Euler and Lagrangian description of a fluid motion

Eulerian description consists of giving the velocity field $\mathbf{u}(\mathbf{x}, t)$: the velocity of the particle that at time t is in the position $\mathbf{x}$.

Lagrangian description, of giving $\Phi(\mathbf{x}_0, t)$: gives the position at each time t of the particle that at a reference time t_0 was occupying the position $\mathbf{x}_0$. Sometimes written as $\Phi_t(\mathbf{x}_0)$.

To pass from one description to the other one has to solve the ODE system

$$\mathbf{x}'(t) = \mathbf{u}(\mathbf{x}(t), t)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

and then define $\Phi(\mathbf{x}_0, t) = \mathbf{x}(t)$. So,

$$\frac{\partial \Phi}{\partial t} = \mathbf{u}(\Phi, t).$$

Theorem: (Reynolds Transport Theorem) *If* $\Omega(t)$ *represents a set of points that move along the integral trajectories of a velocity vector field* $\mathbf{u}(\mathbf{x}, t)$*, and* $f(\mathbf{x}, t)$ *is a function of the space and time, then*

$$\frac{d}{dt}\int_{\Omega(t)} f(\mathbf{x}, t)\, d\mathcal{V} = \int_{\Omega(t)} \left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\right)\, d\mathcal{V}.$$

Problem 7.1: Prove it. Use a change of variables to write all the integrals over the same domain.

Continuity Equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

(special cases: incompressible, incompressible and homogeneous)

Balance of momentum

Newton's Second Law: the time derivative of the total linear momentum of a set of particles must equal the resultant of the forces made upon these particles.

These forces can be of two types: *volume forces* acting on these particles like, for example, gravity fields created by distant masses ($\mathbf{F}(\mathbf{x}, t)$), and *surface* or *continuity forces* that are the forces that the rest of the fluid makes upon this set of particles through the boundary of this set ($\mathbf{S}(\mathbf{x}, t)$).

Inviscid character: $\mathbf{S} = -p\mathbf{n}$.

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho \mathbf{u}\, d\mathcal{V} = \int_{\partial\Omega_0(t)} -p\mathbf{n}\, d\mathcal{S} + \int_{\Omega_0(t)} \rho \mathbf{F}\, d\mathcal{V} =$$

$$\int_{\Omega_0(t)} -\nabla p\, d\mathcal{V} + \int_{\Omega_0(t)} \rho \mathbf{F}\, d\mathcal{V}$$

Aplying Reynolds Theorem to each of the components of these vector integrals, and omitting the integrals, since the domain of integration is arbitrary,

$$\rho_t u^i + \rho u^i_t + u^i \nabla \cdot (\rho \mathbf{u}) + \rho(\mathbf{u} \cdot \nabla) u^i = -p_{x_i} + \rho \mathbf{F}^i,$$

and then, using the continuity equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\rho}\nabla p + \mathbf{F}.$$

Equations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\rho}\nabla p + \mathbf{F}$$

(or $(D/Dt)\mathbf{u} = \rho^{-1}\nabla p + \mathbf{F}$) and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

constitute the system of Euler equations for the motion of an inviscid fluid. They are more popular in the incompressible case, where the continuity equation reduces to

$$\nabla \cdot \mathbf{u} = 0$$

In the compressible case, they are usually closed with a new equation of the form

$$\rho = \rho(p) \qquad (1)$$

which is called *equation of state* (*barotropic flow*).

Theorem: (Kelvin's Circulation Theorem) *If* $\mathbf{u}(\mathbf{x}, t)$ *is a solution of system of the Euler equations for a barotropic flow, with* $\mathbf{F} = \nabla W$ *(conservative forces) and* c_t *is a closed curve that moves in time following the integral curves of* $\mathbf{u}(\mathbf{x}, t)$*, then*

$$\frac{d}{dt}\, circ\,[c_t] = \frac{d}{dt}\int_{c_t} \mathbf{u}(\mathbf{x}, t)\cdot d\ell = 0.$$

Proof: $$\frac{d}{dt}\int_{c_t(s)} \mathbf{u}\cdot d\ell = \frac{d}{dt}\int_0^1 \mathbf{u}(\Phi_t(c_0(s)), t)\cdot \frac{d}{ds}\Phi_t(c_0(s))\, ds$$

$$= \int_0^1 \left(\frac{D\mathbf{u}}{Dt}\cdot\frac{d}{ds}\Phi_t(c_0(s)) + \mathbf{u}\cdot\frac{d}{dt}\frac{d}{ds}\Phi_t(c_0(s))\right)\, ds$$

$$= \int_0^1 \left(-\frac{\nabla p}{\rho} + \nabla W\right)\frac{d}{ds}\Phi_t(c_0(s))\, ds = 0,$$

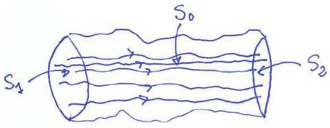
since $\mathbf{u}\cdot\frac{d}{dt}\frac{d}{ds}\Phi_t(c_0(s)) = \frac{1}{2}\frac{d}{ds}(\mathbf{u}\cdot\mathbf{u})$.

Kelvin's Theorem is the basis of the importance given to *potential flows* $\mathbf{u} = \nabla\phi$ (ϕ is called the *velocity potential*).

In the stationary, incompressible and homogeneous case Euler equations for are satisfied for $\mathbf{u} = \nabla\phi$ with $p = -\frac{1}{2}\rho\mathbf{u}^2 + C$ (this is Bernoulli equation, use $\rho \equiv$const. and $u^i_j = u^j_i$. Observe that one merely needs $\mathbf{u}$ to be irrotational, $\nabla \times \mathbf{u} = 0$, a bit less tan potential. This will be more important in the two-dimensional case).

The incompressibility gives $\nabla^2\phi = 0$ (Laplace's equation), and giving $\nabla\phi \cdot \mathbf{n}$ at $\partial\Omega$ one fixes the mass flux across the boundaries of Ω.

You have an irregular cylinder S_0 limited by two lids S_1 and S_2, of areas A_1 and A_2. You want to model a flow with a total flux of Q units of volume per unit of time from left to right. Under the hypothesis of potential flow, one would have to solve $\nabla^2\phi = 0$ in the interior of the cylinder, with $\partial\phi/\partial\mathbf{n} = 0$ on S_0, $\partial\phi/\partial\mathbf{n} = -Q/A_1$ on S_1 and $\partial\phi/\partial\mathbf{n} = Q/A_2$ at S_2.



- the total flux Q must be the same on the two lids.
- with these boundary conditions we are not imposing the velocities at the two lids, but merely their normal components.
- we have chosen the constant value Q/A_i for the normal velocity at the lids, as the simplest choice: any other function with integral Q would be also admissible.

Theorem: Let Ω be a bounded region in $\mathbb{R}^n$ ($n = 2, 3$) and let be given a (compatible) flux $f(\mathbf{x})$ at $\partial\Omega$. Then, among all the possible stationary solutions of the incompressible and homogeneous Euler equations there is one and only one that is a potential flow. This solution $\mathbf{u} = \nabla\phi$ minimizes the kinetic energy $\frac{1}{2}\int_\Omega \rho \mathbf{v}^2 \, d\mathcal{V}$ among al the vector fields $\mathbf{v}$ such that $\nabla \cdot \mathbf{v} = 0$, and $\mathbf{v} \cdot \mathbf{n} = f$ at $\partial\Omega$.

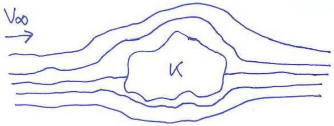
Proof:

$$\frac{1}{2}\int \rho(\mathbf{u}^2 - \mathbf{v}^2) = -\frac{1}{2}\int \rho(\mathbf{u} - \mathbf{v})^2 + \int \rho(\mathbf{u} - \mathbf{v}) \cdot \mathbf{u}$$

$$\leq \int \rho(\mathbf{u} - \mathbf{v})\nabla\phi = 0$$

(in the last step one uses that gradient vector fields are orthogonal in the L^2 sense to divergence-free vector fields that are parallel to the boundary)

The *torrential flow* around a bounded obstacle $K \subset \mathbb{R}^n$ $(n = 2, 3)$ is $\mathbf{u} = \nabla\phi$, where $\nabla^2\phi = 0$ in $\Omega = \mathbb{R}^n - K$, $\nabla\phi \cdot \mathbf{n} = 0$ on ∂K and $\nabla\phi \to \mathbf{V}_\infty$ as $\mathbf{x} \to \infty$.



If $K = B_a(0) \subset \mathbb{R}^3$ then

$$\phi = \frac{a^3}{2|\mathbf{x}|^2}\mathbf{V}_\infty \cdot \frac{\mathbf{x}}{|\mathbf{x}|} + \mathbf{x} \cdot \mathbf{V}_\infty.$$

Theorem (the D'Alembert Paradox): Torrential flows do not exert any force on K (in $\mathbb{R}^3$).

Proof:

(Obs.: $\phi \leftrightarrow -\phi$)
We first accept that $\phi = \mathbf{x} \cdot \mathbf{V}_\infty + \phi_1$ *with* $\phi_1 = O(1/|\mathbf{x}|^2)$ *and* $\nabla \phi_1 = O(1/|\mathbf{x}|^3)$. *Reasons to accept that:*
First, that if $K = B_a(0)$ *then* $\phi_1 = \dfrac{a^3}{2|\mathbf{x}|^2} \mathbf{V}_\infty \cdot \dfrac{\mathbf{x}}{|\mathbf{x}|}$.
Second, note that ϕ_1 *is a solution of* $\nabla^2 \phi_1 = 0$ *on* $\mathbb{R}^3 \setminus K$ *with* $\partial_{\mathbf{n}} \phi_1 = -\mathbf{V}_\infty \cdot \mathbf{n}$ *on* ∂K. *But it is clear that the net flux* $\int_{\partial K} \mathbf{V}_\infty \cdot \mathbf{n} \; d\mathcal{S} = 0$. *We could compare with* $\phi(x) = \frac{1}{|\mathbf{x}|}$, *the electrical potential due to a single charge: it tends to zero at infinity more slowly. But its gradient has a non-zero flux. The dipole potential, instead,* $\phi = \frac{-x}{|\mathbf{x}|^3}$ *has a gradient with zero flux and tends to zero faster, as fast as we expect.*

$$\mathbf{F} = \int_{\partial K} -p\mathbf{n}\, d\mathcal{S} = -\int_{\partial K} p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, d\mathcal{S} = -\int_{\partial B_R(0)} p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, d\mathcal{S}.$$

To justify the second equality we call Ω' the domain between K and $\partial B_R(0)$, for R large. The difference between the third and second integrals can be written as

$$-\int_{\partial\Omega'} p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, d\mathcal{S}$$

and we are going to see that this integral is zero by changing it into a volume integral:

$$\int_{\partial\Omega'} -p\mathbf{n}\, d\mathcal{S} = -\int_{\Omega'} \nabla p\, d\mathcal{V} = \int_{\Omega'} \frac{1}{2}\rho\nabla|\nabla\phi|^2\, d\mathcal{V},$$

and, the other part (looking at each component and using that $\nabla^2\phi = 0$)

$$-\int_{\partial\Omega'} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, d\mathcal{S} = -\int_{\partial\Omega'} \rho\nabla\phi(\nabla\phi\cdot\mathbf{n})\, d\mathcal{S} = -\frac{1}{2}\int_{\Omega'} \rho\nabla|\nabla\phi|^2\, d\mathcal{V}.$$

So, the formula for **F** is correct. Then

$$\mathbf{F} = -\int_{\partial B_R(0)} \left(p_0\mathbf{n} - \frac{\rho}{2}|\mathbf{V}_\infty + \underbrace{\nabla\phi_1}_{\sim\frac{1}{R^3}}|^2\mathbf{n} + \rho(\mathbf{V}_\infty + \underbrace{\nabla\phi_1}_{\sim\frac{1}{R^3}})((\mathbf{V}_\infty + \underbrace{\nabla\phi_1}_{\sim\frac{1}{R^3}})\cdot\mathbf{n}) \right) d\mathcal{S}.$$

And by letting $R \to \infty$ in some terms and using the Divergence Theorem in the others, we obtain $\mathbf{F} = 0$.