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Resonance Tongues and Instability Pockets in the Quasi–Periodic Hill–Schrödinger Equation

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Abstract: This paper concerns Hill's equation with a (parametric) forcing that is real analytic and quasi-periodic with frequency vector $\omega \in \mathbb{R}^d$ and a 'frequency' (or 'energy') parameter a and a small parameter b. The 1-dimensional Schrödinger equation with quasi-periodic potential occurs as a particular case. In the parameter plane $\mathbb{R}^2 = \{a, b\}$, for small values of b we show the following. The resonance "tongues" with rotation number $\frac{1}{2}\langle \mathbf{k}, \omega\rangle$, $\mathbf{k} \in \mathbb{Z}^d$ have C^∞-boundary curves. Our arguments are based on reducibility and certain properties of the Schrödinger operator with quasi-periodic potential. Analogous to the case of Hill's equation with periodic forcing (i.e., $d = 1$), several further results are obtained with respect to the geometry of the tongues. One result regards transversality of the boundaries at $b = 0$. Another result concerns the generic occurrence of instability pockets in the tongues in a reversible near-Mathieu case, that may depend on several deformation parameters. These pockets describe the generic opening and closing behaviour of spectral gaps of the Schrödinger operator in dependence of the parameter b. This result uses a refined averaging technique. Also consequences are given for the behaviour of the Lyapunov exponent and rotation number in dependence of a for fixed b.

Contents

1. Introduction, Main Results

Consider Hill's equation

$$x'' + (a + bq(t))x = 0, \tag{1}$$

where a, b are real parameters and the real analytic function q is quasi-periodic in t, with a fixed frequency vector $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$. If the function q is even, Hill's equation is reversible, but for the main result of this paper we shall consider general, including non-reversible, Hill's equations. The quasi-periodicity means that if $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ is the d-dimensional torus, there exists a real analytic function $Q : \mathbb{T}^d \to \mathbb{R}$ such that, $q(t) = Q(t\omega)$. The frequency-vector ω, moreover, is assumed to be Diophantine with constants $c > 0$ and $\tau \geq d - 1$, i.e., that $|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau}$, for all $\mathbf{k} \in \mathbb{Z}^d - 0$. The set of such $\omega \in \mathbb{R}^d$ is known to have large measure as c is small, e.g., see [7].

The objects of our interest are resonance tongues as they occur in the parameter space $\mathbb{R}^2 = \{a, b\}$. Our main result states that in the present analytic case, for small $|b|$, the tongue boundaries are infinitely smooth curves. This result is used to study the geometry of the resonance tongues. Here we restrict to reversible near-Mathieu cases, which are a small perturbation of the exact Mathieu equation where $q(t) = \sum_{i=1}^{d} c_i \cos(\omega_i t)$, with $c_1, \dots, c_d$ real constants. In the first remark of Sect. 1.2.3 a geometric reason is given for restricting to reversible systems when looking for instability pockets. An example of a near-Mathieu case with $d = 2$ and a deformation parameter ϵ is given by

$$q_\varepsilon(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \varepsilon \cos(\omega_1 + \omega_2)t. \tag{2}$$

It is shown that the occurrence of instability pockets is generic in the reversible setting and a concise description of its complexity is given in terms of singularity theory. We shall draw several consequences regarding the spectral behaviour of the corresponding Schrödinger operator, in particular regarding the effect of instability pockets on the collapsing of gaps. We develop examples where collapsed spectral gaps occur in a way that is persistent for perturbation of the b-parametrized, reversible family.

The set-up of this paper is similar to Broer & Simó [15], where certain cases of Hill's equation with quasi-periodic forcing of two frequencies were studied in a more experimental way. Unlike in the periodic case, smoothness of the tongue boundaries is not easy to obtain. This novel result uses a reducibility result by Eliasson [22]. This makes an analysis possible as in the periodic case. However due to accumulation of tongues we need a delicate averaging technique.

1.1. Background and motivation. Our motivation on the one hand rests on the analogy with the periodic Hill equation, where several results in the same direction were known. On the other hand, the present results were motivated by the interest they have for certain spectral properties of the Schrödinger operator.

1.1.1. Hill's equation and the quasi-periodic Schrödinger operator. Hill's equation with quasi-periodic forcing is a generalization and extension of the classical, periodic Hill equation. Both the periodic and the quasi-periodic case occur as a first variation equation in the stability analysis of periodic solutions and lower dimensional tori in the Hamiltonian with few degrees of freedom. It was devised by George Hill in the 19th century to study the motion of the Moon [32].

For fixed $b \in \mathbb{R}$, Hill's equation shows up as the eigenvalue equation of the one-dimensional quasi-periodic Schrödinger operator

$$(H_{bV}x)(t) = -x''(t) + bV(t)x(t), \tag{3}$$

where $V(t) = -q(t)$, which is an essentially self-adjoint operator on $L^2(\mathbb{R})$. In this setting, the parameter a is called the energy- or the spectral-parameter. Indeed, the eigenvalue equation has the format $H_{bV}x = ax$. For general reference see [45, 18, 40]. Quasi-periodic Schrödinger operators occur in the study of electronic properties of solids [5]. Moreover, these operators are important for solutions of KdV equations with quasi-periodic initial data [30].

Presently, b is not considered as a constant, but as a parameter. This will give a better understanding of certain spectral phenomena as these were observed for fixed values of b. One example concerns the fact that generically no collapsed gaps occur, as shown by Moser & Pöschel [34]. Including b as a parameter gives a deeper insight in the generic opening and closing behaviour of such gaps in the dependence of b. Therefore our main interest is with the quasi-periodic analogue of the stability diagrams as these occur for the periodic Hill equation in the parameter plane $\mathbb{R}^2 = \{a, b\}$. The spectral properties of the quasi-periodic Schrödinger operator obtained in this paper are related to the dynamical properties of Hill's equation.

1.1.2. The periodic Hill equation revisited. We briefly reconsider Hill's equation with periodic forcing (the case $d = 1$), compare Broer & Levi [12], Broer & Simó [16], who study resonances in the near-Mathieu equation

$$x'' + (a + b\,q(t))\,x = 0, \; q(t + 2\pi) \equiv q(t), \tag{4}$$

with q even and where a and b are real parameters. As is well-known, in the (a, b)-plane, for all $k \in \mathbb{N}$, resonance tongues emanate from the points $(a, b) = ((\frac{k}{2})^2, 0)$. Inside these tongues, or instability domains, the trivial periodic solution $x = x' = 0$ is unstable. Compare Van der Pol & Strutt [47], Stoker [46], Hochstadt [28], Keller & Levy [31], Magnus & Winkler [32] or Arnol′d [3, 2, 1]. For related work on nonlinear parametric forcing, see Hale [27] and Broer et al. [9–11, 6, 14, 8]. For nonlinear discrete versions see [37, 38].

The stability properties of the trivial solution of the periodic Hill equation are completely determined by the eigenvalues of the linear period map, also called stroboscopic or Poincaré map $P_{a,b}$. Note that due to the conservative character of Hill's equation we have $P_{a,b} \in Sl(2, \mathbb{R})$, the space of 2×2-matrices with determinant 1. In fact, elliptic eigenvalues correspond to stability and hyperbolic eigenvalues to instability.

The geometry of the tongue boundaries was studied in [12] and [16]. It turns out that generically the boundaries of a given tongue may exhibit several crossings and tangencies, thereby also creating instability pockets, see Fig. 1. This term was coined by Broer-Levi prompted by the term 'instability interval' [31] as it occurs for fixed values of b. In [12, 16] normal forms and averaging techniques provide a setting for singularity theory. It turns out that in the near-Mathieu case close to the $k : 2$ resonance, one can have between 0 and $k - 1$ instability pockets, with all kinds of intermediate tangencies: the whole scenario has at least the complexity of the singularity $\mathbb{A}_{2k-1}$, compare [4].

Remark. For a description and analysis of more global phenomena in the periodic case, see [13]. A singularity theory approach of resonances in a general dissipative context is given in [17].

1.2. Towards the main result. In this section we formulate our Main Theorem regarding the smoothness of the boundaries of resonance tongues in Hill's equation with quasi-periodic forcing.

1.2.1. The quasi-periodic Hill equation: Rotation number, spectral gaps and resonance tongues. Preliminary to formulating our main result, we need some definitions. We start rewriting the quasi-periodic Hill equation (1) as a vector field on $\mathbb{T}^d \times \mathbb{R}^2 = \{\theta, (x, y)\}$, where $\theta = (\theta_1, \dots, \theta_d)$ are angles counted mod 2π. This yields a vector field $\mathcal{X}$, in system form given by

$$\begin{aligned} \theta' &= \omega, \\ x' &= y, \\ y' &= -(a + bQ(\theta))x. \end{aligned} \tag{5}$$

Observe that evenness of Q leads to time-reversibility, which here is expressed as follows: if $R : \mathbb{T}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}$ is given by $R(\theta, x, y) = (-\theta, x, -y)$, then $R_*(\mathcal{X}) = -\mathcal{X}$. Reversibility will not be assumed for the main result.

Our interest is with the invariant d-torus $\mathbb{T}^d \times \{(0, 0)\} \subseteq \mathbb{T}^d \times \mathbb{R}^2$, which carries quasi-periodic dynamics with frequency vector ω, where we study properties of the normal linear behaviour. Each trajectory inside the d-torus densely fills this torus, from which it may be clear that, unlike in the periodic case $d = 1$, for $d \geq 2$ no appropriate two-dimensional Poincaré map is defined.

Nevertheless resonance tongues can be defined by means of the rotation number $\operatorname{rot}(a, b)$, a concept living both in the periodic and the quasi-periodic setting. We freely quote from [30]. The rotation number of Eq. (1) is defined as

$$\mathrm{rot}(a,b) = \lim_{T\to+\infty} \frac{\arg\big(x'(T)+ix(T)\big)}{T},$$

where x is any non-trivial solution of Eq. (1). This number exists and is independent of the particular solution. The map

$$(a,b) \in \mathbb{R}^2 \mapsto \mathrm{rot}(a,b)$$

is continuous and, for fixed b, is a non-decreasing function of a. Also $\mathrm{rot}(a,b) = 0$ if a is sufficiently small. Moreover, the spectrum of the Schrödinger operator H_{bV} is the set of points a for which the map $a \mapsto \mathrm{rot}(a,b)$ is not locally constant.

We recall that the complement of the spectrum is called a resolvent set. The open intervals where the rotation number is constant are called spectral gaps. In these gaps, the rotation number must be of the form

$$\alpha = \frac{\langle \mathbf{k}, \omega\rangle}{2},$$

where $\mathbf{k} \in \mathbb{Z}^d$ is a suitable multi-integer such that $\langle \mathbf{k}, \omega\rangle \geq 0$. This is referred as the Gap Labelling Theorem [30]. The set

$$\mathcal{M}_+(\omega) = \left\{ \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathbb{R} \,\middle|\, \mathbf{k} \in \mathbb{Z}^d \text{ and } \langle \mathbf{k}, \omega\rangle \geq 0 \right\}$$

is called the module of positive half-resonances of ω. When for a certain resonance $\alpha \in \mathcal{M}_+(\omega)$ the corresponding spectral gap vanishes, the unique a for which $\mathrm{rot}(a,b) = \alpha$ gives rise to the collapsed gap $\{a\}$.

Now we can define the resonance tongue, the object of our main present interest.

Definition 1. *Let* $\mathbf{k} \in \mathbb{Z}^d$*. The resonance tongue of the quasi-periodic Hill equation* (1) *associated to* $\mathbf{k}$ *is the set*

$$\mathcal{R}(\mathbf{k}) = \left\{ (a,b) \in \mathbb{R}^2 \,\middle|\, \mathrm{rot}(a,b) = \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle \right\}.$$

This statement means that, for any fixed b_0 and any resonance $\frac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+(\omega)$, the set of all a for which (a, b_0) belongs to the resonance tongue $\mathcal{R}(\mathbf{k})$ is precisely the closure of the spectral gap of H_{b_0V} (either collapsed or non-collapsed) corresponding to this resonance by the Gap Labelling Theorem. See Fig. 1 for illustration.

1.2.2. Formulation of the Main Theorem. As said before, the present paper is concerned with the geometry and regularity of their boundaries for the quasi-periodic Hill equation (1) in the parameter plane $\mathbb{R}^2 = \{a, b\}$, where the function q is fixed.

For $\alpha_0 \in \mathcal{M}_+(\omega)$, if $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega\rangle$, let $\hat{\mathcal{R}}(\alpha_0) = \mathcal{R}(\mathbf{k})$. Each tongue $\hat{\mathcal{R}}(\alpha_0)$ has the form

$$\hat{\mathcal{R}}(\alpha_0) = \left\{ (a,b) \in \mathbb{R}^2 \mid a_-(b;\alpha_0) \leq a \leq a_+(b;\alpha_0) \right\} \tag{6}$$

and $a_+(0;\alpha_0) = a_-(0;\alpha_0) = \alpha_0^2$. Indeed, if $b = 0$ and $a > 0$, the solutions of (1), which now is autonomous, are linear combinations of $e^{\pm i\sqrt{a}t}$. By the above definition of the rotation number it follows that $\mathrm{rot}(a, 0) = \sqrt{a}$.

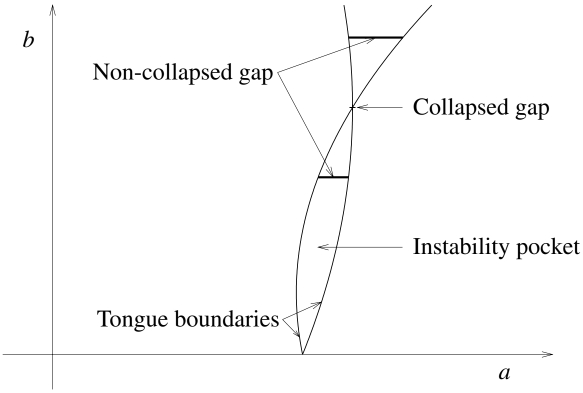

Fig. 1. Resonance tongue with pocket in the (a, b)-plane giving rise to spectral gaps on each horizontal line with constant b. Note how collapses of gaps correspond to crossings of the tongue-boundaries at the extremities of an instability pocket

Mostly the value of α_0 is fixed, in which case we suppress its occurrence in the boundary functions $a_\pm$. Note that in (6) one can ask, in general, for not more than continuity of the mappings $a_\pm : \mathbb{R} \to \mathbb{R}$, since we are imposing that $a_- \leq a_+$. Compare with the periodic case [12, 16].

Nevertheless, recall that in the periodic case $d = 1$ there exist real analytic boundary curves $a_{1,2} = a_{1,2}(b)$ such that $a_- = \min\{a_1, a_2\}$ and $a_+ = \max\{a_1, a_2\}$. For the present case $d \geq 2$ we have the following result.

Theorem 1 (Smoothness of tongue boundaries)**.** *Assume that in Hill's equation* (1)

$$x'' + (a + bq(t))x = 0$$

with $a, b \in \mathbb{R}$, the function q is real analytic and quasi-periodic with Diophantine frequency vector $\omega \in \mathbb{R}^d$, with $d \geq 2$. Then, for some constant $C = C(q, \omega)$ and for any $\alpha_0 \in \mathcal{M}_+(\omega)$, there exist C^∞-functions $a_1 = a_1(b)$ and $a_2 = a_2(b)$, defined for $|b| < C$, satisfying

$$a_- = \min\{a_1, a_2\} \text{ and } a_+ = \max\{a_1, a_2\}.$$

Remark. In the sequel, beyond a proof of this theorem, constructive methods are given to obtain C^r-approximations of the tongue boundaries. These methods can be applied, a fortiori, to the periodic case $d = 1$.

1.2.3. Instability pockets, collapsed gaps and structure of the spectrum. We sketch the remaining results of this paper, regarding instability pockets and the ensuing behaviour of spectral gaps.

In the quasi-periodic Hill equation, instability pockets can be defined as in the periodic case. The fact that a tongue has a boundary crossing at (a_0, b_0) means that $\{a_0\}$ is a collapsed gap for the Schrödinger operator (3) with $b = b_0$. An example of this occurs at the tongue tip $b = 0$.

Moser & Pöschel [34] show that, for small analytic quasi-periodic potentials with Diophantine frequencies, collapsed gaps can be opened by means of arbitrarily small perturbations. This implies that it is a generic property to have no collapsed gaps for fixed values of b. In this paper we go one step beyond, studying how gaps behave when the system is depending on the parameter b in a generic way.

By Theorem 1 we know that for analytic forcing (potential), for small $|b|$ and for a Diophantine frequency vector ω, the tongue boundaries are infinitely smooth. From this it follows that the computational techniques regarding normal forms and singularity theory, for studying the tongue boundaries, carry over from the periodic to the quasi-periodic setting. In particular this leads to a natural condition for the tongue boundaries to meet transversally at the tip $b = 0$, implying that there are no collapsed gaps for small $|b| \neq 0$. As a result we find, that for reversible Hill equations of near-Mathieu type, after excluding a subset of Diophantine frequency vectors ω of measure zero, the situation is completely similar to the periodic case. Compare with the description given before in Sect. 1.1.1.

We shall present examples of families of reversible quasi-periodic Hill equations of near-Mathieu type with instability pockets. These examples are persistent in their (reversible) setting. To our knowledge, so far the existence of collapsed gaps in quasi-periodic Schrödinger operators has only been detected by De Concini and Johnson [20] in the case of algebraic-geometric potentials. These potentials only have a finite number of non-collapsed gaps, while all other gaps are collapsed. In view of the present paper, this is a quite degenerate situation. See Fig. 2 for an actual instability pocket for which normal form methods are needed up to second order, the results of which are compared with direct numerical computation. The techniques just described are useful when studying a fixed resonance. We note, however, that for investigating 'all' resonance tongues at once, even in a concrete example, we will use certain direct methods, which amount to refined averaging techniques. Compare with the periodic case [16].

Remarks. 1. In the non-reversible case generically no instability pockets can be expected. To explain this, consider the classical periodic case $d = 1$, compare [12]. Recall that here the stability diagram can be described in terms of Hill's map, which assigns to every parameter point (a, b) the Poincaré matrix $P_{a,b} \in Sl(2, \mathbb{R})$, which is the 3-dimensional Lie group of 2×2-matrices of determinant 1. The tongue boundaries just are pull-backs under Hill's map of the unipotent cone, which has dimension 2 (except for singularities at $\pm\mathrm{Id}$). In the 3-dimensional matrix space the surfaces formed by the cone and the image under Hill's map of the (a, b)-plane generically meet in a transversal way. However, the intersection curves (which correspond to the tongue boundaries) generically do not meet away from the tip $b = 0$. Boundary crossings however do occur generically under the extra condition of reversibility, which reduces the dimension of the ambient matrix space to 2.

2. At this moment we comment on global aspects of the geometry, as related to the spectrum of the corresponding Schrödinger operator H_V. Unlike in the periodic case the union of resonance tongues is a dense subset of $\mathbb{R}^2 = \{a, b\}$. This is due to the fact that the module of positive half-resonances, $\mathcal{M}_+(\omega)$ is dense in $\mathbb{R}_+$ and the fact that the rotation number is continuous. So if no collapsed gaps occur, we can even say that the union of interiors of the tongues forms an open and dense subset. Since the latter set is contained in the complement of the spectrum, it is natural to ask whether this spectrum itself, for fixed b, is a Cantor set. Johnson et al. [29, 23] show that for generic pairs of $(\omega, \tilde{V}) \in \mathbb{R}^d \times C^\delta(\mathbb{T}^d)$, with $0 \leq \delta < 1$, the spectrum of H_V, see (3), with $V(t) = \tilde{V}(\omega t)$, indeed is a Cantor set. In the analytic case, Eliasson [22] using KAM theory, proves that for Diophantine frequencies and small potentials, Cantor spectrum has generic occurrence. Notably, here the Cantor set has

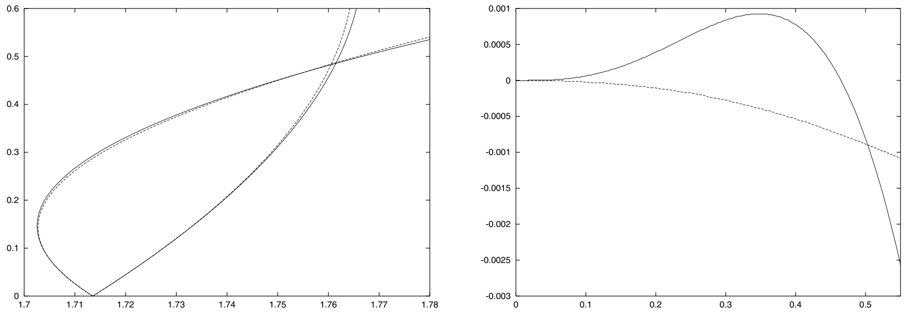

Fig. 2. *Left*: Numerical computation of the instability pocket of the near-Mathieu equation with $q_\varepsilon(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \epsilon \cos(\omega_1 + \omega_2)t$, see (2), in the (a, b)-plane with $\omega_1 = 1, \omega_2 = (1 + \sqrt{5})/2$, and $\epsilon = 0.3$. Solid lines correspond to the approximation of the boundaries by second order averaging in (a, b); Dashed lines correspond to direct numerical computation. *Right*: Difference between the averaging and the direct numerical approximation as a function of b. Solid lines correspond to the tongue boundary that for small b turns to the left, dashed lines to the boundary as it turns to the right

positive measure. In a 2-dimensional strip where $|b|$ is sufficiently small, this gives a Cantor foliation of curves in between the dense collection of resonance tongues. For general background about this antagonism between topology and measure theory in Euclidean spaces, see [39].

3. Quasi-periodic Hill's equations can be written as a Hamiltonian with one degree of freedom. In a similar way one can consider linear Hamiltonian equations with quasi-periodic coefficients with more degrees of freedom. For the regularity of the boundaries where changes of stability occur in that case see [42].

1.2.4. Outline of the paper. Let us briefly outline the rest of this paper. In Sect. 2 we present the ingredients for our proof of Theorem 1, including the notion of reducibility. Only a sketch of this proof is presented, a detailed proof is postponed to Sect. 4. In fact, most of the proofs are postponed to the latter section.

Sect. 3 contains applications of Theorem 1. For the criterion for transversality of the tongue boundaries at the tip see Sect. 3.1. A more thorough asymptotics at the tongue tip $b = 0$ and the ensuing creation of instability pockets is studied in a class of reversible near-Mathieu equations which is contained in Sect. 3.2. A proof is given in Appendix B. The zero measure set of Diophantine frequency vector ω to be excluded for this analysis, is considered in Appendix C. A concrete example with instability pockets is studied in Sect. (3.3). Finally in Appendix A a Lipschitz property of the tongue boundaries is given under very general conditions.

2. Towards a Proof of the Main Theorem 1

We consider parameter values (a_0, b_0) at a tongue boundary, i.e., at an endpoint of a spectral gap, which may possibly be collapsed. At a boundary point (a_0, b_0) the rotation number $\mathrm{rot}(a_0, b_0) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle$, i.e. it is 'rational'. This suggests a Van der Pol (covering) transformation, leading to a system of co-rotating coordinates, e.g., compare [8, 30]. In the co-rotating coordinates the tongue boundary near (a_0, b_0) gets a simpler form, that can even be further simplified by repeated time-averaging, where the time-dependence is pushed to higher order in the localized parameters (a, b).

2.1. Dynamical properties. Reducibility and rotation numbers. Recall from Sect. 1.2.1 the system form (5)

$$\begin{aligned}\theta' &= \quad \omega,\\ x' &= \quad y,\\ y' &= -(a + bQ(\theta))x\end{aligned}$$

of the quasi-periodic Hill equation (1), which is a vector field $\mathcal{X}$ on $\mathbb{T}^d \times \mathbb{R}^2 = \{\theta, (x, y)\}$. Also recall that evenness of Q leads to time-reversibility.

Since this is a linear equation with quasi-periodic coefficients, a main tool to study its dynamical behaviour is its possible reducibility to constant coefficients by a suitable transformation of variables. We always require that the transformation is quasi-periodic with the same basic frequencies as the original equation (or a rational multiple of these). The reduced matrix, which is not uniquely determined, is called the Floquet matrix. Note that for $d = 1$ reduction to Floquet form is always possible [27, 32, 41].

Remark. In the present setting generically, for Liouville-type rotation numbers (i.e., which are neither rational nor Diophantine) the normal behaviour of the invariant torus $\mathbb{T}^d \times \{(0, 0)\}$ is irreducible. In fact, there exist nearby solutions that are unbounded, where the growth is less than linear [22]. We recall that the Liouville-type rotation numbers form a residual subset (dense G_δ, second Baire category) of the positive half line. Notably, for large values of $|b|$ and not too large a irreducibility holds for a set of positive measure in the parameter plane $\mathbb{R}^2 = \{a, b\}$ [24].

In the case of Hill's equation, whenever (5) is reducible, again due to the conservative character of the system, the Floquet matrix can be chosen in $sl(2, \mathbb{R})$, i.e., with trace zero. Even if the Floquet matrix of a reducible system is not unique, the real parts of the eigenvalues are; indeed they exactly coincide with the Lyapunov exponents. It can be shown that for a smooth quasi-periodic potential, (a, b) is in the interior of a resonance tongue if and only if it is reducible to a hyperbolic Floquet matrix see [34]. In the case of quasi-periodic differential equations on $sl(2, \mathbb{R})$ (which includes Hill's equation) this latter property is equivalent to exponential dichotomy; see Sacker & Sell [43] for other equivalent definitions.

Reducibility outside of and at the boundary of resonance tongues in general does not have to hold, see the above remark. The following result by Eliasson [22] however proves reducibility for small values of b and suitable conditions on the forcing. Also compare with previous results by Dinaburg & Sinai [21] and Moser & Pöschel [34].

Theorem 2 ([22])**.** *Consider the quasi-periodic Hill equation* (1)*, or, equivalently the system form* (5)*. Assume that the following conditions hold:*

- *The frequency vector ω is Diophantine with constants $c > 0$ and $\tau \geq d - 1$, i.e., $|\langle \mathbf{k}, \omega\rangle| \geq c|\mathbf{k}|^{-\tau}$, for all $\mathbf{k} \in \mathbb{Z}^d - \{0\}$.*
- *The function $Q : \mathbb{T}^d \to \mathbb{R}$ is analytic on a strip around the d-torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d) \subset \mathbb{C}^d/(2\pi\mathbb{Z}^d)$, given by $|\mathrm{Im}\,\theta| < \sigma$.*

Then there exists a constant $C = C(\tau, \sigma) > 0$ such that if

$$|b| \sup_{|\mathrm{Im}\,\theta| < \sigma} |Q(\theta)| < C, \tag{7}$$

while the rotation number of (1) *is either rational or Diophantine, then there exists an analytic map*

$$\theta \in \mathbb{T}^d \mapsto Z(\theta) \in GL(2, \mathbb{R}),$$

and a constant matrix $B(a,b) \in sl(2,\mathbb{R})$, *such that a fundamental solution of* (5) *can be written as*

$$X(t) = Z\,(\omega t/2)\exp\left(B(a,b)t\right).$$

Moreover, if condition (7) *is satisfied then*

1. $B(a,b)$ *is nilpotent and non-zero if and only if* a *is an endpoint of a non-collapsed spectral gap;*
2. $B(a,b)$ *is zero if and only if* $\{a\}$ *is a collapsed gap.*

Remarks. 1. At first sight it may seem that this theorem implies that for small values of $|b|$ the situation is more or less similar to the periodic case $d = 1$. That this is not true follows from the above remark on irreducibility for Liouville-type rotation numbers.

2. Theorem 2 only gives analytic dependence of the reducing transformation on the angles θ, but not on the parameters (a,b). However, as we shall argue below, smoothness of the tongue boundaries can be derived from it anyway. We shall use Theorem 2 only to arrive at a suitable perturbative setting around any point (a_0,b_0) at a tongue boundary. We shall construct a formal power series for the tongue boundary, which is shown to be the actual Taylor expansion at (a_0,b_0). Here we make direct use of the definition of the derivative as a differential quotient. Limits are taken by constructing a series of shrinking wedge-like neighbourhoods of the curve, with increasing order of tangency at (a_0,b_0). The construction of the wedges uses dynamical properties of Hill's equation, e.g., concerning the variation of the rotation number outside the tongue and its constancy in the interior. Note that this will determine regions of exponential dichotomy in the interior of the tongue.

Using the fundamental solution provided by Theorem 2, we turn to co-rotating coordinates associated to parameter values at the tongue boundaries. Let $(a_0,b_0) \in \mathbb{R}^2$ be at a tongue boundary. Then Theorem 2 provides us with a fundamental matrix of the form

$$X\left((a_0,b_0)\right)(t) = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \begin{pmatrix} 1 & ct \\ 0 & 1 \end{pmatrix}, \tag{8}$$

where c is a constant and where $z_{ij}(t) = Z_{ij}(\omega t/2)$. Observe that $c = 0$ if and only if (a_0,b_0) is an extreme point of an instability pocket. Also observe that $z_{ij}(t)$, $1 \le i, j \le 2$, is quasi-periodic with frequency vector $\frac{1}{2}\omega$.

To construct the co-rotating coordinates, again consider Hill's equation (1)

$$x'' + (a + bq(t))x = 0$$

and perform the linear, t-dependent change of variables

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \phi, \tag{9}$$

where $\phi \in \mathbb{R}^2$. Observe that the change of variables in t is quasi-periodic with rotation number $\mathrm{rot}(a_0,b_0) = \frac{1}{2}\langle \mathbf{k}, \omega\rangle$. The differential equation for ϕ reads

$$\phi' = \left(\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + \delta_\mu q \begin{pmatrix} z_{11}z_{12} & z_{12}^2 \\ -z_{11}^2 & -z_{11}z_{12} \end{pmatrix}\right)\phi, \tag{10}$$

where $\mu = (a - a_0, b - b_0)$ is the new local multiparameter and where $\delta_\mu q = (a - a_0) + (b - b_0)q$.

Proposition 1. *In the above circumstances:*

1. *The functions $z_{11}z_{12}$, z_{11}^2 and z_{12}^2 are quasi-periodic in t with frequency vector ω.*
2. *In the case of even q we have*

$$
\begin{aligned}
X(-t;(a,b)) &= X^{-1}(t;(a,b)) \text{ and, thus,}\\
z_{12}(-t) &= -z_{12}(t),\\
z_{11}(-t) &= \quad z_{11}(t).
\end{aligned}
$$

3. *Let $\delta\mathrm{rot}(a,b)$ be the rotation number of* (10). *Then*

$$\delta\mathrm{rot}(a,b) = \mathrm{rot}(a,b) - \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle.$$

   Moreover, the tongue boundaries coincide with the boundaries of the set where $\delta\mathrm{rot}(a,b) = 0$.
4. *We can choose $Z \in Sl(2,\mathbb{R})$.*

We omit a proof, since this is elementary. The third item again uses [22].

2.2. Proof of Theorem 1. By repeated averaging we recursively push the time dependence of Eq. (10) to higher order in the local parameter $\mu = (a - a_0, b - b_0)$. See [16, 8, 44] for details. As before, by θ we denote the angular variables in $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$.

From Theorem 2 recall that the function $Q = Q(\theta)$ is complex analytic on a strip $|\mathrm{Im}\ \theta| < \sigma$. At each averaging step, for some $\sigma_1 < \sigma$, a linear change of variables

$$\psi = (I + R(\theta,\mu))\phi,$$

is found, which is complex analytic in θ on the strip $|\mathrm{Im}\ \theta| < \sigma_1$ and in the local parameters μ in a neighbourhood of 0. Furthermore, if the system is reversible, then the change of variables preserves this reversibility. To be more precise we have

Proposition 2. *In the above situation, after r steps of averaging system* (10) *takes the form*

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & c + S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -S_3^{(r)}(\mu) \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi, \tag{11}$$

where

$$S_i^{(r)}(\mu) = \sum_{1 \le s \le r} D_{s,i}^{(r)}(\mu)$$

for $i = 1, 2, 3$ and where the functions $D_{s,i}^{(r)}$, for $i = 1, 2, 3$ and $1 \le s \le r$, have the following properties:

1. *$D_{s,i}^{(r)}(\mu)$ are homogeneous polynomials of degree s in μ;*
2. *$D_{s,i}^{(t)} = D_{s,i}^{(s)}$ for $s \le t \le r$;*
3. *$D_{1,i}^{(1)} = (a - a_0)[z_{1i}^2] + (b - b_0)[q z_{1i}^2]$, where $[\,\cdot\,]$ denotes the time average, for $i = 1, 2$, and $D_{1,3}^{(1)} = (a - a_0)[z_{11}z_{12}] + (b - b_0)[q z_{11}z_{12}]$;*

4. *The remainder $M^{(r)}(\theta, \mu)$ is complex analytic in both θ and μ, (when $|\mathrm{Im}\ \theta|$ and $|\mu|$ are sufficiently small) while it is of order $r+1$, that is, the function*

$$(\theta, \mu) \mapsto \left| M_{ij}^{(r)}(\theta, \mu) \right| \big/ |\mu|^{r+1},$$

$1 \le i, j \le 2$, *is bounded on a neighbourhood of* $\mu = 0$.

In the case of even q we have $S_3^{(r)} \equiv 0$.

In the application of this result, the key idea is that the equation

$$S_1^{(r)}(\mu) \left(c + S_2^{(r)}(\mu) \right) - S_3^{(r)}(\mu)^2 = 0, \tag{12}$$

which is the determinant of the averaged part of Eq. (11), determines the derivatives of the tongue boundaries up to order r. In the analysis of (12) we distinguish between the cases $c \neq 0$ (non-collapsed gap) and $c = 0$ (collapsed gap). This will be done next.

2.2.1. Non-collapsed gap ($c \neq 0$). We first treat the case $c \neq 0$ of a non-collapsed gap. We will assume that $c > 0$, which means that (a_0, b_0) is at the right boundary of a resonance tongue. The case of $c < 0$ can be treated similarly. Let

$$G^{(r)}(\mu) \equiv S_1^{(r)}(\mu) \left(c + S_2^{(r)}(\mu) \right) - S_3^{(r)}(\mu)^2.$$

We solve the equation $G^{(r)}(\mu) = 0$ by the Implicit Function Theorem, which provides a polynomial

$$a^{(r)}(b) = a_0 + \sum_{1 \le k \le r} \nu_k (b - b_0)^k.$$

The coefficients ν_k, $1 \le k \le r$, are uniquely determined by the functions $D_{s,i}^{(r)}$, $1 \le s \le r$, $i = 1, 2, 3$ and $G^{(r)}((a^{(r)}(b) - a_0, b - b_0)) = O_{r+1}(b - b_0)$. Here and in what follows, $g(\xi) = O_m(\xi)$, means that

$$\left| \frac{g(\xi)}{|\xi|^m} \right|$$

is bounded around $\xi = 0$.

In order to apply the Implicit Function Theorem, we compute

$$\frac{\partial}{\partial a} G^{(r)}(\mu)_{|\mu=0} = c[z_{11}^2] > 0.$$

This yields a unique polynomial $a^{(r)} = a^{(r)}(b)$ with the properties stated above.

Our next purpose is to show that, if $b \mapsto a(b)$ is a tongue boundary with $a(b_0) = a_0$, then

$$\lim_{b \to b_0} \frac{|a(b) - a^{(r)}(b)|}{|b - b_0|^r} = 0.$$

More precisely we have

Proposition 3. *Consider Eq.* (11) *with* $c > 0$*. There exist positive constants* N *and* Δ*, such that if* a_{N_+} *and* a_{N_-} *are defined by*

$$a_{N_\pm}(b) = a^{(r)}(b) \pm N|b - b_0|^{r+1},$$

the following holds. For $0 < |b - b_0| < \Delta$*,*

1. *the rotation number* $\mathrm{rot}(a_{N_+}(b), b)$ *is different from* $\mathrm{rot}(a_0, b_0)$,
2. *the system* (11) *(or equivalently* (5)*) for* $\mu = (a_{N_-}(b) - a_0, b - b_0)$ *has zero rotation number.*

The proof is postponed to Sect. 4. As a direct consequence we have

Corollary 1. *Let* (a_0, b_0) *be at the tongue boundary as above and assume that* $\{a_0\}$ *is not a collapsed gap. Then, there exists a function* $b \mapsto a(b)$ *defined in a small neighbourhood of* $b = b_0$*, such that in this neighbourhood,*

1. $(a(b), b)$ *is at the tongue boundary of the same tongue as* (a_0, b_0),
2. *The map* $b \mapsto a(b)$ *at* b_0 *is* r*-times differentiable at* b_0 *and can be written as*

$$a(b) = a_0 + \sum_{1 \le k \le r} \nu_k (b - b_0)^k + O_{r+1}(b - b_0).$$

Proof. From now on, assume that $0 < |b - b_0| < \Delta$. Then, by Proposition 3, the set

$$\left\{(a_{N_-}(b), b) : 0 < |b - b_0| < \Delta\right\}$$

is a subset of the tongue's interior. Again by Proposition 3, for each $0 < |b - b_0| < \Delta$, the set $\left\{(a_{N_+}(b), b) : 0 < |b - b_0| < \Delta\right\}$ is a subset of the complement of the tongue. Now, for each fixed b, the map $a \mapsto \delta\mathrm{rot}(a, b)$ is monotonous, while, moreover, $\mathrm{rot}(a, b)$ is continuous in a and b. Therefore, for each $0 < |b - b_0| < \Delta$ there exists a unique $a(b)$ such that $(a(b), b)$ is at the tongue boundary.

Putting $a(b_0) = a_0$, the map $b \mapsto a(b)$ is continuously extended to $b = b_0$. The above argument also implies that for $0 < |b - b_0| < \Delta$,

$$a_{N_-}(b) \le a(b) \le a_{N_+}(b), \tag{13}$$

and as $a_{N_-}(b_0) = a(b_0) = a_{N_+}(b_0) = a_0$, this inequality directly extends to $b = b_0$. Thus, due to the form of both a_{N_+} and a_{N_-}, we have that for $|b - b_0| < \Delta$,

$$\left|a(b) - a^{(r)}(b)\right| \le N|b - b_0|^{r+1},$$

from which the corollary follows. □

Remark. The case where the Floquet matrix has non-zero element below the diagonal runs similarly.

2.2.2. Collapsed gap ($c = 0$). In the case $c = 0$ of a collapsed gap, system (11) reads

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -S_3^{(r)}(\mu) \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi. \tag{14}$$

Thus, the analogue of (12) now is

$$G^{(r)}(\mu) = S_1^{(r)}(\mu) S_2^{(r)}(\mu) - S_3^{(r)}(\mu)^2 = 0.$$

We will see in Sect. 4 that there exist two polynomials of order r, $a_1^{(r)}(b)$ and $a_2^{(r)}(b)$ such that

$$G^{(r)}\left(a_i^{(r)}(b) - a_0, b - b_0\right) = O_{r+1}(b - b_0)$$

and, using the same tools as in the case of a non-collapsed gap the following result is true, whose proof is postponed to Sect. 4,

Proposition 4. *Under the above assumptions, there exist positive constants N and Δ, such that if $|b - b_0| \le \Delta$, then*

$$|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| \le N|b - b_0|^{r+1} \text{ and}$$
$$|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| \le N|b - b_0|^{r+1}.$$

As a direct consequence we now have

$$\lim_{b \to b_0} \frac{|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0 \tag{15}$$

and

$$\lim_{b \to b_0} \frac{|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0 \tag{16}$$

and we can choose a_+ and a_- in such a way (skipping the restriction $a_- \le a_+$) that both maps are continuous and r times differentiable at b_0. Moreover their Taylor expansions up to order r at b_0 are given by $a_1^{(r)}$ and $a_2^{(r)}$. Compare Corollary 1 of Proposition 3.

2.2.3. Conclusion of Proof of Theorem 1. Summarizing we conclude that Theorem 1 follows from the previous subsections, since we have shown that the tongue boundaries are infinitely smooth. Indeed, by Eliasson's Theorem 2 a positive constant C exists only depending on ω and Q (see (5)), such that for any $r \in \mathbb{N}$ the following holds. For any $|b_0| < C$, polynomials $a_1^{(r)}$ and $a_2^{(r)}$ exist of degree r in $(b - b_0)$, such that (15) and (16) hold.

This indeed proves Theorem 1. Moreover these subsections provide a method to compute the Taylor expansions of the tongue boundaries, provided that a certain number of harmonics of the reducing matrix Z is known, compare Theorem 2.

3. Applications and Examples

In this section the methods and results of Sect. 2 are applied to study the geometric structure of resonance tongues in Hill's equations with quasi-periodic forcing (1). In the previous section we saw that the tongue boundaries are smooth around $b = 0$. Also we found that their Taylor expansions around a certain point can be obtained by an averaging procedure for which one needs to know the reducing matrix at that point. In general this is not known unless $b = 0$, i.e., when the system has constant coefficients. In this section this fact will be used to obtain generalizations of results as these hold for Hill's equations with periodic coefficients, compare [12, 16].

3.1. A criterion for transversality at the tongue tip. The first application will be a criterion for the transversality of the tongue boundaries at the origin, i.e., at the tongue tip. In the periodic case it is known [2, 12, 16] that the two boundaries of a certain resonance tongue are transversal at $b = 0$ if, and only if, the corresponding harmonic (or Fourier coefficient) of q does not vanish. In the quasi-periodic case, the situation is the same.

Proposition 5. *In the quasi-periodic Hill equation*

$$x'' + (a + b\, q(t))\, x = 0,$$

where $q(t) = Q(\omega t)$ and $Q : \mathbb{T}^d \to \mathbb{R}$, assume that Q is real analytic and that the frequency vector $\omega \in \mathbb{R}^d$ is Diophantine. Then the tongue boundaries of the $\mathbf{k}^{\text{th}}$ resonance, $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+(\omega)$, $\alpha_0 \neq 0$, meet transversally at $b = 0$ if and only if the $\mathbf{k}^{\text{th}}$ harmonic of Q does not vanish.

Proof. Let $a_0 = \alpha_0^2$ and $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+(\omega)$. Then a fundamental solution for $a = a_0$ and $b = 0$ is given by

$$X(t) = \begin{pmatrix} \cos(\alpha_0 t) & \frac{1}{\alpha_0}\sin(\alpha_0 t) \\ -\alpha_0 \sin(\alpha_0 t) & \cos(\alpha_0 t) \end{pmatrix}. \tag{17}$$

Following the notation of the previous section, let

$$z_{11}(t) = \cos(\alpha_0 t) \qquad \text{and} \qquad z_{12}(t) = \frac{1}{\alpha_0}\sin(\alpha_0 t).$$

Then

$$\begin{aligned} z_{11}^2(t) &= \cos^2(\alpha_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha_0 t), \\ z_{12}^2(t) &= \frac{1}{a_0}\sin^2(\alpha_0 t) = \frac{1}{2a_0} - \frac{1}{2a_0}\cos(2\alpha_0 t), \\ z_{11}(t) z_{12}(t) &= \frac{1}{2\alpha_0}\sin(2\alpha_0 t). \end{aligned}$$

Denoting the tongue boundaries by $a_i = a_i(b)$, for $i = 1, 2$, their derivatives at $b = 0$ are obtained averaging once and considering the equation

$$S_1^{(1)}(a - a_0, b - b_0) S_2^{(1)}(a - a_0, b - b_0) - S_3^{(1)}(a - a_0, b - b_0)^2 = 0.$$

Using Proposition 2, it is seen by means of a computation that

$$a_1'(0) = -Q_0 + |Q_{\mathbf{k}}|,$$
$$a_2'(0) = -Q_0 - |Q_{\mathbf{k}}|,$$

from which follows that $a_1'(0) \neq a_2'(0)$ if, and only if, $Q_{\mathbf{k}} \neq 0$. This concludes our proof. □

3.2. Order of tangency at the tongue tip and creation of instability pockets. We now focus on a special class of quasi-periodic Hill equations of the reversible near-Mathieu type:

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^*, \omega\rangle t\right)\right)\right) x = 0, \tag{18}$$

compare with Sect. 1. Here ε is a small deformation parameter and $\omega = (\omega_1, \ldots, \omega_d)^T$ is a Diophantine frequency vector. Also we take $c_j \neq 0$ for all $j = 1, \ldots, d$ and fix $\mathbf{k}^* = (k_1^*, \ldots, k_d^*)^T$ a non-zero vector in $\mathbb{Z}^d$. We often abbreviate $\langle \mathbf{k}\rangle = \langle \mathbf{k}, \omega\rangle$.

Theorem 3. *Consider the reversible near-Mathieu equation with quasi-periodic forcing* (18) *as above. Then*

(i) If $\varepsilon = 0$, the order of tangency at $b = 0$ of the $\mathbf{k}^{*\text{th}}$ resonance tongue is greater or equal than $|\mathbf{k}^*|$ and it is exactly $|\mathbf{k}^*|$ if, and only if, ω does not belong to $\mathcal{A}(\mathbf{k}^*)$, where $\mathcal{A}(\mathbf{k}^*)$ is a subset of the Diophantine frequency vectors of measure zero.
(ii) If $\varepsilon \neq 0$, $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is Diophantine and $|\varepsilon|$ is small enough, there exists at least one pocket at the $\mathbf{k}^{*\text{th}}$ resonance tongue with ends $b = 0$ and $b = b(\varepsilon) \neq 0$. Here ε needs to have a suitable sign.

Remarks. 1. Note that the above result only applies to quasi-periodic near-Mathieu equations of the type (18). For more general quasi-periodic forcings, the problem of the order of tangency of the tongue boundaries at $b = 0$ is at least as complicated as in the periodic case, see [16].
2. The sets $\mathcal{A}(\mathbf{k}^*)$ are not empty in general. For examples and some properties of these sets, see Appendix C.
3. Instead of fixing ε one can also fix $|b_0|$ sufficiently small and show that, for a suitable value of $\varepsilon = \varepsilon(b_0)$, one can create an instability pocket in Eq. (18) with ends at $b = 0$ and $b = b_0$. A suitable choice of the components of c also allows several pockets (associated to different $\mathbf{k}^*$) with ends at $b = 0$ and $b = b_0$ and for the same value of ε. In general the same complexity holds here as in the periodic case, compare with the general discussion in Sect. 1.1.2.

A proof of Theorem 3 is given in Appendix B. One consequence of the theorem is

Corollary 2. *Assume that in Hill's equation*

$$x'' + (a + bq(t))\, x = 0,$$

the forcing q is an even function, real analytic, quasi-periodic and with Diophantine frequency vector ω. Suppose that, for some $\mathbf{k}^* \neq \mathbf{0}$, the $\mathbf{k}^{*\text{th}}$ harmonic of q does not vanish and that $\omega \notin \mathcal{A}(\mathbf{k}^*)$. Then, the following equation

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + q(t)\right)\right) x = 0 \tag{19}$$

has a pocket at the $\mathbf{k}^{*\text{th}}$ resonance tongue provided that the $|c_j|$ are sufficiently large.

Proof. Let $\varepsilon > 0$ be a small parameter and define $\tilde{c}_j = c_j\varepsilon$, for $j = 1, \ldots, d$. Writing $\tilde{b} = b/\varepsilon$, Eq. (19) reads

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \varepsilon q(t)\right)\right) x = 0.$$

Since the $\mathbf{k}^{*\text{th}}$ harmonic $q_{\mathbf{k}^*}$ of q does not vanish, this even function can be split into

$$q(t) = q_{\mathbf{k}^*} \cos(\langle \mathbf{k}^*, \omega\rangle t) + \tilde{q}(t),$$

where $\tilde{q}$ is an even function whose $\mathbf{k}^{*\text{th}}$ harmonic vanishes. Let $\tilde{\varepsilon} = \varepsilon q_{\mathbf{k}^*}$. In these new parameters Eq. (19) gets the form

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \tilde{\varepsilon}\cos(\langle \mathbf{k}^*, \omega\rangle t) + \frac{\tilde{\varepsilon}}{q_{\mathbf{k}^*}}\tilde{q}(t)\right)\right) x = 0.$$

The only difference of the latter equation with (18) is the term $\tilde{q}$. But since its $\mathbf{k}^{*\text{th}}$ harmonic vanishes, the conclusions of Theorem 3 concerning the existence of pockets hold here, provided $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is Diophantine, $\tilde{c}_j$ do not vanish and ε is sufficiently small. The latter condition is equivalent to the c_j being sufficiently large. □

3.3. A reversible near-Mathieu example with an instability pocket. In this section the following concrete example of a reversible near-Mathieu equation with quasi-periodic forcing is investigated:

$$x'' + (a + b(\cos t + \cos\gamma t + \epsilon\cos(1+\gamma)t))\, x = 0. \tag{20}$$

Here γ is a Diophantine number and ε a deformation parameter. We consider the resonance $\alpha_0 = \frac{1}{2}(1+\gamma)$, which means that (a, b, ϵ) will be near

$$\left(\left(\tfrac{1}{2}(1+\gamma)\right)^2, 0, 0\right).$$

Since γ is strongly incommensurable with 1 and the forcing is entire analytic, there exists a constant $C = C(\gamma)$ such that if $|b| < C$ and $|\epsilon| < 1$, then there is reducibility at the tongue boundaries [22]. Compare Sect. 2. After a twofold averaging and other suitable linear transformations which do not affect the resonance domains, the system is transformed into

$$\phi' = \left(\begin{pmatrix} 0 & X(\nu) + Y(\nu) \\ -X(\nu) + Y(\nu) & 0 \end{pmatrix} + M^{(2)}(\nu)\right)\phi,$$

where $\nu = (a - a_0, b, \epsilon)$ and where X and Y are given by the following expansions in ν:

$$X(\nu) = -\frac{1}{1+\gamma}\left(\frac{a-a_0}{1+\gamma} - \frac{(a-a_0)^2}{(1+\gamma)^3} - \frac{b^2}{4(1+\gamma)^2}C(\epsilon)\right)$$

and

$$Y(\nu) = -\frac{b}{1+\gamma}\left(\frac{-\epsilon}{2(1+\gamma)} + \frac{\epsilon(a-a_0)}{(1+\gamma)^3} + \frac{b}{2(1+\gamma)^2}\left(1+\frac{1}{\gamma}\right)\right).$$

Here

$$C(\epsilon) = \frac{\epsilon^2}{2(1+\gamma)} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} + 1 + \frac{1}{\gamma}.$$

Hence, in the notation of Sect. 2 we have $S_1^{(2)}(\nu) = X(\nu) - Y(\nu)$ and $S_2^{(2)}(\nu) = X(\nu) + Y(\nu)$. Thus $S_1^{(2)}(\nu) = 0$ (resp. $S_2^{(2)}(\nu) = 0$) if and only if $X(\nu) = Y(\nu)$ (resp. $X(\nu) = -Y(\nu)$). The Taylor expansions of the tongue boundaries up to second order in (b, ϵ) are given by

$$a_1^{(2)}(b,\epsilon) = \left(\frac{1+\gamma}{2}\right)^2 - \frac{b\epsilon}{2} + \frac{b^2}{4(1+\gamma)}\left(3 + \frac{3}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma}\right)$$

and

$$a_2^{(2)}(b,\epsilon) = \left(\frac{1+\gamma}{2}\right)^2 + \frac{b\epsilon}{2} + \frac{b^2}{4(1+\gamma)}\left(-1 - \frac{1}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma}\right).$$

Therefore the second order Taylor expansions of the tongue boundaries have a transversal crossing both at $(b, \epsilon) = (0, 0)$ and at the point $(b, \epsilon) = (\gamma\epsilon, \epsilon)$ if $\epsilon \neq 0$. By Theorem 1 we know that the boundary functions $a_1^{(2)}$ and $a_2^{(2)}$ are of class C^∞ in b. With little more effort, one also establishes this same degree of smoothness in the parameter ϵ. Following the argument of the previous subsection, one has

Corollary 3. *For the reversible near-Mathieu equation* (20) *there exists a positive constant C such that, if $|b| < C$ and $|\epsilon| < 1$, then the tongue boundaries of the resonance corresponding to $\alpha_0 = (1+\gamma)/2$ are C^∞ functions of (b, ϵ), while*

1. *for $\epsilon \neq 0$ the tongue boundaries have two transversal crossings, one at $(a, b) = (a_0, 0)$ and the other at $(a_1^{(2)}(\gamma\epsilon, \epsilon) + O_3(\epsilon), \gamma\epsilon + O_2(\epsilon))$,*
2. *for $\epsilon = 0$ the tongue boundaries at $b = 0$ have a second order tangency.*

Remark. Note that Corollary 3 exactly describes the $\mathbb{A}_3$-scenario, compare [4]. For the periodic analogue see [12], where Hill's map has a Whitney cusp singularity. Compare with Sect. 1.1.2.

4. Proofs

The main aim of this section is to prove Propositions 3 and 4 of Sect. 2. We recall the setting of Sect. 2. Around a point $(a_0, b_0) \in \mathbb{R}^2$ with $|b_0|$ sufficiently small, at the boundary of a resonance zone by Theorem 2 and Proposition 1 a fundamental matrix exists of the form (8) where the Z matrix is symplectic. Let $\mu = (a - a_0, b - b_0) =: (\alpha, \beta)$ be the new local parameters and hence $\delta_\mu q = \alpha + \beta q$. The change of variables (9) reduces the equation for $\phi = (\phi_1, \phi_2)$ to the form (10).

The corresponding Hamiltonian, written in autonomous form by introducing new momenta $J \in \mathbb{R}^d$, reads

$$K(\phi_1, \phi_2, \theta, J) = \langle J, \omega \rangle + \frac{1}{2} c \phi_2^2 + \delta_\mu Q \left(\frac{1}{2} z_{11}^2 \phi_1^2 + z_{11} z_{12} \phi_1 \phi_2 + \frac{1}{2} z_{12}^2 \phi_2^2 \right).$$

The first two terms of the right-hand side form the unperturbed Hamiltonian K_0, the last one is K_1.

The rotation number of this Hamiltonian is $\delta\mathrm{rot}(a, b) = \mathrm{rot}(a, b) - \frac{1}{2}\langle \mathbf{k}, \omega \rangle$. The tongue boundaries are the boundaries of the set $\delta\mathrm{rot}(a, b) = 0$. After r steps of averaging, the system takes the form of (11),

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & c + S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -S_3^{(r)}(\mu) \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi,$$

see Proposition 2. In what follows, the expression of the previous equation in polar coordinates will be used. Writing $\varphi = \arg(\phi_2 + i\phi_1)$, the differential equation for φ becomes

$$\varphi' = (S_1^{(r)} + M_1^{(r)}) \sin^2 \varphi + 2(S_3^{(r)} + M_3^{(r)}) \sin \varphi \cos \varphi + (c + S_2^{(r)} + M_2^{(r)}) \cos^2 \varphi, \tag{21}$$

which is a quadratic form with matrix

$$\begin{pmatrix} S_1^{(r)} & S_3^{(r)} \\ S_3^{(r)} & c + S_2^{(r)} \end{pmatrix} + \begin{pmatrix} M_1^{(r)} & M_3^{(r)} \\ M_3^{(r)} & M_2^{(r)} \end{pmatrix}. \tag{22}$$

We recall that $M_j^{(r)} = O_{r+1}(|\mu|)$ uniformly in θ in a complex neighbourhood of $\mathbb{T}^d$. It is now important to distinguish between the cases of a non-collapsed gap ($c \neq 0$) and of a collapsed gap ($c = 0$).

4.1. Non-collapsed gap. Suppose we are in the case of a non-collapsed gap, i.e., with $c \neq 0$. The present aim is to prove Proposition 3 which deals with the case $c > 0$. The case $c < 0$ is treated similarly. Recall that in Sect. 2.2.1 for any $r \geq 1$ we obtained a polynomial of order r in $b - b_0$, $a^{(r)}(b)$ such that, if

$$G^{(r)}(\mu) = S_1^{(r)}(\mu) \left(c + S_2^{(r)}(\mu) \right) - S_3^{(r)}(\mu)^2,$$

then

$$G^{(r)} \left(a^{(r)}(b) - a_0, b - b_0 \right) = O_{r+1}(b - b_0).$$

In order to prove Proposition 3 we will show that there exist constants $N > 0$, sufficiently large, and $\Delta > 0$, sufficiently small, such that if $0 < |b - b_0| < \Delta$,

1. Equation (11) for $(a, b) = (a^{(r)}(b) + N|b - b_0|^{r+1}, b)$ has rotation number strictly different from zero.
2. Equation (11) for $(a, b) = (a^{(r)}(b) - N|b - b_0|^{r+1}, b)$ has zero rotation number.

In what follows we write again $(\alpha, \beta) = (a - a_0, b - b_0)$ and $\alpha^{(r)}(\beta) = a^{(r)}(b)$. Let, for some $N > 0$,

$$R_j^{\pm}(\beta) = S_j^{(r)}(\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta), \qquad j = 1, 3,$$

$$R_2^{\pm}(\beta) = c + S_2^{(r)}(\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta)$$

and $M^{\pm}(\theta, \beta) = M^{(r)}\left(\theta, (\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta)\right)$. With these definitions, matrix (22) becomes

$$\begin{pmatrix} R_1^{\pm} & R_3^{\pm} \\ R_3^{\pm} & c + R_2^{\pm} \end{pmatrix} + \begin{pmatrix} M_1^{\pm} & M_3^{\pm} \\ M_3^{\pm} & M_2^{\pm} \end{pmatrix}. \tag{23}$$

Let $R^{\pm}$ be the first term of the previous expression. First of all, note that, since

$$\frac{\partial G^{(r)}}{\partial \alpha}|_{\mu=0} = c[z_{11}^2],$$

then

$$\det R^{\pm}(\beta) = R_1^{\pm}(\beta)\left(c + R_2^{\pm}(\beta)\right) - \left(R_3^{\pm}(\beta)\right)^2 = \pm\left(cN[z_{11}^2] + A\right)|\beta|^{r+1} + O_{r+2}(\beta),$$

being the time-dependent term A uniformly bounded for all $\theta \in \mathbb{T}^d$. This means that N and β_0 can be chosen so that

$$|\det R^{\pm}(\beta)| \geq \frac{cN}{2}[z_{11}^2]|\beta|^{r+1},$$

provided $|\beta| < \beta_0$, and the sign of $\det R^{\pm}$ is $\pm$. The elements of the time-depending part, the $M_j^{\pm}(\theta, \beta)$, can be uniformly bounded by $\frac{N}{4}[z_{11}^2]$ if N and β_0 are suitably modified. The modulus of the eigenvalues of $R^{\pm}$ can be bounded from below by $\frac{N}{3}|\beta|^{r+1}$ and $2c/3$. Now we distinguish between the cases of R^+ and R^-.

In the case of R^+, the symmetric matrix (23) is definite positive and for all $\theta \in \mathbb{T}^d$, φ' in (21) is bounded from below by $\frac{N}{12}|\beta|^{r+1}$, since the minimum of φ', ignoring the contribution of the time-dependent part, is $\frac{N}{3}|\beta|^{r+1}$. This implies that the rotation number is different from zero, if $0 < |\beta| < \beta_0$.

In the case of R^-, the time independent part of (23) has a positive eigenvalue bounded from below by $2c/3$ and a negative one bounded from above by $-\frac{N}{12}|\beta|^{r+1}$.

In particular, if $0 < |\beta|$ is small enough, there exist φ_1 and φ_2, independent of θ, such that the right-hand side of (21) is positive and negative, respectively, uniformly for all $\theta \in \mathbb{T}^d$. In particular, the rotation number must be zero. □

Remark. The Normal Form for $c \neq 0$ can be obtained without terms in $\phi_1\phi_2$ and without changing the term in ϕ_2^2. Indeed, at each step of normalization the homological equation is of the form $[G, H_0] = M$, where M contains known terms of the form $\phi_1^{j_1}\phi_2^{j_2}\exp(\mathrm{i}\langle \mathbf{k}, \theta\rangle)$ with $j_1 + j_2 = 2$. Let us see the system to solve for a fixed $\mathbf{k}$. Let $T_1\phi_1^2 + T_3\phi_1\phi_2 + T_2\phi_2^2$ be the terms having $\exp(\mathrm{i}\langle \mathbf{k}, \theta\rangle)$ as a factor in the expression of

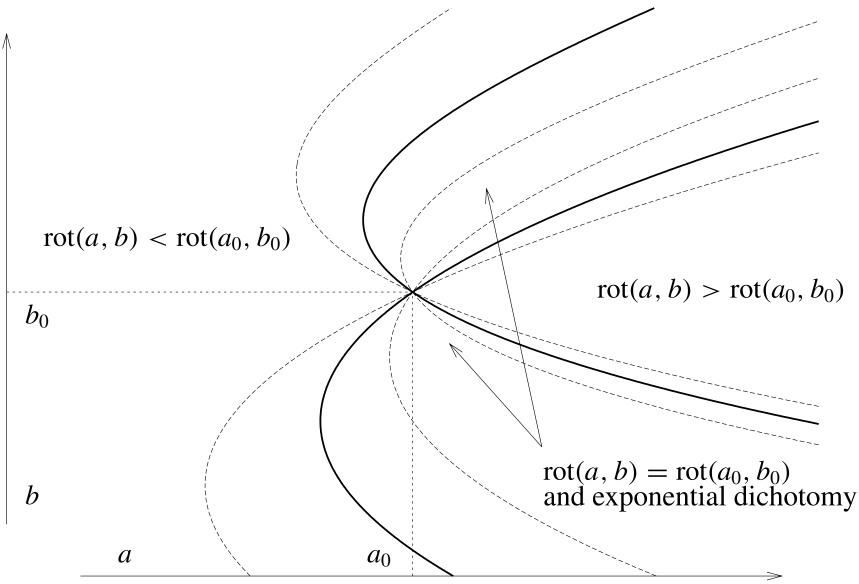

Fig. 3. Areas of exponential dichotomy inside a resonance tongue as guaranteed by Lemma 2 and areas with rotation number different from $\mathrm{rot}(a_0, b_0)$ outside the tongue as guaranteed by Lemma 2. Solid lines denote tongue boundaries

M and $A_1\phi_1^2 + A_3\phi_1\phi_2 + A_2\phi_2^2$ be the corresponding terms to be found in G. In matrix form we have

$$\begin{pmatrix} \mathrm{i}\langle \mathbf{k}, \omega\rangle & 0 & 0 \\ 2c & \mathrm{i}\langle \mathbf{k}, \omega\rangle & 0 \\ 0 & c & \mathrm{i}\langle \mathbf{k}, \omega\rangle \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}.$$

If $\mathbf{k} \neq 0$ the matrix is invertible. If $\mathbf{k} = 0$ one can not cancel T_1, which must be kept in the Normal Form, but the terms T_2 i T_3 can be cancelled by suitable choices of A_1, A_2. The value of A_3 is arbitrary.

4.2. Collapsed gap. Present aim is to prove Proposition 4, i.e., assuming that $c = 0$. Here we follow ideas similar to the above case $c \neq 0$. We shall see that the tongue boundaries can be divided over sectors, determined by whether the modulus of the modified rotation number is greater than some constant or whether the rotation number is zero and there is exponential dichotomy, compare with Fig. 3. From this we obtain the tangency of the required order.

Recall that the first step of averaging gives

$$\langle J, \omega\rangle + \left(\frac{1}{2}S_1^{(1)}\phi_1^2 + S_3^{(1)}\phi_1\phi_2 + \frac{1}{2}S_2^{(1)}\phi_2^2\right) + O_2(\mu, \phi, \theta),$$

where O_2 denotes terms which are $O(|\mu|^2)$ (and quadratic in ϕ and depending on time through θ) and

$$S_1^{(1)} = \alpha[z_{11}^2] + \beta[Qz_{11}^2], \quad S_2^{(1)} = \alpha[z_{12}^2] + \beta[Qz_{12}^2],$$

$$S_3^{(1)} = \alpha[z_{11}z_{12}] + \beta[Qz_{11}z_{12}],$$

see Proposition 2. Hence the coefficients of α in $S_1^{(1)}$ and $S_2^{(1)}$ are positive and $[z_{11}z_{12}]^2 < [z_{11}^2] \times [z_{12}^2]$, a key fact in what follows. To order r the coefficient $S_j^{(1)}$ is replaced by $S_j^{(r)}$ for $j = 1, 2, 3$, of the form described before, and O_2 by O_{r+1}.

After r steps of normalization the matrix of the system is

$$\begin{pmatrix} S_3^{(r)} & S_2^{(r)} \\ -S_1^{(r)} & -S_3^{(r)} \end{pmatrix} + \begin{pmatrix} M_3 & M_2 \\ -M_1 & -M_3 \end{pmatrix},$$

where the M_j terms depend on θ analytically on the same domain as Q and are of order $r+1$ in α, β.

First we analyze the part coming from the Normal Form. As it is well-known, the boundaries of the resonance zone correspond to μ-values such that the determinant of the system

$$G(\alpha, \beta) := S_1^{(r)} S_2^{(r)} - S_3^{(r)2}$$

is equal to zero. As the terms of degree 1 in α in the S_j give rise to a positive definite part in the Hamiltonian, there exists a canonical change of variables (a rotation and scalings) such that $S_1^{(r)}$ and $S_2^{(r)}$ start as $n\alpha$ (for some $n > 0$) and $S_3^{(r)}$ contains no linear term in α. By scaling G we can assume $n = 1$ in the previous expressions. Hence, we are left with

$$\begin{aligned} S_1^{(r)} &= \alpha + \sigma_1(\beta) + \alpha\rho_1(\alpha, \beta), \\ S_2^{(r)} &= \alpha + \sigma_2(\beta) + \alpha\rho_2(\alpha, \beta), \\ S_3^{(r)} &= \qquad \sigma_3(\beta) + \alpha\rho_3(\alpha, \beta), \end{aligned}$$

where σ_j are polynomials in β of maximal degree r and starting, in principle, with linear terms and ρ_j are polynomials in α, β of maximal degree $r-1$. If $\sigma_j \not\equiv 0$ let k_j be the minimal degree in σ_j, for $j = 1, 2, 3$. Otherwise we set $k_j = \infty$. Using Newton's polygon arguments (see, e.g., [25]) to look to the relevant terms of the zero set of G, one can neglect the ρ_j terms.

Assume first $k = \min\{k_1, k_2, k_3\} \le r$. Introducing the change of variables $\alpha = \gamma\beta^k$ the function G can be written as

$$\beta^{2k}\left(\gamma^2 + (m_{1,k} + m_{2,k})\gamma + m_{1,k}m_{2,k} - m_{3,k}^2 + O(\beta)\right),$$

where $m_{j,k}$ denotes the coefficient of degree k in σ_j, $j = 1, 2, 3$ (some of them can be zero, but not all). Factoring out β^{2k} and neglecting the $O(\beta)$ term the zeros, γ_1 and γ_2, of the equation for γ are simple, unless $m_{1,k} - m_{2,k}$ and $m_{3,k} = 0$. Hence, the Implicit Function Theorem implies that there are two different analytic functions

$$g_j(\beta) = \beta^k(\gamma_j + O(\beta)), \quad j = 1, 2$$

in the zero set of G, which differ at order $k \le r$.

If $m_{1,k} - m_{2,k}$ and $m_{3,k} = 0$ let us introduce $\hat{\alpha} = \alpha + m_{1,k}\beta^k$ and rewrite G in terms of $\hat{\alpha}, \beta$. We rename $\hat{\alpha}$ again as α. Then the new equation for α, β is as before where k is at least replaced by $k+1$ and where the maximal degree of the σ_j and ρ_j polynomials also can increase. If the equation for the new γ has two different roots one obtains two curves $g_j(\beta)$ in the zero set of G, as before. Otherwise the procedure is iterated and ends when two different curves are obtained or when a value $k > r$ is reached.

If $k = \infty$ the procedure is stopped immediately. In this case, or when we reach $k > r$ in the iterative process, after a change of variables $\hat{\alpha} = \alpha - P(\beta)$, where P is a polynomial of degree r, the problem is equivalent to the initial one. Here the $S_j^{(r)}$ polynomials are replaced by S_j^*, where

$$\begin{aligned} S_1^* &= \hat{\alpha} + \sigma_1^*(\beta) + \hat{\alpha}\rho_1^*(\hat{\alpha}, \beta), \\ S_2^* &= \hat{\alpha} + \sigma_2^*(\beta) + \hat{\alpha}\rho_2^*(\hat{\alpha}, \beta), \\ S_3^* &= \qquad \sigma_3^*(\beta) + \hat{\alpha}\rho_3^*(\hat{\alpha}, \beta), \end{aligned}$$

and where the minimal degree of the σ_j^* is at least $k+1$. Hence, after a finite number of steps we obtain

Lemma 1. *Consider the Normal Form after r steps of normalization in the case* $c = 0$. *Let* $G(\alpha, \beta) = 0$ *be the defining equation of a boundary of the resonance zone. Then there exists* $\beta_0 > 0$ *such that, for* $|\beta| < \beta_0$, *one of the following statements holds:*

a) The zero set of G consists of two analytic curves $\alpha = g_j(\beta)$, $j = 1, 2$, *with* $g_2(\beta) - g_1(\beta) = d\beta^k(1 + O(\beta)), k \le r,\ d > 0$. *Furthermore*

$$G(\alpha, \beta) = (\alpha - g_1(\beta))(\alpha - g_2(\beta))F(\alpha, \beta),$$

where F is an analytic function with $F(0, 0) > 0$.

b) There exists a curve $\alpha = P(\beta)$, *with P a polynomial of degree r, and a constant* $L > 0$ *such that the zero set of G is contained in the domain bounded by* $P(\beta) \pm L|\beta|^{r+1}$.

Proof. To complete the proof of the first item it is only necessary to remark that, from the previous discussion, only two branches of $G = 0$ can emerge from $(0, 0)$. Hence

$$\frac{G(\alpha, \beta)}{(\alpha - g_1(\beta))(\alpha - g_2(\beta))}$$

is an invertible function. The fact that $F(0, 0) > 0$ follows from the positive definite character of the linear terms in α.

Concerning the second item, using the variable $\hat{\alpha} = \alpha - P(\beta)$ one can work with the S_j^* functions. Let us denote as $G^*(\hat{\alpha}, \beta)$ the expression $S_1^* S_2^* - S_3^{*2}$, that is, the value of G in the new variables. Replacing $\hat{\alpha}$ by $\pm L|\beta|^{r+1}$ the function G^* becomes positive if L is large enough. It remains to show that the zero set is not empty, but this will be an immediate consequence of Lemma 2. □

Next we consider the variations of the rotation number in different domains of the parameter plane. That is, we want to estimate $\delta\mathrm{rot}(a, b)$ which in the current parameters will be denoted simply by $\mathrm{rot}(\alpha, \beta)$. The differential equation for $\varphi = \arg(\phi_2 + \mathrm{i}\phi_1)$, i.e., Eq. (21) for $c = 0$, reads

$$\varphi' = (S_1^{(r)} + M_1)\sin^2\varphi + 2(S_3^{(r)} + M_3)\sin\varphi\cos\varphi + (S_2^{(r)} + M_2)\cos^2\varphi.$$

Lemma 2. *Consider the rotation number* $\rho := \mathrm{rot}(\alpha, \beta)$ *of the differential equation* (21) *in co-rotating coordinates. Then there exist constants* $N, \beta_0 > 0$ *such that, for* $0 < |\beta| < \beta_0$, *a) In case a) of Lemma 1 let*

$$g_-(\beta) = \min\{g_1(\beta), g_2(\beta)\}, \quad g_+(\beta) = \max\{g_1(\beta), g_2(\beta)\}.$$

Then one has

$$\rho < 0 \quad \textit{if} \quad \alpha < g_-(\beta) - N|\beta|^{r+1}; \qquad \rho > 0 \quad \textit{if} \quad \alpha > g_+(\beta) + N|\beta|^{r+1};$$

$$\rho = 0 \quad \textit{if} \quad g_-(\beta) + N|\beta|^{r+1} < \alpha < g_+(\beta) - N|\beta|^{r+1}.$$

b) In case b) of Lemma 1 one has

$$\rho < 0 \quad \textit{if} \quad \alpha < P(\beta) - N|\beta|^{r+1}; \qquad \rho > 0 \quad \textit{if} \quad \alpha < P(\beta) + N|\beta|^{r+1}.$$

Proof. Let us consider the quadratic form in the expression of φ', i.e., Eq. (22) for $c = 0$, in case a) obtained by skipping the M_j terms and where α is taken equal $g_\pm(\beta)$. For definiteness let $\tilde{S}_j = S_j^{(r)}(g_\pm(\beta), \beta)$. This quadratic form is degenerate. As, in general, the discriminant of the quadratic form is $-G$, the form is indefinite for α in (g_-, g_+) and definite outside $[g_-, g_+]$. If $\alpha = g_-$ the form is negative definite everywhere except at one direction. Similarly, if $\alpha = g_+$ it is positive definite everywhere except at one direction.

We want to see the effect of adding the M_j terms and the change in the value of α. From the expression of the $S_j^{(r)}(\alpha, \beta)$ one has

$$S_1^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_1(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) < \tilde{S}_1 - \frac{N}{2}|\beta|^{r+1},$$

$$S_2^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_2(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) < \tilde{S}_2 - \frac{N}{2}|\beta|^{r+1},$$

$$|S_3^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_3(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) - \tilde{S}_3| < \frac{N}{4}|\beta|^{r+1},$$

uniformly in θ, if N is large enough and $0 < |\beta| < \beta_0$ for some β_0. The current quadratic form is bounded from above by

$$\tilde{S}_1 \sin^2\varphi + 2\tilde{S}_3 \sin\varphi\cos\varphi + \tilde{S}_2\cos^2\varphi - \frac{N}{2}|\beta|^{r+1}(\sin^2\varphi + \sin\varphi\cos\varphi + \cos^2\varphi).$$

Hence $\varphi' < -\frac{N}{4}|\beta|^{r+1}$, proving the first of the assertions in a). The second assertion is proved in the same way.

To prove the third statement in a) it is better to shift α by $g_-(\beta)$. Let $\hat{\alpha} = \alpha - g_-(\beta)$. Then the S_j functions are of the form

$$\begin{aligned}
\hat{S}_1 &= \hat{\alpha} + \hat{\sigma}_1(\beta) + \hat{\alpha}\hat{\rho}_1(\hat{\alpha}, \beta),\\
\hat{S}_2 &= \hat{\alpha} + \hat{\sigma}_2(\beta) + \hat{\alpha}\hat{\rho}_2(\hat{\alpha}, \beta),\\
\hat{S}_3 &= \qquad \hat{\sigma}_3(\beta) + \hat{\alpha}\hat{\rho}_3(\hat{\alpha}, \beta).
\end{aligned}$$

It is clear that when $\hat{\alpha} = 0$ we have $G = 0$ by construction, and the other root is $g_+(\beta) - g_-(\beta) = d|\beta|^k(1 + O(\beta))$, $d > 0$. Therefore, $\hat{\sigma}_1(\beta) + \hat{\sigma}_2(\beta) = -d|\beta|^k(1 + O(\beta))$

and $\hat{\sigma}_1(\beta)\hat{\sigma}_2(\beta) = (\hat{\sigma}_3(\beta))^2$. Furthermore the $\hat{\sigma}_j$ functions have k as minimal degree for $j = 1, 2, 3$. For definiteness let $\hat{\sigma}_j(\beta) = h_j|\beta|^k\,(1 + O(\beta))$, with $h_j \neq 0$.

We set now $\hat{\alpha} = N|\beta|^{r+1}$ and add the M_j terms to the $\hat{S}_j$ functions. The new determinant is of the form

$$\left((N+A)|\beta|^{r+1} + \hat{\sigma}_1\right)\left((N+B)|\beta|^{r+1} + \hat{\sigma}_2\right) - \left(C|\beta|^{r+1} + \hat{\sigma}_3\right)^2,$$

where $|A|, |B|, |C|$ are uniformly bounded for all θ by quantities which are $O_0(\beta)$. Therefore the determinant is uniformly bounded from above by $-dN|\beta|^{k+r+1}/2$ if N is large enough. This shows that the quadratic form is indefinite for all θ.

Furthermore, when $\hat{\alpha} = N|\beta|^{r+1}$ the $\hat{S}_j$ functions are $O(|\beta|^k)$. This, combined with the bound on the discriminant and the different terms contributing to the $\hat{S}_j$ shows that the slopes of the directions in the (ϕ_1, ϕ_2)-plane for which $\varphi' = 0$ are of the form

$$c_1(\beta) \pm |\beta|^{\frac{r+1-k}{2}}(c_2 + c_3(\beta, \theta)),$$

where c_1 and c_3 are analytic functions of their arguments and

$$c_1(0) = -\frac{h_3}{h_1} \neq 0,\ c_2 = \frac{\sqrt{dN}}{|h_1|}$$

and $|c_3(\beta, \theta)| < c_2/2$, uniformly in θ. The time dependence appears only in the c_3 term. One of the directions is attracting for the dynamics of φ in $\mathbb{S}^1$ and the other is repelling. We recall that these directions depend on t. However the slopes of both directions are bounded away from $c_1(\beta)$ uniformly in θ and therefore in t. Let $\varphi_r^*(t)$ be the argument of a repelling direction. Any value of the form $\varphi_r^*(t) + m\pi$ is also repelling. Consider two consecutive repelling curves. For any fixed β with $|\beta| < \beta_0$ small enough, they are contained in a strip of the form $(\arg(c_1(\beta) - 2c_2), \arg(c_1(\beta) + 2c_2 + \pi)$. Any initial condition (ϕ_1, ϕ_2) between these repelling curves remains in the strip for all t. This shows that $\rho = 0$, as desired.

To prove the assertion for $\alpha = g_+(\beta) - N|\beta|^{r+1}$ one proceeds in a symmetric way. Then it follows for the full interval as in the statement, by monotonicity of ρ with respect to a.

Finally, we proceed to case b). By introducing $\hat{\alpha} = \alpha - P(\beta)$ one obtains S functions like the S_j^* defined above, with $\sigma_j^*(\beta) = O_{r+1}(\beta)$. Then

$$S_1^*(-N|\beta|^{r+1}, \beta) + M_1(\alpha, \beta, \theta) < -\frac{N}{2}|\beta|^{r+1},$$

$$S_2^*(-N|\beta|^{r+1}, \beta) + M_2(\alpha, \beta, \theta) < -\frac{N}{2}|\beta|^{r+1},$$

$$|S_3^*(-N|\beta|^{r+1}, \beta) + M_3(\alpha, \beta, \theta)| < \frac{N}{4}|\beta|^{r+1},$$

uniformly in θ, if N is large enough and $|\beta| < \beta_0$ for some β_0. The current quadratic form is bounded from above as in the a) case by $-\frac{N}{4}|\beta|^{r+1}$. This proves the first assertion in b) and the second one is proved in a similar way. Furthermore, as was announced in Lemma 1, the zero set of G is contained between these two curves because the rotation number passes from < 0 to the left to > 0 to the right.

This finishes the proof of Lemma 2 and the last part of Lemma 1, case b). □

Proposition 4 is now immediate. Indeed, let $a_1^{(r)}$ and $a_2^{(r)}$ be the Taylor expansions up to order r in $b - b_0$ of $a_0 + g_1(b - b_0)$ and $a_0 + g_2(b - b_0)$ respectively. Then, letting $\Delta = \beta_0$, there is a constant N, given by the previous lemma, such that if $a_+(b)$ and $a_-(b)$ denote the right and left boundary of the tongue, then

$$|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| \le N|b - b_0|^{r+1} \text{ and}$$
$$|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| \le N|b - b_0|^{r+1},$$

for $|b - b_0| < \Delta$, as we wanted to show.

4.3. Differentiability of rotation number and Lyapunov exponent for a fixed potential. In this section we fix the parameter b_0 in a sufficiently small neighbourhood of the origin, to ensure reducibility according to [22], see Sect. 2. In this case we study rotation number $\rho = \rho(a)$ and the (maximal) Lyapunov exponent $\lambda = \lambda(a)$ of the quasi-periodic Hill equation (1), or equivalently (5), in dependence of the parameter a. The results in this setting are completely analogous to the periodic case, and proofs can be obtained from those of the previous section.

Corollary 4. *In the above situation, let a_0 be an endpoint of a spectral gap. Then*

1. *If a_0 is in the left (resp. right) endpoint of a non-collapsed spectral gap, then the functions*

$$\alpha \in (-1, 1) \mapsto \rho(a_0 - \alpha^2) \text{ and } \alpha \in (-1, 1) \mapsto \lambda(a_0 + \alpha^2)$$

(resp. $\alpha \in (-1, 1) \mapsto \rho(a_0 + \alpha^2)$ and $\alpha \in (-1, 1) \mapsto \lambda(a_0 - \alpha^2)$) are differentiable at zero.

2. *If $\{a_0\}$ is a collapsed spectral gap, then the functions*

$$a \mapsto \rho(a) \text{ and } a \mapsto \lambda(a)$$

are differentiable at a_0.

In particular, in any non-collapsed spectral gap $[a_-, a_+]$ the function $a \mapsto w(a) := -\lambda(a)^2$ is analytic in (a_-, a_+) and has lateral derivatives at $a = a_-, a_+$.

The same result was obtained in [35, 36] in more general contexts (e.g., for the Schrödinger equation with almost periodic or ergodic potential). Our method of proof is very similar to [35].

Remark. With a little more effort, one can recover the fact that for fixed, small potential the function $a \mapsto w(a)$ in a gap $[a_-, a_+]$ is of class $C^\omega((a_-, a_+)) \cap C^\infty([a_-, a_+])$, see Moser and Pöschel [34].

5. Conclusions and Outlook

Summarizing, this paper studies the geometric structure of resonance tongues in a class of Hill equations with quasi-periodic forcing $bQ(\omega t)$. Several results were obtained, analogous to the periodic case, regarding smoothness of tongue boundaries and the occurrence of instability pockets. Here we used reducibility of the equations at the tongue boundaries for small $|b|$.

5.1. Larger $|b|$. According to numerical ([15]) and analytical work ([24]), it seems that for each tongue boundary there exists a critical value b_{crit}, such that for $|b| < b_{\text{crit}}$ reducibility holds, while for $|b| > b_{\text{crit}}$ not even continuous reduction (to Floquet form) seems possible.

Resonance tongues, however, can be defined by the rotation number. Since this definition is independent of reducibility, we can still speak of tongue boundaries for larger $|b|$. Therefore the problem remains, whether and to what extent the possible non-reducibility of the system affects the regularity of the tongue boundaries. In Appendix A we show that tongue boundaries are always globally Lipschitz, but is this the best possible result in general? Is it possible that the tongue boundaries be continued with some degree of regularity above the critical value?

5.2 Analyticity? It should be noted that our Main Theorem 1 on the regularity of the tongue boundaries for small analytic potentials just expresses infinite differentiability, whereas in the periodic case analyticity holds.

In the present quasi-periodic case, analyticity of the tongue boundaries would follow if the averaging process of Sect. 2.2 were convergent. However generically this is not the case. Indeed, such convergence of the averaging process in a point at a tongue boundary would imply the existence of a spectral interval (that is, a whole interval in the spectrum) besides the point in the tongue boundary, where generically we have the Cantor spectrum, see [33, 29, 22].

Of course these considerations do not forbid analyticity of the tongue boundaries and it is interesting to know whether analyticity is the case or not. Work in this direction, in a more general context, is in progress (see [42]).

Appendix

A. Lipschitz Property of Tongue Boundaries in the Large

In the paper we approached the regularity of the tongue boundaries using reducibility. However, there exists numerical [15] and analytical evidence [24], that in cases far from constant coefficients this approach cannot be used. Presently we reconsider the quasi-periodic Hill equation (1), or equivalently (5), where we only assume the components of ω to be rationally independent (i.e., not necessarily Diophantine) and where the function $Q : \mathbb{T}^d \to \mathbb{R}$ is just continuous.

Proposition 6. *In the above situation, let*

$$C = \sup_{\theta \in \mathbb{T}^d} |Q(\theta)|,$$

and $b \in \mathbb{R} \mapsto a(b) \in \mathbb{R}$ be a (left or right) tongue boundary. Then for all $b, b' \in \mathbb{R}$ we have

$$\left|a(b) - a(b')\right| \leq C \left|b - b'\right|,$$

Our proof is based on Sturm-like arguments for the oscillation of the zeroes of a second order linear differential equation. First we review some useful properties of the rotation number [30, 36].

For any non-trivial solution $x(t)$ of Eq. (1) we define a function $N(T; x)$, which is the number of zeroes of $x(t)$ in the interval $[0, T]$. From the form that (1) takes in polar coordinates (by the so-called Prüfer transformation), it follows that all the zeroes of $x(t)$ are simple and that the limit

$$\lim_{T\to\infty} \frac{\pi N(T; x)}{T}$$

exists for all initial conditions, agreeing with the rotation number of (1), or equivalently, the system (5).

Our idea to prove Proposition 6 is to use a suitable Sturm Oscillation Theorem to control the zeroes of a variation $Q + \delta Q$ of the original potential Q, with the property that $\delta Q(\theta)$ is either positive or negative for all $\theta \in \mathbb{T}^d$.

Lemma 1. *Assume that the maps $Q, \delta Q : \mathbb{T}^d \to \mathbb{R}$ are continuous and that $\delta Q(\theta) > 0$ for all $\theta \in \mathbb{T}^d$. Let ρ_1 be the rotation number of*

$$x'' + Q(\theta)x = 0, \quad \theta' = \omega$$

and ρ_2 the rotation number of

$$y'' + (Q(\theta) + \delta Q(\theta))\, y = 0, \quad \theta' = \omega.$$

Then $\rho_1 \le \rho_2$.

Lemma 1 is a direct consequence of the Sturm Comparison Theorem, see, e.g., [19]. Indeed, by this result, the number of zeroes $N(T; x)$ in the interval $[0, T]$ is less than or equal to the number of zeroes $N(T, y)$ of y in the same interval, assuming that we have the same initial conditions $x(0) = y(0)$, $x'(0) = y'(0)$. Therefore, by the above considerations

$$\rho_1 = \lim_{T\to\infty} \frac{\pi N(T; x)}{T} \le \lim_{T\to\infty} \frac{\pi N(T; y)}{T} = \rho_2,$$

as was to be shown.

We proceed showing how Lemma 1 can be used to check the Lipschitz condition stated in Proposition 6. First, note that, if the condition $\delta Q > 0$ is replaced by $\delta Q < 0$, then we have $\rho_1 \ge \rho_2$.

In the setting of Proposition 6, condition

$$\delta a - C|\delta b| > 0 \tag{24}$$

implies that $\delta Q(\theta) = \delta a + \delta b Q(\theta) > 0$ for all $\theta \in \mathbb{T}^d$ and thus, by Lemma 1, that

$$\operatorname{rot}(a, b) \le \operatorname{rot}(a + \delta a, b + C\delta b).$$

Now, if (a, b) is at the boundary of a certain tongue (for simplicity assume a is the right endpoint of the corresponding spectral gap), this means that for arbitrarily small perturbations in the a direction, the rotation number is strictly larger than that of the original equation. That is, for any $\delta' a > 0$,

$$\operatorname{rot}(a, b) < \operatorname{rot}(a + \delta' a, b).$$

The lemma then yields that, if $(\delta a, \delta b)$ satisfies (24), also

$$\operatorname{rot}(a, b) < \operatorname{rot}(a + \delta' a + \delta a, b + C\delta b).$$

As $\delta' a$ may be arbitrarily small the perturbations $(\delta a, \delta b)$ in the sector defined by condition (24) do not contain any point in the boundary of the same tongue as (a, b). Therefore, in our proof of Proposition 6 we have

$$a(b_1) - a(b_2) \leq C\,|b_1 - b_2|\,.$$

In order to prove the remaining inequality, observe that perturbations in the sector

$$\delta a + C|\delta b| < 0 \tag{25}$$

contain no points in the left boundary of the tongue of (a, b). By contradiction assume that such a point in the left boundary exists and let $(\delta a, \delta b)$ satisfying (25) be the corresponding perturbation. Then, due to the openness of the above condition, there exists a positive constant $\delta' a$ such that $(\delta a + \delta' a, \delta b)$ still satisfies (25). Moreover, as we are assuming $(a + \delta a, b + \delta b)$ to be in the endpoint of the left spectral gap and $\delta' a > 0$,

$$\mathrm{rot}(a, b) = \mathrm{rot}(a + \delta a, b + \delta b) < \mathrm{rot}(a + \delta' a + \delta a, b + \delta b).$$

On the other hand, the perturbation $(-\delta a - \delta' a, -\delta b)$ satisfies condition (24) and therefore

$$\mathrm{rot}(a + \delta' a + \delta a, \delta b) \leq \mathrm{rot}(a, b),$$

which implies $\mathrm{rot}(a, b) < \mathrm{rot}(a, b)$. This is the desired contradiction, whereby Proposition 6 is proved.

Remark. The Lipschitz property in Proposition 6 regarding tongue boundaries also holds in the periodic case, where the proof runs exactly the same, and where this is referred to as the directional convexity of stability and instability domains, see [49]. The property also provides a bound on the derivatives of the tongue boundaries whenever they exist. This bound coincides with the one obtained in the averaging process of Sect. 2.2.

B. Proof of Theorem 3

Our proof follows from an analysis of the normal form to order $|\mathbf{k}^*|$. There are several normalization techniques and any such method for arbitrary $|\mathbf{k}^*|$ can be cumbersome. Therefore we only use the format of the normal form of order $|\mathbf{k}^*|$ to find out which terms are relevant. Subsequently, the coefficients of those terms are obtained by an alternative, recurrent and simpler method.

Let us set $\alpha_0 = \langle \mathbf{k}^* \rangle / 2$ and $a = \alpha_0^2 + \alpha$. Next, a scaling

$$x = \frac{\xi}{\sqrt{\alpha_0}}, \quad y = \eta\sqrt{\alpha_0},$$

and passage to complex coordinates

$$\xi = \frac{q + ip}{\sqrt{2}}, \quad \eta = \frac{iq + p}{\sqrt{2}},$$

give the following form for the time-dependent Hamiltonian:

$$H(q,p,t) = \alpha_0 iqp + \frac{q^2-p^2+2iqp}{2}\left(\frac{\alpha}{2\alpha_0}+\frac{b}{4\alpha_0}\left(\sum_{j=1}^{d} c_j\cos(\omega_j t)+\varepsilon\cos\left(\langle\mathbf{k}^*\rangle t\right)\right)\right). \quad (26)$$

Now let J be canonically conjugate to the time t, and let $\zeta_j = \exp(i\ \mathrm{sign}(k_j^*)\omega_j t)$ for $j = 1,\dots,d$. Then (26) can be written as the sum of an integrable part H_0 and a perturbation H_1,

$$H_0 = J+\alpha_0 iqp,$$
$$H_1 = \hat{b}\left(q^2-p^2+2iqp\right)\left(\hat{\alpha}+\sum_{j=1}^{d} c_j\left(\zeta_j+\zeta_j^{-1}\right)+\varepsilon\left(\zeta^{\mathbf{k}^*}+\zeta^{-\mathbf{k}^*}\right)\right), \quad (27)$$

where

$$\hat{\alpha} = 2\alpha/b, \qquad \hat{b} = b/(8\alpha_0)$$

act as perturbation parameters.

To carry out the normalization (averaging) one can use any Lie series method, for instance the Giorgilli-Galgani algorithm [26] as was done in [16]. Starting with $H_{0,0} = H_0$ and $H_{1,0} = H_1$, the terms

$$H_{j,k} = \sum_{l=1}^{k}\frac{l}{k}\left[G_l, H_{j,k-l}\right], \qquad j = 0,1, \quad k>0,$$

where $[\cdot,\cdot]$ denotes the Poisson bracket, are computed recurrently. A term as $H_{j,k}$ contains $\hat{b}^{j+k}$ as a factor. The functions G_n are determined for cancelling the time dependence as far as possible, i.e., if no resonances occur. To be precise, assume that $G_1,\dots G_{n-1}$ are already computed. Then all terms in $H_{1,n-1}+H_{0,n}$ are known except the ones coming from $[G_n, H_{0,0}]$. Let K_n contain the known terms at order n. Then G_n is determined by requiring $K_n+[G_n,H_{0,0}]$ not to contain terms in the ζ_j variables. The transformed Hamiltonian then is $N = N_0+N_1+N_2+\cdots$, where $N_0 = H_{0,0}$ and $N_n = H_{1,n-1}+H_{0,n}$. In particular, N_n is of order n with respect to $\hat{b}$.

It directly follows that

$$\left[H_{0,0}, q^r p^{2-r}\zeta^{\mathbf{k}}\right] = q^r p^{2-r}\zeta^{\mathbf{k}} i\left(\alpha_0(2-2r)-\langle\mathbf{k}\rangle\right), \qquad r = 0,1,2,$$

and this shows that all terms with $\alpha_0(2-2r)-\langle\mathbf{k}\rangle$ different from zero can be cancelled to any finite order. Proceeding by induction one observes that, if $j+k = m$ then $H_{j,k}$ has the form

$$H_{j,k} = q^2 d_1 - p^2 d_2 + iqp(d_3+d_4),$$

with the corresponding G_m of the form

$$G_m = i(q^2 d_1 + p^2 d_2) + qp(d_3-d_4).$$

Here d_1 contains the terms with real coefficients of the form $\hat{\alpha}_0^{m-s}\zeta^{\mathbf{k}}$ with $|\mathbf{k}| = r$, where s and r have the same parity. The terms in d_2 can be obtained from d_1 by a replacement of $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$. Similarly, the expression of d_3 is identical to that of d_4 but replacing $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$.

Summing up, by a canonical change of variables, the Hamiltonian $H = H_0 + H_1$, up to a remainder of higher order in $\hat{b}$, can be reduced to the normal form

$$NF = J + a_0 iqp + \mathrm{coef}_1 iqp + \mathrm{coef}_2 \left(q^2 \zeta^{-\mathbf{k}^*} - p^2 \zeta^{\mathbf{k}^*}\right), \tag{28}$$

where

- $\mathrm{coef}_1 = \hat{\alpha} + r_1$, where r_1 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$, and containing some power of $\hat{b}$ as a factor;
- $\mathrm{coef}_2 = \varepsilon \hat{b} f_2(\hat{b}, \hat{\alpha}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} \times r_2$, where r_2 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$ and where $f_2(0, 0, 0, c) \neq 0$ does not depend on c;
- the order of the remainder in $\hat{b}$ is larger than $|\mathbf{k}^*|$.

Truncating away the remainder and passing to co-rotating coordinates (u, v) defined by

$$u = q \exp(-i\langle \mathbf{k}^* \rangle t), \qquad v = p \exp(i\langle \mathbf{k}^* \rangle t)$$

yields the system

$$u' = i\,\mathrm{coef}_1\, u - 2\,\mathrm{coef}_2\, v, \qquad v' = -2\,\mathrm{coef}_2\, u - i\,\mathrm{coef}_1\, v.$$

Therefore, up to the $|\mathbf{k}^*|$th order the tongue boundaries are given by the equation

$$\mathrm{coef}_1 = \pm 2\,\mathrm{coef}_2.$$

So if $r_2(0, 0, 0, c) \neq 0$ for $\varepsilon = 0$ there is a $|\mathbf{k}^*|$th order tangency at $b = 0$, while for $\varepsilon \neq 0$ there is an instability pocket. Hence, our proof of Theorem 3 is concluded by checking when $r_2(0, 0, 0, c)$ vanishes.

To find out whether $r_2(0, 0, 0, c)$ vanishes or not, it is only necessary to consider Eq. (18) for $\varepsilon = 0$ at the exact resonance

$$x'' + \left(\frac{\langle \mathbf{k}^* \rangle}{4} + \frac{b}{2} \left(\sum_{j=1}^{d} c_j \left(\zeta_j + \zeta_j^{-1} \right) \right) \right) x = 0. \tag{29}$$

Note that

$$r_2(0, 0, 0, c) = R(\omega, \mathbf{k}^*) c^{\mathbf{k}^*},$$

where now R does not depend on c. Therefore one may assume that $c_j = 1$ for $j = 1, \ldots, d$.

According to the normal form (28), any non-trivial solution $x(t)$ of (29) can be expanded in powers of b, the first $K - 1$ coefficients of which are quasi-periodic functions and where the K^{th} coefficient is also quasi-periodic if and only if $R(\omega, \mathbf{k}^*)$ vanishes. We are now going to compute this expansion directly from the differential equations, instead of using the Hamiltonian formulation.

Since we are interested in the $\mathbf{k}^{*\text{th}}$ power in ζ, we first consider Eq. (29) only for positive powers of the ζ_j :

$$x'' + \left(\frac{\langle \mathbf{k}^* \rangle^2}{4} + \frac{b}{2} \sum_{j=1}^{d} \zeta_j \right) x = 0. \tag{30}$$

Scaling time by $t = 2\tau$ turns (30) into

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^2 \right) x = 0, \tag{31}$$

where the dot denotes derivation with respect to τ and where $\mu = 2b$ is the new perturbation parameter. Also note that after this change we have $\zeta_j = z_j^2$, where $z_j = \exp(i \operatorname{sign}(k_j^*)\omega_j \tau)$. Any solution of Eq. (31) can be expanded in powers of μ as follows:

$$x^{(1)} = x_0 + \mu x_1 + \mu^2 x_2 + \cdots + \mu_K x_K + O(\mu^{K+1}),$$

where $K = |\mathbf{k}^*|$. Substitution of this expansion into (31) leads to the following recursive relations:

$$\ddot{x}_r + \langle \mathbf{k}^* \rangle^2 x_r = -\left(\sum_{j=1}^{d} z_j^2 \right) x_{r-1}$$

for $r = 1, \ldots, K$ and

$$\ddot{x}_0 + \langle \mathbf{k}^* \rangle^2 x_0 = 0$$

for $r = 0$. One of the two fundamental solutions of the latter equation is $x_0 = z^{-\mathbf{k}^*}$, so that the equation for x_1 becomes

$$\ddot{x}_1 + \langle \mathbf{k}^* \rangle^2 x_1 = -\sum_{j_1=1}^{d} z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}},$$

where $\mathbf{e}_j$ is the j^{th} element of the canonical basis of $\mathbb{R}^d$. A solution of the latter equation is given by

$$x_1 = -\sum_{j_1=1}^{d} \frac{z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}}}{\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{e}_{j_1} \rangle^2}.$$

This recursive process can be continued up to any order. By induction it directly follows that at the r^{th} step

$$x_r = (-1)^r \sum_{j_1, \ldots, j_r = 1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \ldots + \mathbf{e}_{j_r}\right)\right)}}{\prod_{l=1}^{r} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2 \right)},$$

where $\mathbf{s}_l = \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_l}$ for $l = 1, \ldots, r$. Note that when $|r| < |\mathbf{k}^*|$, the denominator of the above expression never vanishes. At the order $K = |\mathbf{k}^*|$ the equation for x_K reads

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + (-1)^{K-1} \sum_{j_1, \ldots, j_K = 1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \ldots + \mathbf{e}_{j_K}\right)\right)}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2 \right)} = 0.$$

In the summation we are only interested in terms with $z^{\mathbf{k}^*}$. Indeed, all other terms can be removed by a procedure similar to the one used for the previous $x_1, \ldots, x_{K-1}$.

The remaining terms can be indexed by the paths of length K joining $\mathbf{0}$ and $\mathbf{k}^*$ in the lattice $\mathbb{Z}^d$. The set of these paths will be denoted by $\Gamma(\mathbf{k}^*)$, and for every path $\gamma \in \Gamma(\mathbf{k}^*)$ we consider the intermediate position vectors $\mathbf{s}_r(\gamma)$. In this way the equation for x_K becomes

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + z^{\mathbf{k}^*} F(\omega, \mathbf{k}^*) = 0,$$

where

$$\begin{aligned} F(\omega, \mathbf{k}^*) &= \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l(\gamma) \rangle^2 \right)} \\ &= \frac{1}{4} \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \langle \mathbf{k}^* - \mathbf{s}_l(\gamma) \rangle \langle \mathbf{s}_l(\gamma) \rangle}, \end{aligned}$$

which has non-vanishing denominators for all irrational frequency vectors ω. The latter equation has as a non-trivial solution

$$x_K(\tau) = -2i\tau \langle \mathbf{k}^* \rangle F(\omega, \mathbf{k}^*) z^{\mathbf{k}^*},$$

provided that $F(\omega, \mathbf{k}^*) \neq 0$, and this solution is clearly not quasi-periodic.

Next we proceed with the other fundamental solution $x^{(2)}$ of Eq. (31), starting with the zero order term $z^{\mathbf{k}^*}$. However it is better to study that solution via the conjugate equation

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^{-2} \right) x = 0. \tag{32}$$

This leads to a recursive process as before for obtaining the coefficients of the expansion of $x^{(+)}$ in terms of μ. Now, taking $x^{(\pm)} = \frac{1}{2}\left(x^{(1)} \pm x^{(2)}\right)$ as fundamental solutions, we get the following equation:

$$\ddot{x} + \left(\langle \mathbf{k} \rangle^2 + \mu \sum_{j=1}^{d} \left(z_j^2 + z_j^{-2} \right) \right) x = 0 \tag{33}$$

which, undoing the changes in τ and μ, can be transformed into (29).

In this way we have found two linearly independent solutions x^+ and x^- of this system, the expansion of which in powers of b have quasi-periodic coefficients in time up to order $K - 1$ and where the K^{th} order coefficient is a function of the form $tz^{\mathbf{k}^*}$ times $F(\omega, \mathbf{k}^*)$. By comparison of coefficients it follows that $F(\omega, \mathbf{k}^*)$ and $R(\omega, \mathbf{k}^*)$ are identical except for a non-zero factor.

Note that $F(\omega, \mathbf{k}^*)$ is a rational function. We denote its numerator by $N(\omega, \mathbf{k}^*)$ and its denominator by $D(\omega, \mathbf{k}^*)$. Define $\mathcal{A}(\mathbf{k}^*)$ as the set of ω's for which $N(\omega, \mathbf{k}^*)$ is non-zero. We claim that $\mathcal{A}(\mathbf{k}^*)$ has measure zero, which follows from the fact that $N(\cdot, \mathbf{k}^*)$ is not identically zero. To check this first note that if $\omega = (1, \dots, 1)^T$, then $D(\omega, \mathbf{k}^*)$ does not vanish. Second we resort to the periodic case [16], noting that the equation now can be transformed to the classical Mathieu equation. It thereby follows that $N(\omega, \mathbf{k}^*)$ is non-zero for this value and, hence that $N(\cdot, \mathbf{k}^*)$ is not identically zero for any $\mathbf{k}^*$. Therefore the set $\mathcal{A}(\mathbf{k}^*)$, given by the zeroes of $N(\omega, \mathbf{k}^*)$, is a zero measure set and the theorem follows. □

Summarizing, the tongue boundaries at the $\mathbf{k}^{*\text{th}}$ resonance, up to order $|\mathbf{k}^*|$, are given by the equation

$$\text{coef}_1 = \pm 2\,\text{coef}_2$$

which, in terms of $\hat{\alpha}$, $\hat{b}$, ε and c becomes

$$\hat{\alpha} + r_1(\hat{\alpha}, \hat{b}, \varepsilon, c) = \pm 2\left(\varepsilon \hat{b} f_2(\hat{\alpha}, \hat{b}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} r_2(\hat{\alpha}, \hat{b}, \varepsilon, c)\right).$$

The tongue boundary crossings up to order $|\mathbf{k}^*|$ correspond to

$$\text{coef}_2 = 0$$

and a further analysis requires to distinguish between the cases of even and odd K.

When K is even, then for any $0 < |\varepsilon| \ll 1$ there is a pocket ending at $b = 0$ and at

$$b_{\text{tip}} = \left(\frac{-\varepsilon f_2(0, 0, 0, c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}\right)^{\frac{1}{K-1}} + \cdots,$$

where the dots denote higher order terms in ε. If K is odd then the sign of ε must be selected such that

$$\frac{-\varepsilon f_2(0, 0, 0, c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}$$

is positive. If this is the case, then there are two instability pockets with ends at $b = 0$ and at

$$b_{\text{tip}}^{\pm} = \pm\left(\frac{-\varepsilon f_2(0, 0, 0, c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}\right)^{\frac{1}{K-1}} + \cdots.$$

C. Structure of the Sets $\mathcal{A}(\mathbf{k})$

An interesting question related to Theorem 3 is whether the set of Diophantine frequency vectors in $\mathcal{A}(\mathbf{k})$, for a fixed resonance $\mathbf{k}$, is empty or not.

When $d = 2$, we can always assume that $\omega = (1, \gamma)$, where γ is a real number. Note that any real irrational γ for which $N(\omega, \mathbf{k}) = 0$ for some $\mathbf{k}$, is Diophantine, since it is algebraic. Direct computations, performed on $F(\omega, \mathbf{k})$, yield that if the order of the resonance is less than 5, all the roots of $N((1, \gamma), \mathbf{k})$ are either rational or complex (i.e., nonreal). However, for $\mathbf{k} = (3, 2)$,

$$N\left((1, \gamma), (3, 2)\right) = 24 + 172\gamma + 454\gamma^2 + 505\gamma^3 + 232\gamma^4 + 49\gamma^5 + 4\gamma^6,$$

which has real irrational zeroes. Direct computation also shows that the same happens for all resonances $6 \le |\mathbf{k}| \le 9$ with $k_1 \ne 1$, $k_2 \ne 1$. For $d \ge 3$ the situation is even simpler, since for $\mathbf{k}^* = (1, 1, 1)$ the polynomial $N(\omega, \mathbf{k}^*)$ has real Diophantine zeroes.

There is one case when $|\mathbf{k}^*|$th order tangency at the $\mathbf{k}^{*\text{th}}$ resonance always can be granted:

Proposition 7. *In the Mathieu equation*

$$x'' + (a + b\,(\cos(t) + \cos(\gamma t)))\,x = 0, \tag{34}$$

where $2\gamma \neq 0$ *is not a negative integer, the order of tangency at* $b = 0$ *of the resonance tongue boundaries corresponding to* $\mathbf{k}^* = (K, 1)$*, for any* K*, exactly is* $|\mathbf{k}^*| = K + 1$.

Proof. In this case the number of paths of minimal length in $\mathbb{Z}^2$ joining $(0, 0)$ and $\mathbf{k}^*$ exactly is $K + 1$ and any of these can be labelled by an integer between 0 and K. These paths will be denoted by $\sigma_0, \dots, \sigma_K$. To show that the order of tangency is exactly $K+1$ we must compute $F((1, \gamma), (K, 1)) =: f(\gamma, K)$, which amounts to

$$f(\gamma, K) = \frac{(-1)^K}{4} \sum_{j=0}^{K} \frac{1}{\prod_{l=1}^{K} \langle \mathbf{k}^* - \mathbf{s}_l(\sigma_j)\rangle \langle \mathbf{s}_l(\sigma_j)\rangle} \tag{35}$$

and show that for $\gamma \notin \mathbb{Z}$, this does not vanish. For each of the paths σ_j, $j = 0, \dots, K$, let α_j be the contribution to the sum in (35). Then

$$\alpha_j = \frac{1}{(K - 1 + \gamma)1} \frac{1}{(K - 2 + \gamma)2} \cdots \frac{1}{(K - j + \gamma) j} \frac{1}{(K - j)(j + \gamma)} \cdots \frac{1}{1(K - 1 + \gamma)},$$

where the total number of factors is K. Using the Gamma-function it follows that

$$\sum_{j=0}^{K} \alpha_j = \sum_{j=0}^{K} \frac{\Gamma(K - j + \gamma)\Gamma(j + \gamma)}{j!(K - j)!\Gamma(K + \gamma)\Gamma(K + \gamma)}.$$

Since

$$d(\gamma, K) = D((1, \gamma), (K, 1)) = (K - 1 + \gamma)1 \cdot \ldots \cdot (\gamma + 1)(K - 1) \cdot (\gamma)(K),$$

it is clear that

$$d(\gamma, K) = \frac{\Gamma(K + \gamma)K!}{\Gamma(\gamma)},$$

implying that

$$\begin{aligned} d(\gamma, K) \sum_{j=0}^{K} \alpha_j &= \sum_{j=0}^{K} \binom{K}{j} \frac{\Gamma(K - j + \gamma)\Gamma(j + \gamma)}{\Gamma(K + \gamma)\Gamma(\gamma)} \\ &= \frac{1}{\Gamma(K + \gamma)\Gamma(\gamma)} \frac{\Gamma(K + 2\gamma)\Gamma(\gamma)^2}{\Gamma(2\gamma)}, \end{aligned}$$

where the last identity is an application of Pochhammer's formula, see [48]. Therefore, the relevant coefficient is

$$f(\gamma, K) = \frac{(-1)^K}{4} \sum_{j=0}^{K} \alpha_j = \frac{1}{D} \frac{\Gamma(\gamma)^2}{\Gamma(K + \gamma)\Gamma(\gamma)\Gamma(2\gamma)} \Gamma(K + 2\gamma),$$

which, if 2γ is not a negative integer, is different from zero. $\square$

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