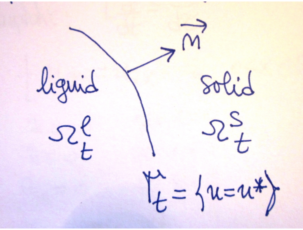
The Stefan Problem (a free boundary problem)



Here **n** is the exterior unit normal vector to the liquid phase and v is the (local) speed of the interphase Γ_t in the direction of **n**.

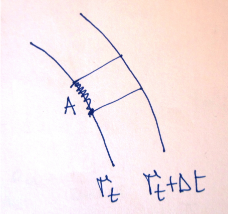
$$\rho^\ell c^\ell u^\ell_t = k^\ell \nabla^2 u^\ell \text{ in } \Omega^\ell_t,$$

$$\rho^s c^s u^s_t = k^s \nabla^2 u^s \text{ in } \Omega^s_t,$$

$$u^\ell = u^* \text{ on } \Gamma_t,$$

$$u^s = u^* \text{ on } \Gamma_t$$

$$+ \text{ heat balance across } \Gamma_t.$$



$$\Delta t\, A[\ \underbrace{-k^\ell u^\ell_{\mathbf{n}}}_{\text{heat leaving } \Omega^\ell} \ - \ \underbrace{(k^s u^s_{(-\mathbf{n})})}_{\text{heat entering } \Omega^s}\ ] = L\rho^s A v \Delta t$$

($v = 0$ means perfect thermal contact, $v > 0$ means melting, $v < 0$ (with ρ^ℓ instead of ρ^s in the r.h.s.) would mean freezing). This is better written as

$$[k u_{\mathbf{n}}]^s_\ell = \rho^s L v,$$

where L is the latent heat.

Exemple 1: A 1-d problem (of melting)

Suppose $\rho^s = \rho^\ell = c^s = c^\ell = k^s = k^\ell = 1$. The liquid phase is $0 < x < s(t)$ and the solid phase $s(t) < x < 1$. We impose the boundary conditions $u(0,t) = 1$ and $u_x(1,t) = 0$, and the initial condition $u(x,0) \equiv \theta < 0$.

$$\begin{cases} u_t = u_{xx} \text{ for } 0 < x < s(t) \text{ and } s(t) < x < 1 \\ u(0,t) = 1 \text{ and } u_x(1,t) = 0 \\ u(x,0) = \theta < 0 \\ u(s(t),t) = 0 \\ [u_x(s(t),t)]_\ell^s = Ls'(t) \\ s(0) = 0 \end{cases}$$

Not easy to solve...

Suppose $L \gg 1$ so $s(t)$ grows slowly. We change to $\tau = t/L$ to make things to happen faster. Then $d/dt = (1/L)d/d\tau$ and so $(1/L)u_\tau = u_{xx}$ becomes $0 = u_{xx}$ (quasi-static!) and $[u_x(s(t),t)]_\ell^s = Ls'(t)$ becomes $[u_x(s(\tau),\tau)]_\ell^s = s'(\tau)$.

The solution is (disregarding the initial condition for u!) $u(x,\tau) = 1 - x/s(\tau)$ for $0 < x < s(\tau)$ and $u(x,\tau) = 0$ for $s(\tau) < x < 1$. Then we get the ode $s'(\tau) = 1/s(\tau)$ and with the initial condition $s(0) = 0$ we conclude $s(\tau) = \sqrt{2\tau}$ and $s(t) = \sqrt{2t/L}$. So, for large L the solid gets completely melted in a total time of $L/2$.

Let's study the other limiting case, $L \to 0$. We have to solve $u_t = u_{xx}$ for $0 < x < 1$ with $u(0, t) = 1$, $u_x(1, t) = 0$ and $u(x, 0) = \theta < 0$, and then the interface $x = s(t)$ will be the isotherm $u(x, t) = 0$. The solution is

$$u = 1 - \sum_{n=0}^{\infty} \frac{4(1-\theta)}{(2n+1)\pi} e^{-\frac{(2n+1)^2\pi^2}{4}t} \sin(\frac{(2n+1)\pi}{2}x)$$

$$\simeq 1 - \frac{4(1-\theta)}{\pi} e^{-\frac{\pi^2}{4}t} \sin(\frac{\pi}{2}x).$$

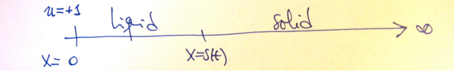
Now we write $u(1, T) = 0$ to calculate the time T needed to melt the whole bar of ice, and one gets

$$e^{-\frac{\pi^2}{4}T} = \frac{\pi}{4(1-\theta)}$$

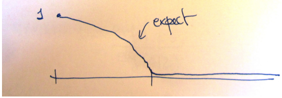
and

$$T = -\frac{4}{\pi^2} \log(\frac{\pi}{4(1-\theta)}).$$

Exemple 2: The one-phase Stefan problem. The Neumann solution.



$$\begin{cases} u_t = u_{xx} \text{ for } 0 < x < s(t) \text{ and } u(x,t) = 0 \text{ for } s(t) < x < \infty \\ u(x,0) = 0 \text{ for } 0 < x < \infty \\ u(0,t) = 1 \\ -u_x(s(t)(-),t) = Ls'(t) \end{cases}$$



Problem 5.1: (Neumann solution)
Look for the solution of the form $u(x,t) = F(x/\sqrt{t})$, for $0 < x < s(t)$, obtain $s(t) = A\sqrt{t}$, $u(x,t) = 1 - \dfrac{\operatorname{erf}(\frac{x}{2\sqrt{t}})}{\operatorname{erf}(A/2)}$ and deduce the relation $A = A(L)$.

Black-Scholes

• Suppose we are a bank. At time $t = t_0$ a costumer can come saying he wants to warranty the possibility (the *option*) of buying at $t = T$ ($T \geq t_0$) a particular good or *asset* (the *underlying asset*, typically a good, a stock, a share) at a price E (the *strike* of the option). The costumer will not get obliged to buy the asset at the price E when $t = T$, he will merely have the *option* to do it (these kind of options are called *European options*).

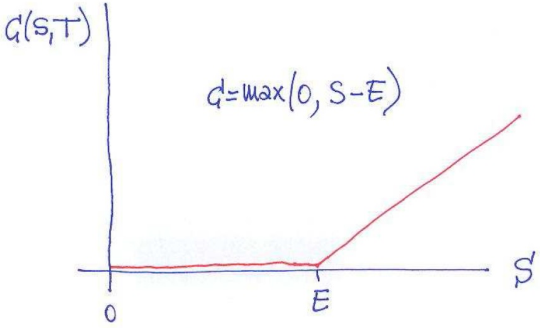
• We want to tell this costumer the price of this service. We want to look at the price at this moment of the asset, say S_{t_0}, and be able to tell the client that in order to have this option he has to pay us (at time t_0) a quantity $C(S_{t_0}, t_0)$.

The mathematical task is now to find the function $C(S, t)$ for $0 < S$ and $t \leq T$. We consider as given and fixed the other parameters of the system, as the strike E and the *execution time* T as well as other general properties of the market. The unknown function $C(S, t)$ will turn out to satisfy a partial differential equation like the heat equation but backwards in time, and the final condition instead of the initial condition will be given.

Example: our costumer wants to buy 20.000 Brendt oil barrels next Dec. 31 at a price of 62$ each.
$t_0 =$ today, in days. (LINK)
$T = 365$, in days
$E = 62$, in dollars.
$S = S_{t_0} =$ today price, in dollars (LINK)
We want to tell him the price of this option, say 1.5$ for each barrel.

The first thing that is easy to see is that the final condition will be

$$C(S, T) = \max(0, S - E).$$



It is clear that if $S \leq E$ we would not ask the costumer to pay anything for such and disadvantageous (and useless) thing for him. It is also clear that if $S \geq E$ we would ask him to pay exactly the difference $S - E$.

The risk-free interest rate

• We suppose that in the market there is a financial product (German or USA government debt obligations, or something similar) without risk and with a (continuous) interest ρ constant.

$$dR = R\rho dt$$

• All the financial operations we, as a bank, are willing to do should give exactly this interest.

If we wanted to get a higher return, for sure another bank would appear that would offer our costumers better conditions (this hypothesis is called *no-arbitrage*). Equally, if we offer lower prices that will be a really bad deal for us.

Evolution of the asset price

The price S_t of the asset is not deterministic. We accept that it is a random process S_t whose variations are given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

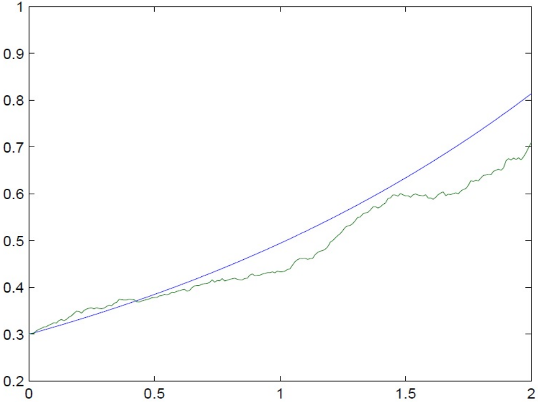
where μ is the instantaneous average rate (deterministic) or *drift rate*, σ is the standard deviation of the instantaneous rate (*volatility*) and W_t is a standard Wiener process (or *Brownian motion*):

$$W_t - W_s \sim \mathcal{N}(0, t-s)$$

for $t > s$.

Recall that $X \sim \mathcal{N}(0,1) \Rightarrow \sqrt{t-s}X \sim \mathcal{N}(0, t-s)$. Recall also $\mathcal{N} = \mathcal{N}(\mu_0, \sigma_0^2)$: μ_0 average, σ_0^2 variance, σ_0 standard deviation (root mean square) and pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

Recall also that if a particle jumps by $\pm h$ each Δt and we define $k = h^2/(2\Delta t)$ then its position $X(t)$ gives $dX = \sqrt{2k}dW_t$.



Plots with $\mu = .5$. Cases $\sigma = 0$ (deterministic) and $\sigma = .07$.

(MATLAB) $>>$ $Dt = .01; y(1) = .3;$ for $k = 1 : 200; y(k+1) \ldots$
$= y(k) + .5 * Dt * y(k) + \text{sqrt}(Dt) * .07 * \text{randn}(1) * y(k);$ end

Ito's Lemma (Ito-Doeblin)

For $C = C(S_t, t)$ we have (Taylor's formula):

$$dC = \partial_S C\, dS_t + \partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, (dS_t)^2 + \partial_S \partial_t C\, dS_t dt + \frac{1}{2}\partial_t^2 C\, (dt)^2 + \cdots ,$$

and we already knew that

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t = \mu S_t\, dt + \sigma S_t X_t\, \sqrt{dt}$$

where $X_t \sim \mathcal{N}(0,1)$. So, in comparison with dt one has $dS_t dt = (dt)^2 = 0$ and so $(dS_t)^2 = \sigma^2 S_t^2 X_t^2\, dt$.
But this last expression can also be simplified in comparison with dS_t: the variance of $[\sigma^2 S_t^2 X_t^2\, dt]$ is of the order of $(dt)^2$ and that of dS_t is dt (larger!). This means that the first can be considered as deterministic in comparison with the second, and can be substituted by its mean value $\sigma^2 S_t^2 dt$. And we finally get

$$dC = \underbrace{\partial_S C\, dS_t}_{\text{random}} + \underbrace{\partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, S_t^2 \sigma^2 dt}_{\text{deterministic}}$$

Portfolio management, 1 (hedging, *cobertura*)

We (the bank) have a portfolio with n options and h assets. Its value is $R = nC + hS$, and the variation of its value

$$dR = n\,dC + h\,dS + C\,dn + S\,dh.$$

We follow the self-financed strategy: if we sell assets then we buy options by the same amount, and viceversa, so: $C\,dn + S\,dh = 0$. This relation allows us to obtain the proportion between the changes in n (dn) and the changes in h (dh) but still does not allow us to decide dn and dh without any ambiguity.

Portfolio management, 2 (hedging, *cobertura*)

So, for the time being,

$$dR = n\, dC + h\, dS = n(\partial_S C\, dS + \partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, S^2\sigma^2 dt) + h\, dS =$$

$$= \underbrace{(n\partial_S C + h)\, dS}_{\text{random}} + \underbrace{n(\partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, S^2\sigma^2 dt)}_{\text{deterministic}}$$

and now we choose at each time n and h in such a way that $n\partial_S C + h = 0$, and the random term disappears, $h = -n\partial_S C$ and we get

$$dR = n(\partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, S^2\sigma^2 dt).$$

Black-Scholes equation

We said that the evolution of the value of our portfolio cannot be different from the evolution of the risk-free asset. So,

$$dR = n(\partial_t C\, dt + \frac{1}{2}\partial_S^2 C\, S^2\sigma^2 dt) = R\rho dt = n(C - \partial_S C\, S)\rho\, dt,$$

and we get the (backwards) parabolic equation

$$-C_t = \frac{1}{2}\sigma^2 S^2 C_{SS} + \rho S C_S - \rho C$$

for $S > 0$ i $t < T$, and the final condition $C(S,T) = \max(0, S - E)$. The changes $x = \log S - Vt$, where $V = \rho - \sigma^2/2$, $\tau = T - t$ and $u(x,\tau) = e^{-\rho(T-\tau)}C(S,t)$ converts the equation into the (forward) diffusion equation

$$u_\tau = \frac{\sigma^2}{2}u_{xx}$$

with the initial condition $u(x,0) = e^{-\rho T}\max(0, e^{x+Vt} - E)$.

Black-Scholes formula

Problem 5.2: Check that

$$C(S,t) = N(d_1(S,t))\,S - N(d_2(S,t))Ee^{-\rho(T-t)}$$

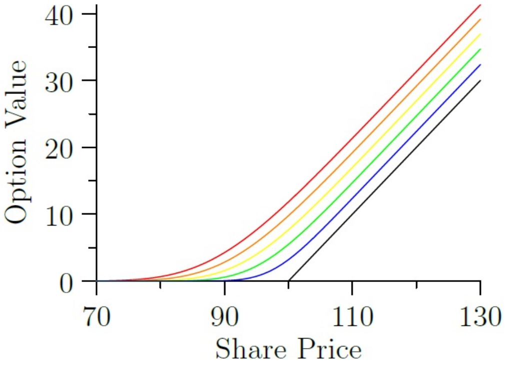
where

$$d_1 = \frac{\log(S/E) + (\rho + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\,dz.$$

Make a broad estimate of the parameters for our first example (Brendt barrels), and plot some reasonable solution curves.



Plots of solutions for different times.

(after http://www.math.unl.edu/ sdunbar1/MathematicalFinance/Lessons/BlackScholes/Solution/solution.pdf)