Commun. Math. Phys. 267, 735–740 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-0071-8

Communications in
**Mathematical
Physics**

Cantor Spectrum and KDS Eigenstates

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Received: 3 November 2005 / Accepted: 17 February 2006
Published online: 19 August 2006 –

Abstract: In this note we consider KDS eigenstates of one-dimensional Schrödinger operators with ergodic potential, which are a class of generalized eigenfunctions including Bloch eigenstates. We show that if the spectrum, restricted to an interval, has zero Lyapunov exponents and is a Cantor set, then for a residual subset of energies, KDS eigenstates do not exist. In particular, we show that the quasi-periodic Schrödinger operators whose Schrödinger quasi-periodic cocycles are reducible for all energies have a limit band-type spectrum.

1. Introduction. Main results

The aim of this note is to relate the existence of KDS eigenstates (from Kotani, Deift and Simon), which generalize Bloch eigenstates, for one-dimensional Schrödinger operators with ergodic potential to the Cantor structure of the spectrum. More specifically, we consider a probability measure space (Ω, μ), a measure preserving invertible ergodic transformation T, and a bounded measurable real-valued function $V : \Omega \to \mathbb{R}$. We let H_ω be the operator on $l^2(\mathbb{Z})$ defined by

$$(H_\omega x)_n = x_{n+1} + x_{n-1} + V(T^n \omega) x_n, \quad n \in \mathbb{Z}. \tag{1}$$

Our primary interest is with almost periodic and quasi-periodic operators, which are included this formulation. Most of the arguments can be transported to the continuous case, with straightforward adaptions, although we restrict to the discrete case for the sake of definiteness.

As it is well-know, an operator like (1) can exhibit different spectral types, depending on V, ω and T. These different types are very much related to the behaviour of solutions of the corresponding eigenvalue equation

$$x_{n+1} + x_{n-1} + V(T^n \omega) x_n = a x_n, \quad n \in \mathbb{Z}, \tag{2}$$

being a the *energy*. To measure the exponential growth of solutions in the spectrum, which is relevant for the spectral decomposition, one can introduce the *upper Lyapunov exponent* as the limit

$$\gamma(a) = \lim_{N\to+\infty} \frac{1}{N} \int_\Omega \log \left\| A_{a,V}(T^{N-1}\omega) \cdots A_{a,V}(\omega) \right\| d\mu(\omega),$$

where

$$A_{a,V}(\omega) = \begin{pmatrix} a - V(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

and whose existence is granted by the subadditive ergodic theorem [Kin68]. Outside the spectrum of H_ω, which is μ-a.e. independent of ω and we write as Σ, the Lyapunov exponent is always positive.

Ishii-Pastur-Kotani theory, see Simon [Sim83] for the discrete version, relates the absolutely continuous spectrum to the set of zero Lyapunov exponents

$$\mathcal{A}_0 = \{a \in \Sigma;\ \gamma(a) = 0\}.$$

If $\mathcal{A}_0$ has positive measure, its essential closure is the support of the absolutely continuous part of the spectrum of H_ω, which is μ-a.e. constant [KS81]. Moreover, for almost every $a \in \mathcal{A}_0$ (in the Lebesgue sense), Eq. (2) has a pair of independent solutions of the form

$$x_n^+ = e^{\mathrm{i}\varphi(n,\omega)} \psi(T^n\omega)$$

and

$$x_n^- = e^{-\mathrm{i}\varphi(n,\omega)} \psi(T^n\omega),$$

where $\varphi(n,\omega)$ is measurable and $\psi \in L^2(\Omega)$, which we will call *KDS eigenstates*, as showed, almost simultaneously, by Kotani [Kot84] and Deift & Simon [DS83]. In the almost-periodic case, the norm of these solutions is an L^2-almost-periodic function with the same frequency module as the potential V. In this note we address the possible existence of KDS eigenstates for the remaining energies and the connection with the existence of gaps in the spectrum. Therefore we define the set

$$\mathcal{A}_1 = \{a \in \Sigma; \text{ there are KDS eigenstates}\}.$$

It is easy to see that

$$\mathcal{A}_1 \subset \mathcal{A}_0 \subset \Sigma.$$

The last inclusion is strict, since the Lyapunov exponent can be positive in the spectrum (e.g. [Her83, SS91, Bou05, Bje05]). In this note we will characterize when the first inclusion is strict.

De Concini & Johnson [DCJ87] considered the case where $\mathcal{A}_0$ contains nonvoid intervals. Let I be one of these open maximal intervals. Then, they show that for *all* $a \in I$, KDS eigenfunctions do exist. At endpoints of I KDS eigenfunctions cannot exist, as we will see later on, but these form (at most) a countable set in any case. The content of our main theorem is that whenever endpoints of gaps are dense in $\mathcal{A}_0$ (therefore being a Cantor set), energies without KDS eigenstates are topologically abundant (although with Lebesgue zero measure according to Kotani theory).

Theorem 1. *Let I be an open interval in $\mathbb{R}$ such that*

$$I \cap \Sigma = \{a \in I; \gamma(a) = 0\} = \mathcal{A}_0 \cap I,$$

and it is a nonvoid Cantor set. Then $(\mathcal{A}_0 \backslash \mathcal{A}_1) \cap I$ is a residual G_δ of $\mathcal{A}_0 \cap I$.

This theorem generalizes a result in [Pui06] in the context of quasi-periodic skew-products and is similar to arguments in circle maps relating Cantor structure of the hyperbolic zones to the existence of non-regular dynamics [Arn61]. Cantor spectrum has been derived for several models, most notably the Almost Mathieu, $V(\theta) = b \cos\theta$ and an irrational frequency. In fact, this work is inspired by some methods in [Rie03] to treat this case, although in a different sense. Moreover, in the Almost Mathieu Lyapunov exponent has been shown to be 0 in the spectrum if, and only if, $|b| \leq 2$. Therefore, we have the following immediate consequence:

Corollary 2. *In the Almost Mathieu operator, with irrational frequency and nonzero coupling, there is a G_δ-set of energies in the spectrum without KDS eigenstates.*

Remark 3. The fact that the Almost Mathieu model is invariant under Fourier transform (*Aubry duality*), allows to try to produce the same result using a theorem by Jitomirskaya & Simon [JS94] who prove that, under the same hypothesis as Corollary 2, there is a residual G_δ of energies which are not point eigenvalues (in $l^2(\mathbb{Z})$). Then using Aubry duality, the dual set of energies could not have L^2 quasi-periodic Bloch waves, which are a particular case of our result. Note that for the existence of KDS eigenstates no control is imposed on the phase of the sequence (only that its modulus follows the dynamics of T), and for Bloch waves dynamics are imposed also in the phase, see the discussion following Theorem 7.1 in [DS83].

Finally, we would like to state a result concerning the *reducibility* of quasi-periodic Schrödinger cocycles to constant coefficients. In this case Ω is a suitable d-dimensional torus and T is a quasi-periodic translation defined by a frequency vector $\alpha \in \mathbb{R}^d$ whose components are rationally independent. A quasi-periodic Schrödinger operator is reducible to constant coefficients if there is a continuous quasi-periodic transformation, with the same basic frequencies, which renders it to a constant matrix (called the *Floquet matrix*). If a Schrödinger cocycle whose energy a is not at the endpoint of a gap is reducible to constant coefficients then the Floquet matrix can be chosen in $SO(2, \mathbb{R})$ and therefore a has KDS eigenstates. This implies that we can get a sort of "inverse" result.

Theorem 4. *If a quasi-periodic Schrödinger cocycle is reducible to constant coefficients for all energies then the spectrum consists of spectral bands* (*nonvoid closed intervals in the spectrum*) *and accumulation points of these.*

Proof. If a Schrödinger cocycle is reducible to constant coefficients and the Lyapunov exponent is positive, then it has an exponential dichotomy and the corresponding energy belongs to the resolvent set [Joh82]. Thus if a Schrödinger cocycle is reducible to constant coefficients for all energies in the spectrum then the Lyapunov exponent must be zero in the spectrum. If in addition there is a component of the spectrum which is a Cantor set, we are under the hypothesis of Theorem 1 and there do not exist Bloch waves for a G_δ set of energies. Even if at endpoints of gaps the cocycle is reducible to constant coefficients and there is only a single Bloch wave, these endpoints form a countable set. So the cocycle is still nonreducible to constant coefficients for a residual set of energies. □

2. Proof of Theorem 1

Take I an open interval in $\mathbb{R}$ such that

$$K := I \cap \Sigma = \{a \in I; \gamma(a) = 0\} = \mathcal{A}_0 \cap I.$$

We must show that K contains a residual set of energies without KDS eigenstates.

It is worth noting that, with these hypotheses, the Lyapunov exponent is a continuous function on I. Indeed, continuity at the resolvent set follows from general principles and continuity at points of K, where γ vanishes, is a consequence of the upper semi-continuous character of the Lyapunov exponent [CS83].

The existence of KDS eigenstates at some energy a_0 implies that there is a fundamental matrix of the first-order system associated to the eigenvalue equation whose norm, at any time, is bounded by a square integrable function. Using the definition of the Lyapunov exponent it is easy to show that it satisfies a Lipschitz condition at this energy.

Lemma 5. *If $a_0 \in K$ is an energy with KDS eigenstates, then the map*

$$a \in \mathbb{R} \mapsto \gamma(a)$$

is Lipschitz at a_0.

When a_0 lies at the endpoint of a gap in K, then one cannot have Lipschitz continuity at a_0.

Lemma 6. *Assume that a_0 is the endpoint of an open gap in the spectrum with $\gamma(a_0) = 0$. Then*

$$\sup_{a \neq a_0} \left| \frac{\gamma(a) - \gamma(a_0)}{a - a_0} \right| = \infty. \tag{3}$$

Proof. The Lyapunov exponent can be expressed through *Thouless formula* [Tho72, AS83, CS83],

$$\gamma(a) = \int_{\mathbb{R}} \log|\lambda - a| d\kappa(\lambda),$$

where $d\kappa$ stands for the integration with respect to the density of states measure (supported on the spectrum). Introducing the so-called *w-function* or *Floquet exponent*,

$$w(a) = -\int_{\mathbb{R}} \log(\lambda - a) d\kappa(\lambda),$$

then Re $w(z) = -\gamma(z)$. If Γ denotes an open spectral gap then a suitable choice of the branch of the logarithm makes it analytic through $\Pi_+ \cup \Gamma \cup \Pi_-$, where Π_+ (resp. Π_-) denotes the upper (resp. lower) half plane. Its derivative,

$$w'(a) = \int_{\mathbb{R}} \frac{1}{\lambda - a} d\kappa(\lambda) \tag{4}$$

is a single-valued function on $\mathbb{C} \backslash \Sigma$ which is never zero in $\mathbb{C} \backslash \mathbb{R}$.

Let us now show that if a_0 is an endpoint of Γ the Lyapunov exponent has the asymptotics given by Eq. (3). For the sake of simplicity, let a_0 be the leftmost endpoint of the spectrum so that the corresponding gap is $\Gamma = (-\infty, a_0)$. Take the determination of the logarithm which makes w analytic and conformal at $\mathbb{C}\backslash[a_0, +\infty)$. With this choice, $w(a)$ is real and negative if $a \in (-\infty, a_0)$.

Since γ is analytic in $(-\infty, a_0)$, with γ' negative and γ'' nonzero there (see Eq. (4)), and γ is continuous at a_0, the limit

$$\lim_{a\to a_0^-, a\in\mathbb{R}} w'(a) = -\lim_{a\to a_0^-} \gamma'(a) = -\lim_{a\to a_0^-} \frac{\gamma(a)-\gamma(a_0)}{a-a_0}$$

exists and is either $+\infty$ or C, a finite positive constant. Let us now see that the latter case is impossible. Montel's theorem (eg. [Sch93]) shows that in that case, w' possesses angular limits at a_0 when approaching from $\mathbb{C}\setminus[a_0, +\infty)$ and they are all equal to C. Therefore, since w is conformal at $\mathbb{C}\setminus[a_0, +\infty)$ and $C \neq 0$ then w must preserve angles at a_0. However, the image of the upper half plane under w is in the region

$$\operatorname{Im} z \in [0, \pi] \quad \text{and} \quad \operatorname{Re} z < 0.$$

At $w(a_0) = 0$ the boundaries of this region form an internal angle of $\pi/2$, and therefore, w cannot preserve angles at a_0.

The argument when a_0 is an endpoint of any other open spectral gap with $\gamma(a_0) = 0$ is very similar. The image of the upper half-plane under w forms an internal angle of $\pi/2$ at $w(a_0)$, because the boundary value of Im w is the IDS and thus constant in the gap, while its real value is minus the Lyapunov exponent. The argument for the lowest gap can now be used, since the Lyapunov exponent has a negative second derivative on the gap and limits of γ' at the endpoints are either $\pm\infty$ or a finite constant. □

We now turn to the proof of Theorem 1. For any $a \in K$ we define

$$m(a) = \sup_{\lambda\neq a, \lambda\in I} \left| \frac{\gamma(a)-\gamma(\lambda)}{a-\lambda} \right|,$$

which is either a positive real number or $+\infty$. If a has a KDS eigenstate then $m(a) < \infty$ according to Lemma 5 and, if a is at the endpoint of a gap in the spectrum, then $m(a) = \infty$ due to Lemma 6. We will show that there is a residual of energies in K with $m(a) = +\infty$. In particular, these cannot only be endpoints of gaps (at most a countable set).

Let, for any $n \in \mathbb{N} \cup \{0\}$,

$$U(n) = \{a \in K; m(a) > n\}.$$

This is an open set of K (due to the continuity of γ on the interval I) which is also dense, because it includes endpoints of gaps in K. Therefore

$$U(\infty) = \bigcap_{n>0} U(n) = \{a \in K; m(a) = \infty\},$$

is a residual G_δ subset in K without KDS eigenstates. □

Acknowledgement. The author wishes to thank Russell Johnson and the anonymous referee for useful comments and suggestions. This work has been supported by MCyT/FEDER Grant BFM2003-09504-C02-01 and CIRIT 2001 SGR-70

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Communicated by B. Simon