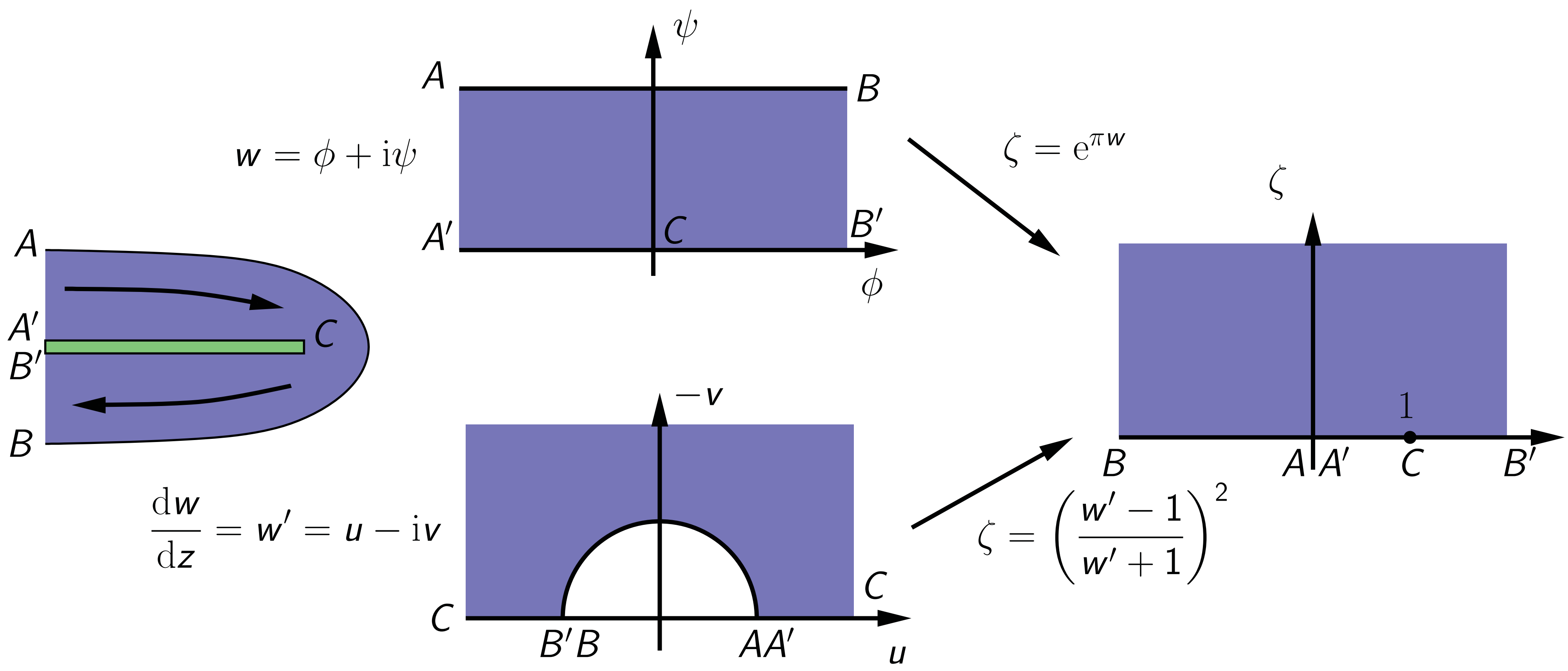
Example 1. Teapot flow



$$\left(\frac{w'-1}{w'+1}\right)^2 = e^{\pi w} \quad \Rightarrow \quad w' = \frac{1+e^{\pi w/2}}{1-e^{\pi w/2}} = -\coth \pi w/4$$

$$\Rightarrow \quad \cosh \pi w/4 = e^{-\pi z/4}.$$

The free streamline is given by $\psi = 1$,

$$e^{-\pi x/4}\left(\cos\frac{\pi y}{4} - i\sin\frac{\pi y}{4}\right) = \cosh\left(\frac{\pi\phi}{4} + \frac{i\pi}{4}\right) = \frac{1}{\sqrt{2}}\cosh\frac{\pi\phi}{4} + \frac{i}{\sqrt{2}}\sinh\frac{\pi\phi}{4}.$$

Thus, parametrically,

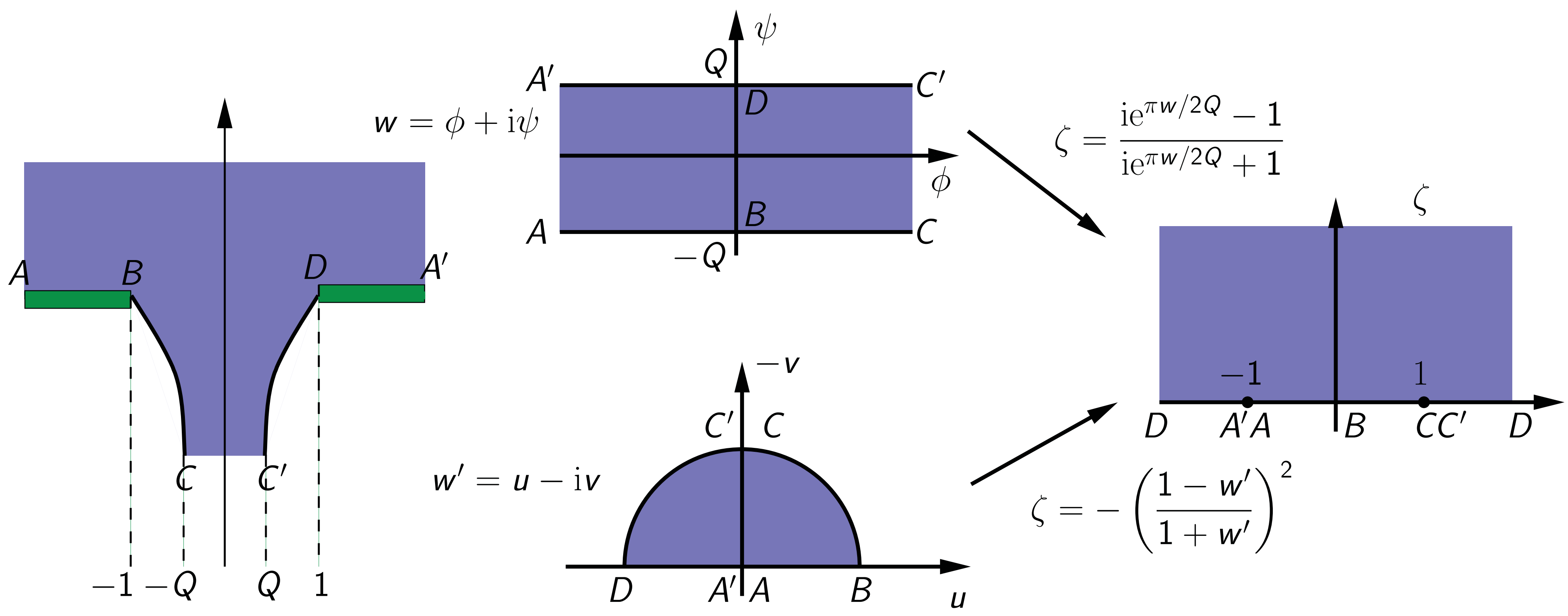
$$e^{-\pi x/4}\cos\frac{\pi y}{4} = \frac{1}{\sqrt{2}}\cosh\frac{\pi\phi}{4}, \qquad e^{-\pi x/4}\sin\frac{\pi y}{4} = -\frac{1}{\sqrt{2}}\sinh\frac{\pi\phi}{4}.$$

Or, eliminating ϕ,

$$\frac{1}{2} = e^{-\pi x/2}\left(\cos^2\frac{\pi y}{4} - \sin^2\frac{\pi y}{4}\right) = e^{-\pi x/2}\cos\frac{\pi y}{2}.$$

At the nose $y = 0$ gives $x = \frac{2\log 2}{\pi}$—the "thinning factor".

Example 2. Flow out of a slot



$$\left(\frac{1-w'}{1+w'}\right)^2 = \frac{1-\mathrm{i}\,\mathrm{e}^{\pi w/2Q}}{1+\mathrm{i}\,\mathrm{e}^{\pi w/2Q}}.$$

This is possible—but complicated—to solve.

If we just want the free surface shape there is a short cut to a parametric form. The key step is to use the natural parametrisation $w' = \mathrm{e}^{-\mathrm{i}\theta}$ on the free surface (θ is the angle the surface makes with the x-axis). E.g. on BC $-\pi/2 < \theta < 0$ as $0 < \zeta < 1$. Then

$$\zeta = -\left(\frac{1-\mathrm{e}^{-\mathrm{i}\theta}}{1+\mathrm{e}^{-\mathrm{i}\theta}}\right)^2 = \tan^2\theta/2.$$

We also use

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\mathrm{d}z}{\mathrm{d}\theta}\Big/\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta}.$$

Then

$$\zeta = \frac{\mathrm{i}e^{\pi w/2Q}-1}{\mathrm{i}e^{\pi w/2Q}+1} \quad \Rightarrow \quad w = \frac{2Q}{\pi}\log\left(-\mathrm{i}\frac{1+\zeta}{1-\zeta}\right) \quad \Rightarrow \quad \frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{4Q}{\pi(1-\zeta^2)}.$$

Thus

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = e^{\mathrm{i}\theta}\cdot\frac{4Q}{\pi(1-\tan^4\theta/2)}\cdot\sec^2\frac{\theta}{2}\tan\frac{\theta}{2} = \frac{2Q}{\pi}e^{\mathrm{i}\theta}\frac{\sin\theta}{\cos\theta}.$$

This is an equation for the free surface.

Written out we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} + \mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2Q}{\pi}\tan\theta\,(\cos\theta + \mathrm{i}\sin\theta)\,.$$

Equating real and imaginary parts

$$x = -1 + \frac{2Q}{\pi}\int_0^\theta \sin t\,\mathrm{d}t = -1 + \frac{2Q}{\pi}(1-\cos\theta),$$

$$y = \frac{2Q}{\pi}\int_0^\theta \frac{\sin^2 t}{\cos t}\,\mathrm{d}t.$$

In particular $x(-\pi/2) = -Q = -1 + 2Q/\pi$, giving

$$Q = \frac{\pi}{\pi+2}.$$