Intuitive review of Vector Calculus: gradient, divergence, curl and laplacian

• Frequently, x is a scalar and $\mathbf{x}$ is a vector. A column vector, mostly: $\mathbf{x} = (x_1, \ldots, x_n)^T$, or $\mathbf{x} = (x, y, z)^T$ when $n = 3$.
• The *nabla* operator of Hamilton $\nabla = (\partial_x, \partial_y, \partial_z)^T$ is a vector operator. If $u(x)$ is a scalar function, then

$$\nabla u = (\partial_x u, \partial_y u, \partial_z u)^T = \text{grad } u$$

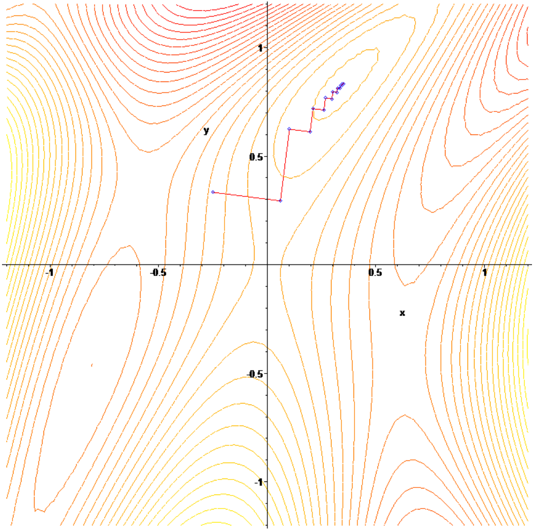
to mean its gradient vector field.
• Taylor's formula: $u(\mathbf{x}_0 + \mathbf{h}) = u(\mathbf{x}_0) + \nabla u(x_0) \cdot \mathbf{h} + O(||h||^2)$.
• For a given $||\mathbf{h}||$, the maximal increase in $u(\mathbf{x}_0 + \mathbf{h}) - u(\mathbf{x}_0)$ happens when $\mathbf{h}$ is collinear with $\nabla u(\mathbf{x}_0)$ ($||\mathbf{h}||$ small).
• $\nabla u(\mathbf{x}_0)$ points in the direction of steepest ascent of u near $\mathbf{x}_0$.
• $||\nabla u(\mathbf{x}_0)||$ is larger if u is steeper near $\mathbf{x}_0$ (closer level surfaces or lines). Also, $\nabla u(\mathbf{x}_0)$ is orthogonal to the surface $u(\mathbf{x}) = u(\mathbf{x}_0)$.

Problem 1.1: It is sometimes said that if $u(x, y)$ represents the height of a mountain (say), then the vector field $-\nabla u(\mathbf{x})$ would be a vector field collinear with the velocities of a water drop falling downhill. That velocity would even be larger at the points where the gradient is larger. Is there any justification for that? A solid ball rolling down would do the same?

Problem 1.2: Suppose that A is an $n \times n$ symmetric positive-definite matrix, and define $u(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x}$. Check that $\nabla u(\mathbf{x}) = A\mathbf{x}$. Prove the following: if $\mathbf{x}(t)$ is the solution of $\mathbf{x}' = -\nabla u(\mathbf{x})$ with $\mathbf{x}(0) = \mathbf{x}_0$, then $\mathbf{x}(t)$ is the unique absolute minimum of

$$J[\mathbf{y}] = \int_0^T \left(\frac{1}{2}\mathbf{y}(t)^T A\mathbf{y}(t) + \frac{1}{2}\mathbf{y}'(t)^T A^{-1}\mathbf{y}'(t) \right) \, dt + \frac{1}{2}|y(T)|^2$$

among all curves $\mathbf{y} : [0, T] \to \mathbb{R}^n$ such that $\mathbf{y}(0) = \mathbf{x}_0$. Prove also a similar result when $u(\mathbf{x})$ is a more general function such that $D^2 u > 0$.



The steepest descent method of minimization

Divergence

If we have a vector field $\mathbf{v} = (v^x, v^y, v^z)^T$, its divergence is the scalar field

$$\text{div } \mathbf{v}(\mathbf{x}_0) = \nabla \cdot \mathbf{v}(x_0) = \partial_x v^x(\mathbf{x}_0) + \partial_y v^y(\mathbf{x}_0) + \partial_z v^z(\mathbf{x}_0)$$

$$= \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = (\partial_x, \partial_y, \partial_z) \cdot \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}$$

Also,

$$\text{div } \mathbf{v}(\mathbf{x}_0) = \text{Trace } D\mathbf{v}(\mathbf{x}_0) = \text{Trace } \begin{pmatrix} v^x_x & v^x_y & v^x_z \\ v^y_x & v^y_y & v^y_z \\ v^z_x & v^z_y & v^x_z \end{pmatrix}$$

The *Divergence Theorem* says that

$$\int_\Omega \nabla \cdot \mathbf{v} \, d\mathcal{V} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, d\mathcal{S}$$

Problem 1.3: Using the Divergence Theorem, prove the following vector identity: $\int_{\partial\Omega} -p\mathbf{n}\, d\mathcal{S} = \int_{\Omega} -\nabla p\, d\mathcal{V}$ and deduce from it Archimedes Principle, when p is the hydrostatic pressure $p = \rho g h$.
Deduce also the following classical (and nice) form of the divergence theorem for a scalar funtion u and a direction i:

$$\int_{\Omega} \partial_i u\, d\mathcal{V} = \int_{\partial\Omega} u n_i\, d\mathcal{S},$$

where n_i is the i-th *direction cosine* of **n**. Deduce also the *integration by parts* version of the previous formula, for two scalar functions u and v:

$$\int_{\Omega} (\partial_i u) v\, d\mathcal{V} = -\int_{\Omega} u(\partial_i v)\, d\mathcal{V} + \int_{\partial\Omega} uv\mathbf{n}_i\, d\mathcal{S}$$

Suppose $\mathbf{v}(\mathbf{x}, t)$ is a velocity. Particles moving with this velocity field will describe trajectories solution of

$$\begin{cases} \mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t), t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

that we represent with the (flow) map $\mathbf{x}_0 \mapsto \mathbf{x}(t) = T_{t_0,t}(\mathbf{x}_0)$, which can be seen as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ (when t_0 and t are fixed). Note that $T_{t_0,t_0} = Id$ and $T_{s,t} \circ T_{r,s} = T_{r,t}$.

Given a set $\Omega \subset \mathbb{R}^3$ we ask ourselves about the volume of $T_{t_0,t}(\Omega)$ or, more precisely, about $\frac{d}{dt}[\text{vol } T_{t_0,t}(\Omega)]_{t=t_0}$.

$$\text{vol } T_{t_0,t}(\Omega) = \int_{T_{t_0,t}(\Omega)} d\mathcal{V} = \int_\Omega JT_{t_0,t}(\mathbf{x})\, d\mathcal{V}.$$

$$T_{t_0,t_0+\varepsilon}(\mathbf{x}) = \mathbf{x} + \varepsilon\mathbf{v}(\mathbf{x}, t_0) + O(\varepsilon^2),$$

$$DT_{t_0,t_0+\varepsilon}(\mathbf{x}) = I + \varepsilon D\mathbf{v}(\mathbf{x}, t_0) + O(\varepsilon^2)$$

$$\det DT_{t_0,t_0+\varepsilon}(\mathbf{x}) = 1 + \varepsilon \text{Trace } D\mathbf{v}(\mathbf{x}) + O(\varepsilon^2),$$

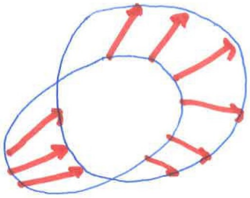
$$\text{vol } T_{t_0,t}(\Omega) = \int_\Omega [1 + \varepsilon \text{Trace } D\mathbf{v}(\mathbf{x}) + O(\varepsilon^2)]\, d\mathcal{V},$$

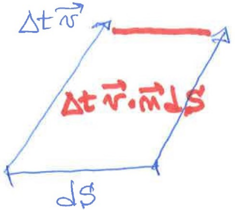
$$\frac{d}{dt}[\text{vol } T_{t_0,t}(\Omega)]_{t=t_0} = \int_\Omega (\nabla \cdot \mathbf{v})\, d\mathcal{V}.$$

So, $\nabla \cdot \mathbf{v}(\mathbf{x}_0, t_0)$ is the rate of variation with respect to time of the volume of a set near $(\mathbf{x}_0, t_0)$ that moves with velocity **v.**

Problem 1.4: Show that if I_n is the $n \times n$ identity matrix and A is any $n \times n$ matrix, then $\det(I_n + \varepsilon A) = 1 + \varepsilon \text{Trace } A + O(\varepsilon^2)$.

But we could also compute the changes in volume of Ω just by looking at the changes at the boundary.





This is the statement of the Divergence Theorem,

$$\int_{\Omega} \nabla \cdot \mathbf{v} \, d\mathcal{V} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, d\mathcal{S}$$

A new view of Divergence and a look to Curl (dimension 3)

$$\frac{d}{dt}\mathbf{x} = \mathbf{v}(x,t) \simeq \mathbf{v}(\mathbf{x}_0,t_0) + \partial_t \mathbf{v}(\mathbf{x}_0,t_0)(t-t_0) + D\mathbf{v}(\mathbf{x}_0,t_0)(\mathbf{x}-\mathbf{x}_0) + \cdots$$

We write $t' = t - t_0$ and $\mathbf{x}' = \mathbf{x} - \mathbf{x}_0$, we drop the primes and consider

$$\frac{d}{dt}\mathbf{x} = \underbrace{\mathbf{v}_0}_{\text{constant velocity}} + \underbrace{\mathbf{v}_1}_{\text{constant acceleration}} t + \underbrace{M}_{n \times n \text{ matrix}} \mathbf{x}$$

$$\frac{d}{dt}\mathbf{x} = M\mathbf{x} = \underbrace{\frac{1}{2}(M + M^T)}_{\text{symmetric}}\mathbf{x} + \underbrace{\frac{1}{2}(M - M^T)}_{\text{anti-symmetric}}\mathbf{x}$$

We are going to look at the three effects separately.

$\frac{1}{2}(M + M^T)$ diagonalizes along 3 orthogonal axes x_1, x_2 and x_3 with real eigenvalues.

$$\begin{cases} \frac{d}{dt}x_1 = \lambda_1 x_1 \\ \frac{d}{dt}x_2 = \lambda_2 x_2 \\ \frac{d}{dt}x_3 = \lambda_3 x_3 \end{cases}$$

which is a very simple deformation. The volume of a deformed orthogonal box would be given in a first approximation by

$$(1+\Delta t\lambda_1)(1+\Delta t\lambda_2)(1+\Delta t\lambda_3) \simeq 1+\Delta t(\lambda_1+\lambda_2+\lambda_3) = 1+\Delta t\, \nabla\cdot\mathbf{v}.$$

This tells us again that the divergence is the local rate of volume change.

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + bz \\ -ax + cz \\ -bx - cy \end{pmatrix} = \begin{pmatrix} -c \\ b \\ -a \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So, let us study $\frac{d}{dt}\mathbf{x} = \mathbf{h} \times \mathbf{x}$. Its solution is a rotation around the axis $\mathbf{h}$ with angular speed $|\mathbf{h}|$ in the positive sense.

$$\frac{1}{2}\left(D\mathbf{v}-(D\mathbf{v})^T\right)=\frac{1}{2}\begin{pmatrix} 0 & v_y^x - v_x^y & v_z^x - v_x^z \\ v_x^y - v_y^x & 0 & v_z^y - v_y^z \\ v_x^z - v_z^x & v_y^z - v_z^y & 0 \end{pmatrix}.$$

So, the anti-symmetric part of $D\mathbf{v}$ means a rotation given by

$$\mathbf{h}=\frac{1}{2}\begin{pmatrix} v_y^z - v_z^y \\ v_z^x - v_y^z \\ v_x^y - v_y^x \end{pmatrix}=\frac{1}{2}\nabla\times\mathbf{v}=\frac{1}{2}\text{ Curl }\mathbf{v} \qquad (1)$$

Problem 1.5: Check the statements we did: the constant-coefficients system of odes

$$\frac{d}{dt}\mathbf{x}=\frac{1}{2}\left(D\mathbf{v}-(D\mathbf{v})^T\right)\mathbf{x}$$

can be written as $(d/dt)\mathbf{x}=\mathbf{h}\times\mathbf{x}$ where $\mathbf{h}=\frac{1}{2}$ Curl $\mathbf{v}$ and its solution consists on a rotation around the axis $\mathbf{h}$ with angular speed $|\mathbf{h}|$ in the positive sense. (Hint: for the second part, suppose $\mathbf{h}=(\omega,0,0)^T$ and solve explicitly.)

Stokes Formula:

$$\int_{\Sigma} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, d\mathcal{S} = \int_{\partial\Sigma} \mathbf{v} \cdot \, d\ell$$

Relations:

$$\nabla \times (\nabla u) \equiv 0,$$

$$\nabla \cdot (\nabla \times \mathbf{v}) \equiv 0,$$

$$\nabla \cdot (\nabla u) = \nabla^2 u = \Delta u = \sum_{i=1}^{n} \partial^2_{x_i x_i} u$$

Integration by parts formula:

$$\nabla \cdot (f\mathbf{X}) = \nabla f \cdot \mathbf{X} + f \nabla \cdot \mathbf{X}$$

so

$$\int_{\partial\Omega} f\mathbf{X} \cdot \mathbf{n} \, d\mathcal{S} = \int_{\Omega} \nabla f \cdot \mathbf{X} \, d\mathcal{V} + \int_{\Omega} f \nabla \cdot \mathbf{X} \, d\mathcal{V}.$$

In dimension 3 we have that

$$\text{[scalar fields]} \xrightarrow{\nabla} \text{[3D vector fields]} \rightarrow$$

$$\xrightarrow{\nabla\times} \text{[3D vector fields]} \xrightarrow{\nabla\cdot} \text{[scalar fields]}$$

and $(\nabla\times)\circ(\nabla) = 0$ and $(\nabla\cdot)\circ(\nabla\times) = 0$.
Poincaré's lemma says that if $\Omega \subset \mathbb{R}^3$ is an open set that is the image by an homeomorphism of the open unit ball, then $\nabla \times \mathbf{u} = 0$ in Ω implies the existence of a scalar field f such that $\mathbf{u} = \nabla f$, and also $\nabla \cdot \mathbf{w} = 0$ implies the existence of a vector field $\mathbf{v}$ such that $\nabla \times \mathbf{w} = \mathbf{v}$.
This is more often stated in terms of differential forms.

If $\mathbf{v}(\mathbf{x}, t)$ is a vector field in $\mathbb{R}^3$ defined on a domain Ω, then $\mathbf{v}$ is determined by its divergence, its curl and its values on $\partial\Omega$. There are formulas for that (*Helmholtz* decomposition, sometimes called the *Fundamental Theorem of Vector Analysis*). When $\Omega = \mathbb{R}^3$ and $\mathbf{v}$ vanishes at infinity faster than $1/|\mathbf{x}|$, these formulas are $\mathbf{v} = -\nabla\Phi + \nabla \times \mathbf{A}$ with

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla' \cdot \mathbf{v}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}\mathcal{V}',$$

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla' \times \mathbf{v}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}\mathcal{V}'.$$

Problem 1.6: Show something weaker, also in this direction: if two vector fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}'(\mathbf{x})$ in $\mathbb{R}^3$ share the same divergence and the same curl, and both tend to zero at infinity, then they must be equal.

Interpretation of the Laplacian: The laplacian of a function measures the difference between the average of this function on a small sphere and the value of the function at the center of the sphere.
Claim:

$$\lim_{r\to 0^+} \frac{\left(\frac{1}{r^{n-1}|S_{n-1}|}\int_{||\mathbf{h}||=r} u(\mathbf{x}_0+\mathbf{h})\, d\mathcal{S}_{\mathbf{h}}\right) - u(\mathbf{x}_0)}{r^2} = \frac{1}{2n}\nabla^2 u(\mathbf{x}_0)$$

Proof: $u(\mathbf{x}_0+\mathbf{h}) = a + \sum_{i=1}^n b_i h_i + \sum_{i,j=1}^n c_{ij} h_i h_j + \dots$ Then $r^{1-n}|S_{n-1}|^{-1}\int_{||\mathbf{h}||=r} a = u(\mathbf{x}_0)$, $\int b_i h_i = 0$, because of the symmetry $h_i \leftrightarrow -h_i$, $\int c_{ij} h_i h_j = 0$ when $i \neq j$, because of the same reason. Then, obviously,

$$\int_{||\mathbf{h}||=r} h_i^2 = \int_{||\mathbf{h}||=r} h_j^2,$$

$$\int_{||\mathbf{h}||=r} h_i^2 = n^{-1}\int_{||\mathbf{h}||=r} r^2 = n^{-1} r^2 r^{n-1} |S_{n-1}|.$$

Problem 1.7: Make similar calculations to interpret the Laplacian as a measure of the difference between the average of the function on a small **ball** (instead of a sphere) and the value of the function at the center of the ball.

The transport of a density

$u(\mathbf{x}, t)$ is an (unknown) density that is convected by a fluid with velocity $\mathbf{v}(\mathbf{x}, t)$. We want to derive the equation of its time evolution.

Ω is an arbitrary domain. The mass **flux** that crosses $\partial\Omega$ (units of mass per units of time, from inside to outside) is $\int_{\partial\Omega} u\mathbf{v} \cdot \mathbf{n} \, d\mathcal{S}$.

Balance of mass + divergence theorem:

$$\frac{d}{dt} \int_{\Omega} u \, d\mathcal{V} = - \int_{\partial\Omega} u\mathbf{v} \cdot \mathbf{n} \, d\mathcal{S} = - \int_{\Omega} \nabla \cdot (u\mathbf{v}) \, d\mathcal{V},$$

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{v}) \right) \, d\mathcal{V} = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{v}) = 0 \qquad \left(\frac{\partial u}{\partial t} = -\nabla u \cdot \mathbf{v} - u \nabla \cdot \mathbf{v} \right)$$

- In 1-D, with $v = c$ (constant): $u_t + cu_x = 0$
- Transport + reaction

$$u_t + \nabla \cdot (u\mathbf{v}) = f(u, x, t)$$

that comes from

$$\frac{d}{dt}\int_{\Omega} u \, d\mathcal{V} = -\int_{\partial\Omega} u\mathbf{v} \cdot \mathbf{n} \, d\mathcal{S} + \int_{\Omega} f \, d\mathcal{V}$$

Example of transport + reaction: 1-D persistent motion with random orientation in animal search strategies

(V. Méndez et al., Stochastic Foundations in Movement Ecology, Sect. 6.1.2)

$$\rho_t^+ + c\rho_x^+ = \lambda(\rho^- - \rho^+)$$

$$\rho_t^- - c\rho_x^- = -\lambda(\rho^- - \rho^+)$$

for $0 < x < L$, and boundary conditions $\rho^+(0, t) = 0$, $\rho^-(L, t) = 0$.

$\rho^+(x, t)$ (resp., $\rho^-(x, t)$) is the pdf for a predator to be at position x at time t while walking to the right (resp. left) with speed c.

Two prays are supposed to be located at $x = 0$ and $x = L$.

$\lambda > 0$ is the switching rate: the probability of switching orientation between t and $t + \Delta t$ is $\lambda\Delta t$.

References:

https://en.wikipedia.org/wiki/Vector_calculus

A.J. Chorin, J.E. Marsden: A mathematical Introduction to Fluid Mechanics. Springer. (§ 1.2)