EXISTENCE AND NON-EXISTENCE OF FINITE-DIMENSIONAL GLOBALLY ATTRACTING INVARIANT MANIFOLDS IN SEMILINEAR DAMPED WAVE EQUATIONS*

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0. Introduction

This paper is concerned with the dynamical system generated by certain semilinear damped wave equations. In §1 we reproduce a result obtained in a previous paper (Mora[1986]), which shows that, when the damping is sufficiently large this dynamical system has the property that its global attractor is contained in a finite-dimensional local invariant manifold of class C^1. In the present paper, we will show that, on the other hand, when the damping is small, it is a fairly generic fact that there is no finite-dimensional local invariant manifold of class C^1 containing the global attractor. The exact result obtained in this connection is stated in Theorem 4.1 In the way towards this result, we have developed some auxiliary results which have some interest by themselves, namely, a result giving optimal inner products for linear wave equations (Theorem 2.1), and a C^1 lineari-

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zation theorem (Theorem 3.1) .

The reason why small damping makes difficult the existence of finite-dimensional (local) invariant manifolds of class C^1 is mainly linear. When the damping is small, the linear part of the equation at a given stationary point easily has all its eigenvalues on the same vertical line of the complex plane. This immediately implies that there is no normally hyperbolic invariant manifold of class C^1 containing that point. If the eigenvalues are all simple, then we can prove that, even dropping the condition of normal hyperbolicity, one has only a countable family of finite-dimensional invariant manifolds of class C^1 . In §2 , this crucial fact is established for the linear problem. By using the C^1 linearization theorem of §3 , this fact can then be translated to the neighbourhood of a stationary state of a nonlinear problem.

Let us now consider a nonlinear problem with a heteroclinic orbit from ϕ to ψ, where ψ is a stationary state with a linearization of the type described above. Certainly, the global attractor of the system must contain the connecting orbit. But, on the other hand, it seems extremely casual that this orbit arrives at the neighbourhood of ψ by precisely one of the few finite-dimensional invariant manifolds of class C^1 which contain ψ. If actually it does not do so, then we can conclude that the global attractor is not contained in any finite-dimensional invariant manifold of class C^1. In §4 , we exhibit a family of equations depending on a parameter which varies over an open ball of a certain Banach space, for which family we have been able to prove that the property of non-existence of a finite-dimensional (local) invariant manifold of class C^1 containing the global attractor is indeed generic.

Finally, let us remark that this example of non-existence differs from the one given by Mallet-Paret, Sell [1986] for parabolic equations in that we do not require our manifolds to be normally hyperbolic.

1. The equations and a result of existence

We shall be considering evolution problems of the following form, where u is a function of $x \in (0,\pi) =: \Omega$ and $t \in \mathbb{R}$ with values in $\mathbb{R}$:

$$u_{tt} + 2\alpha u_t = u_{xx} + f(x,u) \qquad (1.1)$$

$$u_x|_{x=0} = u_x|_{x=\pi} = 0 \tag{1.2$_N$}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = v_0 \tag{1.3}$$

or the analogous one where $(1.2)_N$ is replaced by

$$u|_{x=0} = u|_{x=\pi} = 0 \tag{1.2$_D$}$$

Here, α is a non-negative real parameter, called the damping coefficient, and f is a function $\Omega\times\mathbb{R}\to\mathbb{R}$. The function f is assumed to satisfy the following conditions:

(F1a) $f(\cdot,u)$ is measurable for all $u\in\mathbb{R}$; $f(x,\cdot)$ is of class $C^{1+\eta}$ for almost all $x\in\Omega$ and some $\eta>0$ independent of x.

(F1b) For every bounded open interval $J\subset\mathbb{R}$, the quantity

$$\|f\|^2_{(J)} := \int_\Omega \left[\left(\sup_{u\in J}|f(x,u)|\right)^2 + \left(\sup_{u\in J}|f_u(x,u)|\right)^2 + \left(\sup_{\substack{u,v\in J\\ u\neq v}} \frac{|f_u(x,u)-f_u(x,v)|}{|u-v|^\eta}\right)^2\right]dx \tag{1.4}$$

is finite (i.e. the mapping $x\longmapsto f(x,\cdot)$ from Ω to the Banach space $C^{1+\eta}(\bar{J})$ belongs to L_2 in the sense of Bochner)

(F2) $$\limsup_{|u|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} f(x,u)}{u} < 0 \tag{1.5}$$

Thereafter, we shall consider variations in the function f. For our purpose, it will suffice to restrict our attention to variations of the following form

$$f(x,u) = f_0(x,u) + g(x,u) \tag{1.6}$$

where f_0 is a fixed function satisfying (F1a),(F1b),(F2) , and the perturbation g varies within a space of the form

$$\mathcal{G}(J) := \left\{ g:\Omega\times\mathbb{R}\to\mathbb{R} \;\middle|\; \begin{array}{l} g \text{ satisfies (F1a),(F1b) , and} \\ g(x,u)=0 \text{ whenever } u\notin J \end{array} \right\} \tag{1.7}$$

where J is a fixed bounded open interval of $\mathbb{R}$. One easily verifies that $\mathcal{G}(J)$ is a Banach space with norm given by (1.4) .

Problem (1.1),$(1.2)_B$,(1.3) (B stands for either N or D) will be viewed as a second order evolution problem for a functional variable $u : \mathbb{R} \longrightarrow L_2 := L_2(\Omega)$, namely

$$\ddot{u} + 2\alpha\dot{u} + Au = Fu \tag{1.8}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0 \tag{1.9}$$

where A and F denote the operators on L_2 given by

$$Au := -u_{xx} \tag{1.10}$$

$$(Fu)(x) := f(x, u(x)) \tag{1.11}$$

with domains respectively equal to H^2_B and H^1_B. Here, H^k_B $(k=1,2)$ denote the closures in H^k of the set $\{u : \Omega \rightarrow \mathbb{R} \mid u \in C^\infty(\bar{\Omega})$ and satisfies $(1.2)_B \}$.

Problem (1.8),(1.9) can be rewritten as a first order evolution problem for the pair $(u,\dot{u})$ as a variable with values in $H^1_B \times L_2$. Instead of this, we shall find more convenient to use the pair

$$U := (u,w) := (u, \alpha u + \dot{u}) \tag{1.12}$$

The problem takes then the following form, where $U := (u,w)$ represents a variable with values in $H^1_B \times L_2$:

$$\dot{U} = -\alpha U + \mathbb{A}U + \mathbb{F}U \tag{1.13}$$

$$U(0) = U_0 \tag{1.14}$$

Here, $\mathbb{A}$ and $\mathbb{F}$ denote the operators on $H^1_B \times L_2$ given by

$$\mathbb{A}(u,w) := (w, \alpha^2 u - Au) \tag{1.15}$$

$$\mathbb{F}(u,w) := (0, Fu) \tag{1.16}$$

with domains respectively equal to $H^2_B \times H^1_B$ and $H^1_B \times L_2$. When necessary, the dependence of things with respect to q will be made explicit by writing q as a subindex, like in f_q , F_q , or $\mathbb{F}_q$.

It is a standard fact that the operator $\mathbb{A}$ is the generator of a group on $H^1_B \times L_2$. In fact, this will follow as a lateral result from the estimates obtained in §2 . On the other hand, the form of conditions (F1a) and (F1b) implies that, for every bounded open interval $J \subset \mathbb{R}$, the mapping $(U,g) \mapsto \mathbb{F}_g U$ goes from $(H^1_B \times L_2) \times \mathcal{G}(J)$ to $H^1_B \times L_2$, and it is of class $C^{1+\eta}$ uniformly on bounded sets. With this, the preceeding problem fits in the standard theory of semilinear evolution equations, which allows us to obtain the following result:

Theorem 1.1 . Assume that f satisfies (F1a),(F1b),(F2) . Then, the preceeding problem generates a group $T(t)\ (t \in \mathbb{R})$ (of nonlinear operators) on $H^1_B \times L_2$ with the following properties: (i) For every $U_0 \in H^1_B \times L_2$, the mapping $T(\cdot)U_0 : \mathbb{R} \longrightarrow H^1_B \times L_2$ is continuous; if $U_0 \in H^2_B \times H^1_B$, then this mapping is continuously differentiable. (ii) For each compact interval $[T_0,T_1] \subset \mathbb{R}$, the mapping $H^1_B \times L_2 \ni U_0 \longmapsto T(\cdot)U_0 \in C([T_0,T_1], H^1_B \times L_2)$ is of class $C^{1+\eta}$ uniformly on bounded sets. (iii) There is a compact global attractor in the sense of Babin,Vishik[1983] and Hale[1985] . Assume now that f has the form (1.6) , where f_0 satisfies (F1a),(F1b),(F2) , and g varies over $\mathcal{G}(J)$, J being a fixed bounded open interval of $\mathbb{R}$. Let $T_g(t)\ (t \in \mathbb{R})$ denote the group corresponding to a given g . Then, for each compact interval $[T_0,T_1] \subset \mathbb{R}$, the mapping $(H^1_B \times L_2) \times \mathcal{G}(J) \ni (U_0, g) \longmapsto T_g(\cdot)U_0 \in C([T_0,T_1], H^1_B \times L_2)$ is of class $C^{1+\eta}$ uniformly on bounded sets. ■

The proof of this theorem is fairly standard, so that we shall give only a summary with references.

Summary of the proof. As usual, the curves $T(\cdot)U_0 \in C([0,\infty), H^1_B \times L_2)$ are looked for as solutions of the integral equation resulting from the variation of constants formula. The local existence and uniqueness of solutions of this equation is obtained by means of a contraction mapping argument (see for instance Tanabe[1979] (Thm.6.1.4)). The proof that these solutions can be extended to the whole interval $(-\infty,+\infty)$ is based upon suitable a-priori estimates. These follow easily from the existence and properties of the Lyapunov functional

$$\Phi(u,w) := \int_\Omega \left[\tfrac{1}{2}\dot{u}^2 + \tfrac{1}{2}u_x^2 - q(\cdot,u(\cdot))\right], \quad \text{where} \quad q(x,u) := \int_0^u f(x,\xi)\,d\xi \tag{1.17}$$

(here and in the following $\dot{u}$ stands for $w-\alpha u$). It is a well-known fact that $\dot{\Phi}$, the derivative of Φ along a solution of (1.13) , is given by $\dot{\Phi} = -2\alpha\int_{\Omega}\dot{u}^2$. The a-priori estimates mentioned above follow from the following twoproperties of Φ , whose derivation uses the fact that f satisfies (F2) : (i) the level sets of Φ , $\{U \mid \Phi(U) \leq c\}$ $(c \in \mathbb{R})$, are bounded in $H^1_B \times L_2$; (ii) $\dot{\Phi}$ satisfies an inequality of the form $-4\alpha(\Phi + M) \leq \dot{\Phi} \leq 0$. The proof that $T(\cdot)U_0$ is continuously differentiable when $U_0 \in H^2_B \times H^1_B$ can be found for instance in Tanabe [1979] (Thm.6.1.3) . The $C^{1+\eta}$ dependence of solutions with respect to the initial state U_0 and the parameter q can be obtained by proceeding as in Henry [1981] (Thm.3.4.4) . Finally, the proof of statement (iii) can be found in Hale [1985] (§3, Thm.6.1) (see also Babin,Vishik [1983] (Thm.6.1)) .

In the following we shall refer to the group $T(t)$ $(t \in \mathbb{R})$ as the dynamical system generated by problem (1.1),$(1.2)_B$,(1.3) .

Next we reproduce a slightly generalized version of a result obtained in Mora [1986] , which establishes the fact that, for large values of α, the global attractor is contained in a finite-dimensional local invariant manifold of class C^1 . A previous result ofn this direction has been obtained by Solà-Morales, València [1986] , who, for a spatially homogeneous problem with Neumann boundary conditions, give sufficient conditions on the coefficients which ensure that all the flow is attracted by the invariant subspace formed by the spatially homogeneous states.

Theorem 1.2 . Assume that f satisfies (F1a),(F1b),(F2) and $f(x,0) = 0$, and consider the dynamical system on $H^1_B \times L_2$ generated by problem (1.1), $(1.2)_B$,(1.3) . There exists a finite constant ℓ such that, for every integer n satisfying the following condition

$$2n+1 > 8\ell, \qquad \alpha^2 > (n+1)^2 + \frac{16\ell^2}{(2n+1)-8\ell} \tag{1.17}$$

there is a local invariant submanifold of class C^1 and dimension n (for $B=D$) or $n+1$ (for $B=N$) which contains the global attractor. ■

Corollary 1.3 . There exists a finite constant α^* such that, for $\alpha > \alpha^*$, the global attractor lies in a finite-dimensional local invariant submanifold of class C^1 . ■

The proof of Theorem 1.2 is a trivial generalization of the one given in

Mora [1986] . We only remark here that it is crucially based upon the use of the inner products presented in §2.1 below.

2. The linear problem

In this section we deal with an abstract linear evolution problem which includes the linear damped wave equation as well as the linearization of a semilinear equation about a stationary state. The problem under consideration has the form

$$\ddot{u} + 2\alpha\dot{u} + Au = 0 \tag{2.1}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0 \tag{2.2}$$

where u is now a variable with values in a general Hilbert space E , α is a real number (not necessarily negative), and A is a self-adjoint operator on E with numerical range bounded from below; i.e.

$$\mu_1 := \inf_{u \in Dom(A)} \frac{\langle Au,u\rangle}{\langle u,u\rangle} > -\infty \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of E . As it is well-known, μ_1 coincides with the smallest element of the spectrum of A . In the following, $E^{1/2}$ will denote the Hilbert space consisting of the domain of the operator $(A+\xi I)^{1/2}$ endowed with the inner product

$$\langle u,\hat{u}\rangle_{1/2} := \langle (A+\xi I)^{1/2}u, (A+\xi I)^{1/2}\hat{u}\rangle \tag{2.4}$$

where ξ is a real number greater than $-\mu_1$. Different choices of $\xi > -\mu_1$ result in the same vector space with different but equivalent inner products.

Similarly as before, problem (2.1),(2.2) will be reconsidered as a first order evolution problem for the pair $U := (u,w) := (u,\alpha u+\dot{u})$; i.e.

$$\dot{U} = -\alpha U + \mathbb{A}U \tag{2.5}$$

$$U(0) = U_0 \tag{2.6}$$

where $\mathbb{A}$ will have the form (1.15) . Here, U will be considered as taking values in $\mathbb{E} := E^{1/2} \times E$, and $\mathbb{A}$ will be considered as an operator on

$\mathbb{E}$ with domain $\mathrm{Dom}(\mathbb{A}) := \mathrm{Dom}(A) \times E^{1/2}$.

2.1 Choice of inner products

The inner product on $\mathbb{E} := E^{1/2} \times E$ will be taken as the direct sum of one of the inner products on $E^{1/2}$ given by (2.4) and the inner product on E . Among all the possible values of ξ , we will choose a particular one which makes the numerical range of $\mathbb{A}$ to be contained in a vertical strip as small as possible. As it is shown by the following theorem, this is obtained for $\xi = \max(-\alpha^2, \alpha^2 - 2\mu_1)$, which amounts to take

$$\langle U, \hat{U} \rangle_{\mathbb{E}} := \begin{cases} \langle (A-\alpha^2 I)^{1/2} u, (A-\alpha^2 I)^{1/2} \hat{u} \rangle + \langle w, \hat{w} \rangle & \text{if } \alpha^2 < \mu_1 \\ \langle (A+(\alpha^2-2\mu_1) I)^{1/2} u, (A+(\alpha^2-2\mu_1) I)^{1/2} \hat{u} \rangle + \langle w, \hat{w} \rangle & \text{if } \alpha^2 > \mu_1 \end{cases} \tag{2.7}$$

This choice of ξ has been inspired by the work of Solà-Morales, València [1986] .

Theorem 2.1 . For $\alpha^2 < \mu_1$, the numerical range of $\mathbb{A}$ is contained in the imaginary axis. For $\alpha^2 > \mu_1$, it is contained in the strip $|\mathrm{Re}\,\lambda| \leq \sqrt{\alpha^2 - \mu_1}$. ■

Proof . We have to estimate $|\mathrm{Re}\langle \mathbb{A}U, U\rangle_{\mathbb{E}}| / \langle U, U\rangle_{\mathbb{E}}$ when $U = (u, w)$ is a general element of $\mathrm{Dom}(\mathbb{A}) = \mathrm{Dom}(A) \times E^{1/2}$.

(a) Case $\alpha^2 < \mu_1$. It suffices to notice that

$$\begin{aligned} \langle \mathbb{A}U, U \rangle_{\mathbb{E}} &= \langle (A-\alpha^2 I)^{1/2} w, (A-\alpha^2 I)^{1/2} u \rangle + \langle \alpha^2 u - Au, w \rangle \\ &= \langle w, Au - \alpha^2 u \rangle + \langle \alpha^2 u - Au, w \rangle \end{aligned}$$

which implies

$$\mathrm{Re}\, \langle \mathbb{A}U, U \rangle_{\mathbb{E}} = 0$$

(b) Case $\alpha^2 > \mu_1$. In this case we have

$$\begin{aligned} \langle \mathbb{A}U, U \rangle_{\mathbb{E}} &= \langle (A+(\alpha^2-2\mu_1)I)^{1/2} w, (A+(\alpha^2-2\mu_1)I)^{1/2} u \rangle + \langle \alpha^2 u - Au, w \rangle \\ &= \langle w, Au + \alpha^2 u - 2\mu_1 u \rangle + \langle \alpha^2 u - Au, w \rangle \\ &= 2 \langle w, (\alpha^2 - \mu_1) u \rangle + \langle w, Au - \alpha^2 u \rangle + \langle \alpha^2 u - Au, w \rangle \end{aligned}$$

which implies that

$$\mathrm{Re}\langle \mathbb{A}U,U\rangle_{\mathbb{E}} = 2\,\mathrm{Re}\,\langle w,(\alpha^2-\mu_1)u\rangle$$

From this we derive that

$$|\mathrm{Re}\langle \mathbb{A}U,U\rangle_{\mathbb{E}}| \leq 2\,|\langle w,(\alpha^2-\mu_1)u\rangle| = 2\sqrt{\alpha^2-\mu_1}\,|\langle w,\sqrt{\alpha^2-\mu_1}\,u\rangle|$$
$$\leq \sqrt{\alpha^2-\mu_1}\,[(\alpha^2-\mu_1)\langle u,u\rangle + \langle w,w\rangle]$$

where in the last step we have used Schwarz inequality. Now, we only have to verify that the quantity in square brackets is less than or equal to $\langle U,U\rangle_{\mathbb{E}}$. Indeed, we have

$$\langle U,U\rangle_{\mathbb{E}} = \langle Au+(\alpha^2-2\mu_1)u,\ u\rangle + \langle w,w\rangle$$
$$\geq (\alpha^2-\mu_1)\langle u,u\rangle + \langle w,w\rangle$$

where in the last step we have used (2.3) . Q.E.D.

From the properties of A one easily derives that $\mathrm{Range}(\mathbb{A}-\lambda I) = \mathbb{E}$ for any real λ with $|\lambda| > \sqrt{\alpha^2-\mu_1}$. By applying the theory of dissipative operators (see for instance Pazy [1983] (§1.4, Thm.4.3)) one then obtains the following result.

Corollary . For $\alpha^2<\mu_1$, $\mathbb{A}$ is the generator of a group $J(t)$ $(t\in\mathbb{R})$ of unitary operators. For $\alpha^2>\mu_1$, $\mathbb{A}$ is the generator of a group $J(t)$ $(t\in\mathbb{R})$ satisfying the bound

$$\|J(t)\| \leq \exp(\sqrt{\alpha^2-\mu_1}\,|t|) \qquad (\forall t\in\mathbb{R}) \quad \blacksquare \tag{2.8}$$

In order to see that the preceeding estimates are optimal, it suffices to notice that $\mu_1\in\mathrm{Spec}(A)$ implies $\pm(\alpha-\mu_1)^{1/2}\in\mathrm{Spec}(\mathbb{A})$, which shows that the vertical strip which contains the numerical range is the smallest possible.

2.2 The infinite-dimensional whirl

In this paragraph we assume that A has compact resolvent. In the following, e_k and μ_k $(k=1,2,\dots)$ denote respectively a complete orthonormal system of eigenfunctions of A and the corresponding sequence of eigenvalues, which sequence is assumed to be non-decreasing. Finally, E_k will denote the one-dimensional space generated by e_k. We then have the orthogonal decomposition invariant by A $E=\bigoplus_{k=1}^{\infty} E_k$. Correspondingly, the space $\mathbb{E}:=E^{1/2}\times E$ has the orthogonal decomposition $\mathbb{E}=\bigoplus_{k=1}^{\infty}\mathbb{E}_k$, where $\mathbb{E}_k:=E_k\times E_k$. This decomposition of $\mathbb{E}$ is invariant by $\mathbb{A}$ and also by the group $J(t):=e^{\mathbb{A}t}$ $(t\in\mathbb{R})$.

Let us now assume that $\alpha^2<\mu_1$. Then the effect of $J(t)$ on $\mathbb{E}_k$ consists in a rotation of angle $\omega_k t$, where $\omega_k:=(\mu_k-\alpha^2)^{1/2}$. From this fact it follows that, for every $U\in\mathbb{E}$, the function $\mathbb{R}\ni t\longmapsto J(t)U\in\mathbb{E}$ is almost periodic.

In the following we consider the group $L(t)$ $(t\in\mathbb{R})$ generated by (2.5),(2.6). Obviously, $L(t)=e^{-\alpha t}J(t)$.

Proposition 2.3. Assume that A has compact resolvent, $\alpha>0$, and $\alpha^2<\mu_1$. If a positive semiorbit of $L(t)$ is contained in a submanifold M of $\mathbb{E}$ differentiable at the origin, then it is contained also in the tangent subspace of M at the origin. ∎

Proof. Let U_0 be a point of the semiorbit which is assumed to be contained in M. Let F be the tangent subspace of M at the origin. Finally, let P denote the orthogonal projection of $\mathbb{E}$ onto F, and $Q:=I-P$. The fact that F is tangent to M at the origin means that

$$\|QU\| = o(\|PU\|) \quad \text{as} \quad U\to 0 \quad \text{on } M$$

In particular, this implies that

$$\|QL(t)U_0\| = o(\|PL(t)U_0\|) \quad \text{as} \quad t\to+\infty$$

or, equivalently since $L(t)=e^{-\alpha t}J(t)$,

$$\|QJ(t)U_0\| = o(\|PJ(t)U_0\|) \quad \text{as} \quad t\to+\infty$$

Using the fact that $J(t)U_0$ is an almost periodic function of t, one can then derive that $QU_0=0$, i.e. $U_0\in F$. Q.E.D.

Corollary 2.4. Under the hypotheses of Proposition 2.3, the only $L(t)$-invariant submanifolds of $\mathbb{E}$ differentiable at the origin are the invariant closed linear subspaces. ∎

Proposition 2.5. Assume that A has compact resolvent, all its eigenvalues are simple, and $\alpha^2<\mu_1$. Then, the only $L(t)$-invariant closed linear subspaces of $\mathbb{E}$ are those of the form $\mathbb{E}_K = \bigoplus_{k\in K} \mathbb{E}_k$, where K is a subset of $\mathbb{N}\setminus\{0\}$. ■

Proof . Let F be an invariant closed linear subspace. By linearity, invariance by $L(t)$ is equivalent to invariance by $J(t) = e^{\alpha t} L(t)$. On the other hand, using the fact that the operators $J(t)$ $(t\in\mathbb{R})$ are unitary, we see that the invariance of F by $J(t)$ implies the same property for $F^{\perp}$. In order to prove the proposition, it suffices to show that, for any $k\in\mathbb{N}\setminus\{0\}$ one has either $\mathbb{E}_k \subset F$ or $\mathbb{E}_k \subset F^{\perp}$. Now, since both F and $F^{\perp}$ are invariant and, on the other hand, the $\mathbb{E}_k$ do not contain proper invariant subspaces, the preceeding alternative is equivalent to the following statement :

$$\text{for any } k\in\mathbb{N}\setminus\{0\}, \text{ it happens that } \quad F\cap\mathbb{E}_k \neq \{0\} \text{ or } F^{\perp}\cap\mathbb{E}_k \neq \{0\} \tag{2.9}$$

In proving (2.9) , we shall use the fact that

$$U\in\mathbb{E}_k \iff S(t)U := \tfrac{1}{2}[J(t)+J(-t)]U = (\cos\omega_k t)\,U \tag{2.10}$$

which follows from the fact that $J(t)$ restricted to $\mathbb{E}_k$ is a rotation of frequency ω_k , and the hypothesis that the eigenvalues μ_k , and therefore the frequencies ω_k , are all different. In order to prove (2.9) , we shall take an arbitrary $U\in\mathbb{E}_k$ and show that it belongs to either F or $F^{\perp}$. For this, we decompose $U\in\mathbb{E}_k\setminus\{0\}$ in its F and $F^{\perp}$ components:

$$U = V + W, \quad \text{where } V\in F \text{ and } W\in F^{\perp}$$

By applying the operators $S(t) := \frac{1}{2}[J(t)+J(-t)]$, and using the invariance and linearity of F and $F^{\perp}$, one obtains that

$$S(t)U = S(t)V + S(t)W \quad , \text{ where } S(t)V\in F \text{ and } S(t)W\in F^{\perp}$$

By using (2.10) , one immediately obtains that both $V\in F$ and $W\in F^{\perp}$ must belong to $\mathbb{E}_k$. Since V and W cannot be simultaneously zero, this proves (2.9) . Q.E.D.

Corollary 2.6 . Under the hypotheses of Proposition 2.5 , $\mathbb{E}$ has only a countable family of finite-dimensional $L(t)$-invariant closed linear subspaces. ■

By combining the preceeding facts, we can state the following result :

Theorem 2.7 . Assume that A has compact resolvent, all its eigenvalues are simple, $\alpha > 0$, and $\alpha^2 < \mu_1$. Then the group generated by (2.5),(2.6) has only a countable family of finite-dimensional invariant submanifolds containing the origin and being differentiable at it. ∎

Remark . The result is no longer true when the condition of differentiability at the origin is dropped. If the frequencies ω_k satisfy some linear relation with integer coefficients, then one can have continuous families of finite-dimensional invariant Lipschitzian submanifolds containing the origin and being differentiable everywhere except at the origin.

3. A C^1 linearization theorem

In this section we give a C^1 linearization theorem which is applicable to certain stationary states of semilinear damped wave equations. In the finite-dimensional case, our result is included essentially in that of Hartman [1960] (Thm.(I)) , which instead of our condition (3.2) requires only that L be a contraction.

Theorem 3.1 . Let U be an open subset of a Banach space X , and T a C^1 map $U \to X$ with a fixed point p . Let L be the Fréchet derivative of T at p , i.e. $L := DT(p)$. Assume that L has a bounded inverse, and that the following properties are satisfied for some $\eta > 0$:

$$DT(p+x) - L = o(\|x\|^\eta) \quad \text{as } x \to 0 \tag{3.1}$$

$$\|L^{-1}\| \, \|L\|^{1+\eta} < 1 \tag{3.2}$$

Then, there exist V , neighbouhood of p in U , with $T(V) \subset V$, and R , a C^1 diffeomorphism onto its image, with $R(p) = 0$, $DR(p) = I$, and

$$DR(p+x) - I = o(\|x\|^\eta) \quad \text{as } x \to 0 \tag{3.3}$$

such that the following equation holds :

$$R\,T = L\,R \tag{3.4}$$

Such a map is unique in the following sense: if V' and R' satisfy also the preceeding properties, then R and R' coincide in any ball centered at p and contained in $V \cap V'$. ■

Remarks . (i) Condition (3.2) implies that L is a contraction.
(ii) The exponent η is by no means restricted to be less than 1 ; increasing η makes condition (3.2) less restrictive, but then condition (3.1) requires T to be closer to linear.

Proof . Without loss of generality, we assume $p=0$. Let us rewrite equation (3.4) in the equivalent form

$$R = L^{-1} R T \tag{3.5}$$

By writing $T=L+\gamma$ and $R=I+\rho$, this equation for R transforms into the following equation for ρ :

$$\rho = L^{-1} \rho (L+\gamma) + L^{-1}\gamma \tag{3.6}$$

In the sequel, the right-hand side of (3.6) will be denoted by $K(\rho)$, and its first term will be denoted by $K_1(\rho)$:

$$K_1(\rho) := L^{-1} \rho (L+\gamma) , \qquad K(\rho) := K_1(\rho) + L^{-1}\gamma \tag{3.7}$$

The existence and uniqueness of ρ satisfying (3.6) will be obtained by verifying that the transformation K is a contraction in an appropriate Banach space $\mathcal{R}$. In the following, X_δ denotes the open ball of radius δ in the space X . The Banach space $\mathcal{R}$ will consist of the mappings $\rho \in C^1(\overline{X}_\delta, X)$ satisfying $\rho(0)=0$, $D\rho(0)=0$, and $D\rho(x) = o(\|x\|^\eta)$ as $x \to 0$. One can check that this is a Banach space when endowed with the norm

$$\|\rho\|_{\mathcal{R}} := \sup_{x \in X_\delta \setminus \{0\}} \|x\|^{-\eta} \|D\rho(x)\| \tag{3.8}$$

We claim that, for δ sufficiently small, K is a contraction of $\mathcal{R}$. To have this property, we first need that $K(\rho)$ be defined on the same domain as ρ , which amounts to ask that $T=L+\gamma$ map X_δ into itself. This is true for δ sufficiently small, because L is a contraction , and γ is C^1 with $\gamma(0)=0$ and $D\gamma(0)=0$. In fact,

$$\|(L+\gamma)x\| \leq (\|L\| + \varepsilon(\delta)) \|x\| \qquad (\forall x \in X_\delta), \tag{3.9}$$

where

$$\varepsilon(\delta) := \sup_{x \in X_\delta} \| D\gamma(x) \| , \tag{3.10}$$

which has the property that $\varepsilon(\delta) \downarrow 0$ when $\delta \downarrow 0$. Now we can verify that K maps $\mathcal{R}$ into itself. Indeed, $L^{-1}\gamma \in \mathcal{R}$ because $\gamma \in \mathcal{R}$, and $K_1(\rho) \in \mathcal{R}$ because $\rho \in \mathcal{R}$; the fact that $D(K_1(\rho))(x) = o(\|x\|^\eta)$ as $x \to 0$ follows from the estimate

$$\begin{aligned} \|x\|^{-\eta} \| D(K_1(\rho))(x) \| &= \|x\|^{-\eta} \| L^{-1} D\rho((L+\gamma)x) \; (L + D\gamma(x)) \| \\ &\leq \|L^{-1}\| \frac{\|D\rho((L+\gamma)x)\|}{\|(L+\gamma)x\|^\eta} \frac{\|(L+\gamma)x\|^\eta}{\|x\|^\eta} \|L + D\gamma(x)\| \\ &\leq \|L^{-1}\| \left(\|L\| + \varepsilon(\delta)\right)^{1+\eta} \frac{\|D\rho((L+\gamma)x)\|}{\|(L+\gamma)x\|^\eta} \qquad (\forall x \in X_\delta \setminus \{0\}) \end{aligned} \tag{3.11}$$

where $\varepsilon(\delta)$ is the quantity defined by (3.10) . Finally, from (3.11) follows that

$$\| K(\rho) - K(\sigma) \|_{\mathcal{R}} = \| K_1(\rho - \sigma) \|_{\mathcal{R}} \leq \|L^{-1}\| \left(\|L\| + \varepsilon(\delta)\right)^{1+\eta} \|\rho - \sigma\|_{\mathcal{R}} \tag{3.12}$$

which shows that, if condition (3.2) is satisfied and δ is sufficiently small, K is a contraction of $\mathcal{R}$.

In order to complete the proof, it only remains to notice that, if δ is small enough, then R will be a C^1 diffeomorphism onto its image, which follows from the inverse function theorem. The theorem is thus established with $V = X_\delta$. Q.E.D.

<u>Corollary 3.2</u> . Let X be a Banach space, and $T(t)$ $(t \in \mathbb{R})$ a group of diffeomorphisms of X with a fixed point p . Let $L(t)$ $(t \in \mathbb{R})$ be the group of bounded linear operators on X given by $L(t) := D(T(t))(p)$. Assume that, for some $\tau \in \mathbb{R}$ and some $\eta > 0$, $T := T(\tau)$ and $L := L(\tau)$ satisfy properties (3.1) and (3.2) . Then there exist V , neighbourhood of p , and $R : V \to X$, a C^1 diffeomorphism onto its image, with $R(p) = 0$, $DR(p) = I$, and (3.3) , such that, for every $t \in \mathbb{R}$, the equation

$$R \; T(t) = L(t) \; R \tag{3.13}$$

holds in some ball centered at p and contained in V . Such a ball can be chosen independently of t when t varies over any interval of the form $[t_0,+\infty)$ with t_0 finite. ■

Proof . By the preceeding theorem, there exist a neighbourhood V of p and $R: V \to X$, a C^1 diffeomorphism onto its image, satisfying $R(p)=0$, $DR(p)=I$, (3.3) , and

$$L(-\tau)\ R\ T(\tau) \ = \ R$$

Now, one easily verifies that the preceeding properties are also satisfied if V and R are replaced by $V' := T(-t)V$ and $R' := L(-t)RT(t)$, where t is any real number. By the uniqueness statement of Theorem 3.1 , this implies that equation $L(-t)RT(t)=R$ holds in any ball contained in $V \cap T(-t)V$. Finally, to establish the last statement of the corollary, it suffices to show that $\bigcap_{t\geq t_0} T(-t)V$ is a neighbourhood of p . Using the fact that $T(-\tau)V \supset V$, this reduces to see that $\bigcap_{t_0\leq t\leq t_0+\tau} T(-t)V$ is a neighbourhood of p , i.e. there exists a $\delta>0$ such that $T(-t)V \supset X_\delta$ for all $t\in[t_0,t_0+\tau]$, or equivalently $V \supset T(t)X_\delta$ for all $t\in[t_0,t_0+\tau]$, which follows from the joint continuity of the mapping $\mathbb{R}\times X \ni (t,x) \longmapsto T(t)x \in X$ at the points of the compact set $[t_0,t_0+\tau]\times\{p\}$. Q.E.D.

Corollary 3.3 . Let us consider problem (1.1),$(1.2)_B$,(1.3) with the hypotheses of Theorem 1.1 . Let $u=u^*(x)$ be a stationary state, and let μ_1 be the lowest eigenvalue of the differential operator $-\partial_x^2 - f_u(x,u^*(x))$ with boundary conditions $(1.2)_B$. If $\alpha^2<\mu_1$, then, near this stationary state, the flow $T(t)\ (t\in\mathbb{R})$ is C^1-equivalent to its linearization. ■

Proof . Let $L(t)\ (t\in\mathbb{R})$ be the group of bounded linear operators obtained by linearizing $T(t)\ (t\in\mathbb{R})$ at the stationary state $u=u^*$. This coincides with the group generated by equation (2.5) or (2.1) with A being given by the differential operator $-\partial_x^2 - f_u(x,u^*(x))$ with the boundary conditions $(1.2)_B$. Let $\mathbb{E}=H_B^1\times L_2$ be endowed with the inner product (2.7) . According to Corollary 2.2 , $J(t) = e^{\alpha t}L(t)\ (t\in\mathbb{R})$ is a unitary group. Corollary 3.2 can now be applied with no matter which $\tau>0$ and small $\eta>0$, since $T(\tau)$ is $C^{1+\eta}$ and $\|L(\tau)\| = e^{-\alpha\tau}$, $\|L(-\tau)\| = e^{\alpha\tau}$. Q.E.D.

4. Exhibiting non-existence

Our example of non-existence belongs to problem (1.1),$(1.2)_N$,(1.3) with f of the form (1.6) , where f_0 will be fixed and q variable.

The fixed function f_0 will be taken to be independent of x ; accordingly, we shall write $f_0(u)$ for $f_0(x,u)$. This function f_0 is assumed to fulfil the general conditions (F1a),(F1b),(F2) , and also the following particular ones :

$$f_0(0) = f_0(1) = 0 \quad ; \quad 1 \text{ is the only positive zero of } f_0 \tag{4.1}$$

$$0 < f_0'(0) < 1 \tag{4.2}$$

$$f_0'(1) < -\alpha^2 \tag{4.3}$$

Since f_0 is independent of x , and the boundary conditions are of Neumann type, the dynamical system on $\mathbb{E} = H_B^1 \times L_2$ corresponding to $q = 0$ has a two-dimensional invariant linear subspace consisting of the states which are spatially homogeneous (i.e. constant with respect to x); on this subspace, (1.1) reduces to a second-order ordinary differential equation. In the following, $\bar{0}$ and $\bar{1}$ denote the points of this subspace given respectively by $u=0, \dot{u}=0$ and $u=1, \dot{u}=0$. Conditions (4.1)-(4.3) imply the following facts :

Both $\bar{0}$ and $\bar{1}$ are hyperbolic stationary states. (4.4)

$\bar{0}$ has a one-dimensional unstable maniflod,
and $\bar{1}$ is asymptotically stable. (4.5)

There is an heteroclinic orbit from $\bar{0}$ to $\bar{1}$. (4.6)

In fact, the heteroclinic orbit which connects $\bar{0}$ to $\bar{1}$ lies on the subspace of spatially homogeneous states. In the $(u,\dot{u})$-plane this orbit looks like shown in Fig.1 below. In the following, the corresponding solution of (1.1) (which is unique except for translations) will be denoted by $u_0(t)$ $(t \in \mathbb{R})$. As it is easily verified, there exists a time t_0^* such that

On $(-\infty, t_0^*]$, u_0 and $\dot{u}_0$ are both strictly increasing. (4.7)

On $[t_0^*, +\infty)$, u_0 remains $\geq u_0(t_0^*) =: a_0$ (4.8)

We now introduce a perturbation q which will break this special situation occurring for $q=0$. This perturbation q will be allowed to vary

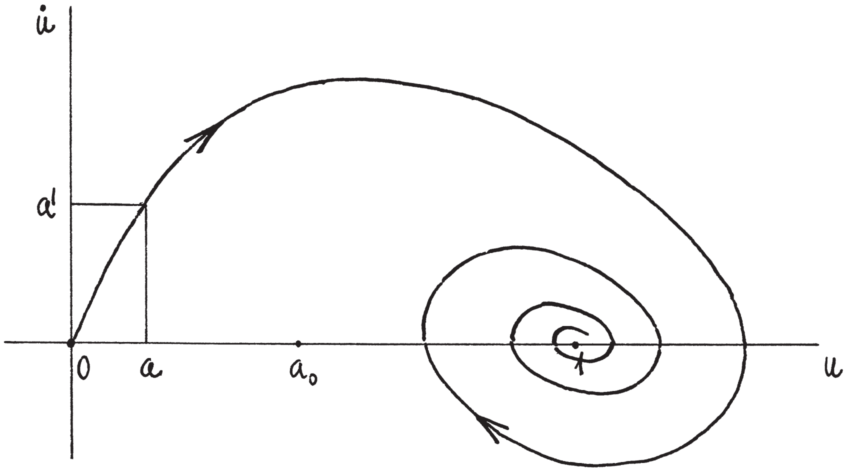

Fig.1

within a certain ball of the Banach space $\mathcal{G}(J)$ defined in (1.7) , where $J := (a,b)$ is a fixed open interval with $0<a<b<1$ and $a<a_0$, where a_0 is the quantity appearing in (4.8) . In the following, the space $\mathcal{G}(J)$ will be denoted simply by $\mathcal{G}$, and its ball of radius δ will be denoted by $\mathcal{G}_\delta$. Clearly, for every $q\in\mathcal{G}$, the corresponding flow still satisfies (4.4) and (4.5) . In fact, these perturbed flows remain unchanged inside the open set $\mathcal{J} := \{U=(u,w) \mid u(x) \notin \bar{J} \ (\forall x\in\Omega)\} \subset \mathbb{E}$.

Let us look at the flow in the neighbourhood of the stationary state $\bar{1}$, where we know it does not depend on q . Since this stationary state is sta-ble, every neighbourhood contains another one which is positively invariant (i.e. $T(t)$ maps it into itself for every $t\geq 0$). Condition (4.3) implies that this stationary state satisfies the hypotheses of Corollary 3.3 . Therefore, we are ensured that $\mathcal{J}$ contains a positively invariant neighbourhood of $\bar{1}$ where the flow $T_0(t) \ (=T_q(t)) \ (t\in\mathbb{R})$ is C^1-equivalent to its linearization at $\bar{1}$, $L(t) \ (t\in\mathbb{R})$. In the following, V denotes a small neighbourhood of $\bar{1}$ with this property. On the other hand, the group $L(t) \ (t\in\mathbb{R})$ coincides with the one generated by equation (2.5) , or (2.1) , with A being the operator given by $-\partial_x^2 - f_0'(1)$ with boundary conditions $(1.2)_N$, which clearly satisfies the hypotheses of Theorem 2.7 . Therefore, we can conclude that V includes only a countable family of fini-te-dimensional local invariant manifolds of class C^1 containing $\bar{1}$.

Let us now consider the orbit that departs form $\bar{0}$ towards the positive u direction. Before leaving $\mathcal{J}$, this orbit will coincide with that corresponding to $q=0$. Therefore, by suitably choosing the time origin, we can assume that the corresponding solution of (1.1), which we shall denote by $u_g(t,x)$, satisfies the following relations :

$$u_g(t,x) = u_0(t) \quad (\forall t \leq 0), \qquad u_g(0,x) = a \tag{4.9}$$

where a is the left end of J. In the following, U_g will denote the curve $\mathbb{R} \to \mathbb{E}$ given by $U_g(t) := (u_g(t,\cdot), \hat{u}_g(t,\cdot))$, where $\hat{u}_g := \alpha u_g + \dot{u}_g$. Now, since $U_0(t) \to \bar{\lambda}$ as $t \to +\infty$, Theorem 1.1 ensures that, if δ is small enough, then the following property will hold :

$$\exists T > 0 \text{ such that, for every } g \in \mathcal{G}_\delta, \; U_g(T) \in V \tag{4.10}$$

In particular, this implies that, for $g \in \mathcal{G}_\delta$, $U_g(t) \to \bar{\lambda}$ as $t \to +\infty$, i.e. the corresponding dynamical system satisfies also (4.6). From now on, we assume δ small enough for property (4.10) to hold.

Let Γ denote the C^1 mapping

$$\Gamma : \mathcal{G}_\delta \ni g \longmapsto U_g(T) \in V \subset \mathbb{E} \tag{4.11}$$

where T is the quantity appearing in (4.10). Our purpose is to see that there are many $g \in \mathcal{G}_\delta$ for which $\Gamma(g) = U_g(T)$ does not belong to any of the countably many finite-dimensional local invariant manifolds of class C^1 containing $\bar{\lambda}$. In this case, one can conclude that there is no finite-dimensional local invariant manifold of class C^1 containing the global attractor. In fact, we will prove the following result :

<u>Theorem 4.1</u> . Consider problem (1.1),$(1.2)_N$,(1.3) with f_0 independent of x and satisfying (F1a),(F1b),(F2),(4.1),(4.2),(4.3) , and g belonging to $\mathcal{G} := \mathcal{G}(J)$, where $J := (a,b)$ is a bounded open interval with $0 < a < b < 1$ and $a < a_0$ (a_0 is the quantity appearing in (4.8) , which depends on f_0). There is a $\delta > 0$ and a residual subset $\mathcal{R}$ of $\mathcal{G}_\delta$ such that if $g \in \mathcal{R}$ then there is no finite-dimensional local invariant manifold of class C^1 containing the global attractor. ∎

The proof of Theorem 4.1 will be based upon the following fact, whose proof will be given afterwards :

<u>Proposition 4.2</u> . Under the hypotheses of Theorem 4.1 , there is a $\delta > 0$ and a dense subset $\mathcal{D}_\delta$ of $\mathcal{G}_\delta$ such that if $g \in \mathcal{D}_\delta$ then $\mathrm{Range}\,(D\Gamma(g))$ is infinite-dimensional. ∎

<u>Proof of Theorem 4.1</u> . Let M_K (K varying among the finite subsets of $\mathbb{N}\setminus\{0\}$) be the countable family of finite-dimensional local invariant manifolds containing $\bar{\Lambda}$. We will see that, for every K , there is a open and dense subset $\mathcal{R}_K$ of $\mathcal{G}_\delta$ such that $g\in\mathcal{R}_K \Longrightarrow \Gamma(g)\notin M_K$. From this the theorem will follow by a cathegory argument. The openness of $\mathcal{R}_K$ is an immediate consequence of the continuous dependence of solutions with respect to g (Theorem 1.1) . The denseness of $\mathcal{R}_K$ follows from Proposition 4.2 : Indeed, if $\mathcal{R}_K$ were not dense, there would be some open set $\mathcal{U}\in\mathcal{G}_\delta$ such that $\Gamma(g)\in M_K$ $(\forall g\in\mathcal{U})$. But this would imply that, for every $g\in\mathcal{U}$, $\mathrm{Range}\,(D\Gamma(g))\subset T(M_K)_{\Gamma(g)}$, which contradicts Proposition 4.2 since $T(M_K)_{\Gamma(g)}$ is finite-dimensional. Q.E.D.

<u>Proof of Proposition 4.2</u> . We will take $\mathcal{D}_\delta:=\mathcal{D}\cap\mathcal{G}_\delta$, where $\mathcal{D}:=\bigcup\limits_{I\subset J}\mathcal{G}(I)$, which is easily verified to be dense in $\mathcal{G}=\mathcal{G}(J)$. Let $g\in\mathcal{D}_\delta$, and assume that it belongs to $\mathcal{G}(I)$ with $I=(a_g,b_g)$ $a<a_g<b_g<b$. In order to see that $\mathrm{Range}\,(D\Gamma(g))$ is infinite-dimensional, we shall see that there is a linearly independent family $h_n\,(n\in\mathbb{N})$ of elements of $\mathcal{G}$ such that the images $D\Gamma(g)\,h_n\,(n\in\mathbb{N})$ are linearly independent elements of $\mathbb{E}$. For every $h\in\mathcal{G}$, the value of $D\Gamma(g)h$ is given by

$$D\Gamma(g)h \;=\; Y(T) \tag{4.12}$$

where

$$Y(t) := \left(y(t,\cdot),\hat{y}(t,\cdot)\right), \quad \hat{y} = \alpha y+\dot{y} \tag{4.13}$$

and $y(t,x)$ is the solution of the first variation equation

$$y_{tt} + 2\alpha y_t = y_{xx} + f'(u_g)y + g_u(x,u_g)y + h(x,u_g) \tag{4.14}$$

$$y_x|_{x=0} = y_x|_{x=\pi} = 0 \tag{4.15}$$

$$y|_{t=0} = y_t|_{t=0} = 0 \tag{4.16}$$

In the following, y_n and Y_n denote the particular y and Y that are obtained when $h=h_n$.

Our choice of the functions h_n will be based upon the following fact, whose proof is given at the end of this section.:

<u>Lemma 4.3</u> . If $\delta>0$ is small enough, then, for every $g\in\mathcal{D}_\delta$, we can find $c_g\in(a,a_g]$ and $t_g^*>0$ such that

$$t \leq t_g^* \Rightarrow u_g(t,x) = u_0(t) \leq a_g \quad (\forall x \in \Omega) \tag{4.17}$$

$$t \geq t_g^* \Rightarrow u_g(t,x) \geq c_g \quad (\forall x \in \Omega) \quad ■ \tag{4.18}$$

We shall use this fact by taking the functions h_n in such a way that

$$h_n(x,u) = 0 \quad \text{for} \quad u \notin (a,c_g) \tag{4.19}$$

By doing so, it results that, in equation (4.14) , the term $g_u(x,u_g)\,y$ vanishes for $t \leq t_g^*$, and the term $h(x,u_g)$ vanishes for $t \geq t_g^*$.

According to this fact, $D\Gamma(g)h_n$ will be given by

$$D\Gamma(g)h_n = Y_n(T) = \Phi(T,t_g^*)\,Y_n(t_g^*) \tag{4.20}$$

where $\Phi(t,s)$ $(s \leq t)$ denotes the system of evolution operators of the linear problem obtained from (4.14),(4.15) when dropping the last term of (4.14) . However, these operators are isomorphisms of the Hilbert space $\mathbb{E}$. Therefore, our problem reduces to show that the $h_n\ (n \in \mathbb{N})$, which must satisfy (4.19) , can be chosen in such a way that the resulting $Y_n(t_g^*)$ be linearly independent elements of $\mathbb{E}$.

To study this question, equation (4.14) needs to be considered only in the interval $0 \leq t \leq t_g^*$, where, by Lemma 4.3 , we know that the term $g_u(x,u_g)\,y$ vanishes, and also that $u_g(t,x)$ is independent of both x and g . We shall take advantage of these facts by choosing the functions h_n in the factorial form $h_n(x,u) = c_n(x)\,\varphi_n(u)$, which makes possible to solve the equation by separation of variables. More specifically, we shall take

$$h_n(x,u) = \varphi_n(u)\cos nx \tag{4.21}$$

where, for every $n \in \mathbb{N}$, φ_n will be a function $\mathbb{R} \to \mathbb{R}$ of class $C^{1+\eta}$ with support contained in the interval $[a,c_g]$. By separating variables, we obtain that, for $0 \leq t \leq t_g^*$,

$$y_n(t,x) = \gamma_n(t)\cos nx \tag{4.22}$$

where γ_n is the solution of the ordinary differential equation problem

$$\ddot{\gamma}(t) + 2\alpha\dot{\gamma}(t) + n^2\gamma(t) - f_0'(u_0(t))\,\gamma(t) = \varphi_n(u_0(t)) \tag{4.23}$$

$$\gamma(0) = \dot{\gamma}(0) = 0 \tag{4.24}$$

In the following, the expression appearing at the left-hand side of (4.23) will be denoted by $L_n(\gamma;t)$.

Obviously, to attain our purpose, it will suffice to find a family $\varphi_n\ (n\in\mathbb{N})$ of functions $\mathbb{R}\to\mathbb{R}$ of class $C^{1+\eta}$ supported in $[a,c_g]$ such that the corresponding solutions of (4.23),(4.24) satisfy

$$\gamma_n(t_g^*)\neq 0 \quad (\forall n\in\mathbb{N}) \tag{4.25}$$

If f_0 is of class $C^{2+\eta}$, this can be easily accomplished by taking

$$\varphi_n(u) := L_n(\sigma\zeta_n;\, u_0^{-1}(u)) \tag{4.26}$$

where, for each $n\in\mathbb{N}$, ζ_n is a solution of the homogeneous equation $L_n(\zeta;t)=0$ satisfying $\zeta(t_g^*)=1$, and σ is a function $\mathbb{R}\to\mathbb{R}$ of class C^∞ such that $\sigma(t)=0$ for $t\leqslant 0$ and $\sigma(t)=1$ for $t\geqslant u_0^{-1}(c_g)$. By introducing (4.26) into (4.23) , one immediately obtains that $\gamma_n=\sigma\zeta_n$, which obviously satisfies $\gamma_n(t_g^*)=1$. If f_0 is not of class $C^{2+\eta}$ but merely $C^{1+\eta}$, then the resulting functions φ_n need not be of class $C^{1+\eta}$; however, any functions $\hat{\varphi}_n$ of class $C^{1+\eta}$ sufficiently near φ_n in the sup norm will serve our purpose. Q.E.D.

<u>Proof of Lemma 4.3</u> . In order to prove Lemma 4.3 , we shall need the following estimate

$$|u_g(t,x)-u_0(t)| \leqslant K(1-e^{-\alpha t})\,\|g\|_{(J)} \quad (\forall x\in\Omega,\ \forall t\in[0,T]) \tag{4.27}$$

This is an improvement of an estimate given by Theorem 1.1 , namely

$$|u_g(t,x)-u_0(t)| \leqslant K_0\,\|g\|_{(J)} \qquad (\forall x\in\Omega,\ \forall t\in[0,T]) \tag{4.28}$$

In order to derive (4.27) , we proceed as follows. Let us define $v(t,x):=u_g(t,x)-u_0(t)$, and $V(t):=(v(t,\cdot),\hat{v}(t,\cdot))$, where $\hat{v}:=\alpha v+\dot{v}$. By the variation of constants formula, $V:\mathbb{R}\to\mathbb{E}$ is given by

$$V(t) = \int_0^t e^{-\alpha(t-s)}\,J(t-s)\left(\hat{\mathbb{F}}_g(U_g(s))-\hat{\mathbb{F}}_0(U_0(s))\right)ds \tag{4.29}$$

where $J(t)\ (t\in\mathbb{R})$ is the group of unitary operators generated by $\mathbb{A}$ (see § 2) when A is the operator $-\partial_x^2-f_0'(1)$ with boundary conditions $(1.2)_N$, and $\hat{\mathbb{F}}_g$ denote the mappings $H_B^1\times L_2\to H_B^1\times L_2$ corresponding to the functions $\hat{f}_g$ defined by $\hat{f}_g(x,u):=f_g(x,u)-f_0'(1)\,u$.

From (4.29) one obtains that

$$\|V(t)\|_{\mathbb{E}} \leq \int_0^t e^{-\alpha(t-s)}\left[\|\hat{F}_0(u_g(s)) - \hat{F}_0(u_0(s))\|_{L_2} + \|G(u_g(s))\|_{L_2}\right]ds$$

$$\leq \int_0^t e^{-\alpha(t-s)}\left[\left(\|f_0\|_{(M)} + |f_0'(1)|\right)\|u_g(s) - u_0(s)\|_{L_\infty} + \|g\|_{(J)}\right]ds \quad (4.30)$$

where $u_g(t) := u_g(t,\cdot) \in H_B^1$, $\hat{F}_0$ and G denote the functions $H_B^1 \to L_2$ corresponding respectively to $\hat{f}_0$ and g , and M is some bounded open interval of $\mathbb{R}$. From here, the desired estimate (4.27) follows by simply introducing (4.28) into the right-hand side of (4.30) .

Let us proceed with the proof of Lemma 4.3 . We start from (4.7) and (4.8) , where we remark that $t_0^* > 0$, because $a < a_0$ and we have chosen $t = 0$ when $u_0 = a$. From (4.7) , it follows that

$$0 \leq s < t \leq t_0^* \implies u_0(s) < u_0(t) - a'(t-s) \quad (4.31)$$

where $a' := \dot{u}_0(0)$. We now claim that, by virtue of (4.27) , (4.31) and (4.8) imply the following fact :

(A) If δ is small enough, then, for every $g \in \mathcal{G}_\delta$

$$0 < t \leq t_0^* \implies u_0(\tfrac{t}{2}) < u_g(t,x) \quad (\forall x \in \Omega) \quad (4.32)$$

$$t_0^* \leq t < +\infty \implies u_0(\tfrac{t_0^*}{2}) < u_g(t,x) \quad (\forall x \in \Omega) \quad (4.33)$$

Specifically, it suffices that

$$\delta \leq \frac{a'}{2K}\min\left(t_0^*, \frac{1}{\alpha}\right) \quad (4.34)$$

In fact, for $0 < t \leq t_0^*$, (4.27), (4.34), and (4.31) imply that

$$u_g(t,x) \geq u_0(t) - K(1-e^{-\alpha t})\delta > u_0(t) - K\alpha t\delta$$

$$\geq u_0(t) - a'\tfrac{t}{2} > u_0(\tfrac{t}{2})$$

On the other hand, for $t_0^* \leq t \leq T$, (4.27), (4.34), (4.8), and (4.31) imply that

$$u_g(t,x) \geq u_0(t) - K(1-e^{-\alpha t})\delta > u_0(t) - K\delta$$

$$\geq u_0(t) - a'\frac{t_0^*}{2} \geq u_0(t_0^*) - a'\frac{t_0^*}{2} > u_0(\tfrac{t_0^*}{2})$$

Furthermore, it is clear that, if the neighbourbhood V is small enough, then (4.33) will also be true for $t>T$, because for such t $U_g(t,\cdot)$ remains always inside V .

Finally, the statement of Lemma 4.3 is easily obtained from (A) by taking

$$t_g^* := \begin{cases} t_0^* & \text{if } a_g \geq a_0 \\ u_0^{-1}(a_g) & \text{if } a_g \leq a_0 \end{cases}, \qquad c_g := u_0\left(\frac{t_g^*}{2}\right) \tag{4.35}$$

which quantities satisfy

$$0 < t_g^* \leq t_0^*, \qquad a < c_g \leq a_g \tag{4.36}$$

In fact, for $-\infty < t \leq t_g^*$, (4.8) implies that

$$u_g(t,x) = u_0(t) \leq u_0(t_g^*) \leq a_g$$

On the other hand, for $t_g^* \leq t \leq t_0^*$, (4.22) and (4.8) imply that

$$u_g(t,x) \geq u_0\left(\frac{t}{2}\right) \geq u_0\left(\frac{t_g^*}{2}\right) =: c_g$$

Finally, for $t_0^* \leq t < +\infty$, (4.23) and (4.8) imply that

$$u_g(t,x) > u_0\left(\frac{t_0^*}{2}\right) \geq u_0\left(\frac{t_g^*}{2}\right) =: c_g$$

Q.E.D.

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