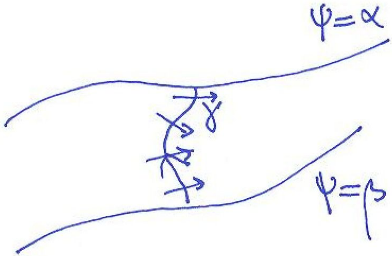
- *Plane-parallel* motion of a fluid: $\mathbf{u}(x, y, z) = (u(x, y), v(x, y), 0)$. The zero-divergence is $u_x + v_y = 0$. If (u, v) is a gradient then it is irrotational: $-u_y + v_x = 0$.
- These are the Cauchy-Riemann equations for $\bar{\mathbf{u}} = u - iv$, the *conjugate velocity*.
- On every simply connected open set, the function $\bar{\mathbf{u}}$ will have a complex primitive $\Omega(z)$, where $z = x + iy$ that is called the *Complex Potential*. The real part of Ω is the *velocity potential* $\phi(x, y)$ and its imaginary part $\psi(x, y)$ is called the *stream function*. So, $u = \phi_x = \psi_y$ and $v = \phi_y = -\psi_x$.

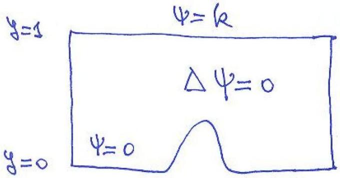
- Observe that when the domain W occupied by the fluid is not simply connected, then the existence of $\Omega(z)$ can be only local.
- Our original hypotheses after Kelvin's Theorem are that ϕ must always exist, but even in that case, the harmonic conjugate function ψ needs not to exist if the domain is not simply connected.
- Observe also that the (local) existence of ψ comes from the divergence zero condition, and so its existence has nothing to do with the potentiality hypothesis.

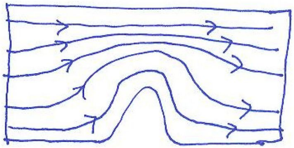
Theorem: Suppose that (u, v) are time independent and divergence-free. If $\psi_x = -v$ and $\psi_y = u$, then
i) ψ is a first integral of the particles' motion
ii) $\alpha - \beta$ is the net flux that crosses a curve γ from left to right if γ connects the curve $\psi(z) = \beta$ with $\psi(z) = \alpha$ (see figure).



Problem 8.1: Prove it.

• The stream function must be also harmonic. The boundary value conditions for ψ must be of Dirichlet type, instead of Neumann, as they were for the velocity potential.
• If one wants to calculate the flow on a rectangle of unit height when there is an irregularity in the bottom surface and the total flux is (say) equal to K, then, one has to solve the equation $\nabla^2\psi = 0$ in the domain with: $\psi = 0$ in the bottom line, $\psi = K$ in the top line, and $\psi(x, y) = Ky$ in the two lateral lines.



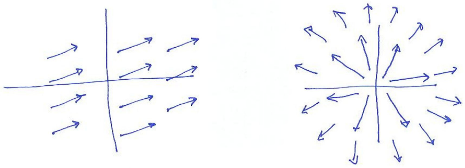


• These lateral boundary conditions are somehow arbitrary.

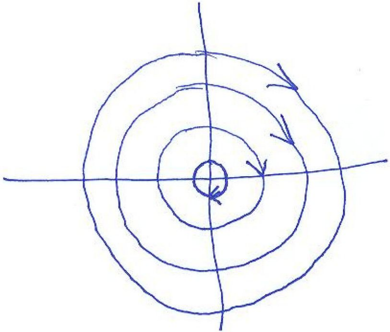
Stationary flows defined by simple holomorphic complex potentials

• The simplest is $\Omega(z) = (a + ib)z$. The complex velocity is in this case $\Omega'(z) = a + ib$ and the flux is a homogeneous velocity $(a, -b)$. The velocity potential is $\phi = ax - by$ and the stream function is $\psi = ay + bx$.

• Second, $\Omega(z) = k \ln z$. If $k > 0$ it is a source, and if $k < 0$ it is a sink. The derivative is $\Omega'(z) = k/z$, so the velocity is $u = kx/(x^2 + y^2)$, $v = ky/(x^2 + y^2)$. The domain of definition is $\mathbb{C} \setminus 0$, that is no simply connected. This is why the stream function $\psi = k \arg(z)$ is not single-valued. The total flux crossing a curve $|z| = r$ is $Q = 2\pi k$.



- Third, $\Omega(z) = ik \ln z$, *Potential vortex*: particles rotate around $z = 0$ with a speed that decreases linearly with the distance to the origin. $\phi = -k \arg(z)$ is not single-valued, $\psi = k \ln |z|$, $u = ky/(x^2 + y^2)$, $v = -kx/(x^2 + y^2)$. Rotates clockwise when $k > 0$. The circulation along a closed curve surrounding the origin is $-2\pi k$.

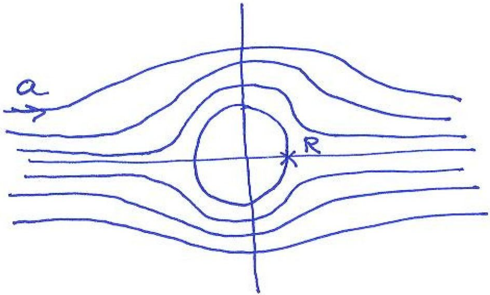


• The single expression $\Omega(z) = \frac{\Gamma + iQ}{2\pi i} \ln z$ means a vortex-source of circulation Γ and flux Q. Streamlines are spirals that in polar coordinates are written in the form $r = Ce^{Q\theta/\Gamma}$.

Problem 8.2: Let $\Omega(z) = \frac{\Gamma + iQ}{2\pi i} \ln z$, and calculate the velocity vector field $\mathbf{u} = (u, v)$. Then, calculate $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$. To make these calculations one needs distribution theory, to deal with the singularity $z = 0$ (appearence of delta functions).

Problem 8.3: Calculate the complex potential due to a source (flux Q) and a sink (flux $-Q$) at the points $z = -M$ and $z = M$, respectively, and a velocity a (> 0) at infinity. Prove the existence of a closed streamline (the *Rankine oval*) separating the paths of the particles that come from infinity and go to infinity, from the particles that come from the source and go to the sink.

• Fourth: $\Omega(z) = a(z + \frac{R^2}{z})$, $\Omega'(z) = a(1 - \frac{R^2}{z^2})$. This is a flux around the circle $|z| \leq R$ with a given velocity $(a, 0)$ at infinity. It is represented in the next figure. The stream function is $\psi(x, y) = a(y - R^2 y/(x^2 + y^2))$ the circle $|z| = R$ together with the straight line $y = 0$ are streamlines with $\psi = 0$.



Problem 8.4:

Prove Milne-Thompson Circle Theorem:

Let $\Omega(z)$ be the complex potential of a fluid flow with no boundaries, and no singularities (sources, vortices...) within $|z| = R$. If a circular cylinder $|z| = R$ is placed into that flow, the complex potential for the new flow is given by $\Omega(z) + \bar{\Omega}(R^2/\bar{z})$.

Problem 8.5:

Prove the following statement:

The real and imaginary parts of the integral

$$\int_\gamma \left(\frac{d\Omega}{dz} \right) dz,$$

for an arbitrary complex potential $\Omega(z)$ and an arbitrary closed curve γ are $\Gamma + iQ$, where Γ is the circulation along γ and Q is the outflux. (Here γ needs not to be the boundary of an obstacle)

Theorem: *(of Blasius) For a potential flow of complex potential $\Omega(z)$ around a bounded obstacle K, the total force* **F** *is given by*

$$F = \left(\rho \frac{i}{2} \int_{\partial K(+)} \left(\frac{d\Omega}{dz} \right)^2 \, dz \right)^*,$$

where the $*$ means complex conjugate.

Proof: Let $\gamma(s)$ be a parametrization by the arc length of $\partial K(+)$. Then the exterior unit normal is $-i\gamma'(s)$ and

$$F = -\int_0^L p\mathbf{n} \, ds = \int_0^L ip\gamma'(s) \, ds = i \int_0^L (p_0 - \frac{1}{2}\rho|\frac{d\Omega}{dz}|^2)\gamma'(s) \, ds.$$

Since the velocity $(d\Omega/dz)^*$ is parallel to ∂K we have that $(d\Omega/dz)^* = q(s)\gamma'(s)$ and $|q(s)| = |d\Omega(\gamma(s))/dz|$.
So,

$$F = i \underbrace{\int_0^L p_0 \gamma'(s) \, ds}_{=0} - i \int_0^L \frac{1}{2}\rho q^2(s) \underbrace{|(\overline{\gamma'(s)})^2|}_{=1} \gamma'(s) \, ds = -\frac{i\rho}{2} \int_0^L q^2(s)\gamma'(s) \underbrace{\gamma'(s)\overline{\gamma'(s)}}_{=1} \, ds$$

$$= -\frac{i\rho}{2} \int_0^L \left(\frac{d\Omega}{dz} \right)^{*2} \overline{\gamma'(s)} \, ds = \left(\rho \frac{i}{2} \int_{\partial K(+)} \left(\frac{d\Omega}{dz} \right)^2 \, dz \right)^*,$$

as we wanted to prove.

Corollary: *(Kutta-Joukowski Theorem) If (u, v) is the velocity field of an ideal fluid in irrotational motion around an obstacle K and $(u, v) \to V_\infty$ as $z \to \infty$, then the resultant force is*

$$F = -i\rho V_\infty \Gamma,$$

where Γ is the circulation along a closed curve (with positive orientation) surrounding the obstacle.

Proof: Laurent series

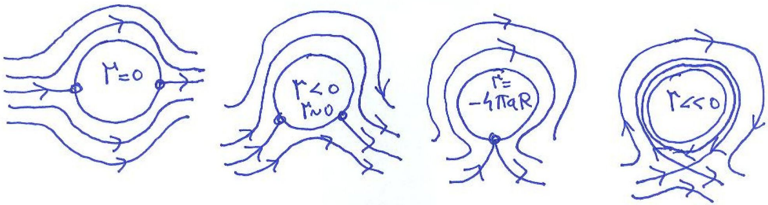
$$u - iv = \overline{V}_\infty + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$$

$$(u - iv)^2 = \overline{V}^2_\infty + 2\overline{V}_\infty \frac{c_1}{z} + O(\frac{1}{z^2}).$$

By Blasius' Theorem and the method of residues $F = 2\pi\rho V_\infty \overline{c_1}$. From the other side $c_1 = \frac{1}{2\pi i} \int_{\partial K}(u - iv)\, dz$ and by Problem 8.5 this integral is $\Gamma + iQ$, where Γ and Q are the circulation and the outflux. Since $Q = 0$, then $c_1 = \Gamma/(2\pi i)$.

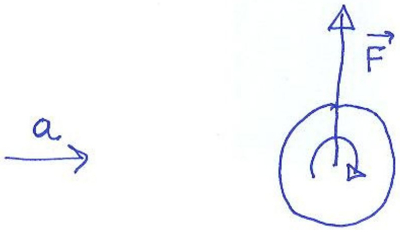
- D'Alambert Paradox in two dimensions is a consequence of Blasius and Kutta-Joukowsky Theorem. If the flow is a potential flow, then the circulation along any closed curve is always zero.
- The exterior of a bounded obstacle in two dimensions is not a simply connected domain, as it is in dimension three. There is a distinguished class of closed curves, those that surround the obstacle, that we could admit that have nonzero circulation, possibly produced by phenomena happening on the boundary of the obstacle.
- We are going to exploit this possibility in the subsequent two examples: the rotating cylinder and Joukowski's wing profile.
- In any case, the force predicted by Kutta-Joukowski Theorem is always perpendicular to the velocity at infinity, so it is never a drag force, in any case it is a lift force.

The rotating cylinder flow is given by $\Omega(z) = a(z + \frac{R^2}{z}) + \frac{\Gamma}{2\pi i} \ln z$. Here the cylinder has radius R and the velocity at infinity is $(a, 0)$. It is the superposition of the torrential flow and a vortex of circulation Γ. The conjugate velocity is $u - iv = a(1 - \frac{R^2}{z^2}) + \frac{\Gamma}{2\pi i}\frac{1}{z}$.

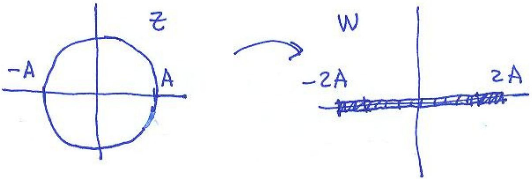


If $\Gamma = 0$ the flow is symmetric in the vertical direction. And there are two rest points (*stagnation points*) at $z = \pm R$. When Γ decreases, these two points start to move downwards, until they collide for $\Gamma = -4\pi aR$. Below this value, a separatrix curve appears, and the fluid inside the separatrix rotates in the negative sense without being melted with the fluid outside it. For $\Gamma > 0$ the flows are similar.

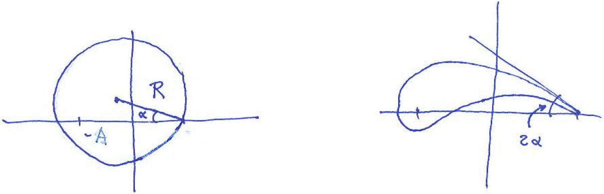
The pictures show clearly that one can imagine this circulation to be produced by the rotation of the cylinder itself, and that the fluid is carried out by this motion because of some viscosity effect, that was not taken into account when we deduced the equations. If we apply Kutta-Joukowski Theorem to the rotating cylinder flow we obtain $F = -i\rho a\Gamma$, that is directed it is represented in the next picture. This force is an explanation of the so-called *Magnus Effect*.



The *Joukowski Transformation* is $w(z) = z + \frac{A^2}{z}$ that maps the exterior of the circle $|z| = A$ to the exterior of $[-2A, A]$.



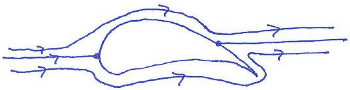
This map was used by Joukowski to define wing profiles as the images by $w(z)$ of circles passing through $z = A$. These wing profiles have a *cusp* at the *trailing edge* (single tangent).

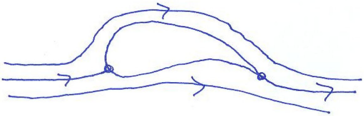


Then this map transforms the exterior of a disc to the exterior of a wing profile, so by conformal mapping one can translate the flows discussed above around the cylinder into flows around the wing. Let us take as above the velocity at infinity to be $(a, 0)$. If we take $\Gamma = 0$ one gets a flow like the one below at the left, where we see an stagnation point at the *leading edge* but another one in a very artificial place. If one takes the right value of Γ to move the second stagnation point to the trailing edge, as it is represented in the figure on the right, then one gets a natural circulation that induces a corresponding lift force. This lift force has the following magnitude

$$|F| = 4\pi R a^2 \sin(\alpha)$$

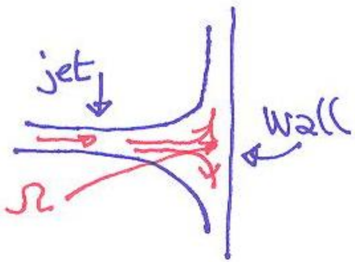
and it is directed upwards (if $a > 0$). Observe that R is a kind of measure of the size of the wing and α is the half of the angle between the tangent to the trailing edge with the velocity at infinity.



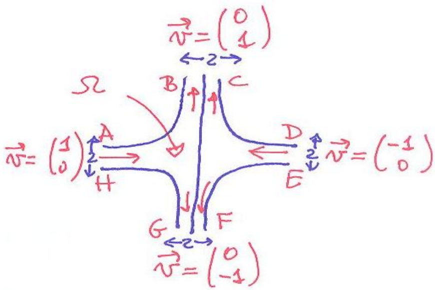


A free boundary problem for the velocity potential

Problem: *Find the form of a two-dimensional jet colliding perpendicularly to a wall*



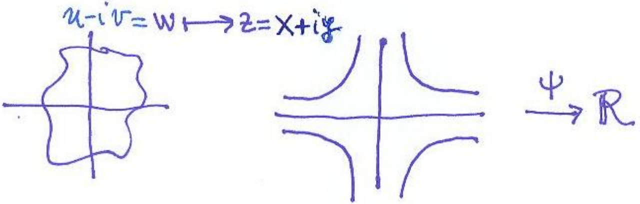
The domain Ω where the PDE has to be solved is part of the problem (the stated problem, in fact).



$$\begin{cases} \nabla^2 \psi = 0 \text{ in } \Omega & \text{(irrotational motion)} \\ \psi = 1 \text{ on AB and EF} & \\ \psi = -1 \text{ on CD and GH} & \text{(flux values)} \\ \psi_x^2 + \psi_y^2 = 1 \text{ in } \partial\Omega & \text{(Bernoulli)} \end{cases}$$

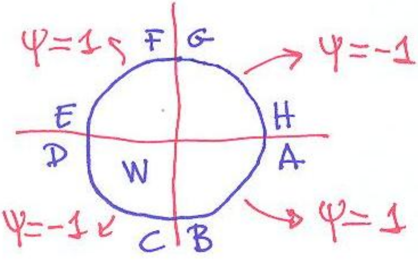
Bernoulli: $p = p_0 - \frac{1}{2}\rho|\vec{\mathbf{v}}|^2$.

Instead of the independent variables (x, y) we will use the variables $(u, -v)$ (remember $\vec{\mathbf{v}} = u + iv$).

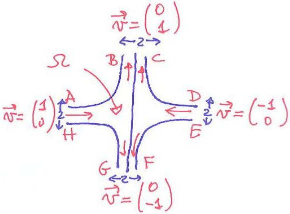


The inverse map of a complex analytic map is also complex analytic, and the composition of an harmonic map after a complex analytic is also harmonic. The image of $\partial\Omega$ in the new variables is $u^2 + (-v)^2 = 1$.

We have to solve $\nabla^2\psi(u,-v) = 0$ in $u^2+v^2 < 1$ under discontinuous boundary conditions.



This problem can be solved by separation of variables by using polar coordinates: $\psi = \frac{a_0}{2} + \sum r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$. The solution is $\psi = \text{ Im } \Omega(u - iv)$ where $\Omega(w) = \frac{2}{\pi}\log\left(\frac{1-w^2}{1+w^2}\right)$.



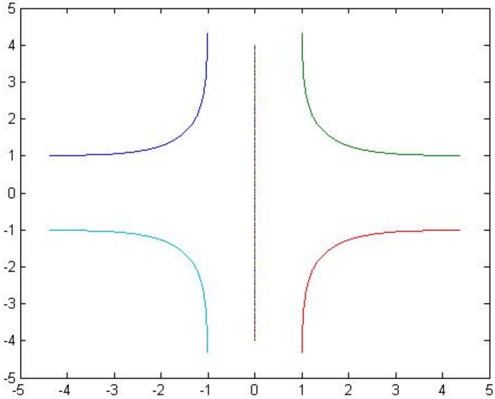
To obtain the curves AB and HG we observe that $\vec{\mathbf{v}}^* = \frac{d\Omega}{dz} = \frac{d\Omega}{dw}\frac{dw}{dz}$, so $\frac{dz}{dw} = \frac{d\Omega}{dw}\frac{1}{w}$ and its primitive turns out to be

$$z(w) = \frac{2}{\pi}\left(\log\left(\frac{1-w}{1+w}\right) + i\log\left(\frac{1+iw}{1-iw}\right)\right).$$

AB will be the image under $z(w)$ of the arc $w = e^{-i\theta}$, for $0 < \theta < \pi/2$. In parametric form:

$$\begin{cases} x(\theta) = \frac{2}{\pi}\left(\log\tan\left(\frac{\theta}{2}\right) - \frac{\pi}{2}\right) \\ y(\theta) = \frac{2}{\pi}\left(\frac{\pi}{2} - \log\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) \end{cases}$$

The solution: the free boundary



Problem 8.6: Use separation of variables with polar coordinates to obtain: $\psi = \frac{a_0}{2} + \sum r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$. Obtain also that $\psi =$ Im $\Omega(u - iv)$ where $\Omega(w) = \frac{2}{\pi} \log\left(\frac{1-w^2}{1+w^2}\right)$. And justify that

$$\begin{cases} x(\theta) = \frac{2}{\pi}\left(\log\tan\left(\frac{\theta}{2}\right) - \frac{\pi}{2}\right) \\ y(\theta) = \frac{2}{\pi}\left(\frac{\pi}{2} - \log\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) \end{cases}$$

Problem 8.7: Find the shape of the free surface in the *Teapot Flow* (see 20141022JonChapman.pdf, pages 19 and 20.
Problem 8.8: Find the shape of the free surface in the *Flow out of a slot* (see 20141022JonChapman.pdf, pages 21-23.