Abstracts

Nonperturbative reducibility and irreducibility

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In this talk we consider discrete, one-dimensional Schrödinger operators with real analytic potentials and Diophantine frequencies

$$(H_{V,\omega,\theta}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \theta)x_n, \qquad n \in \mathbb{Z}. \tag{1}$$

and the relation with the reducibility problem of the associated *quasi-periodic cocycle*, which is the following dynamical system on $SL(2,\mathbb{R}) \times \mathbb{T}^d$,

$$\begin{array}{rccc} (A_{a,V},\omega): & SL(2,\mathbb{R}) \times \mathbb{T}^d & \longrightarrow & SL(2,\mathbb{R}) \times \mathbb{T}^d \\ & (X,\theta) & \mapsto & (A(\theta)X, \theta + 2\pi\omega)\,. \end{array} \tag{2}$$

with

$$A_{a,V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

As many recent works have outlined, there is a close relationship between spectral properties of (1) and dynamics of the cocycle (2) which arise from the fact that the latter is the matrix first-order system associated to the eigenvalue equation of (1),

$$x_{n+1} + x_{n-1} + V\left(2\pi\omega n + \theta\right)x_n = a x_n, \quad n \in \mathbb{Z}.$$

Among dynamical properties which are relevant for the spectral description of (1), conjugacy of cocycles is a key tool, since it allows to classify different dynamical types. Two cocycles (A,ω) and (B,ω) are *conjugated* if there is a *conjugation* $Z : \mathbb{T}^d \to SL(2,\mathbb{R})$ such that $(A,\omega) \circ (Z,0) = (Z,0) \circ (B,\omega)$. The notion of conjugacy, once a regularity class for the transformation has been imposed, allows to classify dynamically quasi-periodic cocycles. A particularly important class is that of *reducible* cocycles, which are those conjugated to a cocycle with constant coefficients, which is called a *Floquet matrix*.

In our case of interest, i.e. when the frequencies are Diophantine, $\omega \in DC(c,\tau)$ for some c, τ,

$$|\langle \mathbf{k}, \omega\rangle| \geq \frac{c}{|\mathbf{k}|^\tau}, \quad \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\},$$

and the potential V is real analytic, $C^a_\rho(\mathbb{T}^d,\mathbb{R})$ for some $\rho > 0$ with the norm

$$|V|_\rho := \sup_{|\Im\theta|<\rho} |V(\theta)| < \infty,$$

Johnson [7] showed that a is not in the spectrum of (1) if, and only if, the cocycle (2) is reducible to constant coefficients with hyperbolic Floquet matrix (possibly halving the frequency). In the spectrum, however, the situation is much more involved.

Eliasson [6], among other results, showed that, once ρ, c and τ positive have been fixed, there is a $\varepsilon_E = \varepsilon_E(c,\tau,\rho) > 0$ such that if $|V|_\rho < \varepsilon_E$, then the cocycle $(A_{a,V},\omega)$ is analytically reducible to constant coefficients for Lebesgue almost all

$a \in \mathbb{R}$. Moreover, for a generic V, with $|V|_\rho < \varepsilon$ the spectrum of (1) is a Cantor set and it contains a subset of a's for which $(A_{a,V}, \omega)$ is not reducible to constant coefficients.

This result is optimal for $d > 1$ due to an example of Bourgain [2] and, in fact, it gives a characterization of the set of "reducible energies a" in terms of their rotation number or integrated density of states. When $d = 1$, we can use localization results by Bourgain & Jitomirskaya [3] and Aubry duality to show the following.

Theorem 1 ([10])**.** *Let $\rho > 0$. Then,there is a $\varepsilon = \varepsilon(\rho) > 0$ such that if*

$$|V|_\rho < \varepsilon$$

and $\omega \in DC(c, \tau)$ then the following holds

(i) $(A_{a,V}, \omega)$ *is* reducible *to constant coefficients for* almost all $a \in \mathbb{R}$.
(1) *For a* generic $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$, *the spectrum $\sigma(V, \omega)$ is a Cantor set.*
(2) *If $\sigma(V, \omega)$ is a Cantor set, there is a residual set in $\sigma(V, \omega)$ for which $(A_{a,V}, \omega)$ is not reducible to constant coefficients.*

The theorem above is not a full extension of Eliasson's result, since the set of reducible energies has no characterization in terms of the IDS (see the talk by Avila and Jitomirskaya on these reports for an answer to this question), but it clarifies the relationship between Cantor spectrum and irreducibility (which was an ingredient in the proof of [6]).

As said before the proof of the first item is an application of [3] for which it suffices to realize that the (analytic) reducibility of a Schrödinger cocycle with Floquet matrix in $SO(2, \mathbb{R})$ is equivalent to the existence a pair of linearly independent *Bloch waves* for the Harper-like equation. Passing to the equation in Fourier space, one needs to find exponentially localized eigenvalues of a long-range operator (with exponentially decaying coefficients) and a small cosine potential. Such solutions are provided by Bourgain & Jitomirskaya. To prove full measure, one only needs to apply the bounds for the growth of the density of states when the Lyapunov exponent is zero [5].

Items (ii) and (iii) in Theorem 1 allow a d-dimensional version. Indeed, using the analyticity of m-functions, as in Avila & Jitomirskaya [1], one can show that, whenever the spectrum has an open interval with zero Lyapunov exponent, then the corresponding cocycle is reducible to constant coefficients and the Floquet matrix is a rotation. In such a case, an application of Moser & Pöschel [8] (cf. [10]) shows that an arbitrarily small and generic perturbation opens up all collapsed gaps in the interval and the spectrum in the interval becomes a Cantor set. When $d = 1$, Cantor spectrum is supposed to be generic without the smallness hypothesis or, even, to hold always when the Lyapunov exponent is positive (see Sinai [11] for results in this direction), but there is no proof in this generality.

The relationship between Cantor spectrum and irreducibility can also be generalized to more general potentials (such as ergodic potentials, see [9]). For definiteness, let us consider continuous quasi-periodic potentials on $\mathbb{T}^d$ with irrational frequency vector ω and Cantor spectrum where the Lyapunov exponent vanishes.

Then, there is a residual G_δ in the spectrum where the corresponding cocycle is not reducible to constant coefficients by a continuous transformation. In fact such a transformation cannot be square integrable either. The latter is related to the existence of *Kotani eigenstates* (pair of solutions with norms given by the absolute values of an L^2 function along the trajectory of the ergodic transformation) which Kotani considered in the set where the Lyapunov exponent vanishes in the ergodic setting. De Concini & Johnson [4] showed that when this set contains an interval then there are Kotani eigenstates for all points in the interval. Our result shows that whenever the spectrum is a Cantor set and the Lyapunov exponent vanishes there, then there is a topologically significant set for which these eigenstates do not exist.

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Schrödinger cocycles at non-perturbatively small coupling

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We are interested in one-dimensional quasiperiodic Schrödinger operators $H = H_{v,\alpha,\theta}$ defined on $l^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n \tag{1}$$

where $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the potential, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\theta \in \mathbb{R}$ is the phase. The most important example is given by the almost Mathieu operator, when $v(x) = 2\lambda\cos(2\pi x)$.

In [6], Eliasson obtained a very precise description of such operators for α Diophantine, in the case of small analytic potentials in the perturbative regime (this means that a smallness condition depends on α, and thus the analysis of a given potential, however small, can only be carried out for a positive Lebesgue