Newton's gravitation law: the gravitational force created on a unit point mass located at a point $\mathbf{x} = (x, y, z)$ by a point mass M located at the point $\mathbf{x}' = (x', y', z')$ is

$$F(x, y, z) = \frac{M(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} = \frac{M(x' - x, y' - y, z' - z)}{((x - x')^2 + (y - y')^2 + (z - z')^2)^{3/2}}.$$

The vector field $F(\mathbf{x})$ is called the *gravitational force field* created by M. We have that $F = \nabla V$, where

$$V(\mathbf{x}) = \frac{M}{|\mathbf{x}' - \mathbf{x}|} = \frac{M}{r},$$

the *gravitational potential*. The *potential energy* is $-V$.
We have taken the gravitational universal constant to be 1.

One can easily check that $\nabla^2 V(\mathbf{x}) = 0$ for all $\mathbf{x} \neq \mathbf{x}'$ (for radial functions of three variables $\nabla^2 = \partial_{rr} + \frac{2}{r}\partial_r$).
But, what is the 'value' of $\nabla^2 V(\mathbf{x}')$? Let's try to calculate using integrals. Ler B be the ball of radius R centered at $\mathbf{x}'$. At least formally,

$$\int_B \nabla^2 V \, d\mathcal{V} = \int_{\partial B} \frac{\partial V}{\partial \mathbf{n}} \, dS = \int_{\partial B} -\frac{M}{R^2} \, dS = -4\pi M,$$

We can imagine $\nabla^2 V(\mathbf{x})$ as a function that is zero everywhere except at the point $\mathbf{x} = \mathbf{x}'$ where it takes a (minus) infinite value that makes its integral to be $-4\pi M$. We can imagine that, after perhaps some precise justification, $\nabla^2 V(\mathbf{x}) = -4\pi M \delta(\mathbf{x} - \mathbf{x}')$.

With point masses $M_1, M_2, \ldots M_n$ located at $\mathbf{x}'_1, \mathbf{x}'_2 \ldots \mathbf{x}'_n$

$$V(\mathbf{x}) = \sum_{i=1}^{n} \frac{M_i}{|\mathbf{x} - \mathbf{x}'_i|}.$$

We also have that $\nabla^2 V(\mathbf{x}) = 0$ except at the masses, and using the Divergence Theorem, one obtains Gauss' Theorem: If S is a closed surface, then

$$\int_{\partial S} \frac{\partial V}{\partial \mathbf{n}}\, dS = \sum_{M_i \in \text{ int } S} -4\pi M_i.$$

Observe that the left hand side represents the flux of the vector field F along the surface S.
Gauss' Theorem is more used when the M_i represent electric charges. The electric potential, that is called Coulomb's potential, is treated in the same way, except that the charges can also have negative signs and so charges of the same sign don't attract, but repel, each other.

The gravitational (or electrical) potential and the force created by a 'continuous' distribution of masses, given by a density function $m(\mathbf{x})$ are

$$V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}', \quad F(\mathbf{x}) = \int_{\mathbb{R}^3} m(\mathbf{x}') \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^3}\, d\mathcal{V}'.$$

The following result holds:

Theorem: *Suppose that both* $||m||_1 := \int_{\mathbb{R}^3} |m(\mathbf{x})|\, d\mathcal{V}$ *and* $||m||_\infty := \sup_{\mathbf{x}\in\mathbb{R}^3} |m(\mathbf{x})|$ *are finite. Then* $V(\mathbf{x}), F(\mathbf{x})$ *are continuous functions,* $V(\mathbf{x}), F(\mathbf{x}) \to 0$ *when* $\mathbf{x} \to \infty$*,* $V(\mathbf{x})$ *is of class* $\mathcal{C}^1$ *and* $F = \nabla V$*. If* $m(\mathbf{x}) \equiv 0$ *in an open set* Ω*, then* $\nabla^2 V \equiv 0$ *in* Ω*.*

This last statement (that the potential is an harmonic function outside the masses) is *Laplace's Law*. The PDE $\nabla^2 u = 0$ is known as *Laplace's Equation* or sometimes *Potential Equation*.

Let us prove something much simpler, but in this direction, namely the pointwise bound

$$|V(\mathbf{x})| \leq \frac{3}{2}(4\pi)^{1/3}||m||_1^{2/3}||m||_\infty^{1/3}.$$

Proof: $V(\mathbf{x}) = \int_{B_R(\mathbf{x})} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}' + \int_{\mathbb{R}^3 - B_R(\mathbf{x})} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'$ and we bound the two integrals independently.
$|\int_{B_R(\mathbf{x})} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'| \leq ||m||_\infty \int_{B_R(\mathbf{x})} \frac{1}{r}\, d\mathcal{V}' = 2\pi R^2 ||m||_\infty$,
$|\int_{\mathbb{R}^3 - B_R(\mathbf{x})} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'| \leq \frac{1}{R}||m||_1$ and so

$$|V(\mathbf{x})| \leq 2\pi R^2 ||m||_\infty + \frac{1}{R}||m||_1.$$

We can now choose $R = ((1/(4\pi))(||m||_1/||m||_\infty))^{1/3}$ and one gets the desired inequality.

Problem 6.1: Does a similar pointwise bound hold for $||F(x)||$?
Problem 6.2: Use the pointwise bound above (and perhaps approximate $m(x)$ by continuous functions) to show the part of the previous theorem that says that $V(\mathbf{x})$ is continuous.
Problem 6.3: Use the pointwise bound above (and perhaps the approximation of $m(x)$ in the L^1 norm by functions of compact support) to show the part of the previous theorem that says that $V(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$.

Gauss' Theorem for continuous (i.e., not discrete) distributions of masses (or charges): *If* $m \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,

$$V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\mathcal{V}'$$

and $\Omega \subset \mathbb{R}^3$ *is an open set, then*

$$\int_{\partial\Omega} \frac{\partial V}{\partial \mathbf{n}} \, dS = \int_{\Omega} -4\pi m(\mathbf{x}') \, d\mathcal{V}'.$$

For the proof one has to make a nontrivial use of Fubini's Theorem to check that

$$\int_{\partial\Omega} \frac{\partial V}{\partial \mathbf{n}} \, dS = \int_{\partial\Omega} \int_{\mathbb{R}^3} m(\mathbf{x}') \frac{(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{n}}{|\mathbf{x} - \mathbf{x}'|^3} \, d\mathcal{V}' \, dS$$

$$= \int_{\mathbb{R}^3} m(\mathbf{x}') \int_{\partial\Omega} \frac{(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{n}}{|\mathbf{x} - \mathbf{x}'|^3} \, dS \, d\mathcal{V}',$$

and we know that the last integral is 0 when $\mathbf{x}'$ is outside the closure of Ω and -4π when $\mathbf{x}'$ is in Ω.

Gravitational potential created at the point $\mathbf{x} = (x, y, z)$ by a solid sphere centered at $(0, 0, 0)$, of constant density m_0, of radius R_1 with a concentrical spherical cavity of radius $R_0 < R_1$.

By symmetry we accept that $V = V(r)$, where $r = |\mathbf{x}|$. By Gauss' Theorem for continuous distributions of masses,

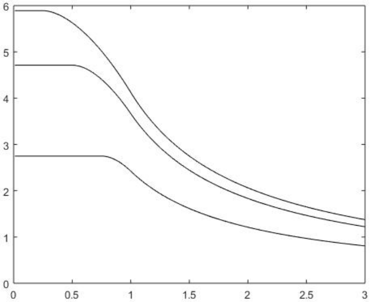
$$\int_{|\mathbf{x}|=r} V'(r)\, dS = \int_{|\mathbf{x}|<r} -4\pi m(\mathbf{x}')\, d\mathcal{V}'$$

$$4\pi r^2 V'(r) = \begin{cases} -4\pi m_0 \frac{4}{3}\pi(R_1^3 - R_0^3) \text{ , if } r > R_1 \\ -4\pi m_0 \frac{4}{3}\pi(r^3 - R_0^3) \text{ , if } R_1 > r > R_0 \\ 0 \text{ , if } z < R_0 \end{cases}$$

and

$$V(r) = \begin{cases} \frac{4}{3}\frac{\pi}{r} m_0 (R_1^3 - R_0^3), \text{ if } r > R_1 \\ \frac{4}{3}\pi m_0 (-\frac{1}{2}r^2 - \frac{R_0^3}{r} + \frac{3}{2}R_1^2), \text{ if } R_1 > r > R_0 \\ 2\pi m_0 (R_1^2 - R_0^2), \text{ if } r < R_0 \end{cases}$$

So, the force field outside the hollow ball coincides with the force field created by the total mass concentrated at the center of the sphere. Inside the spherical shell, coincides with the force field created by the total mass of the shell lying between its distance to the center and the center. And inside the interior cavity the gravitational force is zero (Faraday cage).

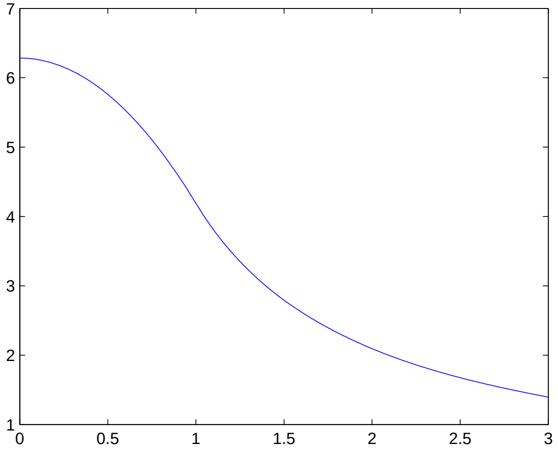


Graphs of $V(r)$ for $m_0 = 1$, $R_1 = 1$ and $R_0 = .75, .5, .25$ (ascending).

The gravitational potential created by a solid sphere of radius R and density m_0 is then

$$V(x,y,z) = \begin{cases} 2\pi m_0 R^2 - \dfrac{2}{3}\pi r^2 m_0, \text{ if } r \leq R \\ \dfrac{4}{3}\pi R^3 \dfrac{m_0}{r}, \text{ if } r \geq R \end{cases}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.
The graph of $V(r)$ shows a parabola and an hyperbola that match at $r = R$ together with their first derivatives, but with a discontinuity in the second derivative. The following is a plot of $V(r)$ versus r for $R = 1$ and $m_0 = 1$.


7
6
5
4
3
2
1
0
0.5
1
1.5
2
2.5
3

For this function $V(r)$ it turns out that $\nabla^2 V(r) = -4\pi m_0$ for $r < R$ and $\nabla^2 V(r) = 0$ for $r > R$. After that, imagining that a general function $m(\mathbf{x})$ could be quite well approximated by linear combinations of characteristic functions of balls, one could anticipate that if

$$V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d\mathcal{V}'$$

then

$$\nabla^2 V(\mathbf{x}) = -4\pi m(\mathbf{x})$$

(*Poisson's formula*), or, more precisely

Theorem: *If $m(\mathbf{x}) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ belongs to the Hölder class $\mathcal{C}^\alpha$, for some $\alpha > 0$ in a neighborhood of $\mathbf{x}_0$, then in this neighborhood the second derivatives of V exist, belong to the same Hölder class and satisfy $\nabla^2 V(\mathbf{x}) = -4\pi m(\mathbf{x})$.*

Observe that if $f(\mathbf{x})$ is a function of $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and locally Hölder, then

$$u(\mathbf{x}) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\mathcal{V}'$$

is a solution, and the unique solution, of the *Poisson problem*

$$\begin{cases} \nabla^2 u = f \\ u \to 0 \text{ as } \mathbf{x} \to \infty. \end{cases}$$

The uniqueness comes from the maximum principle applied to

$$\begin{cases} \nabla^2 (u - v) = 0 \\ (u - v) \to 0 \text{ as } \mathbf{x} \to \infty. \end{cases}$$

Problem 6.4: Prove Poisson's formula in the (simplest) case where $m(\mathbf{x}) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is of class $\mathcal{C}^2$, instead of merely Hölder. To do that, make first a change of variables and write

$$V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x} - \mathbf{x}')}{|\mathbf{x}'|} \, d\mathcal{V}'.$$

Problem 6.5: Prove that if $m(\mathbf{x}) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $V(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}'$, then $\nabla^2 V(\mathbf{x}) = -4\pi m(\mathbf{x})$ in the weak sense, that is

$$- \int_{\mathbb{R}^3} \nabla V(\mathbf{x}) \nabla \phi(\mathbf{x})\, d\mathcal{V} = - \int_{\mathbb{R}^3} 4\pi m(\mathbf{x}) \phi(\mathbf{x})\, d\mathcal{V},$$

for all $\phi \in \mathcal{C}^1_{\text{comp}}(\mathbb{R}^3)$. Prove also that this weak solution is unique if one asks it to tend to zero as $\mathbf{x} \to \infty$.

Problem 6.6: Prove (in $\mathbb{R}^3$) that $\nabla^2_{(\mathbf{x})} \dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ in the sense of distributions, that is

$$\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \nabla^2 \phi(\mathbf{x})\, d\mathcal{V} = -4\pi \phi(\mathbf{x}'),$$

for all $\phi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^3)$.

The role played by the Hölder condition

Consider

$$V = \int_{\mathbb{R}^3} \frac{m(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{V}' = \int_{\mathbb{R}^3} \frac{m(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, d\mathcal{V}'$$

in the particular case $m(0)=0$. We want to obtain its second derivatives at $\mathbf{x}=0$. Proceeding formally, one has that

$$\frac{\partial^2}{\partial x^2}\frac{1}{r} = -\frac{1}{r^3} + \frac{3(x-x')^2}{r^5}$$

and we should study the convergence of the integral (in spherical coordinates centered at $\mathbf{x}=0$)

$$\int_{\mathbb{R}^3} m(\mathbf{x}')(-\frac{1}{\rho^3} + 3\frac{(\rho\sin\theta\cos\phi)^2}{\rho^5})\ \rho^2\sin\theta\ d\rho d\theta d\phi.$$

The convergence of this integral is not clear at all, and that the use of the Hölder condition $|m(\mathbf{x})| = |m(\mathbf{x})-m(0)| \leq K\rho^{\alpha}$ is an almost necessary help.

This is the key point that explains the importance of Hölder classes in the theory of elliptic equations (*Schauder Theory*).

Total Potential W

If we have a finite number of masses $m_1, m_2, \dots m_n$ located at the points $\mathbf{x}^1, \mathbf{x}^2, \dots \mathbf{x}^n$ we can consider the total potential energy of the configuration, that takes account of the reciprocal interaction of the masses. If $V_i(\mathbf{x}) = m_i/|\mathbf{x} - \mathbf{x}_i|$ then this total potential energy is given by

$$W = \sum_{i<j} m_j V_i(\mathbf{x}_j) = \sum_{\substack{i,j=1\dots n \\ i\neq j}} \frac{1}{2} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}.$$

The generalization of this to the continuously distributed case is

$$W = \frac{1}{2} \int_{\mathbb{R}^3} m(\mathbf{x}) V(\mathbf{x})\, d\mathcal{V} = \frac{-1}{8\pi} \int_{\mathbb{R}^3} (\nabla^2 V(\mathbf{x})) V(\mathbf{x})\, d\mathcal{V}$$

$$= \frac{1}{8\pi} \int_{\mathbb{R}^3} ||\nabla V(\mathbf{x})||^2\, d\mathcal{V}.$$

(Dirichlet energy)

Problem 6.7: Prove that the potential $V(\mathbf{x})$ defined by a mass density $m(\mathbf{x})$ is a critical point of the Lagrangian functional

$$L[U] = \int_{\mathbb{R}^3} \left(\frac{1}{8\pi}||\nabla U||^2 - mU\right) d\mathcal{V}.$$

(One possibility is to make rigorous the following idea: the actual potential V is, among all the functions U that tend to zero at infinity, the one (obviously) that minimizes $\int_{\mathbb{R}^3} ||\nabla(U - V)||^2 \, d\mathcal{V}$. But the square under this integral can be developed, and an integration by parts can be performed, and one obtains

$$\int_{\mathbb{R}^3} ||\nabla U||^2 \, d\mathcal{V} - 8\pi \int_{\mathbb{R}^3} mU \, d\mathcal{V} + \int_{\mathbb{R}^3} |\nabla V|^2 \, d\mathcal{V},$$

that has the same extremals as $L[U]$.)

Potential at large distances

The idea is to obtain a first approximation of $V(\mathbf{x})$ for large values of $|\mathbf{x}|$.

Suppose that $\mathbf{y}$ is a fixed point and $\mathbf{x}$ is a variable point that we will suppose that approaches infinity. Let us call $\mu = |\mathbf{y}|/|\mathbf{x}|$ that will play the role of a small dimensionless parameter, and γ the angle between $\mathbf{x}$ and $\mathbf{y}$. Since $|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2(1-2\mu\cos\gamma+\mu^2)$ we have that, as a first approximation

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}|}\frac{1}{\sqrt{1-2\mu\cos\gamma+\mu^2}} \simeq \frac{1}{|\mathbf{x}|}(1+\mu\cos\gamma)$$

$$= \frac{1}{|\mathbf{x}|}\left(1+\frac{|\mathbf{y}|}{|\mathbf{x}|}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}\right) = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^3}$$

Using this approximation with $\mathbf{y} = \mathbf{x}'$ in the expression of $V(\mathbf{x})$ we have that

$$V(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\int_{\mathbb{R}^3} m(\mathbf{x}')\, d\mathcal{V}' + \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int_{\mathbb{R}^3} m(\mathbf{x}')\mathbf{x}'\, d\mathcal{V}' = \frac{M}{|\mathbf{x}|} + M\frac{\mathbf{x}\cdot\mathbf{x}_M}{|\mathbf{x}|^3}$$

where M is the total mass and $\mathbf{x}_M$ is the center of mass of the density $m(\mathbf{x})$. But using again the approximate equality now with $\mathbf{y} = \mathbf{x}_M$ we finally obtain

$$V(\mathbf{x}) \simeq \frac{M}{|\mathbf{x} - \mathbf{x}_M|},$$

that in words means that from far away from the masses the potential is (approximately) as if all the mass was concentrated at the center of mass.

Two-dimensional Newtonian gravitation.

Gravitational potential V created by an infinitely long vertical bar of negligible cross section (the z-axis) and *linear density M* (units of mass per unit of length) on a point $(x,0,0)$, with $x>0$.

$$V(x,0,0)=\int_{-\infty}^{\infty}\frac{M\,dz'}{\sqrt{x^2+(z')^2}}=M\left[\arg\sinh\frac{z'}{x}\right]_{z'=-\infty}^{\infty}=\infty.$$

Another idea:

$$F=\int_{-\infty}^{\infty}M\frac{(0,0,z')-(x,0,0)}{(x^2+(z')^2)^{3/2}}\,dz'$$

$$=\left(\int_{-\infty}^{\infty}M\frac{-x}{(x^2+(z')^2)^{3/2}}\,dz',\ 0\ ,\int_{-\infty}^{\infty}M\frac{z'}{(x^2+(z')^2)^{3/2}}\,dz'\right).$$

The integral in the third component is convergent and it is zero. This is also reasonable from the physical point of view.

The integral on the first component is easy to compute, with the change of variables $t = z'/x$. We have

$$\int_{-\infty}^{\infty} M \frac{-x}{(x^2 + (z')^2)^{3/2}}\, dz' = -\frac{M}{x}\left[t(1+t^2)^{-1/2}\right]_{t=-\infty}^{t=\infty} = -M\frac{2}{x}.$$

Imagine now a universe where the ‘points’ are infinitely long vertical bars, of negligible cross-section, parallel to the z-axis, that attract each other according to Newton’s law. This is a genuine two-dimensional universe: a universe where all the objects are invariant by translations along the z-direction. Each bar can be identified by the coordinates of its point of intersection with the plane $z = 0$.

A bar of linear density M located at the point $\mathbf{x}' = (x', y')$ will exert a force per unit length over a bar located at $\mathbf{x} = (x, y)$, of unit linear density, equal to $-2M/r$, where $r = |\mathbf{x}' - \mathbf{x}|$ is the distance between them, according to the previous calculation. This force is the gradient of the potential function $V(\mathbf{x}) = -\log r^2$, and that new function should be considered as the two-dimensional reduction of the usual three-dimensional gravitational potential.

It is as easy (or as difficult) as in the three dimensional case to see that $\nabla^2_{(\mathbf{x})}(-\log|\mathbf{x}' - \mathbf{x}|^2) = -4\pi\delta(\mathbf{x}' - \mathbf{x})$, where now the ∇^2 is the two-dimensional laplacian.

If we have a distributed density of vertical bars $m(x, y)$ (meaning that the bar over the small square of sides dx and dy over the point (x, y) has a mass per unit vertical length of $m(x, y)\ dxdy$) then the gravitational potential will be

$$V(x, y) = \int_{\mathbb{R}^2} m(x', y')(-\log((x - x')^2 + (y - y')^2))\ dx'dy'.$$

Under some hypotheses, including that m must be locally Hölder continuous, it will hold that $\nabla^2 V(\mathbf{x}) = -4\pi m(\mathbf{x})$ (Poisson's equation). Nevertheless, the property that $V(\mathbf{x})$ tends to zero at infinity will no longer hold in general.

Problem 6.8: Find a smooth function $f(x, y)$, with compact support, such that the following problem

$$\begin{cases} \nabla^2 u = f \text{ in } \mathbb{R}^2 \\ u(x, y) \to 0 \text{ as } (x, y) \to \infty. \end{cases}$$

has no solution. (Hint: take $f = f(r)$.)

Problem 6.9: Suppose that $z = 0$ represents the ground level on a street where an electric cable is buried at $x = 0$ and $z = -D$. The ground is kept at a fixed temperature $u = 0$ and the cable is releasing heat (Joule effect) at Q units of energy per unit time and unit length of the cable.
The cable lies inside a protecting pipe of radius d $(0 < d < D)$. Calculate the steady-state temperature $u = u_0$ of the protecting pipe. We suppose $u_0 =$constant, the conductivity of the pipe is supposed to be high. The result will depend on the thermal conductivity k of the ground.

One-dimensional Newtonian gravitation.

Homogeneous parallel planes (or flat laminae) whose particles attract each other with the law of the inverse of the square of the distance. We suppose that these planes are parallel to the (y, z)-plane. If one of these laminae has a surface density M (units of mass for unit of area) and intersects the x-axis at a point x' we want to calculate the force that produces over another parallel plane, of density equal to 1, situated at the point x of the same axis. We suppose that $x > x'$. The force is

$$\int_{\mathbb{R}^2} \frac{M(x'-x)}{((x-x')^2+(y')^2+(z')^2)^{3/2}}\, dy'dz' =$$

$$M(x'-x)\int_0^{2\pi}\int_0^{\infty} \frac{\rho\, d\rho d\theta}{((x-x')^2+\rho^2)^{3/2}} = -2\pi M.$$

This means the attracting force does not depend on the distance! Not exactly: the force is $-2\pi M$ if $x > x'$ and $2\pi M$ if $x < x'$. So, the potential is $V(x) = -2\pi|x - x'|$, and its laplacian, that is now just the second derivative, is $\nabla^2 V = -4\pi\delta(x - x')$. The potential defined by a distributed density $m(x')$ will then be

$$V(x) = -2\pi \int_{-\infty}^{\infty} m(x')|x - x'|\, dx'.$$

With the function

$$m(x') = \begin{cases} 0, \text{ if } |x'| > L \\ M, \text{ if } |x'| \leq L. \end{cases}$$

that is the gravitational potential produced by a solid slab of length $2L$, it turns out that

$$V(x) = \begin{cases} -4\pi ML|x|, \text{ if } |x| > L \\ -2\pi ML^2 - 2\pi Mx^2, \text{ if } |x| \leq L. \end{cases}$$

Useful, because some earth gravitational anomalies are related to the existence of strata of different densities.

Observe, though, that in this 1-D case the force on the y-direction would be

$$\int_{\mathbb{R}^2} \frac{My'}{((x-x')^2+(y')^2+(z')^2)^{3/2}}\,dy'dz' = 0,$$

as in the z-direction.

Problem 6.10: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and with compact support. Give necessary and sufficient conditions for the problem $u''(x) = f(x)$, $u(x) \to 0$ as $x \to \pm\infty$ to have a solution. Is this solution unique? What can you imagine as a necessary and sufficient condition if f is continuous but merely of $L^1(\mathbb{R})$, instead of being of compact support?

A reference on Classical Gravitation and Elementary Potential Theory:

O.D. Kellog: Foundations of Potential Theory (1929)