All the references are made to the document *06Gravitation* that can be seen in
https://mat.upc.edu/en/people/jc.sola-morales/coses-cosas-things

Problem 6.2. The thing is not so difficult. With a change of variable (that has already been done several times) you can write

$$V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}' = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}'|}\, d\mathcal{V}'.$$

With this second form it is easier to show that V is continuous at a generic point $\mathbf{x}=\mathbf{x}_0$. The only Functional Analysis result you have to use is that if $f \in L^1$ then for each ε there exists a δ such that if $|\mathbf{h}| < \delta$ then $\|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\|_1 < \varepsilon$ (you can use this without proof, but the proof is an easy consequence of the fact that continuous functions of compact support are dense in L^1). The same argument does not hold in L^∞, and the only thing you can say is that $\|f(x) - f(\mathbf{x}+\mathbf{h})\|_\infty \leq 2\|f\|_\infty$

Therefore

$$|V(\mathbf{x}_0) - V(\mathbf{x}_0+\mathbf{h})| = |\int_{\mathbb{R}^3} \frac{m(\mathbf{x}_0-\mathbf{x}') - m(\mathbf{x}_0-\mathbf{x}'+\mathbf{h})}{|\mathbf{x}'|}\, d\mathcal{V}'| \leq$$

$$\leq \frac{3}{2}(4\pi)^{1/3}||m(\cdot) - m(\cdot+\mathbf{h})||_1^{2/3}|||m(\cdot) - m(\cdot+\mathbf{h})||_\infty^{1/3},$$

the first factor tends to zero if $\mathbf{h} \to 0$, and the second remains bounded.

Problem 6.3 When $m(\mathbf{x})$ has compact support, say inside the ball of radius R, then

$$|V(\mathbf{x})| = |\int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'| = |\int_{B_R} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'| \leq$$

$$\leq \frac{1}{|\mathbf{x}| - R} \int_{B_R} |m(\mathbf{x}'|\, d\mathcal{V}'|,$$

if $|\mathbf{x}| > R$. And this quantity tends to zero as $|\mathbf{x}| \to \infty$.

Then, for a general $m(\mathbf{x}')$ you can approximate $m(\mathbf{x}')$ by the truncated functions $m_N(\mathbf{x}')$ that coincide with m inside the ball of radius N and vanish outside. For these functions $\|m - m_N\|_1 \to 0$ as $N \to \infty$, while $\|m - m_N\|_\infty$ remains bounded. Therefore

$$|V(\mathbf{x}) - V_N(\mathbf{x})| \leq \frac{3}{2}(4\pi)^{1/3}||m - m_N||_1^{2/3}|||m - m_N||_\infty^{1/3},$$

a quantity that tends to zero as $N \to \infty$. From this and the fact that each $|V_N|$ vanishes at infinity, it is easily deduced the same property for V.

Problem 6.4 This problem is solved in
Lawrence C. Evans, Partial Differential Equations, American Mathematical Society, 1998
Theorem 1 on page 23. He also assumes that $m(\mathbf{x})$ is of compact support. You can also use this extra assumption, if you wish. Else, you can use an arbitrary ball and express $m = m_1 + m_2$, where both are of class $\mathcal{C}^2$, m_2 vanishes inside this ball, and m_1 has compact support inside a concentric ball of twice the radius. Then you can use the proof of Evans for the potential created by m_1, but you can observe that the potential created by m_2 is harmonic in the ball (*the potential is harmonic outside the masses*). Poisson's equation is satisfied only in the ball, but remember that the ball is arbitrary.

The proof of Evans is excellent and has three steps. In the first step he proves that the potential is of class $\mathcal{C}^2$. Then the conclusion can be derived from Gauss Theorem (page 7 of my slides). That would be an alternative to the steps 2 and 3 of Evans. But the spirit of the problem is more in the direction of the Evans' proof, because in the $\mathcal{C}^2$ case everything can be computed without (say) much work.

Problem 6.5 The way I see it is the following. First start with test functions $\phi \in \mathcal{C}_c^4(\mathbb{R}^3)$ instead of merely $\mathcal{C}_c^2$. Let V' be the potential created by the density $\nabla^2\phi$. Since $\nabla^2\phi$ is of class $\mathcal{C}^2$ you can use the previous problem, and deduce that $\nabla^2 V' = -4\pi\nabla^2\phi$, and using the behaviour at infinity we can deduce (maximum principle for the difference $V' + 4\pi\phi$) that $V' = -4\pi\phi$. Then,

$$-\int \nabla\phi \cdot \nabla V = \int \nabla^2\phi \; V = \int \nabla^2\phi \, (m * 1/|\mathbf{x}|) = \int \nabla^2\phi(\mathbf{x}) \int m(\mathbf{y})(1/|\mathbf{x}-\mathbf{y}|) \; d\mathbf{y} \; d\mathbf{x}$$

$$= \int m(\mathbf{y}) \int \nabla^2\phi(\mathbf{x})(1/|\mathbf{x}-\mathbf{y}|) \; d\mathbf{x} \; d\mathbf{y} = \int m(\mathbf{y})V'(\mathbf{y}) \; d\mathbf{y} = \int m(\mathbf{y})(-4\pi\phi(\mathbf{y})) \; d\mathbf{y},$$

as we wanted to prove.

Now the problem is to pass from $\mathcal{C}_c^4$ to $\mathcal{C}_c^1$. Forget this part, if you are too busy. What I would do is to use *mollifiers* of class $\mathcal{C}_c^\infty$ of the form $\rho_n(\mathbf{x}) = n^3\rho_0(n\mathbf{x})$ and consider $\phi_n = \phi * \rho_n$. We have that the functions ϕ_n are of class $\mathcal{C}^4$, that $\phi_n \to \phi$ in the $\mathcal{C}^1$ norm, and that the ϕ_n and ϕ have a compact supports contained in a common bounded set K. Then,

$$\int \nabla\phi \cdot \nabla V = \int_K \nabla\phi \cdot \nabla V = \lim_{n\to\infty} \int \nabla\phi_n \cdot \nabla V = \lim_{n\to\infty} \int m(\mathbf{x})(4\pi\phi_n(\mathbf{x})) = 4\pi \int m(\mathbf{x})\phi(\mathbf{x}).$$

Problem 6.6 You can use problem 6.5, because $\mathcal{C}_c^\infty \subset \mathcal{C}^1$. The rest of the problem is just using the definition of distributional derivatives, and the definition of the δ function.