Cantor Spectrum for Quasi-Periodic Schrödinger Operators

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Abstract. We present some results concerning the Cantor structure of the spectrum of quasi-periodic Schrödinger operators. These are obtained studying the dynamics of the corresponding eigenvalue equations, specially the notion of reducibility and Floquet theory. We will deal with the Almost Mathieu case, and the solution of the "Ten Martini Problem" for Diophantine frequencies, as well as other models.[1]

In recent years there has been substantial progress in the understanding of the structure of the spectrum of Schrödinger operators with quasi-periodic potential. Here we will concentrate on one-dimensional, real analytic quasi-periodic potentials with one or more Diophantine frequencies.

We will begin with the best studied of such operators: the Almost Mathieu operator and the solution of the "Ten Martini Problem" which asks for the Cantor structure of its spectrum. Secondly we will see how this Cantor structure is generic in the set of quasi-periodic real analytic potentials. We will end introducing a different approach to this problem which is helpful to study the phenomenon of "gap opening".

1 The Almost Mathieu Operator & the Ten Martini Problem

The *Almost Mathieu operator* is probably the best studied model among quasi-periodic Schrödinger operators. It is the following second-order difference operator:

$$(H^{AM}_{b,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \;, \tag{1}$$

where b is a real parameter (a *coupling* parameter, since for $b = 0$ the operator is trivial), ω is the *frequency*, which we assume to be an irrational number (in most of what follows, also Diophantine) and $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ will be called the *phase*.

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Considered as an operator on $l^2(\mathbb{Z})$, the Almost Mathieu operator is bounded and self-adjoint. The reason for its name comes from the fact that its eigenvalue equation, namely

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = a x_n, \qquad n \in \mathbb{Z} ,$$

(sometimes called *Harper's equation*), is a discretization of the classical *Mathieu equation*,

$$x'' + (a + b\cos(t))\, x = 0 .$$

which is a second-order periodic differential equation (see Ince [9]). The analogies between the Harper equation and Mathieu equation are quite striking and their comparison illustrates the differences between periodic and quasi-periodic Schrödinger operators.

1.1 The IDS and the Spectrum

The Integrated Density of States (IDS) is a very convenient object for the description of the spectrum of quasi-periodic Schrödinger operators which can be extended to more general operators. Here we introduce it in the case of the Almost Mathieu operator for the sake of concreteness.

Fix some $b \in \mathbb{R}$ and $\phi \in \mathbb{T}$. For any $L \in \mathbb{N}$ we define $H^{AM,L}_{b,\omega,\phi}$ as the restriction of the Almost Mathieu operator to the interval $\{1, \ldots, L-1\}$ with zero boundary conditions at 0 and L. Let

$$k^{AM,L}_{b,\omega,\phi}(a) = \frac{1}{(L-2)} \# \left\{ \text{eigenvalues of } H^{AM,L}_{b,\omega,\phi} \leq a \right\} .$$

Then Avron & Simon [2] prove that

$$\lim_{L\to\infty} k^{AM,L}_{b,\omega,\phi}(a) = k^{AM}_{b,\omega}(a) ,$$

which is called *integrated density of states*, IDS for short. This limit is independent of the value of ϕ and it is a continuously increasing function of a.

The IDS can be used to describe the spectrum of quasi-periodic Schrödinger operators in a very nice way. Indeed, the spectrum of $H^{AM}_{b,\omega,\phi}$ is precisely the set of points of increase of the map

$$a \mapsto k^{AM}_{b,\omega}(a)$$

so that the intervals of constancy belong to the resolvent set of the operator (and are called the *spectral gaps*). In particular, this characterization shows that the spectrum of the Almost Mathieu operator (and in general any quasi-periodic Schrödinger operator with irrational frequencies) does not depend on ϕ. Therefore, we write

$$\sigma^{AM}_{b,\omega} = \operatorname{Spec}\left(H^{AM}_{b,\omega,\phi}\right) .$$

Remark 1. The IDS for one-dimensional quasi-periodic Schrödinger operators has many other characterizations. It can be linked to the *rotation number* of the corresponding eigenvalue equation. This object was introduced by Johnson & Moser [12] in the continuous case (see Delyon & Souillard [7] for the adaption to the discrete case and Johnson [11] for a review on these different characterizations).

The IDS can also be used to "label" the spectral gaps of the Almost Mathieu operator. This is the contents of the *Gap Labelling Theorem*, by Johnson & Moser [12]: if I is a spectral gap (an interval of constancy of the IDS) then there is an integer $n \in \mathbb{Z}$ such that

$$k_{b,\omega}^{AM}(a) = n\omega, \qquad (\text{modulus } \mathbb{Z})$$

for all $a \in I$. Figure 1 displays the gap labelling for the Almost Mathieu operator at "critical coupling" $b = 2$.

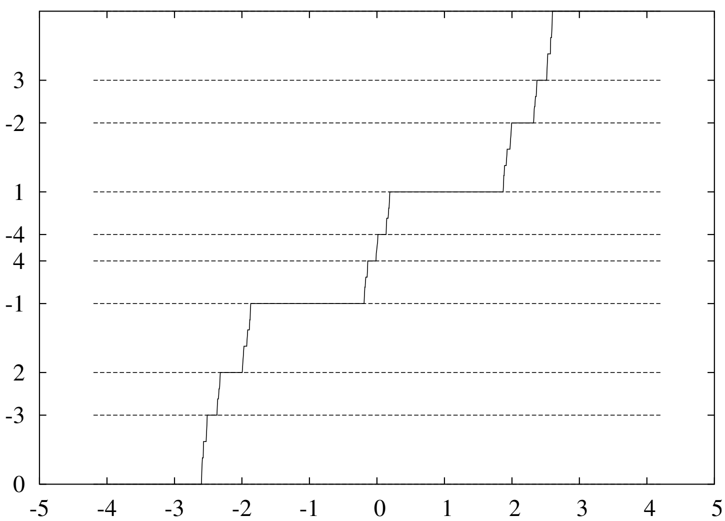

Fig. 1. Schematic view of Gap Labelling for $H_{2,\omega,\phi}^{AM}$ and $b = 2$. The IDS is plot as a function of a. Integers in the vertical direction correspond to values n such that the IDS equals $n\omega$ modulus $\mathbb{Z}$

The Gap Labelling Theorem motivates the following definitions. For any $n \in \mathbb{Z}$ let

$$I(n) = \left\{a \in \mathbb{R}; k_{b,\omega}^{AM}(a) - n\omega \in \mathbb{Z}\right\} .$$

If $I(n) = [a_-^n, a_+^n]$ for some $a_-^n < a_+^n$ then we will say that (a_-^n, a_+^n) is a *non-collapsed* or *open* spectral gap. If $a_-^n = a_+^n$ then we will call $\{a_-^n\}$ a *collapsed* or *closed* spectral gap. Note that noncollapsed spectral gaps are subsets of the resolvent whereas collapsed spectral gaps belong to the spectrum. In both cases, the endpoints of gaps belong to the spectrum.

In the quasi-periodic case, when ω is an irrational frequency, the possible values of the IDS at gaps define the set of labels

$$\mathcal{M}(\omega) = \{m + n\omega, \quad n, m \in \mathbb{Z}\} \cap [0,1] ,$$

which is dense in $[0,1]$. Since the IDS is a continuously increasing function of a, the spectrum of the Almost Mathieu operator is a Cantor set if all spectral gaps are open. For a general quasi-periodic Schrödinger operator, gaps can be collapsed and, in fact, the spectrum may contain intervals. Figure 2 displays a numerical computation of some of the gaps of the Almost Mathieu operator. None of them appears to be collapsed.

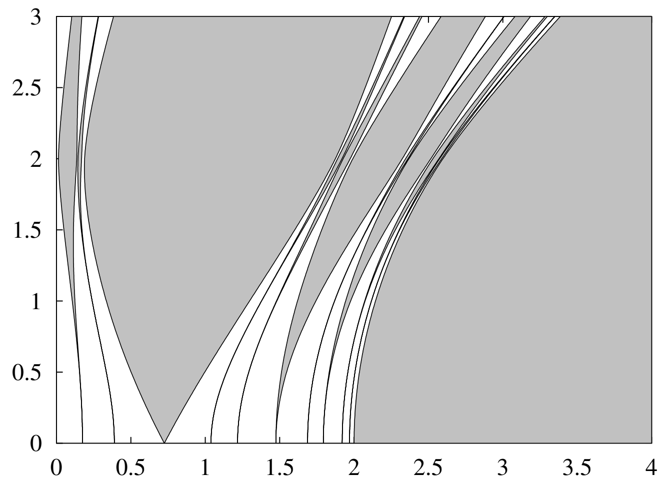

Fig. 2. Numerical computation of the ten biggest spectral gaps for the Almost Mathieu operator. Coupling parameter b is in the vertical direction, whereas the spectral one a is in the horizontal one. Shaded regions correspond to gaps

Due to all this, and to some physical arguments, Simon [18], after an offer by Kac, posed the following problems on the Cantor structure for the spectrum of the Almost Mathieu operator. The first one is the "Ten Martini Problem": for ω irrational and $b \neq 0$ prove that the spectrum of the Almost Mathieu operator is a Cantor set. The second one, which implies the first, is the "Strong (or Dry) Ten Martini Problem" and, under the same hypothesis, asks if all gaps, as predicted by the Gap Labelling Theorem, are open.

Concerning the Ten Martini Problem, we can prove the following [14].

Corollary 1. *Assume that $\omega \in \mathbb{R}$ is Diophantine, that is, there exist positive constants c and $r > 1$ such that*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

for all $n \neq 0$. Then, the spectrum of the Almost Mathieu operator, $\sigma^{AM}_{b,\omega}$, is a Cantor set if $b \neq 0, \pm 2$.

Remark 2. Very recently Avila & Jitomirskaya have proved Cantor structure for *all* irrational frequencies. The set of Diophantine frequencies is a total measure subset of the real numbers.

Concerning the dry version of the Ten Martini Problem, using a reducibility theorem by Eliasson [8], one can also say something in the perturbative regime [14]

Corollary 2. *Let $\omega \in \mathbb{R}$ be* Diophantine. *Then, there is a constant $C = C(\omega) > 0$ such that if $0 < |b| < C$ or $4/C < |b| < \infty$ all the spectral gaps of $\sigma^{AM}_{b,\omega}$ are open.*

In the remaining of this section we will sketch the reason why Corollary 1 is an (almost) direct consequence of the following nonperturbative localization result due to Jitomirskaya [10].

Theorem 1. *Let ω be Diophantine. Then, if $|b| > 2$ the operator $H^{AM}_{b,\omega,0}$ has only pure point spectrum with exponentially decaying eigenfunctions.*

1.2 Sketch of the Proof

Let $b > 2$ and ω Diophantine. Jitomirskaya proves that, in this case, $H^{AM}_{b,\omega,0}$ has pure-point spectrum with exponentially decaying eigenfunctions. In particular (and this is everything that we will need from her result), there exists a dense subset in $\sigma^{AM}_{b,\omega}$ of point eigenvalues of $H^{AM}_{b,\omega,0}$ whose eigenvectors are exponentially localized. Let a be one of these eigenvalues and $\psi = (\psi_n)_{n\in\mathbb{Z}}$ its exponentially localized eigenvector. We are going to see that a is the endpoint of a noncollapsed spectral gap. From this the Cantor structure of the spectrum follows immediately.

By hypothesis $a \in \sigma^{AM}_{b,\omega}$ and $\psi \in l^2(\mathbb{Z})$ satisfy the Harper equation

$$\psi_{n+1} + \psi_{n-1} + b\cos(2\pi\omega n)\psi_n = a\psi_n, \qquad n \in \mathbb{Z}\ ,$$

with some constants $A, \beta > 0$ such that

$$|\psi_n| \leq A\exp(-\beta|n|), \qquad n \in \mathbb{Z}\ .$$

The very special form of the Almost Mathieu operator makes that the Fourier transform of ψ,

$$\tilde{\psi}(\theta) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}, \qquad \theta \in \mathbb{T}\ ,$$

which is real analytic in $|\mathrm{Im}\theta| < \beta$, defines the following *quasi-periodic Bloch wave*

$$x_n = \tilde{\psi}\,(2\pi\omega n + \theta)\,, \qquad n \in \mathbb{Z}\,,$$

and that this, satisfies the equation

$$(x_{n+1} + x_{n-1}) + \frac{4}{b}\cos(2\pi\omega n + \theta)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}\,. \tag{2}$$

for any $\theta \in \mathbb{T}$. Note that this is again a Harper equation whose parameters have changed

$$(a, b) \mapsto \left(\frac{2a}{b}, \frac{4}{b}\right)\,.$$

This invariance of the Almost Mathieu operator under Fourier transform is known as Aubry duality [1]. Although the argument above requires the existence of a point eigenvalue, Avron & Simon [2] proved the following form of Aubry duality in terms of the IDS

$$k^{AM}_{b,\omega}(a) = k^{AM}_{4/b,\omega}\left(\frac{2a}{b}\right)\,.$$

In particular, if we prove that $2a/b$ is the endpoint of a noncollapsed gap of $\sigma^{AM}_{4/b,\omega}$ we are done.

1.3 Reducibility of Quasi-Periodic Cocycles

Our main tool will be to use the dynamics of the eigenvalue equation (2) to prove that a is an endpoint of a non-collapsed gap. To do so, it is convenient to write down (2) as a first order system

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega\,, \tag{3}$$

with $\theta_n \in \mathbb{T}$. Such first-order systems are usually called *quasi-periodic skew-products.* The evolution of the vector $v_n = (x_{n+1}, x_n)^T$ and the angle θ_n can be seen as the iteration of a *quasi-periodic cocycle* on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$(v,\theta) \in \mathbb{R}^2 \times \mathbb{T} \mapsto \left(A^{AM}_{2a/b,4/b,\omega}(\theta), \omega)\right)(v,\theta) = \left(A^{AM}_{2a/b,4/b,\omega}(\theta)v, \theta + 2\pi\omega\right)\,,$$

setting

$$A^{AM}_{2a/b,4/b,\omega}(\theta) = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix}\,.$$

That is,

$$v_{n+1} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos(2\pi\omega n + \phi) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos(\phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot v_0$$

and

$$\theta_n = 2\pi\omega n + \theta_0 \ .$$

When the frequency ω is rational the skew-product is periodic and, thanks to Floquet theory, it can be reduced to a skew-product with constant matrix by means of a periodic transformation. Quasi-periodic reducibility tries to extend this theory to the quasi-periodic case. Let us now introduce some basic notions.

Two cocycles (A, ω) and (B, ω) of $SL(2, \mathbb{R}) \times \mathbb{T}$ (not necessarily associated to the Harper equation) are *conjugated* if there exists a continuous $Z : \mathbb{T} \to SL(2, \mathbb{R})$ such that

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B(\theta), \qquad \theta \in \mathbb{T} \ .$$

In this case the corresponding quasi-periodic skew-products

$$u_{n+1} = A(\theta)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

and

$$v_{n+1} = B(\theta)v_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

are conjugated through the change $u = Zv$.

Particularly important to our purposes is the case of cocycles which are conjugated to a constant cocycles. A cocycle (A, ω) is *reducible to constant coefficients* if it is conjugated to a cocycle (B, ω) with B not depending on θ.

Remark 3. B is called the *Floquet matrix*. Neither B nor Z are unique.

The fundamental solution of a reducible system $X_n(\phi)$ has the following *Floquet representation*:

$$X_n(\phi) = Z(2\pi n\omega + \phi)B^n Z(\phi)^{-1} X_0(\phi) \ . \tag{4}$$

In particular, and this is an important observation, if $B = I$ then all solutions of the corresponding skew-product are quasi-periodic with frequency ω. If the cocycle comes from a Harper's equation, then all the solutions of this equation are quasi-periodic Bloch waves.

Now let us go back to our dual Harper's equation. In terms of $\tilde{\psi}$ we have that the relation

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega + \theta) \\ \tilde{\psi}(2\pi\omega + \theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega + \theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

holds for all $\theta \in \mathbb{T}$. The following Lemma shows that, in this situation, the Almost Mathieu cocycle is reducible to constant coefficients.

Lemma 1. *Let $A : \mathbb{T} \to SL(2, \mathbb{R})$ be a real analytic map and ω be Diophantine. Assume that there is a nonzero real analytic map $v : \mathbb{T} \to \mathbb{R}^2$, such that the relation*

$$v(\theta + 2\pi\omega) = A(\theta)v(\theta)$$

holds for all $\theta \in \mathbb{T}$. *Then, the quasi-periodic cocycle* (A, ω) *is reducible to constant coefficients by means of a quasi-periodic transformation which is real analytic. Moreover the Floquet matrix can be chosen to be of the form*

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{5}$$

for some $c \in \mathbb{R}$.

1.4 End of Proof

Classical Floquet theory for periodic Hill's equation relates endpoints of gaps to the corresponding Floquet matrices (which always exist because the system is periodic). It turns out that, if an Almost Mathieu cocycle is reducible to constant coefficients such characterization also holds. In fact, one can prove that if an Almost Mathieu cocycle (or any other quasi-periodic Schrödinger cocycle), for some a, b, ω fixed, is reducible to constant coefficients with Floquet matrix B then a is at the endpoint of a spectral gap of the operator if, and only if, trace $B = \pm 2$. Moreover the gap is collapsed if and, only if, $B = \pm I$.

Therefore, if

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

then the gap is collapsed if, and only if, $c = 0$. Summing up, we have that $2a/b$ is a noncollapsed spectral gap of $\sigma^{AM}_{4/b,\omega}$ if, and only if, the Floquet matrix of the corresponding cocycle, is the identity.

Now we can use an adaption of Ince's argument [9] for the classical Mathieu equation to our case. If B was the identity then, as we learned from Floquet representation (4), there would be two linearly independent real analytic quasi-periodic Bloch waves of Harper's equation

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos(2\pi\omega n + \phi)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}\ .$$

Passing to the dual, this would tell us that

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n)x_n = ax_n, \qquad n \in \mathbb{Z}\ .$$

has two linearly independent solutions in $l^2(\mathbb{Z})$. This is a contradiction with the limit-point character of the Almost Mathieu operator (or the preservation of the Wronskian for the difference equation).

Therefore $B \neq I$ $(c \neq 0)$ so that $2a/b$ is the endpoint of a noncollapsed gap of $\sigma^{AM}_{4/b,\omega}$. Since such endpoints are dense in the spectrum, this must be a Cantor set for all $b \neq 0, \pm 2$ and Diophantine frequencies.

2 Extension to Real Analytic Potentials

In the proof of the Ten Martini Problem that we have presented above, there are some features which are specific of the Almost Mathieu operator. Some other, however, can be extended to more general potentials. Let us try to reproduce the proof for a real analytic potential $V : \mathbb{T} \to \mathbb{R}$ instead of $b \cos \theta$. The corresponding Schrödinger operators are of the form

$$(H_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n \ .$$

The dual model of this operator is the following *long-range operator*,

$$(L_{V,\omega,\phi}x)_n = \sum_{k\in\mathbb{Z}} V_k x_{n+k} + 2\cos(2\pi\omega n + \phi)x_n$$

so that analytic quasi-periodic Bloch waves of $H_{V,\omega,\phi}$ correspond to exponentially localized eigenvectors of $L_{V,\omega,\phi}$. Bourgain & Jitomirskaya [3] proved that, for some $\varepsilon > 0$, $L_{V,\omega,\phi}$ has pure-point spectrum with exponentially localized eigenfunctions for almost all $\phi \in \mathbb{T}$ if

$$|V|_\rho := \sup_{|\mathrm{Im}\theta|<\rho} |V(\theta)| < \varepsilon$$

and ω is Diophantine.

Using this result and some facts on the IDS one can show [15] that for Lebesgue almost every $a \in \mathbb{R}$, the cocycle

$$(A_{a-V}, \omega) = \left(\begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \omega \right)$$

is reducible to constant coefficients if $|V|_\rho < \varepsilon$ and ω is Diophantine. Also, there exists a dense set of values of a in the spectrum such that the corresponding cocycle is reducible to

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \ .$$

Therefore, these values of a are at endpoints of spectral gaps of $H_{V,\omega,\phi}$. However, we cannot use Ince's argument and it may happen that some of these are collapsed (see Fig. 3). In fact, there are examples of quasi-periodic Schrödinger operators (with V small, ω Diophantine) which do not display Cantor spectrum (see De Concini & Johnson [6]).

Nevertheless, even if c can be zero, Moser & Pöschel [13] showed that, in this reducible setting, a closed gap can be opened by means of an arbitrarily small and generic perturbation of the potential, as it is shown in [15] (the proof by Moser & Pöschel is in the continuous case, although it extends without trouble to the discrete). Here generic is meant in the G_δ-sense, considering the space of real analytic perturbations in some fixed complex strip

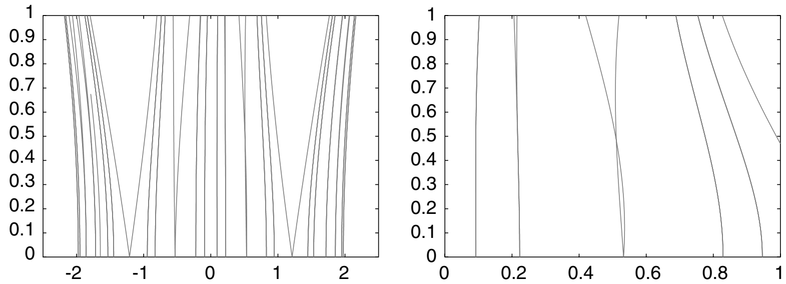

Fig. 3. *Left:* Endpoints of some spectral gaps of $H_{bV,\omega,\phi}$, with $V(\theta) = \cos(\theta) + 0.3\cos(2\theta)$ and several values of b (in the vertical direction) and $\omega = (\sqrt{5}-1)/2$. Note the spectral gap which is collapsed. *Right:* Magnification of the figure around the collapsed gap

furnished with the supremum norm. Since there is, at most, a countable number of collapsed spectral gaps, we can conclude that, a generic potential with $|V|_\rho < \varepsilon$, for some ρ fixed, has Cantor spectrum for Diophantine frequencies (see again [15]). This generalizes nonperturbatively results obtained by Eliasson [8] on the genericity of Cantor spectrum for quasi-periodic Schrödinger operators.

3 Cantor Spectrum for Specific Models

The results in the previous section on the genericity of Cantor spectrum for quasi-periodic Schrödinger operators have the disadvantage that they cannot be applied to specific examples of Schrödinger operators. In this section we will briefly describe how to get Cantor spectrum, and opening of all gaps, for some prescribed families of quasi-periodic Schrödinger operators. This is joint work with Broer & Simó [4, 16].

Here we will consider continuous Schrödinger operators, and for the sake of definiteness, the following *quasi-periodic Mathieu operator*

$$H^{QPM}_{b,\omega,\phi} x = -x'' + b \sum_{j=1}^{d} \cos(\omega_j t) x \ ,$$

where now $x \in L^2(\mathbb{R})$. Let us consider the self-adjoint extension of $H^{QPM}_{b,\omega,\phi}$ to $L^2(\mathbb{R})$ whose spectrum, again, does not depend on ϕ (see Fig. 4). For such operators we can prove the following.

Theorem 2. *Let $d \geq 2$. Then for almost all $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^{\mathrm{d}}$ there is a $C = C(\omega)$ such that for all values of $0 < |b| < C$, except for a countable set, the spectrum of the quasi-periodic Mathieu operator $H^{QPM}_{b,\omega,\phi}$ has all gaps open and, thus, it is a Cantor set.*

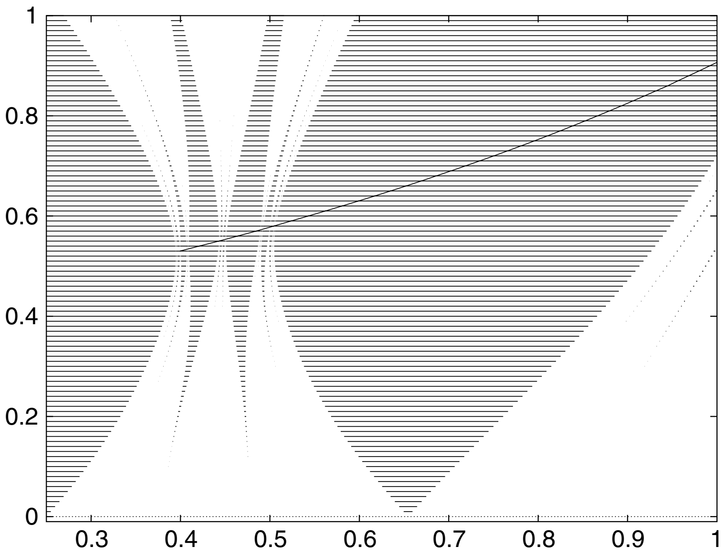

Fig. 4. Spectrum of $H^{QPM}_{b,\omega,\phi}$ for several values of b vertical direction. Shaded regions correspond to gaps. Here $\omega = (1, \gamma)^T$ where $\gamma = (1+\sqrt{5})/2$. Source Broer & Simó [5]

The idea for the proof is based on the study of gap boundaries as functions of the coupling constant b. The set formed by the closure of a certain gap (with fixed label) in the (a, b)-plane will be called a *resonance tongue*. When the boundaries of a certain resonance tongue merge for two different values of b we will speak of an *instability pocket* (see Fig. 5).

In the periodic case it is known (see, e.g. Rellich [17]) that tongue boundaries are real analytic functions. In the quasi-periodic case the same methods cannot be applied, but using KAM techniques it can be seen that tongue boundaries are real analytic if $|b|$ is smaller than a certain constant C which depends on the Diophantine class of ω [16].

Using Birkhoff Normal Form, we show that all these tongue boundaries (which we know are real analytic) have some finite order of contact at $b = 0$ [4]. In particular, each gap can collapse at most a finite number of times. Since the number of gaps is countable we only have to take out a countable subset of $|b| < C$.

Remark 4. This is a perturbative result (the smallness condition on the potential depends on the precise Diophantine conditions on the frequency vector) but it holds irrespectively of the dimension d (contrary to the methods in the first two sections).

Remark 5. The same result holds for any quasi-periodic potential whose Fourier coefficients are all nonzero.

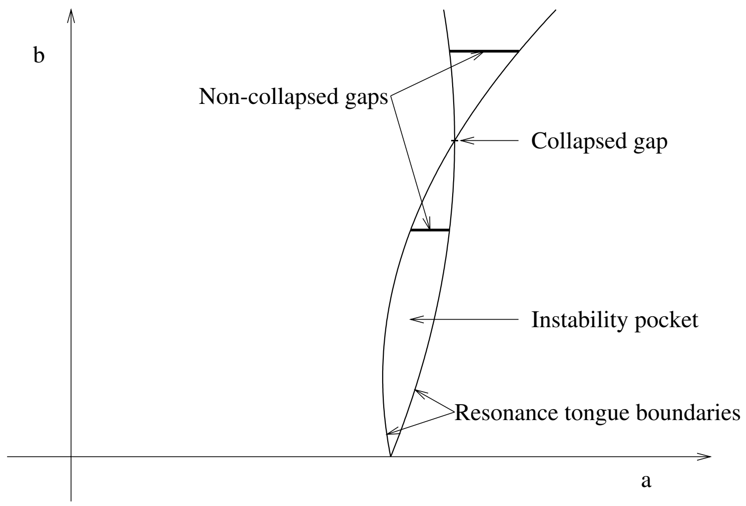

Fig. 5. Resonance tongue with pocket in the (a, b)-plane giving rise to spectral gaps on each horizontal line with constant b. Note how collapse of gaps corresponds to crossings of tongue boundaries at tips of an instability pocket

Remark 6. In [4] it is shown that by means of suitable and arbitrarily small perturbations of the potential of $H^{QPM}_{b,\omega,\phi}$ it is possible to produce pockets at any gap of the operator.

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