Boundary conditions

In a domain Ω one has to add boundary conditions to the heat (or diffusion) equation:

- 1. $u(\mathbf{x}, t) = \phi$ for $\mathbf{x} \in \partial\Omega$. Temperature given at the boundary. Also density given at the boundary. (Dirichlet)
- 2. $ku_{\mathbf{n}}(\mathbf{x}, t) = \phi$ for $\mathbf{x} \in \partial\Omega$. Flux given at the boundary. The most typical is $\phi \equiv 0$, isolated boundaries. (Neumann)
- 3. $ku_{\mathbf{n}}(\mathbf{x}, t) + hu(\mathbf{x}, t) = \phi$ for $\mathbf{x} \in \partial\Omega$. These are called *convection* boundary conditions in the thermal engineering literature: $-k\nabla u \cdot \mathbf{n} = h(u - u_\infty)$ if the temperature of the fluid around the body is u_∞ far away. (third class, Robin, convection, Newton)
- 4. Perfect thermal contact: $-k_1\nabla u_1 \cdot \mathbf{n} = -k_2\nabla u_2 \cdot \mathbf{n}$. ($\mathbf{n}$ is the exterior unit normal to Ω_1).

- 5. Dynamic boundary conditions: $u_t + \alpha u_{\mathbf{n}} = \phi$ at the boundary.

Example: bar $(-1 < x < 1)$ with ends immersed into two fluid containers, at homogeneous temperatures $a(t)$ and $b(t)$.

$$V_f C_f \rho_f \frac{d}{dt} a(t) = \int_A -k u_{\mathbf{n}} \, d\mathcal{S} = A k u_x(-1, t)$$

$$V_f C_f \rho_f \frac{d}{dt} b(t) = -A k u_x(1, t)$$

$a'(t) = \alpha u_x$, $b'(t) = -\alpha u_x$, $u(-1, t) = a(t)$ and $u(1, t) = b(t)$.
So, $u_t(-1, t) + \alpha(-u_x(-1, t)) = 0$ and $u_t(1, t) + \alpha u_x(1, t) = 0$.

Example: (to understand conductivity, or diffusivity) Calculate the steady-state temperature profiles inside a wall $-1 < x < 1$ made of two parts in perfect thermal contact: one is $-1 < x < 0$ with conductivity k and the other is $0 < x < 1$ of conductivity 1. The boundary temperatures are $u = 0$ for $x = -1$ and $u = 1$ for $x = 1$.

$$\begin{cases} u_{xx} = 0, \text{ for } -1 < x < 0 \text{ and } 0 < x < 1, \\ u(-1) = 0, \\ u(0^-) = u(0^+), \quad ku_x(0^-) = u_x(0^+) \\ u(1) = 1. \end{cases}$$

The solution is $u = (x+1)/(1+k)$ for $-1 < x < 0$ and $u = (kx+1)/(1+k)$ for $0 < x < 1$.

Problem 3.1: (lumped approximation) Calculate the steady-state temperature profiles inside a wall $-1 < x < 1$ made of three parts in perfect thermal contact: one is $-1 < x < -\varepsilon$ with conductivity 1, the second is $-\varepsilon < x < \varepsilon$ with conductivity k (small) and the third is $\varepsilon < x < 1$ with conductivity 1. The boundary temperatures are $u = a$ for $x = -1$ and $u = b$ for $x = 1$. Observe that for k small, $u \sim a$ in the first part and $u \sim b$ in the third part. How small (with ε) should k be to keep this property as $\varepsilon \to 0$?

Dissipation and limit behavior (bounded domain)

Imagine $u_\infty = 0$ and the three cases of boundary values for $u_t = D\nabla^2 u$: (1) $u = 0$; (2) $u_{\mathbf{n}} = -(h/k)u$, with $h > 0$; and (3) $u_{\mathbf{n}} = 0$. One has

$$\frac{d}{dt}\int_\Omega \frac{1}{2}u^2 = -D\int_\Omega (\nabla u)^2 + D\int_{\partial\Omega} u u_{\mathbf{n}}.$$

In cases (1) and (2) this is strictly negative, unless $u \equiv 0$. In case (3) this is strictly negative, unless $u \equiv$ constant. Also,

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(\nabla u)^2 = -D\int_\Omega u_t^2 + D\int_{\partial\Omega} u_t u_{\mathbf{n}}.$$

In cases (1) and (3) this is strictly negative, except at equilibria. In case (2) we have $\frac{d}{dt}\left(\int_\Omega \frac{1}{2}(\nabla u)^2 + (Dh/k)\int_{\partial\Omega}\frac{1}{2}u^2\right)$ is strictly negative, except at equilibria.

Problem 3.2: By showing that $\int \frac{1}{2}u^2$ tends exponentially to zero, show that all solutions of $u_t = D\nabla^2 u$ with $u = 0$ at $\partial\Omega$ tend to zero exponentially, provided that Ω is *bounded in one direction*, that is $\Omega \subset \{a < x_i < a + L\}$. *Hint*: show first Poincaré's inequality. If Ω is bounded in some direction and $u(\mathbf{x})$ is a smooth function in Ω that vanishes at $\partial\Omega$, then there exists a constant $C > 0$, independent of u such that $\int_\Omega u^2 \leq C \int_\Omega (\nabla u)^2$.

Can one show, with similar methods, that all the solutions of $u_t = D\nabla^2 u$ with $u_\mathbf{n} = 0$ at $\partial\Omega$ tend exponentially to a steady solution? Given the initial condition $u(\mathbf{x}, 0)$ can one predict which is the value of the constant solution at which $u(\mathbf{x}, t)$ tends as $t \to \infty$?

Dissipation and limit behavior (infinite domain)

Dimension one, $k = 1$. Non-negative solutions with finite mass: $0 \le u(x,0)$, $||u(\cdot,0||_1 < \infty$.

$$0 \le u(x,t) \le \frac{1}{\sqrt{4\pi t}} ||u(\cdot,0)||_1 .$$

Also, we have the particular solutions $G(x,t) = (4\pi t)^{-1/2} e^{-x^2/(4t)}$. So, $t^{-1/2}$ is the optimal power.

Problem 3.3: Suppose $0 \le u(x,0)$, $||u(\cdot,0)||_1 < \infty$, and $u(x,0)$ nonzero. Prove that there is a value x_0 and a constant $C > 0$ such that $u(x_0,t) \ge C/\sqrt{t}$ for t sufficiently large.

For large t every positive solution approaches a Gaussian: Suppose $0 \le u(x,0) \le M_1(1+x^2)^{-\alpha}$ for some $\alpha > 1$, and that $||u(\cdot,0)||_1 = M_2$. Then it can be proved that $||u(\cdot,t) - M_2 G(\cdot,t)||_\infty \sim 1/t$.

Entropy

If an event happens with probability p, its (Shannon) amount of information is $-\log p$. If there are n possible events, with probabilities p_i, the (Shannon) *entropy* H of the system is the expected value of the amount of information: $\sum -p_i \log p_i$ (> 0).

If we have a pdf $u(x)$ for $-\infty < x < \infty$ and a partition $\{ih \leq x < (i+1)x,\ k \in \mathbb{Z}\}$ the entropy would be

$$\sum_i -\left(\int_{ih}^{(i+1)h} u\, dx\right) \log\left(\int_{ih}^{(i+1)h} u\, dx\right) \simeq$$

$$\sum_i -u(ih)h \log(u(ih)h) \simeq -\log h - \int_{-\infty}^{\infty} u \log u\, dx$$

and we define its (*renormalized*) entropy as $h[u] = -\int_{-\infty}^{\infty} u \log u\, dx$. Observe that needs not to be positive.

If u satisfies $u_t = u_{xx}$ then

$$\frac{d}{dt}h[u] = -\int u_{xx}\log u - \underbrace{\int u_{xx}}_{=0} = \int \left(-(u_x \log u)_x + |u_x|^2/u\right) =$$

$$\int |u_x|^2/u > 0,$$

so the entropy always increases. The same is true in n dimensions or in a bounded domain with Neumann boundary conditions. Also, for the gaussian

$$-\int G \log G = -\int \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \left(\log \frac{1}{\sqrt{4\pi t}} - \frac{x^2}{4t}\right)$$

Problem 3.4: Show that the first term tends to infinity as $\log t$ and the second does not depend on t.

Separation of variables in an interval

$$u_t = u_{xx} \ \text{ in } 0 < x < a$$

$$u_x(0) = u_x(a) = 0$$

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n e^{-n^2 \frac{\pi^2}{a^2} t} \cos n \frac{\pi}{a} x$$

$$\alpha_0 = \frac{1}{a} \int_0^a u(x,0)\, dx, \ \ \alpha_n = \frac{2}{a} \int_0^a u(x,0) \cos \frac{n\pi}{a} x \, dx.$$

Separation of variables in a rectangle

$u_t = \nabla^2 u$ in $0 < x < a$, $0 < y < b$ with Neumann boundary conditions.

$$u(x,y,t) = \sum_{n,m=0}^{\infty} \alpha_{nm} e^{-(n^2 \frac{\pi^2}{a^2} + m^2 \frac{\pi^2}{b^2})t} \cos(n\frac{\pi}{a}x)\cos(m\frac{\pi}{b}y) =$$

$$\sum_{n=0}^{\infty} \alpha_{n0} e^{-n^2 \frac{\pi^2}{a^2} t} \cos(n\frac{\pi}{a}x) +$$

$$\sum_{n,m-1=0}^{\infty} \alpha_{nm} \underbrace{e^{-(n^2 \frac{\pi^2}{a^2} + m^2 \frac{\pi^2}{b^2})t}}_{\leq e^{-\pi^2 t/b^2}} \cos(n\frac{\pi}{a}x)\cos(m\frac{\pi}{b}y).$$

So, if $0 < b \ll 1$ u becomes shortly independent of y and satisfies simply $u_t = u_{xx}$ in $0 < x < a$ with Neumann boundary conditions.

Thin (slender) domains

$\Omega_\varepsilon = \{(x, y)|0 < x < a, 0 < y < \varepsilon g(x)\}$, $u_t = D\nabla^2 u$ in Ω_ε, $\partial u/\partial \mathbf{n} = 0$ on $\partial\Omega_\varepsilon$. Suppose $u(x, y, t) = u^0(x, t) + \varepsilon u^1(x, y, t) + O(\varepsilon^2)$, and the same for the x-derivatives.

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_0^{\varepsilon g(x)} u(x, y, t)\, dy\, dx = \int_0^{\varepsilon g(x_2)} Du_x\, dy - \int_0^{\varepsilon g(x_1)} Du_x\, dy$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \varepsilon g(x) u^0(x, t)\, dx = \varepsilon g(x_2) D u_x^0(x_2, t) - \varepsilon g(x_1) D u_x(x_1, t)$$

$$\int_{x_1}^{x_2} g(x) u_t^0(x, t)\, dx = \int_{x_1}^{x_2} (g(x) D u_x^0(x, t))_x\, dx$$

$$u_t^0 = \frac{D}{g(x)}(g(x) u_x^0)_x, \quad u_t^0 = D u_{xx}^0 + D\frac{g'(x)}{g(x)} u_x^0.$$