r+1}(\theta,\xi^{r+1}(0)) - I|.
\end{aligned}
$$

We now estimate all of these terms:

$$
|Z^{r+1}(\theta,\xi^{r+1}(0)) - I| \le \exp(|\chi||X^{r+1}|_{\mathcal{D}_{r+2}}) - 1 \le 2|\chi|\varepsilon_{r+1}^{1/2} = 2|\chi|\sigma_r^{1/2},
$$

$$
|Z_r(\theta,\xi^{r+1}(0))| \le \prod_{j=0}^{\infty} \exp(|\chi|\varepsilon_j^{1/2}) = \exp\left(\sum_{j=0}^{\infty} |\chi|\varepsilon_j^{1/2}\right) < \exp(2|\chi|\varepsilon_0^{1/2}) < \infty,
$$

$$
|Z_r(\theta,\xi^{r+1}(0)) - Z_r(\theta,0)| \le |D_{\eta^{r+1}} Z_r|_{\mathcal{D}^{r+1}} |\xi^{r+1}(0)|.
$$

We now need a bound for $|D_{\eta^{r+1}} Z_r|_{\mathcal{D}^{r+1}}$. To find this we will apply Cauchy estimates

$$
\begin{aligned}
|D_{\eta^{r+1}} Z_r|_{\mathcal{D}^{r+1}} &= |D_{\eta^{r+1}}(Z_{r-1}(\cdot,\xi^r(\cdot))Z^r(\cdot,\xi^r(\cdot)))|_{\mathcal{D}^{r+1}} \\
&\le |D_{\eta^r}(Z_{r-1}Z^r)|_{\{|\mathrm{Im}\,\theta|<\rho_{r+1}\}\times\{|\eta^r|<2\sigma_{r+1}\}} |D\xi^r|_{\sigma_{r+1}} \\
&\le \frac{c'}{\sigma_r/2 - 2\sigma_{r+1}} |Z_{r-1}Z^r|_{\mathcal{D}_{r+1}} |D\xi^r|_{\sigma_{r+1}} \\
&\le \frac{2c'}{\sigma_r - 4\sigma_{r+1}} |Z_{r-1}|_{\mathcal{D}_r} |\exp(\chi X^r)|_{\mathcal{D}_{r+1}} |D\xi^r|_{\sigma_{r+1}} \\
&\le \frac{2c'}{\sigma_r - 4\sigma_{r+1}} \exp(2|\chi|\varepsilon_0^{1/2}) \exp(|\chi|\sigma_{r+1}^{1/2})(1 + c_1\sigma_r{}^{1/2}) \\
&< \frac{c_2}{\sigma_r - 4\sigma_{r+1}},
\end{aligned}
$$

where c_2 is a new constant. Therefore,

$$
|Z_r(\theta,\xi^{r+1}(0)) - Z_r(\theta,0)| < \frac{2c_2\sigma_{r+1}}{\sigma_r - 4\sigma_{r+1}} = \frac{2c_2\sigma_r^{3/2}}{\sigma_r - 4\sigma_r^{3/2}} = \frac{2c_2\sigma_r^{1/2}}{1 - 4\sigma_r^{1/2}} \le c_3\sigma_r^{1/2},
$$

c_3 being another constant. Collecting all of these bounds,

$$
|Z_{r+1}(\theta,0) - Z_r(\theta,0)| < (c_3 + 2|\chi|\exp(2|\chi|\varepsilon_0^{1/2}))\sigma_r^{1/2},
$$

which implies that $(Z_r(\cdot,0))_r$ is a Cauchy sequence on $|\mathrm{Im}\,\theta| < \rho_0/2$ and therefore it converges to some Z^* on $\mathcal{D}^*$. Moreover, as

$$
|Z^*|_{\mathcal{D}^*} < \exp\left(\chi \sum_{r\ge 1} |\chi^r|_{\mathcal{D}^r}\right) < \exp(\chi\tilde{c}(\nu)\varepsilon_0\rho_0^{-\nu}) \tag{84}
$$

and $\varepsilon_0 < 1$, there exists a real analytic map $X^* : \mathcal{D}^* \to g$, with $|X^*|_{\mathcal{D}^*} < \tilde{c}(\nu)\varepsilon_0\rho_0^{-\nu}$, such that $Z^* = \exp(\chi X^*)$. For the second bound in (84) we have used (67) for $r = 1$ and (83) for $r > 1$. This ends the proof of Theorems 4 and 11 disregarding the dependence on external parameters.

Analytic dependence on μ. Up to now we have proved theorems 4 and 11 disregarding the dependence with respect to the external parameters μ. First of all note that these proofs (not yet considering the dependence on μ) can be extended to apply to analytic $P : \mathbb{T}^d \to g_{\mathbb{C}}$ such that

$$|P|_{\rho_0} < \varepsilon_0, \tag{85}$$

and to complex χ with $|\chi| \leq 1$ (in the case of Theorem 11). Here $g_{\mathbb{C}}$ stands for the complexification of the Lie algebra g. Elements of $g_{\mathbb{C}}$ are of the form $P_1 + iP_2$, where P_1 and P_2 belong to g. The bound of (85) holds because the admissibility of (A_0, C, S, ω) (respectively, $(\chi^k A_0, C, S, \omega)$) implies that the equations

$$\partial_\omega X(\theta) = \chi^k([A_0, X(\theta)] + P(\theta) - C(\overline{P})), \quad \overline{X} = S(\overline{P}),$$

for $P : \mathbb{T}^d \to g_{\mathbb{C}}$ with analytic extension to $|\mathrm{Im}\,\theta| < \rho_0$, have a unique analytic solution $X : \mathbb{T}^d \to g_{\mathbb{C}}$ that satisfies the estimates

$$|X|_{\rho_0-\delta} \leq c\frac{|P|_{\rho_0}}{\delta^\nu}$$

for all $0 < \delta \leq \rho$. With this in mind it can be checked that all the other parts of the proof hold.

Let us now consider the dependence with respect to μ. That is, assume that both P and χ depend real analytically on μ in a certain ball around the origin. Again, we deal with Theorems 4 and 11 at the same time. The reader interested only in Theorem 4 can replace χ by 1.

Let $\nu > 0$ such that, if $|\mu| < \nu$, then $|P(\cdot, \mu)|_\rho < \varepsilon$ and $|\chi(\mu)| < 1$. For these complex values of μ there exist $\xi^*(\mu) \in g_{\mathbb{C}}$ (with $C(\xi^*(\mu)) = \xi^*(\mu)$ and $\xi^*(\mu) \in g$ for real values of μ) and $X(\cdot, \mu) : \mathbb{T}^d \to g_{\mathbb{C}}$, with analytic extension to $|\mathrm{Im}\,\theta| < \rho/2$ such that $Z(\theta, \mu) = \exp(\chi(\mu)X(\theta, \mu))$ satisfies

$$\partial_\omega Z(\theta, \mu) = \chi(\mu)^k(A_0 + \chi(\mu)P(\theta, \mu) - \chi(\mu)\xi^*(\mu))Z(\theta, \mu) - Z(\theta, \mu)\chi(\mu)^k A_0,$$

for $|\mathrm{Im}\,\theta| < \rho/2$ and $|\mu| < \nu$. Moreover, if μ is real then P is real analytic in θ and belongs to g, so $\xi^*(\mu) \in g$ and $X(\theta, \mu) \in g$ for real θ. Therefore, we need to show that the dependence of these objects on μ is analytic on $|\mu| < \nu$.

To see this, note that the transformations constructed in the inductive lemma can be made analytic on $\mathcal{D}^r \times \{|\mu| < \nu\}$. For this, it is essential to define P^{r+1} when $\chi = 0$ as (72) to avoid a discontinuity. As the final solution is obtained as the uniform limit (in the complex domain $\mathcal{D}^* \times \{|\mu| < \nu\}$) of the approximations, the limits are analytic there. □

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A. *Appendix. Multiple internal–external resonances*

In this paper we have considered the existence of analytic families of reducible linear quasi-periodic equations with frequency ω and Floquet matrix A_0 satisfying the Diophantine condition (20),

$$\inf_{\lambda \in \mathrm{Spec}(\mathrm{ad})_{A_0}} |\lambda - i\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^{\tau}}, \quad \mathbf{k} \in \mathbb{Z}^d,\ \mathbf{k} \neq \mathbf{0},$$

for some positive constants K, τ. This assumption does not cover the case of *multiple* resonances, which happens when the previous condition holds for all values of $\mathbf{k} \in \mathbb{Z}^d$ except for a finite set of multi-integers. Theorem 4 (and also Theorem 11) can be adapted to the case of multiple resonances, provided suitable conditions are imposed.

We first of all we impose a less restrictive Diophantine condition on the eigenvalues of ad_{A_0} and ω. Assume that there exist positive constants $c, \nu > 0$ and a finite set $\mathcal{R} \subset \mathbb{Z}^d$ such that the estimate

$$\inf_{\lambda \in \mathrm{Spec}(\mathrm{ad}_{A_0})} |\lambda - i\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^{\tau}}, \tag{86}$$

holds for all $\mathbf{k} \in \mathbb{Z}^d - \mathcal{R}$. In particular, zero must belong to this resonant set $\mathcal{R}$.

Second, let us introduce the generalization of the operators S and C. We assume that for all $\mathbf{k} \in \mathcal{R}$ there exist linear operators $S_{\mathbf{k}}, C_{\mathbf{k}}$ of $g_{\mathbb{C}}$ such that $C_{\mathbf{k}}^2 = C_{\mathbf{k}}$ and, for all $P \in g_{\mathbb{C}}$, the identity

$$i\langle \mathbf{k}, \omega\rangle S_{\mathbf{k}}(P_{\mathbf{k}}) = [A_0, S_{\mathbf{k}}(P_{\mathbf{k}})] + P_{\mathbf{k}} - C_{\mathbf{k}}(P_{\mathbf{k}}) \tag{87}$$

holds. Under these two hypothesis, Theorem 4 has to be modified only in the following way. For $\mathbf{k} \in \mathcal{R}$ there exist $\xi^*_{\mathbf{k}} \in g_{\mathbb{C}}$, with $C_{\mathbf{k}}(\xi^*_{\mathbf{k}}) = \xi^*_{\mathbf{k}}$, such that the modified system is

$$x' = \chi^k \left(A_0 + P(\theta) - \sum_{\mathbf{k} \in \mathcal{R}} \xi^*_{\mathbf{k}} \exp(i\langle \mathbf{k}, \theta\rangle) \right) x, \quad \theta' = \omega \tag{88}$$

instead of (14).

Nevertheless, in several practical situations it turns out that it is not needed to make use of this extended version of the theorem, because some preliminary transformations can be performed so that resonances for values of $\mathbf{k}$ different from zero are *removed* and the resonance of zero has a higher multiplicity (see also [**MP84, Eli02, Eli01**] and [**Kri99a**]).

To illustrate this procedure consider a perturbed system

$$x' = (A_0 + P(\theta, \mu))x, \quad \theta' = \omega \tag{89}$$

for which the adjoint operator $\mathrm{ad}_{A_0} : g \to g$ has rational eigenvalues with respect to ω. Assume that we can find matrices $A_0^d, A_0^r \in g$ such that:

(i) $A_0 = A_0^d + A_0^r$;

(ii) A_0^d and ω satisfy the Diophantine condition (20);

(iii) the map $t \mapsto \exp(tA_0^r)$ is quasi-periodic with frequency $\omega/2$; denote by Z its lift to $\mathbb{T}^d$.

If these conditions are fulfilled (an example of this appears in §3) then, the transformation

$$x = \exp(tA_0^r)y$$

sends system (89) to

$$y' = (A_0^d + Q(\theta, \mu))y, \quad \theta' = \omega,$$

where

$$Q(\theta, \mu) = Z(\theta)^{-1} P(\theta, \mu) Z(\theta),$$

which is quasi-periodic with frequency ω and we are under the conditions of Theorem 4.

B. *Appendix. The case of reversible systems*

In practical situations, given a linear differential equation on some Lie algebra, there can be additional symmetries to be taken into account. In this case it is interesting to know whether we can use these symmetries to deduce more properties of the counterterm C, essentially reducing the dimension of the space $C(\xi) = \xi$ in the algebra g. In this section we focus on the reversible case (see [**BHS96**] and references therein).

Definition 30. Given an element $R \in GL(n, \mathbb{R})$, with $R^2 = I$, we will say that a map $Q : \mathbb{R} \to g$ is R-reversible, whenever

$$Q(-t)R = -RQ(t)$$

for all $t \in \mathbb{R}$.

In presence of such a symmetry, the solutions of a linear differential equation have the following properties

PROPOSITION 31. *Consider a reversibility with respect to the involution R. Let $g \subset gl(n, \mathbb{R})$ a Lie sub-algebra. Then the following are true.*

(i) *Let $A_0 \in g$, $Q : \mathbb{R} \to g$, both R-reversible, and let $X : \mathbb{R} \to g$, smooth, such that*

$$X'(t) = [A_0, X(t)] + Q(t)$$

for all $t \in \mathbb{R}$. Then

$$X(-t)R = RX(t)$$

for all $t \in \mathbb{R}$.

(ii) *If $X : \mathbb{R} \to g$ satisfies $X(-t)R = RX(t)$ for all $t \in \mathbb{R}$, then $Z(t) = \exp(X(t))$ also does:*

$$Z(-t)R = RZ(t).$$

(iii) *If $A, X : \mathbb{R} \to g$ satisfy that $X(-t)R = RX(t)$, $A(-t)R = -RA(t)$ and the conjugacy*

$$Z'(t) = A(t)Z(t) - Z(t)B(t),$$

with $Z(t) = \exp(X(t))$, $B(t) \in g$ holds for all $t \in \mathbb{R}$, then B is R-reversible:

$$B(-t)R = -RB(t)$$

for all $t \in \mathbb{R}$.

Proof. The first item follows from the identities

$$(RX(t)R)' = -([A_0, RX(t)R] + Q(t)), \quad (X(-t))' = -([A_0, X(-t)] + Q(t)).$$

As $RX(t)R$ and $X(-t)$ satisfy the same differential equation and they coincide for $t = 0$, then

$$RX(t)R = X(-t)$$

for all $t \in \mathbb{R}$ and the first statement follows. Item (ii) is a direct consequence of the definition of the exponential of a matrix. To prove (iii) we first note that $Z'(t)$ is R-reversible and, as

$$B(t) = Z^{-1}(t)A(t)Z(t) - Z(t)^{-1}Z'(t),$$

then $B(t)$ must be R-reversible because Z^{-1} satisfies

$$Z^{-1}(-t)R = RZ^{-1}(t)$$

for all $t \in \mathbb{R}$. □

With this proposition in mind one can modify Theorems 4 and 11 to obtain additional symmetries of the counterterm C. Here we give only the adaption of Theorem 4 to the reversible case.

THEOREM 32. *Assume that, in addition to the hypothesis of Theorem 4, there is an involution $R \in GL(n, \mathbb{R})$ such that*

$$A_0 R = -RA_0$$

and

$$C(\xi)R = -RC(\xi) \quad S(\xi)R = RS(\xi) \tag{90}$$

hold for all R-reversible $\xi \in g$. Then, if P is R-reversible, the element $\xi^* \in g$ is also R-reversible,

$$\xi^* R = -R\xi^*,$$

and the conjugation X satisfies

$$X(-\theta)R = RX(\theta)$$

for all $\theta \in \mathbb{T}^d$.

As an application, in Hill's equation with quasi-periodic forcing, assume that the quasi-periodic forcing q is even in t, i.e. it satisfies that $q(t) = q(-t)$ for all $t \in \mathbb{R}$. Then the matrix function

$$t \mapsto \begin{pmatrix} 0 & 1 \\ -(a + bq(t)) & 0 \end{pmatrix}$$

is reversible with respect to the involution

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following the construction in §2, the operators C and S clearly satisfy the identities (90). Therefore, the counterterm $\xi^*(\mu)$ is also R-reversible and, thus $\xi^*_{11}(\mu) = 0$, so that the persistence of a collapsed gap is given by the two equations

$$\xi^*_{12}(\mu) = \xi^*_{21}(\mu) = 0.$$

C. *Appendix. A numerical example*

A large set of numerical illustrations of the boundaries of the tongues for Hill's quasi-periodic coefficients like (24) can be found in [**BS98**]. They have been obtained from computations of maximal Lyapunov exponent and rotation number for a set of values of a and b. In this reference one can also find details about how to carry out the computations and an interpretation of the results. In fact, the present paper has been largely motivated by the results in [**BS98**].

Here we show an additional example which concerns the optimality of the results given in §4. In that section we have considered the boundary between a totally hyperbolic situation and another that has just one elliptic mode, the remaining situations being hyperbolic. One can ask if transitions to cases with a higher ellipticity can have analytic boundaries. The example we present here is evidence that this is not the case.

Let us consider the following system

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_1(t) & 0 & 0 & -2 \\ 0 & f_2(t) & 2 & 0 \end{pmatrix} x, \tag{91}$$

where

$$f_1(t) = \lambda_1(1 + e\cos(\omega t)), \quad f_2(t) = \lambda_2(1 + e\cos(t)). \tag{92}$$

The parameters in (91) are λ_1, λ_2, e and the frequency ω. A system like (91), with more involved functions than those shown in (92), appears in the periodic case ($\omega = 1$) in the study of the stability of homographic solutions of the three-body problem with homogeneous potentials (see [**MSS03, MSS06**] and references therein). The parameters λ_1 and λ_2 depend on the masses, m_1, m_2, m_3, of the three bodies and on the degree of homogeneity, $-\alpha$, of the potential. The parameter e denotes a generalized eccentricity.

For some ranges of m_1, m_2, m_3, α there are invariant tori around the homographic solutions (in a suitable rotating frame). In that case the variational equations along these tori can be reduced to the form (91).

For this example we show some results concerning changes of stability for $\lambda_2 = -4$, the frequency fixed to $\omega = (1 + \sqrt{5})/2$ while λ_1 and e are considered as varying parameters. Note that e acts as a perturbation parameter. For $e = 0$ (91) has constant coefficients. Hence (λ_1, e) play a role similar to (a, b) in Hill's equation.

For these parameters in the range $[-4, 0] \times [0 : 1]$ with stepsize 10^{-3}, we have computed the two dominant Lyapunov exponents μ_1, μ_2. Using the change

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} x,$$

system (91) becomes Hamiltonian with

$$H(u, t) = \tfrac{1}{2}((1 - f_1(t))u_1^2 + (1 - f_2(t))u_2^2 + u_3^2 + u_4^2) + u_1u_4 - u_2u_3.$$

Hence, the remaining Lyapunov exponents are obtained from $\mu_3 = -\mu_1$, $\mu_4 = -\mu_2$.

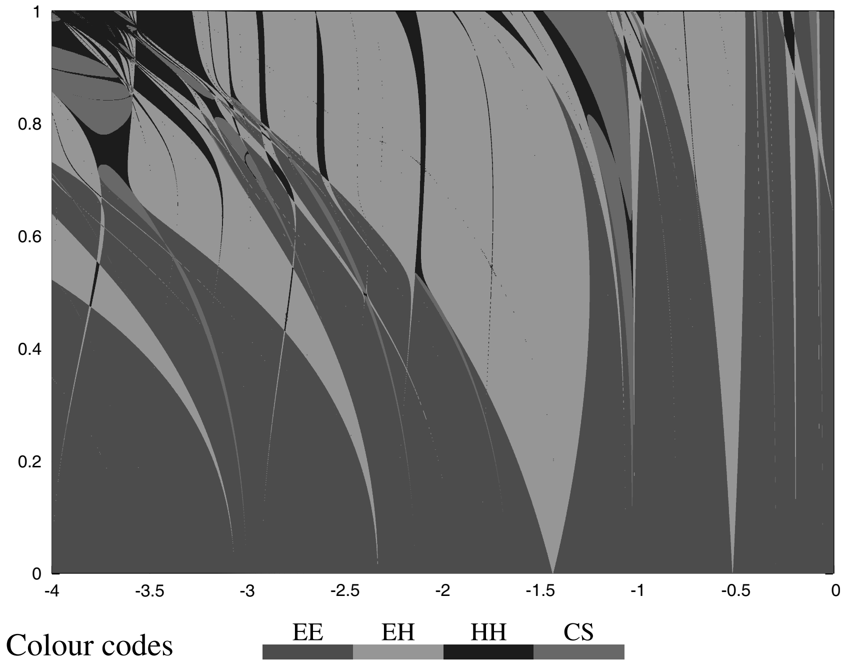

FIGURE C2. Stability properties of (91) with f_j defined in (92) as a function of λ_1 (horizontal variable) and e (vertical variable), for $\lambda_2 = -4$, $\omega = (1 + \sqrt{5})/2$. Colour code: $\mu_1 = \mu_2 = 0$ as EE, $\mu_1 > 0$, $\mu_2 = 0$ as EH, $\mu_1 > \mu_2 > 0$ as HH, $\mu_1 = \mu_2 > 0$ as CS.

To compute the Lyapunov exponents a method to accelerate the convergence has been used. It is different from the one used in [**BS98**]. The method has been introduced in [**CGS03**] and additional details can be found in [**LSSW03**].

Figure C2 presents an overview of the results. Different regions in the plot correspond to different behaviour of the first two Lyapunov exponents, as described in the caption. Replacing ω by a rational approximant the problem reduces to a periodic case and a similar picture is obtained. Then the different grey tones correspond, generically, to elliptic–elliptic, elliptic–hyperbolic, hyperbolic–hyperbolic and complex saddle (or EE, EH, HH, CS for short) as shown in C2, respectively.

The largest EH zone, which opens from $e = 0$ near $\lambda_1 = -1.5$, has boundaries which are of elliptic–parabolic type in the periodic case. In the quasi-periodic one resonances due to the presence on an elliptic part in the boundary can occur. They can be seen as the narrow near-vertical strips of the HH zones. Whereas in the periodic case there is a finite set of such strips, in the quasi-periodic setting they appear in a dense way, despite being individually narrow. This destroys the analyticity of the boundary between EE and EH.

A magnification of the left boundary, which roughly separates the EE and EH domains for this resonance, is shown in Figure C3. The vertical scale has been deformed so that this boundary becomes roughly horizontal. The point where the tongue opens from $e = 0$ can be seen to the right. The white area corresponds to $e < 0$.

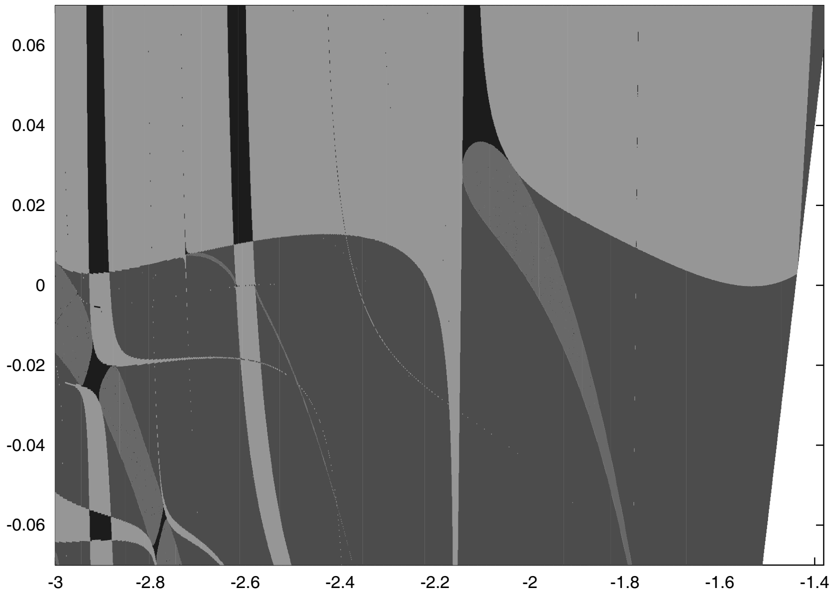

FIGURE C3. A magnification of Fig. C2 followed by a deformation of the vertical variable. The left boundary of the largest EH zone in the previous figure is here seen as roughly horizontal, close to the value 0. Colour code as in Fig. C2.

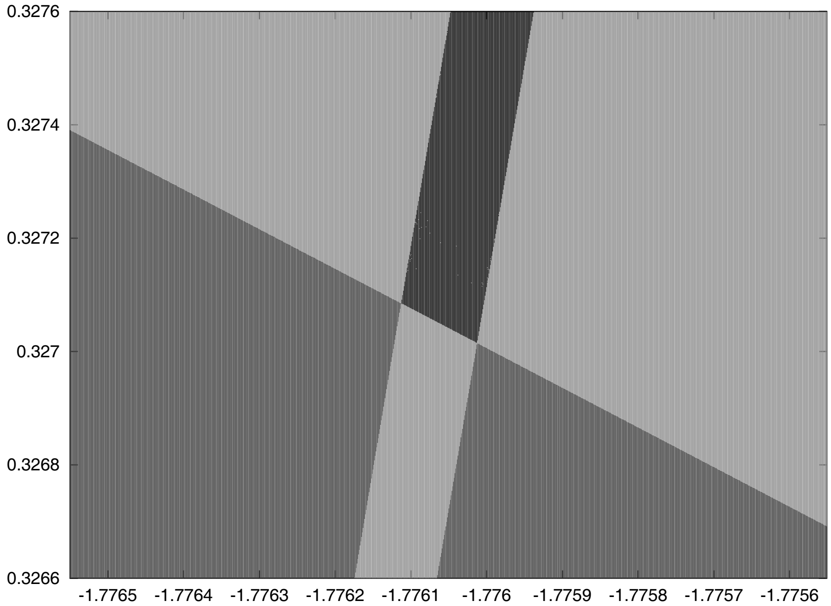

FIGURE C4. Magnification of one of the infinitely many near-vertical strips of HH type inside the EH region and entering the EE region to become HE. The window is $[-1.77655, -1.77555] \times [0.3266, 0.3276]$. This is one of the largest strips. Colour code as in the previous figures.

Finally Figure C4 shows a magnification (again in (λ_1, e) variables) of part of the very narrow almost vertical strip which can be seen in Figure C3 close to $\lambda_1 = -1.8$. Infinitely many of these strips have to be found across the figures. Note that as only the Lyapunov exponents have been computed, there is not enough information to detect the non-reducible systems.

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