A brief excursion to the viscous case

In a fluid, the balance of momentum reads

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho \mathbf{u}\, d\mathcal{V} = \int_{\partial\Omega_0(t)} \boldsymbol{\sigma}\cdot \mathbf{n} d\mathcal{S} + \int_{\Omega_0(t)} \rho \mathbf{G}\, d\mathcal{V},$$

where $\boldsymbol{\sigma}$ is the *stress tensor*. In Euler's equations $\boldsymbol{\sigma} = -p\boldsymbol{I}$. Viscous forces depend on the velocity differences between neighboring particles. In the limit this means that they depend on the 3×3 matrix $D = \nabla \mathbf{u}$, and, more precisely, only on $\frac{1}{2}(D + D^T)$. This is because $D = \frac{1}{2}(D + D^T) + \frac{1}{2}(D - D^T)$ and this second matrix is antisymmetric. Antisymmetric matrices represent the velocity field of a rigid rotation, that does not generate viscous forces. For other reasons (frame independence, see problem below), this viscous part is exactly a scalar multiple of $\frac{1}{2}(D + D^T)$. Finally

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

and μ is called the *dynamic viscosity*.

Problem 9.1: Let M_3 be the linear space of all 3×3 real matrices and S_3 the subspace of the symmetric matrices. Suppose that $\sigma : M_2 \to S_3$ is a linear map with the property that if U is an orthogonal matrix then
$\sigma(U \cdot M \cdot U^{-1}) = U \cdot \sigma(M) \cdot U^{-1}$ for all $M \in M_3$.
Prove that there exist coefficients λ and μ such that σ is of the following form

$$\sigma(M) = \lambda \text{Trace}(M) I_3 + \mu(M + M^T).$$

(Hint: See the book of AJ Chorin and JE Marsden, A Mathematical Introduction to Fluid Mechanics)

The balance of momentum, reads $\dfrac{d}{dt}\int_{\Omega_0(t)} \rho \mathbf{u} \, d\mathcal{V} =$

$$\int_{\partial\Omega_0(t)} -p\mathbf{n} \, d\mathcal{S} + \mu \int_{\partial\Omega_0(t)} (\nabla\mathbf{u} + \nabla\mathbf{u}^T) \cdot \mathbf{n} \, d\mathcal{S} + \int_{\Omega_0(t)} \rho\mathbf{G} \, d\mathcal{V}$$

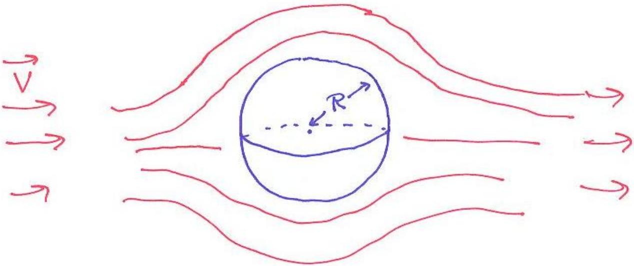
$$= \int_{\Omega_0(t)} -\nabla p \, d\mathcal{V} + \mu \int_{\Omega_0(t)} \nabla \cdot (\nabla\mathbf{u} + \nabla\mathbf{u}^T) \, d\mathcal{V} + \int_{\Omega_0(t)} \rho\mathbf{G} \, d\mathcal{V}.$$

The term $\nabla \cdot (\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ means the divergence, row by row, of the tensor $(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. The i-th row has components $(u^i_j + u^j_i)_{j=1,2,3}$, so its divergence is $\sum_{j=1,2,3}(u^i_{jj} + u^j_{ij})$. By the incompressibility $\sum_{j=1,2,3} u^j_j = 0$, we get that the divergence of the i-th row is $\nabla^2 u^i$. Since all this is true for every control volume Ω_0 we obtain the Navier-Stokes equations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{G},$$

where $\nu = \mu/\rho$ is called the *kinematic viscosity*.

Example: drag force on an obstacle



We want to use the magnitudes R and V to non-dimensionalize the Navier-Stokes Equations.
Non-dimensional formulations are important in fluid dynamics in order to exploit similarity between flows.

Reynolds Number and similarity

Non dimensional variables: $\mathbf{u}' = \mathbf{u}/V$, $p' = p/(\rho V^2)$, $\mathbf{G}' = \mathbf{G}R/V^2$, $t' = tV/R$ and $\mathbf{x}' = \mathbf{x}/R$. Then $\frac{\partial}{\partial t'} = \frac{R}{V}\frac{\partial}{\partial t}$ and $\nabla' = R\nabla$, and we can rewrite the non-dimensional Navier Stokes Equations (dropping the primes) as

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \dfrac{1}{(\mathrm{Re})}\nabla^2\mathbf{u} + \mathbf{G} \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

where $(\mathrm{Re}) = \rho RV/\mu$, and with boundary conditions $\mathbf{u} = 0$ at $|\mathbf{x}| = 1$ and $\mathbf{u} \to (1, 0, 0)$ at infinity.

Stokes Equations

For (Re) $= \rho RV/\mu \to 0$ (very slow, very viscous...) in

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \dfrac{1}{(\mathrm{Re})}\nabla^2\mathbf{u} + \mathbf{G} \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

one gets (at least formally)

$$\begin{cases} 0 = -\nabla p + \dfrac{1}{(\mathrm{Re})}\nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

(Stokes equations) with the same boundary conditions.

Spherical polar coordinates

One takes x as the polar axis and (ρ, θ, ϕ) (θ the polar angle), and want to solve Stokes equations in these coordinates. Axisymmetric motion means $\mathbf{u} = (u^\rho(\rho, \theta), u^\theta(\rho, \theta), 0)$. Incompressibility now means

$$\frac{\partial u^\rho}{\partial \rho} + \frac{2u^\rho}{\rho} + \frac{1}{\rho \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \ u^\theta) = 0$$

and this can be integrated as

$$u^\rho = -\frac{1}{\rho^2 \ \sin\theta} \frac{\partial \psi}{\partial \theta}, \ u^\theta = \frac{1}{\rho \ \sin\theta} \frac{\partial \psi}{\partial \rho}$$

where $\psi = \psi(\rho, \theta)$ is known as the Stokes stream function.

With the help of this auxiliary function $\psi(\rho, \theta)$ the problem of the Stokes flow around the sphere can be explicitly solved and one gets (in the non-dimensional variables)

$$\begin{cases} u^\rho = \cos\theta \left(1 - \frac{3}{2\rho} + \frac{1}{2\rho^3}\right) \\ u^\theta = -\sin\theta \left(1 - \frac{3}{4\rho} - \frac{1}{4\rho^3}\right) \\ p = p_0 - \frac{3\cos\theta}{(\mathrm{Re})\rho^2}. \end{cases}$$

Returning to dimensional variables and calculating the force on the sphere one finally obtains the important Stokes drag formula

$$\mathbf{f} = (f^x, f^y, f^z) = (6\pi\mu R V, 0, 0)$$

(or $\mathbf{f}' = \mathbf{f}/(\rho R^2 V^2) = 6\pi/(\mathrm{Re})$).

Problem 9.2: Prove the existence of the Stokes Stream Function $\psi(\rho, \theta)$ for incompressible axisimetric motions and write Stokes Equations in terms of ψ.

(Hint: See the book of SC Hunter, Mechanics of Continuous Media)

2-D Stokes paradox

The following problem has no solution (Flow around a cylinder):

$$\begin{cases} 0 = -\nabla p + \dfrac{1}{(\mathrm{Re})}\nabla^2 \mathbf{u} \quad \text{in} \quad \mathbb{R}^2 \setminus \{x^2 + y^2 \leq 1\} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = (0,0) \text{ on } x^2 + y^2 = 1 \\ \mathbf{u} \to (1,0) \text{ as } x^2 + y^2 \to \infty \end{cases}$$

Proof:
Taking $\nabla\cdot$ in the first equations and using that $\nabla\cdot\mathbf{u}=0$ we get that $\nabla^2 p = 0$. So, $\nabla^2 p_x = 0$, and if $\mathbf{u}=(u,v)$, then

$$\begin{cases} \nabla^4 u = 0 \text{ in } \mathbb{R}^2 \setminus \{x^2+y^2 \le 1\} \\ u = 0 \text{ on } x^2+y^2=1 \\ u \to 1 \text{ as } x^2+y^2 \to \infty \end{cases} \tag{1}$$

If (1) had a solution, then it would have a solution $u = u(\rho)$ (*angular* averages), and $\nabla^4 u(\rho) = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^2 u(\rho) = 0$, with a general solution $u = A + B\log\rho + C\rho^2 + D\rho^2\log\rho$, which is incompatible with the boundary conditions.

Boundary layer equations of Prandtl, 1

$(\text{Re}) \to \infty$ is a *singular limit* in the N-S equations. One can imagine that far from the boundaries the flow is potential, but in a thin layer (of width δ) near the boundaries the viscous effects are important.
We study the problem in the half-plane $y > 0$ with $(u, v) \sim (1, 0)$ for $y > \delta$. Change of variables to make the layer of width 1: $x' = x, y' = y/\delta, u' = u, v' = v/\delta, t' = t, p' = p$ (and drop the primes).

$$\begin{cases} u_t + uu_x + \delta v \frac{1}{\delta} u_y = -p_x + \frac{1}{(\text{Re})}\left(u_{xx} + \frac{1}{\delta^2} u_{yy}\right) \\ \delta v_t + u\delta v_x + \delta v \frac{1}{\delta} \delta v_y = -\frac{1}{\delta} p_y + \frac{1}{(\text{Re})}\left(\delta v_{xx} + \frac{\delta}{\delta^2} v_{yy}\right) \\ u_x + \frac{\delta}{\delta} v_y = 0. \end{cases}$$

Now we look for $\delta \simeq \delta(\text{Re})$ when $\text{Re} \simeq \infty$ and $\delta \simeq 0$:

Boundary layer equations of Prandtl, 2

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \dfrac{1}{(\mathrm{Re})\delta^2} u_{yy} \\ 0 = -p_y \\ u_x + v_y = 0. \end{cases}$$

If $(\mathrm{Re})\delta^2 \to 0$ when $(\mathrm{Re}) \to \infty$ (δ small), only a viscous term survives, at least in the first equation. This means we are making the same type of simplification we did in the deduction of Stokes equations, so we are looking too close to the boundary. If $(\mathrm{Re})\delta^2 \to \infty$ (δ large) then the viscous effects are excluded. So, we take $\delta \sim \dfrac{C}{\sqrt{(\mathrm{Re})}}$.

Boundary layer equations of Prandtl, 3

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \frac{1}{C} u_{yy} \text{ in } 0 < y < 1 \\ 0 = -p_y \text{ in } 0 < y < 1 \\ u_x + v_y = 0 \text{ in } 0 < y < 1 \\ u = v = 0 \text{ on } y = 0 \\ u = 1, v = 0 \text{ on } y = 1. \end{cases}$$

Problem 9.3: Obtain the *Blasius Boundary Layer Solution* of the previous system and deduce its shear stress on a plate.
(See `https://en.wikipedia.org/wiki/Blasius_boundary_layer`)

Energy dissipation

Recall first that if $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\mathbf{u}$ is a vector field defined on Ω with $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$ at $\partial\Omega$, then, for any scalar function q defined in Ω one has $\int_\Omega \mathbf{u} \cdot \nabla q = 0$.
Observe that we don't need $\mathbf{u}$ to vanish at $\partial\Omega$, but merely $\mathbf{u} \cdot \mathbf{n}$.
If we write an initial and boundary value problem for the full Navier-Stokes equations with conservative forces

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \dfrac{1}{(\mathrm{Re})}\nabla^2 \mathbf{u} + \nabla W & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{u}_0 & \text{for } t = 0 \end{cases}$$

one can estimate the time evolution of the kinetic energy $\frac{1}{2}\int_\Omega \mathbf{u}^2$:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \mathbf{u}^2 = -\frac{1}{(\mathrm{Re})}\int_\Omega (\nabla u)^2 \le -\frac{\mu_0}{(\mathrm{Re})}\int_\Omega \mathbf{u}^2,$$

where $\mu_0 > 0$ is the optimal constant in Poincaré's inequality (depends on Ω, $\mu_0 \ge \pi^2/\ell^2$ if $0 \le \Omega \le \ell$ in some direction). So,

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \mathbf{u}^2\right) \le -\frac{2\mu_0}{(\mathrm{Re})}\left(\frac{1}{2}\int_\Omega \mathbf{u}^2\right)$$

and

$$\left(\frac{1}{2}\int_\Omega \mathbf{u}^2\right) \le e^{-\frac{2\mu_0}{(\mathrm{Re})}t}\left(\frac{1}{2}\int_\Omega \mathbf{u}_0^2\right),$$

(for all $t > 0$ and as long as the solution exists...). The same calculation, but for the Euler equations, where the boundary condition becomes $\mathbf{u}\cdot\mathbf{n} = 0$ and the term $\frac{1}{(\mathrm{Re})}\nabla^2\mathbf{u}$ is dropped from the equation would give that $\frac{1}{2}\int_\Omega \mathbf{u}^2 = \frac{1}{2}\int_\Omega \mathbf{u}_0^2$ for all time.