Structural stability

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OUTLINE

INTRODUCTION

FIRST NAIVE DEFINITION

Roughly speaking, we say that a dynamical system is structurally stable if the qualitative behaviour does not change when sufficiently close systems are considered.

That is: we want to characterize *robust* systems.

However we need to be more precise:

- What does mean qualitative behaviour?
- What does mean sufficiently close systems?

The another important fact is

TO BE OR NOT BE STRUCTURALLY STABLE

The set of structurally stable systems is dense? open?

RECALL:

- Let $A \in \mathcal{M}_{n\times n}$. The linear flow $\dot{x} = Ax$ is hyperbolic if
$$\operatorname{Spec} A \subset \{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \neq 0\}.$$
When a discrete linear dynamical system is considered, i.e. $\bar{x} = Ax$, we say it is hyperbolic if
$$\operatorname{Spec} A \subset \{\lambda \in \mathbb{C} \,:\, |\lambda| \neq 1\}.$$
- A point x is a non-wandering point for the diffeomorphism f (resp. for the flow φ_t) if, given any neighbourhood W of x, there exists some $m > 0$ (resp. $t > t_0 > 0$) for which
$$f^m(W) \cap W \neq \emptyset, \qquad \left(\text{resp. } \varphi_t(W) \cap W \neq \emptyset\right).$$
- In a topological space, we say that that a set is residual if it is the countable intersection of open dense sets.

RECALL:

- We say that a property is generic if it is shared by the elements of a residual set.
- Two diffeomorphisms $f, g : M \to M$ are said to be C^r- conjugate if there is a homeomorphism $h : M \to M$, C^r such that

$$h \circ f = g \circ h.$$

- Two flows φ_t, ψ_t are C^r-equivalent if there exists a C^r homeomorphism that sends orbits of φ_t onto orbits of ψ_t preserving orientation.
- Hartman's theorem. If a dynamical system (either flow or map) has an equilibrium hyperbolic point, then it is topologically conjugated to its linearised part.

LINEAR SYSTEMS

Let A be a $n \times n$ matrix, $A \in L(\mathbb{R}^n)$. We first define an ε-neighbourhood as:

$$N_\varepsilon(A) = \{B \in L(\mathbb{R}^n) \; : \; \|B - A\| < \varepsilon\}.$$

DEFINITION OF STRUCTURALLY STABLE

A linear system (either flow or map) is said to be structurally stable in $L(\mathbb{R}^n)$ if there is an ε-neighbourhood of A, $N_\varepsilon(A)$, such that for every $B \in N_\varepsilon(A)$:

- in the case of flows, e^{tA} and e^{tB} are topologically equivalent;
- in the case of maps, $f(x) = Ax$ and $g(x) = Bx$ are topologically conjugate.

- To be topologically equivalent (or topologically conjugated) will be the definition of the same qualitative behaviour.
- Notice that the definition depends on the set (in this case $L(\mathbb{R}^n)$) we take *a priori*.

CHARACTERIZATION OF STRUCTURALLY LINEAR SYSTEMS

PROPOSITION

A linear flow or diffeomorphism on $\mathbb{R}^n$ is structurally stable in $L(\mathbb{R}^n)$ if and only if it is hyperbolic.

Idea of the proof:

- Prove that if ε is small enough and $B \in N_\varepsilon(A)$, then B is also hyperbolic (you can prove it using Gershgorin's theorem) with the same stability index n^s, i.e., the same number of stable eigenvalues.
- For flows, they are both topologically conjugated to $\dot{x} = -x$, $\dot{y} = y$, $x \in \mathbb{R}^{n^s}$, $y \in \mathbb{R}^{n-n^s}$. In conclusion A is structurally stable.
- For flows, if A is not hyperbolic then it has at least one eigenvalue with real part equal to 0. Then, taking ε small enough $B_\varepsilon^\pm = A \pm \varepsilon I$ is hyperbolic. Since B_ε^+, B_ε^- are not topologically equivalent, A is not structurally stable.
- Do the same for diffeomorphism (first part of Exercise 115).

REMARKS

- The dependence on the set. If we take the subset of linear systems having eigenvalues with real part equal to 0, to be a center is structurally stable. Even more, in the subset of Hamiltonian linear system, also the centers are structurally stable.
- The differential equivalence is too restrictive. If two linear flows are differentiably equivalent, their matrices have to be *almost* similar. Indeed, if $h(e^{At}x) = e^{B\tau(t,x)}h(x)$,

$$Dh(x)Ax = \tau'(0,x)Bh(x) \Longrightarrow Dh(\lambda v)Av = \tau'(0,\lambda v)B\frac{h(\lambda v)}{\lambda} \Longrightarrow Dh(0)A = \tau'(0)BDh(0)$$

  For instance

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \qquad \dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix} x$$

  Are not differentiably equivalent.
- The topological type can be different in the same equivalence class. Take

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \qquad \dot{x} = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} x.$$

  The first is a node (eigenvalue 1) and the second are focus (eigenvalues $1 \pm i\varepsilon$).

OPEN AND DENSE

PROPOSITION

The set $SDD(\mathbb{R}^n)$ of structurally stable linear dynamical systems, is open and dense in $L(\mathbb{R}^n)$. That is the property of being structurally stable is generic.

- Idea of the proof. $SDD(\mathbb{R}^n) = HDD(\mathbb{R}^n)$ with $HDD(\mathbb{R}^n)$ being the set of hyperbolic linear dynamical systems.
- To be open. Let $A \in SDD(\mathbb{R}^n)$, by definition there exists

$$N_\varepsilon(A) \subset L(\mathbb{R}^n), \qquad A \text{ is topologically conjugated to } B \in N_\varepsilon(A).$$

  Then B has to be hyperbolic and consequently B has to be structurally stable.
- To be dense in $L(\mathbb{R}^n)$. If $A \notin HDD(\mathbb{R}^n)$, then $B_\varepsilon = A + \varepsilon Id \in HDD(\mathbb{R}^n)$ if ε is small enough and B_ε is arbitrarily close to A. In fact,

$$\lim_{\varepsilon \to 0} B_\varepsilon = A.$$

- Do the same for diffeomorphism (second part of Exercise 115).

Non linear systems

The first thing we need to do is to define what means close enough. That is a a set of appropriate perturbations.

- For that we first define a norm along with a set. For $U \subset \mathbb{R}^n$, we denote:

$$\mathrm{Vec}^1(U) = \{X : U \to \mathbb{R}^n,\ C^1 \text{ vector fields}\},$$

$$\|X\|_1 = \sup_{x \in U} \sum_{i=1}^{n} |X^i(x)| + \sup_{x \in U} \sum_{i,j=1}^{n} |D_{x_j} X^i(x)|.$$

  Note that $\|X\|$ is small if $|X^i|$ and $|D_{x_j} X^i|$ are small.

- A ε-neighbourhood:

$$N_\varepsilon(X) = \{Y \in \mathrm{Vec}^1(U) \ :\ \|X - Y\| < \varepsilon\}.$$

- We say that Y is $\varepsilon - C^1$ close enough of X if $Y \in N_\varepsilon(X)$.

LOCAL STRUCTURAL STABILITY

DEFINITION

Let $X \in \mathrm{Vec}^1(U)$. We say that X is locally structurally stable if there exists $N_\varepsilon(X) \subset \mathrm{Vec}^1(U)$ such that for any $Y \in N_\varepsilon(X)$, there exist $V, W \subset U$ such that $X_{|V}$ and $Y_{|W}$ are topologically equivalent. That is there exists a homeomorphism $h : V \to W$ such that

$$\varphi_t(h(x)) = h(\psi_{\tau(t)}x), \qquad \varphi_t, \psi_t \text{ flows of } X, Y.$$

The corresponding definition for f diffeomorphisms.

- The question is then: can we characterize the local structurally stable vector fields?
- For arbitrary dimension we only have partial results.
- By the flow box theorem, given a vector field X, in a neighbourhood of a regular point ($X(p) \neq 0$), X is local structurally stable.
- What does happen around a singular point?

Partial results (I)

Proposition

Let $X \in \mathrm{Vec}^1(U)$ have a hyperbolic fixed point x_*. Then, X is locally structurally stable.

Idea of the proof:

- There exists $\hat{V} \subset U$ neighbourhood of x_* and a $N_\varepsilon(X)$ such that if $Y \in N_\varepsilon(X)$ it has a unique hyperbolic fixed point $y_* \in \hat{V}$. In addition the linearised systems $DX(x_*), DY(y_*)$ have stable spaces of the same dimension.
- Then the flows of $DX(x_*), DY(y_*)$ are topologically conjugated.
- By Hartman's theorem, $X_{|V}$ and $Y_{|W}$ are topologically conjugated to $DX(x_*)_{|V}$ and $DY(y_*)_{|W}$.
- By transitivity of topological equivalence the results follows.

We have the corresponding result for maps:

Proposition

Let f be a diffeomorphism having a hyperbolic fixed point x_*. Then f is locally structurally stable.

Do exercise 120.

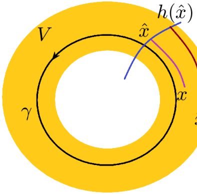
PARTIAL RESULTS (II)

COROLLARY

As a consequence, if we have a vector field $X \in \mathrm{Vec}^1(U)$ with a hyperbolic closed orbit γ, it is locally structurally stable in a neighbourhood of the closed orbit.

Idea of the proof:

- Take $Y \in N_\varepsilon(U)$ with ε sufficiently small, a cross section Σ of X at γ and the associated Poincaré map P_0.
- The flows $\varphi_t^0, \varphi_t^\varepsilon$ of X, Y are ε-close around the periodic orbit.
- The section Σ is a cross section also for Y.



- Consider the map $P_\varepsilon : \Sigma \to \Sigma$ defined by $P_\varepsilon(x) = \varphi^\varepsilon_{\tau(x;\varepsilon)}(x) \in \Sigma$.
- P_ε is $\varepsilon-$ close to P_0. Therefore they both are locally topologically conjugated, by h, around the hyperbolic equilibrium point $x_* = \Sigma \cap \gamma$.
- From this we deduce that the flows φ^0 and φ^ε are topologically equivalent in a neighbourhood V of γ by the homeomorphism defined by

$$x \in V \to \hat{\tau}(x) := \tau(x;0),\ \hat{x} := \varphi^0_{\hat{\tau}(x)}(x) \in \Sigma \to h(\hat{x}) \to \bar{x} = \varphi^\varepsilon_{-\hat{\tau}(x)}(h(\hat{x})).$$

OUTLINE

PRELIMINARIES

We begin by vector fields defined on $\mathbb{R}^2$.

- To guarantee that $\|X\|_1$ is finite we restrict ourselves to the unit disc:

$$\mathbb{D}^2 = \{x \in \mathbb{R}^2 \ : \ \|x\| \leq 1\}$$

- Consider the set $\mathrm{Vec}^1(\mathbb{D}^2)$ defined as the set of vector fields $X \in \mathrm{Vec}^1(U)$ being U an open set which contains $\mathbb{D}^2$.
- For $X \in \mathrm{Vec}^1(\mathbb{D}^2)$, we define

$$\|X\|_1 = \max_{x\in\mathbb{D}^2} \sum_{i=1}^{2} \|X^i(x)\| + \max_{x\in\mathbb{D}^2} \sum_{i=1,j}^{2} \|D_{x_j} X^i(x)\|$$

  which is finite.
- A neighbourhood $N_\varepsilon(X) \subset \mathrm{Vec}^1(\mathbb{D}^2)$ of $X \in \mathrm{Vec}^1(\mathbb{D}^2)$

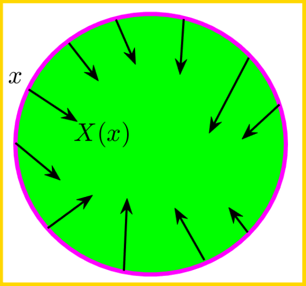
$$N_\varepsilon(X) = \{Y \in \mathrm{Vec}^1(\mathbb{D}^2) \ : \ \|X - Y\|_1 < \varepsilon\}.$$

GLOBAL STRUCTURAL STABILITY

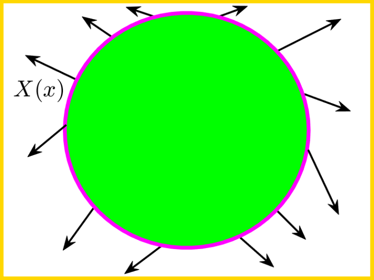
DEFINITION

A vector field $X \in \mathrm{Vec}^1(\mathbb{D}^2)$ is said to be structurally stable if there exists a neighbourhood $N_\varepsilon(X)$ such that any $Y \in N_\varepsilon(X)$ is topologically equivalent to X on $\mathbb{D}^2$.

To prove powerful results we deal with transversal vector fields on $\partial\mathbb{D}^2$.



$\mathrm{Vec}^1_{in}(\mathbb{D}^2)$ the subset of $\mathrm{Vec}^1(\mathbb{D}^2)$ such that $X(x)$ points into $\mathbb{D}^2$ if $x \in \partial\mathbb{D}^2$.



$\mathrm{Vec}^1_{out}(\mathbb{D}^2)$ the subset of $\mathrm{Vec}^1(\mathbb{D}^2)$ such that $X(x)$ points out $\mathbb{D}^2$ if $x \in \partial\mathbb{D}^2$.

OUTLINE

PEIXOTO'S THEOREM ON $\mathbb{D}^2$

THEOREM

Let $X \in \mathrm{Vec}\,^1_{in}(\mathbb{D}^2) \cup \mathrm{Vec}\,^1_{out}(\mathbb{D}^2)$. Then X is structurally stable if and only if its flows satisfies:

A) *All fixed points are hyperbolic.*

B) *All closed orbits are hyperbolic.*

C) *There are no orbits connecting saddle points.*

Even more

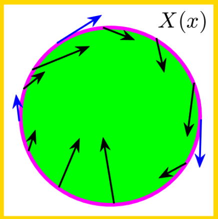
THEOREM

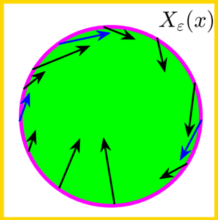
The subset of vector fields in $\mathrm{Vec}\,^1_{in}(\mathbb{D}^2)$ (resp. $\mathrm{Vec}\,^1_{out}(\mathbb{D}^2)$) that are structurally stable is open and dense in $\mathrm{Vec}\,^1_{in}(\mathbb{D}^2)$ (resp. $\mathrm{Vec}\,^1_{out}(\mathbb{D}^2)$). That is, to be structually stable in $\mathrm{Vec}\,^1_{in}(\mathbb{D}^2)$ and $\mathrm{Vec}\,^1_{out}(\mathbb{D}^2)$ is a generic condition.

SOME COMMENTS ABOUT PEIXOTO'S THEOREM

Some comments:

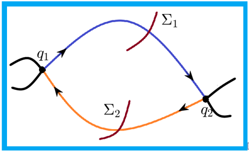
- If X **does not** belong to $\mathrm{Vec}^1_{in}(\mathbb{D}^2) \cup \mathrm{Vec}^1_{out}(\mathbb{D}^2)$ and satisfies some conditions, there is a sequence of vector fields X_{ε_n} belonging to $\mathrm{Vec}^1_{in}(\mathbb{D}^2) \cup \mathrm{Vec}^1_{out}(\mathbb{D}^2)$ and $\|X_{\varepsilon_n} - X\|_1 \to 0$ as $n \to \infty$.

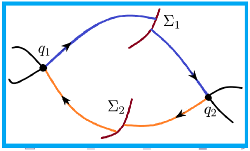




- The conditions a), b) assures that the vector field is locally structurally stable.

- The only global condition is c).





PEIXOTO'S THEOREM ON M

Let M be a two dimensional compact manifold without boundary. We call $\mathrm{Vec}^1(M)$ the set of C^1- vector fields on M with the C^1-norm (the C^1-norm on each of the charts)

THEOREM

A vector field in $\mathrm{Vec}^1(M)$ is structurally stable if and only if its flows satisfies

A) *All fixed points are hyperbolic.*

B) *All closed orbits are hyperbolic.*

C) *There are no orbits connecting saddle points.*

D) *The non-wandering set consists only of fixed points and periodic orbits.*

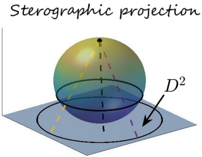
In addition, when M is orientable, the set of structurally stable C^1 vector fields consists on an open dense subset of $\mathrm{Vec}^1(M)$. That is, to be structurally stable is a generic condition.
See the book Geometric Theory of Dynamical Systems. An introduction', by Jacob Palis, Jr. and Welington de Melo.

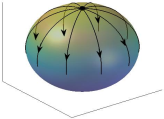
Do exercise 122.

REMARKS:

- Orientable manifold means that two distinct sides of M can be recognised. For instance the sphere, torus, pretzel (see figure)

- Considering the flow on the torus: $\theta(t) = (t, t\omega)$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$ which is unstable and the non wandering set is always $\mathbb{T}^2$. The last condition can not be skipped.
- Since M is compact, flows on it can only have finitely many fixed and periodic points if they are all hyperbolic. This is due to the fact that hyperbolic fixed points are isolated.
- What does happen with the condition that the vector field points *in* or *out*?





- Let $X \in \mathrm{Vec}^1(\mathbb{S}^2)$.
- Any closed cap of $\mathbb{S}^2$ is diffeomorphic to $\mathbb{D}^2$ by the sterographic projection. Let $\hat{X} \in \mathrm{Vec}^1(\mathbb{D}^2)$ the corresponding vector field
- The condition $\hat{X} \in \mathrm{Vec}^1_{in}(\mathbb{D}^2)$ is equivalent for X to have a repellor in the north pole.

STRUCTURAL STABILITY IN NON COMPACT SETS

FIRST IDEA

If a system is structurally unstable in a compact set is unstable at the whole plane.

Show that the vector field, X, defined by

$$X(x, y) = (2x - x^2, -y + xy),$$

is not structurally stable on any compact subset of the plane with the line segment joining the singular points of X in its interior.

SECOND (WRONG) IDEA

If a system is structurally stable in any compact set is structurally stable at the whole plane.

Show that there are arbitrarily large compact subsets of the plane on which the system

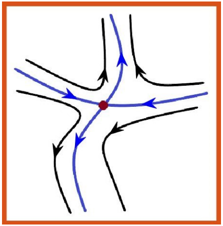
$$\dot{x} = -x, \qquad \dot{y} = \sin(\pi y)e^{-y^2}$$

is structurally stable. However it is not structurally stable in $\mathbb{R}^2$.

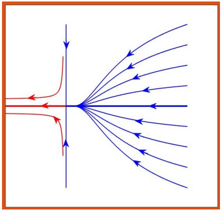
STABLE VECTORS FIELDS IN $\mathbb{R}^2$

We can get them by using the sterographic projection. But they will only have a finite number of fixed points

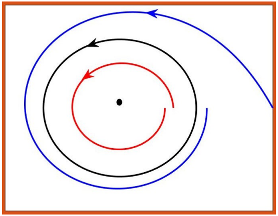
SOME EXAMPLES



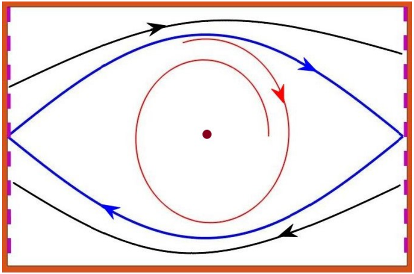
Local structurally stable.



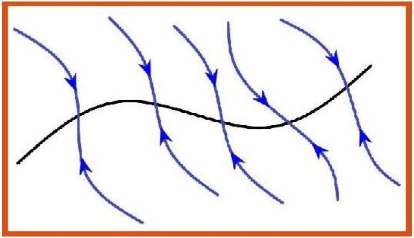
The fixed point is not hyperbolic.



The periodic orbit is not hyperbolic

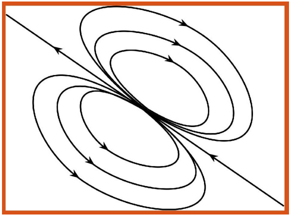


Homoclinic connection in the cylinder

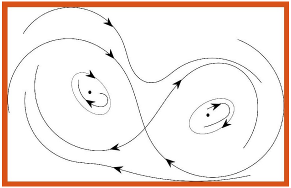


A curve of fixed points

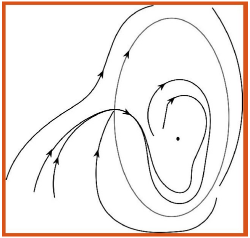
MORE EXAMPLES



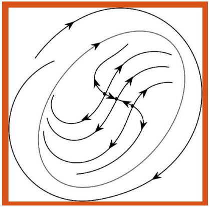
Structurally unstable on compacts.



Structurally stable on $\mathbb{R}^2$ by structurally stable on $\mathbb{S}^2$: The infinity is a repellor.



Structurally unstable on compacts.



Structurally stable on $\mathbb{R}^2$ by structurally stable on $\mathbb{S}^2$: The infinity is a repellor. Note that there are not saddle connections.

OUTLINE

A FIRST RESULT FOR MAPS ON $\mathbb{S}^1$

THEOREM

A diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ ($f \in Diff$) is structurally stable if and only if its non-wandering set consists of finitely many fixed points or periodic orbits all of them hyperbolic.
Moreover, the subset of structurally stable maps is open and dense in $Diff$.

Idea for the proof.
Peixoto's Theorem along with the relationship between two dimensional flows and one dimensional maps by means of Poincaré map.

A COMMENT

Recall that for a diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ to have a periodic orbit means that the rotation number is rational.
As a consequence, if $f \in \mathrm{Diff}$ is structurally stable, then the rotational number is rational. The converse is not true ($x \to x + p/q$) is unstable.
Remember that when the rotation number is irrational, every orbit is dense.

OUTLINE

THE MORSE-SMALE DYNAMICAL SYSTEMS

Morse-Smale dynamical systems satisfy the conditions in Peixoto's theorem:

- There are finitely many hyperbolic equilibrium points and periodic orbits.
- The intersection, if it exists, between the stable and unstable invariant manifolds, is transversal.
- The non-wandering set consists of finitely many hyperbolic equilibrium points and hyperbolic periodic orbits

THEOREM

The Morse-Smale systems are structurally stable.

However:

THEY DO NOT CHARACTERIZE STRUCTURAL STABLE SYSTEMS

There are other systems with this property which are not of Morse-Smale type. In particular there are diffomorphisms on manifolds of dimension $n \geq 2$ that their non-wandering set contains infinitely many hyperbolic periodic orbits: the Anosov diffeomorphism of the torus $\mathbb{T}^n$.
As a consequence the Morse-Smale systems are not generic.

OUTLINE

DEFINITION

Roughly speaking Anosov diffeomorphisms are the ones having *expansion* and *contraction* directions.
The rigourous definition is the following:

DEFINITION

Let $f : M \to M$ be a diffeomorphism defined on a differential manifold. We define the tangent bundle:

$$TM = \{(x, v) \,:\, x \in M,\ v \in T_xM\}.$$

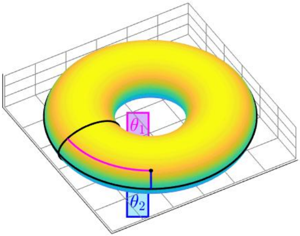
Then, there exists constants $C > 0$ and $0 < \lambda < 1$ such that

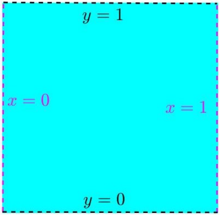
- $TM = E^s \oplus E^u$, $DfE^s = E^s$, $DfE^u = E^u$
- $(x, v) \in E^s$, $\|Df^n(x)v\| \leq C\lambda^n\|v\|$,
- $(x, v) \in E^u$, $\|Df^n(x)v\| \geq C\lambda^{-n}\|v\|$.

We focus on the Anosov diffeomorphisms defined on $M = \mathbb{T}^n$.
They have non-wandering sets having infinitely many periodic points.

LIFTS

We first notice that we can describe a torus $\mathbb{T}^n$ by the square $[0,1]^n$ identifying the sides $x_i = 0$ with $x_i = 1$





As for the maps on $\mathbb{S}^1$ we can describe $f : \mathbb{T}^n \to \mathbb{T}^n$ by means of lifts $\tilde{f}$.

DEFINITION OF LIFTS

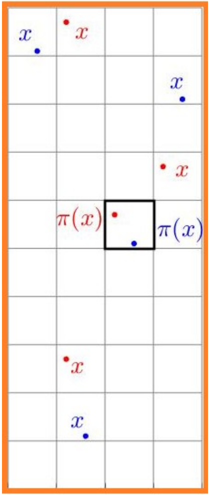
Let $f : \mathbb{T}^n \to \mathbb{T}^n$ a diffeomorphism. Consider the projection

$$\pi : \mathbb{R}^n \to \mathbb{T}^n, \qquad \pi(x) = (x_1 \,(\mathrm{mod} 1), \cdots, x_n \,(\mathrm{mod} 1)) = (\theta_1, \cdots, \theta_n) = \theta.$$

A lift $\tilde{f}$ of f is a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(\pi(x)) = \pi(\tilde{f}(x))$$

PROPERTIES OF THE LIFS



- They are not unique: if $\tilde{f}$ is a lift, $\tilde{f}+1$ is also a lift.
- If $k=(k_1,\cdots,k_n)\in\mathbb{Z}^n$, then for all $x\in\mathbb{R}^n$:

$$\pi(\tilde{f}(x+k)) = f(\pi(x+k)) = f(\pi(x)) = \pi(\tilde{f}(x)).$$

  Since $\tilde{f}$ is continuous,

$$\tilde{f}(x+k) = \tilde{f}(x) + l(k), \qquad l(k)\in\mathbb{Z}^n$$

  with $l(k)$ independent on x.
- $\tilde{f}$ is a lift of f if and only $\tilde{f}^{-1}$ is a lift of f^{-1}. Indeed, take $y=\tilde{f}(x)$ at $f(\pi(x))=\pi(\tilde{f}(x))$:

$$f(\pi(\tilde{f}^{-1}(y))) = \pi(y) \Leftrightarrow \pi(\tilde{f}^{-1}(y)) = f^{-1}(\pi(y)).$$

- $\tilde{f}^q$ is a lift of f^q if $q\in\mathbb{Z}$.

OUTLINE

DEFINITION

DEFINITION

The lift of an Anosov automorphism f is a hyperbolic, linear diffeomorphism, $A:\mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$A:\mathbb{Z}^n \to \mathbb{Z}^n, \qquad \det A = \pm 1.$$

An Anosov automorphism is a diffeomorphism.

Indeed: let A be a lift of f. Then:

- It is clear that $A^{-1}:\mathbb{Z}^n \to \mathbb{Z}^n$.
- A^{-1} satisfies the necessary condition for being a lift:

$$A^{-1}(x+k) = A^{-1}x + l(k).$$

- f^{-1} exists and its lift is A^{-1}. Indeed, define $g(\pi(x)) = \pi(A^{-1}x)$. Since $f(\pi(x)) = \pi(Ax)$,

$$f(g(\pi(x)) = f(\pi(A^{-1}x) = \pi(x).$$

  Changing the role of f, (A) and g, (A^{-1}) we also have $g(f(\pi(x))) = \pi(x)$.
- f, f^{-1} are differentiable since A, A^{-1} are obviously differentiable and π is a local diffeomorphism.

An important property

Proposition

Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov automorphism. A point $\theta \in \mathbb{T}^n$ is a periodic orbit of f if and only if $\theta = \pi(x)$, with $x = (x_1, \cdots, x_n) \in \mathbb{Q}^n$.

Proof. Let f be an Anosov autohomorphism and A one lift.

- Let $q \in \mathbb{N}$ be such that $f^q(\theta) = \theta$. and let $x \in \mathbb{R}^n$ be such that $\pi(x) = \theta$.
- We have that $(A^q - \mathrm{Id})x = m$, with $m \in \mathbb{Z}^n$. Indeed,

$$\pi(x) = \theta = f^q(\theta) = f^q(\pi(x)) = \pi(A^q(x)).$$

- Since A is hyperbolic, A^q has not 1 as eigenvalue so that

$$x = (A^q - \mathrm{Id})^{-1} m, \quad m \in \mathbb{Z}^n$$

- Since A^q is an integer matrix, the elements of $(A^q - \mathrm{Id})^{-1}$ are rational numbers and we conclude.
- We write $x = \left(\frac{p_1^{(0)}}{r}, \cdots, \frac{p_n^{(0)}}{r} \right) \in \mathbb{Q}^n$.
- We notice that, for any $k \in \mathbb{N}$:

$$A^k x = \left(\frac{p_1^{(k)}}{r}, \cdots, \frac{p_n^{(k)}}{r} \right),$$

  with the same denominator r and $p_i^{(k)} \in \mathbb{Z}$.
- There are r^n numbers on $\mathbb{T}^n$ represented by such a numbers.
- Therefore there is $q > 0$ such that $\pi\left(A^q x\right) = \pi(x)$.

Structural stability

The non-wandering set of Anosov automorphism

As a consequence of the previous result, $\Omega(f) = \mathbb{T}^n$ (the non-wandering set).

Notice that the periodic points are dense in the torus and they belong to the non-wandering set.

Theorem (Conjugation result)

Every Anosov diffeomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ having $\Omega(f) = \mathbb{T}^n$, is topologically conjugated to some Anosov automorphism.

$\Omega(f)$ contains infinitely many hyperbolic periodic orbits distributed densely on the torus. But, $\mathbb{T}^n$ is compact, so there are finitely many periodic orbits for each period q, so that there are infinitely many periods.

Theorem (The surprising result)

The Anosov diffeomorphisms on $\mathbb{T}^n$ are structurally stable in Diff.

Even maps having a very complicated dynamics can be structurally stable. This is a big difference with two dimensional flows.

OUTLINE

A well quoted example (Arnold cat map)

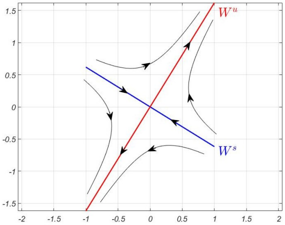
Consider $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

It is clear that it is an Anosov automorphism since

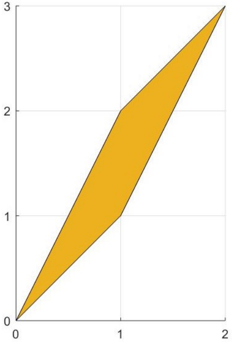
$$A : \mathbb{Z}^2 \to \mathbb{Z}^2, \qquad \det A = \pm 1.$$

The behaviour of A on $\mathbb{R}^2$ is clear

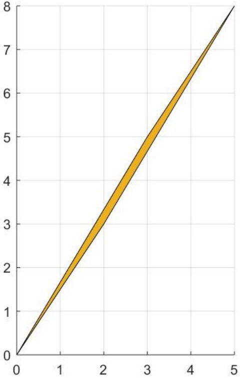


- $x = (0, 0)$ is the unique fixed point.
- The eigenvalues of A are $\lambda_\pm = \dfrac{3 \pm \sqrt{5}}{2}$.
- The eigenvectors are $v_\pm = \left(1, \dfrac{1 \pm \sqrt{5}}{2}\right)$.
- The origin is a saddle point.

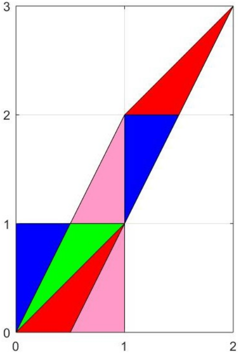
WHAT DOES HAPPEN IN THE TORUS?



The image of the square $A([0,1] \times [0,1])$.



The image of the image $A^2([0,1] \times [0,1])$.



Structure of $A([0,1] \times [0,1])$ in $\mathbb{T}^2$.

The invariant manifolds are dense on the torus

$$\boxed{\overline{W^{u,s}} = \mathbb{T}^2}$$

- Recall that, in $\mathbb{R}^2$, the points of the either stable or unstable manifold are respectively:

$$y = \alpha^- x, \qquad y = \alpha^+ x, \qquad \alpha^\pm = \frac{1 \pm \sqrt{5}}{2}.$$

- The intersection of $y = \alpha^+ x$ with $y = k \in \mathbb{Z}$ are

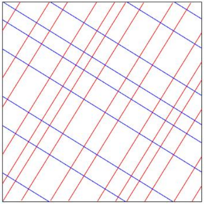
$$\left(\frac{k}{\alpha^+}, k\right) = \left(\frac{k}{\alpha^+}, 0\right), \qquad \pmod 1.$$

- The intersection with $x = m \in \mathbb{Z}$ are

$$(m, \alpha^+ m) = (0, \alpha^+ m), \qquad \pmod 1.$$

- Recall that, for the maps $g_\beta : \mathbb{S}^1 \to \mathbb{S}^1$, $g_\beta(\theta) = \theta + \beta$ with $\beta \in \mathbb{R} \setminus \mathbb{Q}$, orb (0) is dense.
- Then the points $\{k/\alpha^+ \pmod 1\}_{k \in \mathbb{Z}}$ are dense in $[0, 1]$ (apply the previous item for $\beta = 1/\alpha^+$). The same for the points $\{m\alpha^+ \pmod 1\}_{m \in \mathbb{Z}}$.
- We conclude $\overline{W^u} = \mathbb{T}^2$.
- The same for α^-.

THE INVARIANT MANIFOLDS INTERSECT



In blue W^s, in red W^u

- Take one of the branch of W^u inside of the square $[0,1] \times [0,1]$ and let $\varepsilon > 0$ small.

- For every point, x_0^u of this branch, there is a point x^s of W^s such that

$$|x_0^u - x^s| < \varepsilon.$$

- Since, locally, the invariant manifolds are straight lines, we have a transversal intersection between W^u, W^s belonging to

$$B_\varepsilon(x_0^u) = \{x \in [0,1] \times [0,1] \ : \ |x - x_0^u| < \varepsilon\}.$$

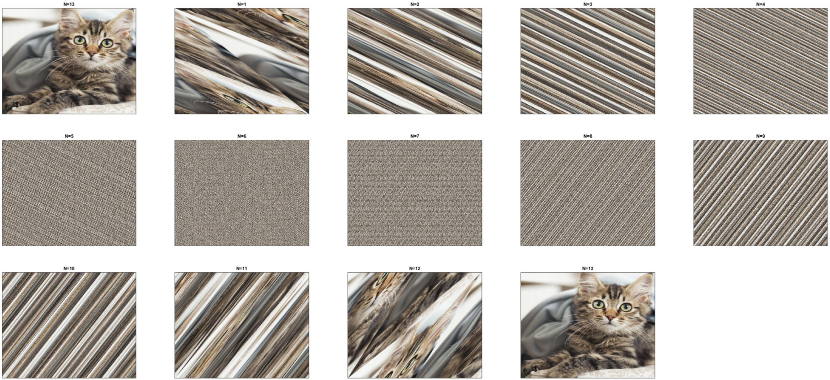
- Since W^u is dense in the torus, the torus is compact and $\varepsilon > 0$ is arbitrary, the transversal intersection points are dense.

HOMOCLINIC POINTS

The homoclinic points are the intersection between the stable and unstable manifolds.
As a consequence the set of homoclinic (transversal) points is dense.

THE ARNOLD'S CAT MAP, HATES CATS

This is what happens when we apply Arnold's cat map to a cat:



PLAY

See the program in Atenea, try to understand what the program does and think about why is this possible? Hint: Periodic points!

Do exercise 128