SMALE'S HORSESHOE

I. Baldomá

QQMDS, 2022

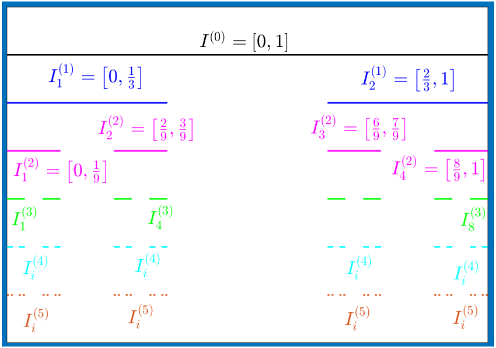
Outline

SOME FACTS WE WILL SEE

- The classical map being structurally stable and having complicated non-wandering set.
- There is a Cantor invariant set.
- We will study what is called symbolic dynamics associated to bisequences.
- The dynamics associated to this Cantor set is extremely complicated. In fact it will be the first example of chaotic dynamics.
- It can be defined in several ways. We are going to study one of them.
- When a map has transversal homoclinic intersection, it exhibits Smale's horseshoe. For that reason the maps with transversal homoclinic intersections are so important. We will see this phenomenon.

CANTOR'S SET

The most popular Cantor set is:



It has a fractal structure and it is not countable. If $C_0 = [0, 1]$ and

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right).$$

Then the Cantor set is $\bigcap_{n \geq 0} C_n$.

DEFINITION

A set is totally disconnected if it has no intervals. A set is perfect if every point of the set is an accumulation point of the set itself.
A set $\Gamma \subset [0, 1]$ is a Cantor's set if it is closed, totally disconnected and perfect of $[0, 1]$.

OUTLINE

BI-INFINITE SEQUENCES

DEFINITION OF BI-INFINITE SEQUENCES

Take the binary symbols

$$\{0, 1\}.$$

A symbol sequence is $\sigma : \mathbb{Z} \to \{0, 1\}$. That is

$$\sigma = \{\sigma_n\}_{n\in\mathbb{Z}} = \{\cdots \sigma_{-2}\sigma_{-1} \cdot \sigma_0\sigma_1\sigma_2 \cdots\}, \qquad \sigma_n \in \{0, 1\}.$$

We call

$$\Sigma = \{\sigma \ : \ \sigma \text{ is a symbol sequence}\} = \{\sigma \ : \ \sigma : \mathbb{Z} \to \{0, 1\}\}.$$

Define the function $d : \Sigma \times \Sigma \to \mathbb{R}$ by:

$$d(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \frac{|\sigma_n - \tau_n|}{2^{|n|}}$$

We can take two other symbols, for instance $\{r, s\}$, $\{*, \circ\}$, etc.

A TOPOLOGY ON Σ

PROPOSITION

The function d defines a distance on Σ, i.e. Σ endowed with the distance d is a metric space. If $d(\sigma,\tau) < 1/2^k$ for some $k \in \mathbb{N}$, then $\sigma_n = \tau_n$ for $|n| \le k$ and if $\sigma_n = \tau_n$ for $|n| \le k$, then $d(\sigma,\tau) \le 1/2^{k-1}$.

Proof

- The distance is well defined since the series is absolutely convergent. In fact $d(\sigma,\tau) \le 3$.
- Properties: $d(\sigma,\tau) \ge 0$, if $d(\sigma,\tau) = 0$, then for any $n \in \mathbb{Z}$, $|\sigma_n - \tau_n| = 0$ which implies that $\sigma = \tau$. Symmetry $d(\sigma,\tau) = d(\tau,\sigma)$ is obvious. Triangular inequality: take $\nu \in \Sigma$:

$$d(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \frac{|\sigma_n - \tau_n|}{2^{|n|}} \le \sum_{n=-\infty}^{\infty} \frac{|\sigma_n - \nu_n| + |\nu_n - \tau_n|}{2^{|n|}} = d(\sigma,\nu) + d(\nu,\tau).$$

- Assume that $d(\sigma,\tau) < 1/2^k$ and that $\sigma_n \neq \tau_n$ for some $|n| \le k$. Then, since $|s_n - t_n| = 1$,

$$\frac{1}{2^k} \le \frac{1}{2^{|n|}} \le \sum_{m=-\infty}^{\infty} \frac{|\sigma_m - \tau_m|}{2^{|m|}} < \frac{1}{2^k}$$

which is a contradiction. Conversely, if $\sigma_n = \tau_n$ for $|n| \le k$,

$$d(\sigma,\tau) = \sum_{|n| \ge k+1} \frac{|s_n - t_n|}{2^{|n|}} \le 2\frac{1}{2^{k+1}}\frac{1}{1-\frac{1}{2}} = \frac{1}{2^{k-1}}$$

OUTLINE

BERNOULLI'S SHIFT

The map $\alpha : \Sigma \to \Sigma$ defined by

$$\tau = \alpha(\sigma) \qquad \text{with} \qquad \tau_n = \sigma_{n-1}, \; n \in \mathbb{Z}$$

is known as the *left* Bernoulli's shift:

$$\alpha(\{\cdots \sigma_{-1} \cdot \sigma_0 \sigma_1 \sigma_2 \cdots\}) = \{\cdots \sigma_{-2} \cdot \sigma_{-1} \sigma_0 \sigma_1 \cdots\}$$

We can also define the *right* Bernoulli's shift by $\beta(\sigma)_n = \sigma_{n+1}$.

PROPOSITION

The map β is an homeomorphism. In addition

- *It has periodic orbits of all periods.*
- *The periodic orbits are dense in Σ.*
- *It has a dense orbit (which is not periodic).*

We also have that $\alpha = \beta^{-1}$ and consequently α has the same properties as β.

The map β is chaotic: that is i) the map is sensitive to initial conditions, ii) the map is topologically transitivity (consequence to have a dense orbit) iii) the periodic points are dense.

PROOF OF CHAOTIC CHARACTER OF β (I)

PERIODIC ORBITS

Notice that σ is a periodic orbit of β (and α) if and only if

$$\beta^q(\sigma) = \sigma \iff \sigma_{n+q} = \sigma_n, \qquad \forall n \in \mathbb{Z}.$$

- First we see that β is an homeomorphism. Notice that it is obvious that $\alpha = \beta^{-1}$ is a bijection. Then we only need to prove that it is continuous. Take $\varepsilon > 0$ and $\sigma^* \in \Sigma$. Let $k \in \mathbb{N}$ be such that $1/2^{k-1} \leq \varepsilon$ and $\delta = 1/2^k$. Then

$$d(\sigma^*, \sigma) < \delta \Rightarrow \sigma^*_n = \sigma_n, \ |n| \leq k \Rightarrow \sigma^*_{n+1} = \sigma_{n+1}, \ |n| \leq k - 1.$$

  Therefore $d(\beta(\sigma^*), \beta(\sigma)) \leq 1/2^{k-1} \leq \varepsilon$ and we are done.
- To get σ q-periodic, we need $\sigma_i = \sigma_{i+q}$; we can repeat the sequence of length q

$$000 \cdots 0001.$$

PROOF OF CHAOTIC CHARACTER OF α (II)

- Take $\sigma \in \Sigma$ and $\varepsilon > 0$. Let q be such that $1/2^{q-1} \leq \varepsilon$ and consider $\sigma^q \in \Sigma$ such that

$$\sigma_n = \sigma_n^q \quad \text{if } |n| \leq q, \qquad \sigma^q \ \text{ having period } 2q+1.$$

  That is

$$\sigma^q = \{\cdots \overline{\sigma_{-q}\sigma_{-q+1}\cdots\sigma_{-1}\cdot\sigma_0\sigma_1\cdots\sigma_q}\cdots\}.$$

  Then $d(\sigma^q, \sigma) \leq 1/2^{q-1} \leq \varepsilon$.

- One dense orbit is the orbit of the sequence $\sigma^* \in \Sigma$:

$$\sigma^* = \{\cdots\sigma_{-2}\sigma_{-1}\cdot 0\,1\,|\,00\,01\,10\,11\,|\,000\,100\cdots\}$$

  composed by all the sequence of finite length. The value of σ_n for $n \leq -1$ **does not matter**. Indeed, given $\varepsilon > 0$ and $\sigma \in \Sigma$ there is m such that

$$\beta^m(\sigma^*)_n = \sigma^*_{n+m} = \sigma_n$$

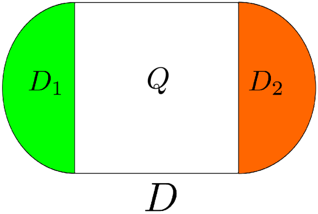
  for $|n| \leq q$ with $1/2^q < \varepsilon$.

Do exercise 135

OUTLINE

DESCRIPTION OF THE SMALE'S HORSESHOE

Consider the domain $D = D_1 \cup Q \cup D_2$ defined in the figure below:



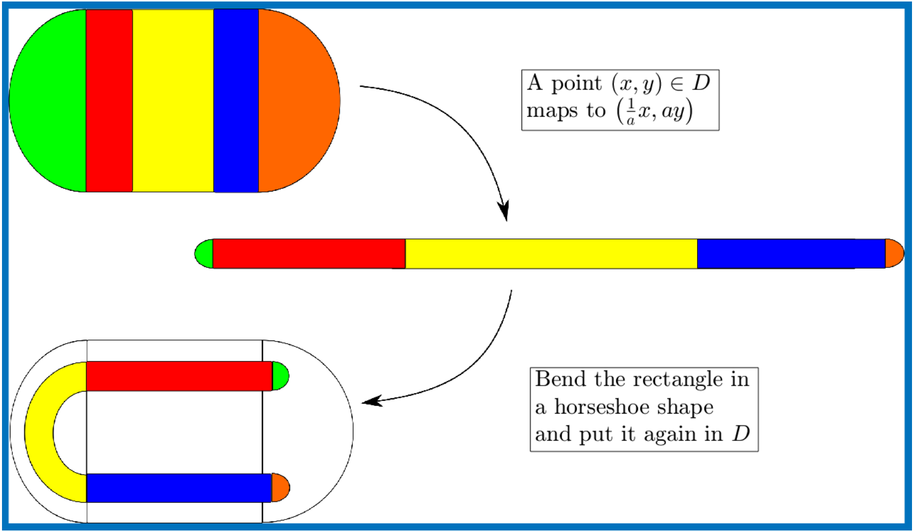
Take $a < 1/2$ and $h \geq 0$. We denote the Smale's horseshoe

$$f : D \to D$$

as the map satisfying:

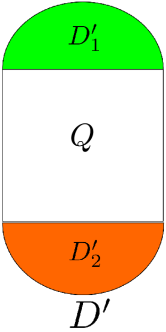
- First, it shrinks in the vertical direction by a and expands in the horizontal direction by $1/a$. A *rectangle* R is produced.
- Second, it bends the central part of R performing the horseshoe shape, that is simetrically with respect $y = 1/2$.
- Third, it places the horseshoe into the domain D leaving a distance h with the top and bottom boundary.

THE SMALE'S HORSESHOE GRAPHICALLY



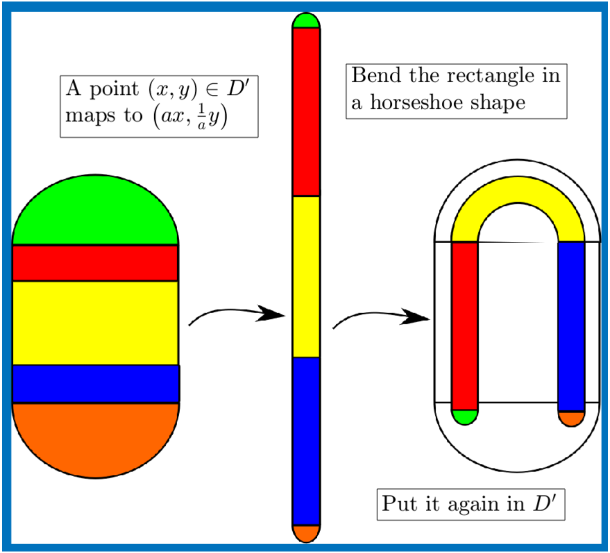
ANOTHER SMALE'S HORSESHOE MAP

$D' = D'_1 \cup Q \cup D'_2$ as below:



We can define a new horseshoe map

$\tilde{f} : D' \to D'$



OUTLINE

REMARKS ABOUT SMALE'S HORSESHOE

- $f, \tilde{f}$ are not diffeomorphisms.
- However $f, \tilde{f}$ are injective maps.
- f and $\tilde{f}$ are well defined if a condition on the contraction factor a and h is satisfied: $2h + 2a < 1$. For instance $a = h = 1/5$.
- In some sense, $\tilde{f}$ is the inverse of f on $f(Q) \cap Q$ (see lemma below)

VERTICAL AND HORIZONTAL RECTANGLES

We say that V is a vertical rectangle if it has width $\alpha \in (0, 1)$ and height 1.
Conversely, we say that H is a horizontal rectangle if it has width 1 and height $\alpha \in (0, 1)$.

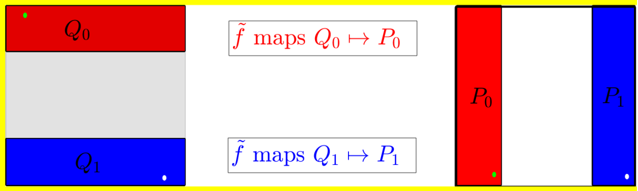
LEMMA

Let f and $\tilde{f}$ be defined as before with $h = 0$. Then

- *$f(Q) \cap Q = Q_0 \cup Q_1$ with Q_0, Q_1 two horizontal rectangles $Q_0 \cap Q_1 = \emptyset$ and Q_0, Q_1 having height a.*
- *$\tilde{f}(Q) \cap Q = P_0 \cup P_1$ with P_0, P_1 two vertical rectangles $P_0 \cap P_1 = \emptyset$, P_0, P_1 having width a.*
- *We can construct $f, \tilde{f}$ such that $f(P_0) = Q_0$, $f(P_1) = Q_1$, $\tilde{f}(Q_0) = P_0$ and $\tilde{f}(Q_1) = P_1$.*
- *If $x \in f(Q) \cap Q$, then $f \circ \tilde{f}(x) = x$.*

The relation between f and $\tilde{f}$

Idea of the proof. The first three items are obvious.



Let $(x_1, x_2) \in Q_1 = [0,1] \times [0,a]$. We have that $\tilde{f}$ acts as:

$$(x_1, x_2) \mapsto \left(ax_1, \frac{1}{a}x_2\right) \mapsto \left(1 - a + ax_1, \frac{1}{a}x_2\right).$$

In addition, f acts on $P_1 = [1-a, 1] \times [0,1]$ as:

$$(x_1', x_2') \mapsto \left(\frac{1}{a}x_1', ax_2'\right) \mapsto \left(\frac{1}{a}x_1' + 1 - \frac{1}{a}, ax_2'\right).$$

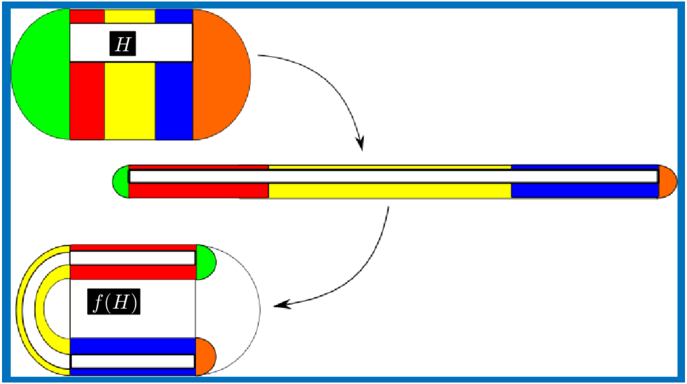
Henceforth, $f \circ \tilde{f}(x) = x$.

Exercise: Do the same for Q_0 .

THE FRACTAL STRUCTURE

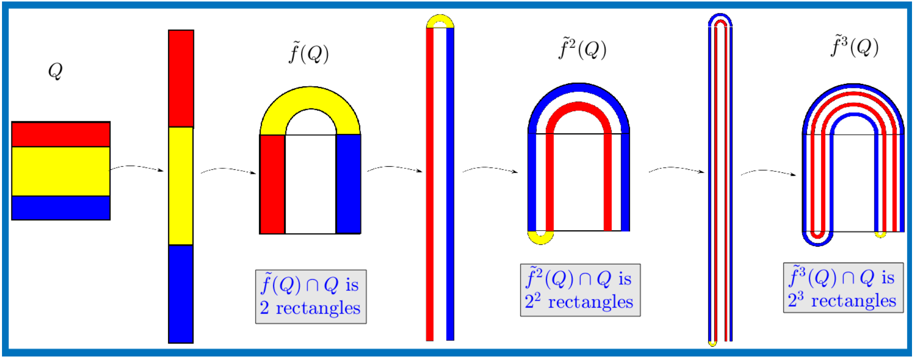
LEMMA

If H is an horizontal rectangle of height α, then $f(H)$ is another horseshoe. In addition $f(H) \cap Q$ consists on two horizontal rectangles, H^0, H^1 of height αa with $H^0 \in Q_0$ and $H^1 \in Q_1$. Analogously for V, a vertical rectangle of width β, $\tilde{f}(V)$ is another horseshoe with $\tilde{f}(V) \cap Q = V^0 \cup V^1$ two vertical rectangles of width βa such that $V^0 \in P_0$ and $V^1 \in P_1$.



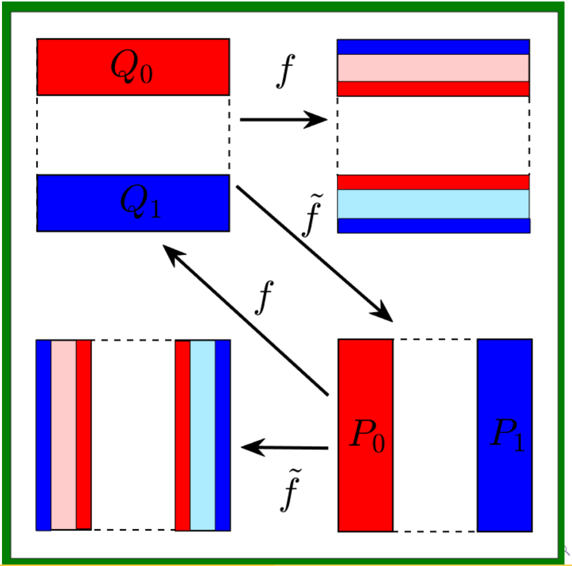
MORE ON THE FRACTAL STRUCTURE

In fact, for instance for $\tilde{f}$, we have:



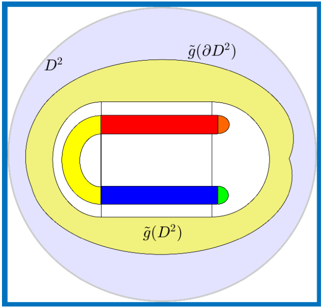
THE BEHAVIOR OF $f, \tilde{f}$ ON RECTANGLES

- In the following diagram we sketch how the maps $f, \tilde{f}$ act on vertical and horizontal rectangles on the rectangle Q.
- We have drawn $f(Q) \cap Q$, $f^2(Q) \cap Q$, $\tilde{f}(Q) \cap Q$ and $\tilde{f}^2(Q) \cap Q$.
- Recall that $\tilde{f} = f^{-1}$ on $Q_0 \cup Q_1$.

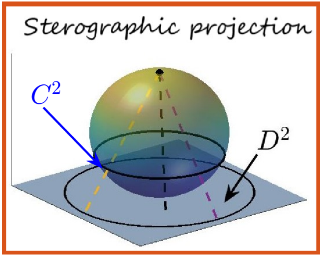


AN INTERESTING COMMENT

- Consider $f : D \to D$ defined as before.
- Take a disc $\mathbb{D}^2$ containing D and define $\tilde{g} : \mathbb{D}^2 \to \mathbb{D}^2$ to be of the form in the figure



- Identify $\mathbb{D}^2$ with a closed cap C^2 in the sphere $\mathbb{S}^2$. Recall that we can do this by using the sterographic projection $h : C^2 \to \mathbb{D}^2$.



- Call $\bar{g} : C^2 \to C^2$, $\bar{g} = h^{-1} \circ \tilde{g} \circ h$
- Extend $\bar{g}$ to a function $g : \mathbb{S}^2 \to \mathbb{S}^2$ by adding a unique, repelling, hyperbolic fixed point in $\mathbb{S}^2 \backslash C^2$.
- g is a diffeomorphism on $\mathbb{S}^2$.
- Then, when restricting to D (respectively D'), $f = h \circ g \circ h^{-1}_{|D}$ and $\tilde{f} = h \circ g^{-1} \circ h_{|D'}$.

OUTLINE

Preliminaries

The dynamics of f on D_1, D_2 as well as the dynamics of $\tilde{f}$ on D_1', D_2' are very simple:

- $f_{|D_2}, \tilde{f}_{|D_2'}$ are contractions. Therefore they have a unique fixed point $q, \tilde{q}$ which are global attractors for $f, \tilde{f}$ respectively.
- Notice also that $f(D_1) \subset D_2$ and $\tilde{f}(D_1') \subset D_2'$.
- As a consequence,

$$p \in D_1 \cup D_2, \quad \lim_{n\to\infty} f^n(p) \to q, \qquad \tilde{p} \in D_1' \cup D_2', \quad \lim_{n\to\infty} \tilde{f}^n(\tilde{p}) \to \tilde{q},$$

- Recall that $\tilde{f} = f^{-1}$ on $Q \cap f(Q)$.

As a consequence, for understanding what happens with the orbits of f we only need to understand the orbits remaining at Q for any $n \in \mathbb{Z}$. That is the set:

$$\Lambda = \{q \in Q : f^n(q) \in Q, \ n \in \mathbb{Z}\}.$$

THE SET Λ

- Consider two horseshoe map defined by $f(Q_0) = P_0$, $f(Q_1) = P_1$ with Q_0, Q_1 two horizontal rectangles and its *inverse* $\tilde{f}$.
- Define the sets $Q^{(1)} = Q_0 \cup Q_1$,
$$Q^{(n+1)} = f(Q^{(n)}) \cap Q, \qquad n \in \mathbb{N}.$$
- Define the sets $Q^{(0)} = P_0 \cup P_1 = \tilde{f}(Q^{(1)}) = \tilde{f}(Q) \cap Q$,
$$Q^{(-n)} = \tilde{f}(Q^{(-n+1)}) \cap Q, \qquad n \in \mathbb{N}.$$
- Then it is clear that
$$\Lambda = \{q \in Q : f^n(q) \in Q,\ n \in \mathbb{Z}\} = \bigcap_{n \in \mathbb{Z}} Q^{(n)}.$$
Indeed, if $q \in \Lambda$, since $f^{-1}(q)$ has to be in Q, $q \in Q_0 \cup Q_1$. In this case $f^{-1}(q) = \tilde{f}(q)$.
- As a consequence Λ is invariant by f and $\tilde{f}$. Notice that, when considering the extension $g : \mathbb{S}^2 \to \mathbb{S}^2$, the corresponding set will satisfy the same topological properties as the ones for Λ.

OUTLINE

Cantor sets in Λ

Proposition

The sets defined by

$$\Lambda_+ = \{q \in Q : f^n(q) \in Q,\ n \in \mathbb{Z}^+\} = \bigcap_{n \in \mathbb{Z}^+} Q^{(-n)}$$

$$\Lambda_- = \{q \in Q : f^{-n}(q) \in Q,\ n \in \mathbb{N}\} = \bigcap_{n \in \mathbb{N}} Q^{(n)}$$

can be written as

$$\Lambda_+ = \Gamma \times [0,1], \qquad \Lambda_- = [0,1] \times \Gamma$$

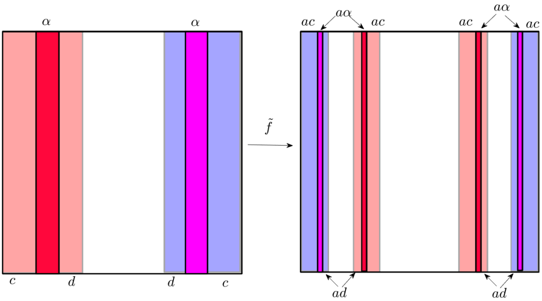
with Γ a Cantor set. In addition Λ_+ is invariant by f and Λ_- is invariant by $\tilde{f} = f^{-1}$.
As a consequence $\Lambda = \Lambda_+ \cap \Lambda_- = \bigcap_{n \in \mathbb{Z}} Q^{(n)}$ is invariant by f and $\tilde{f} = f^{-1}$ and hence $f(\Lambda) = \Lambda$.

PROOF OF PREVIOUS RESULT (I)

- Write
$$\Lambda_+ = \bigcap_{k\in\mathbb{Z}^+} \Lambda_+^k, \qquad \Lambda_+^k = \bigcap_{n=0}^{k} Q^{(-n)}.$$
- The induction hypothesis is that Λ_+^k is the union of 2^{k+1} disjoint vertical rectangles of width a^{k+1}.
- Take $k = 0$, $\Lambda_+^0 = Q^{(0)} = P_0 \cup P_1$. For $k = 1, 2$ recall how the map $\tilde{f}$ acts on vertical rectangles.
- The induction step assume the equality for $k-1$. Then, using that $Q^{(0)} = P_0 \cup P_1$ and the injectivity of $\tilde{f}$ as well:
$$\Lambda_+^k = \bigcap_{n=1}^{k} \tilde{f}(Q^{(-n+1)}) \cap (P_0 \cup P_1) = \tilde{f}\left(\bigcap_{n=1}^{k} Q^{(-n+1)}\right) \cap (P_0 \cup P_1)$$
$$= \tilde{f}(\Lambda_+^{k-1}) \cap (P_0 \cup P_1).$$

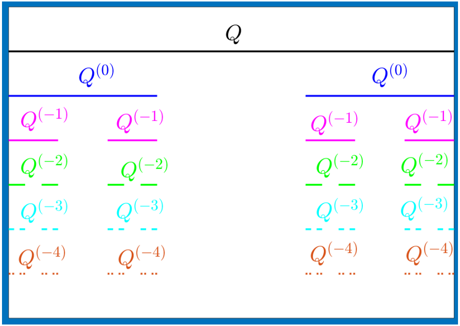
PROOF OF PREVIOUS RESULT (II)

- By induction hypothesis, Λ_+^{k-1} is the union of 2^k disjoint vertical rectangles. Then its image by $\tilde{f}$ is the union of 2^{k+1} vertical rectangles of width aa^k that have to be disjoint because of the injectivity of $\tilde{f}$.
- Take a look again how $\tilde{f}$ acts on vertical rectangles



PROOF OF PREVIOUS RESULT (III)

- Then, by construction, any vertical rectangle in Λ_+^{k-1} contains two and only two rectangles of Λ_+^k. In fact, taking $a = 1/3$, $\Lambda_+ = \Gamma \times [0,1]$ with Γ the Cantor's set introduced before:



- Analogous arguments hold true for Λ_-.
- The remaining properties are obvious.

Some remarks from the proof

- Notice that in the same way we have proven that Λ_+^k are 2^{k+1} vertical rectangles, we can prove that

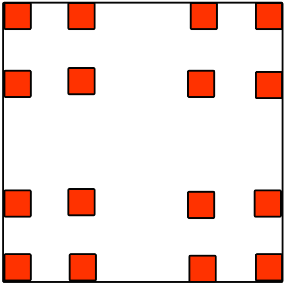
$$\Lambda_- = \bigcap_{k\in\mathbb{Z}^+} \Lambda_-^k, \qquad \Lambda_-^k = \bigcap_{n=1}^{k} Q^{(n)}$$

is the union of 2^k horizontal rectangles of height a^k.

- Then the set $\tilde{\Lambda}^N = \bigcap_{n=-N+1}^{N} Q^{(n)}$ is 2^{2N} disjoint squares of side a^N. In the figure the set for $N = 2$.
- Since $f(Q^{(1)}) \cap Q \subset Q^{(1)}$ and $\tilde{f}(Q^{(0)}) \cap Q \subset Q^{(0)}$, we have that

$$Q^{(n)} \subset Q^{(n-1)}, \qquad Q^{(-n)} \subset Q^{(-n+1)}$$

Therefore, in fact $\Lambda_-^k = Q^{(k)}$ and $\Lambda_+^k = Q^{(-k)}$.



HORSESHOES AND SEGMENTS

Since $\Lambda = \Lambda_+ \cap \Lambda_-$, with

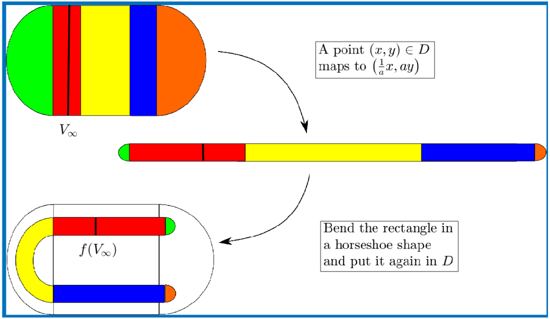
$$\Lambda_+ = \Gamma \times [0,1],\ \Lambda_- = [0,1] \times \Gamma$$

being Γ a Cantor set, we can also write

$$\Lambda_+ = \bigcup_{p\in\Gamma} V_\infty(p),$$

$$\Lambda_- = \bigcup_{p\in\Gamma} H_\infty(p),$$

with $(p,0) \in V_\infty(p)$ a vertical segment and $(0,p) \in H_\infty(p)$ an horizontal segment.



HORSESHOE ON SEGMENTS BELONGING TO $\Lambda_+ \cup \Lambda_-$

Vertical segments

- Recall that $f(\Lambda_+) \subset \Lambda_+ \subset P_0 \cup P_1$.
- Let $V_\infty \subset \Lambda_+$. $f^n(V_\infty)$ is a vertical segment of length a^n belonging to one of the vertical rectangles, P_0 or P_1.

Horizontal segments

- Recall that $f^{-1}(\Lambda_-) \subset \Lambda_- \subset Q_0 \cup Q_1$.
- Let $H_\infty \subset \Lambda_-$. $f^{-n}(H_\infty)$ is an horizontal segment of width a^n belonging to one of the horizontal rectangles, Q_0 or Q_1.

A FINAL COMMENT

Consider now the extension $g : \mathbb{S}^2 \to \mathbb{S}^2$ of f. Define the sets:

$$\mathrm{in}\,(\Lambda) = \{x \in \mathbb{S}^2 \,:\, \mathrm{dist}(g^n(x), \Lambda) \to 0, \text{ as } n \to +\infty\},$$
$$\mathrm{out}\,(\Lambda) = \{x \in \mathbb{S}^2 \,:\, \mathrm{dist}(g^{-n}(x), \Lambda) \to 0, \text{ as } n \to +\infty\}.$$

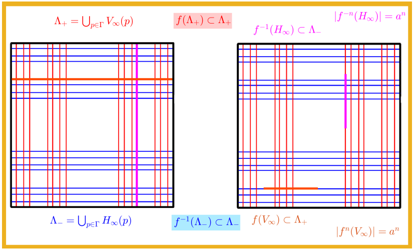
Called the inset and outset of Λ.
Do as a exercise (see exercise 131):

- The set Λ_+ is invariant by g,
- The set Λ_- is invariant by g^{-1},
- The inset of Λ on Q is $\mathrm{in}\,(\Lambda) = \Lambda_+$,
- The outset of Λ on Q is $\mathrm{out}\,(\Lambda) = \Lambda_-$.

Hint: Use the behaviour of f and $\tilde{f}$ on vertical and horizontal segments respectively.

OUTLINE

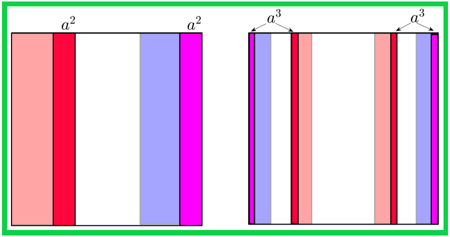
PRELIMINARIES



Recall $\Lambda = \Lambda_+ \cap \Lambda_-$, with

$\Lambda_+ = \Gamma \times [0, 1], \ \Lambda_- = [0, 1] \times \Gamma$

being Γ a Cantor set.



$\tilde{f}$ acting on vertical rectangles. The same for f but with horizontal rectangles.

HORSESHOE ON SEGMENTS BELONGING TO $\Lambda_+ \cup \Lambda_-$

- Recall that $f(\Lambda_+) \subset \Lambda_+ \subset P_0 \cup P_1$ and $f^{-1}(\Lambda_-) \subset \Lambda_- \subset Q_0 \cup Q_1$.
- $f^n(V_\infty) \in P_0 \cup P_1$ and $f^{-n}(H_\infty) \in Q_0 \cup Q_1$.

Outline

BISEQUENCE AND SEGMENTS

Sequence $\sigma^+ = \varphi(V_\infty)$ for any V_∞, vertical segment

$$\text{for } n \geq 0 \qquad \sigma_n^+ = \begin{cases} 1 & f^n(V_\infty) \in P_1 \\ 0 & f^n(V_\infty) \in P_0 \end{cases}$$

Sequence $\sigma^- = \psi(H_\infty)$ for any H_∞, horizontal segment

$$\text{for } n \geq 1 \qquad \sigma^-_{-n} = \begin{cases} 1 & \tilde{f}^{n-1}(H_\infty) = f^{-n+1}(H_\infty) \in Q_1 \\ 0 & \tilde{f}^{n-1}(H_\infty) = f^{-n+1}(H_\infty) \in Q_0 \end{cases}$$

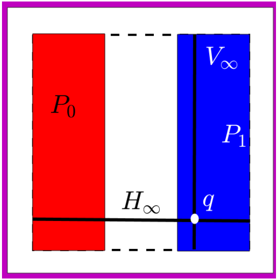
However, because $f^{-1}(Q_0) = P_0$ and $f^{-1}(Q_1) = P_1$ we can write the last assignment by

$$\text{for } n \geq 1 \qquad \sigma^-_{-n} = \begin{cases} 1 & \tilde{f}^{n}(H_\infty) = f^{-n}(H_\infty) \in P_1 \\ 0 & \tilde{f}^{n}(H_\infty) = f^{-n}(H_\infty) \in P_0 \end{cases}$$

BISEQUENCE AND THE CANTOR SET

Let q be a point in Λ. It is clear that there exist two (and only two) H_∞, V_∞ such that

$$q = H_\infty \cap V_\infty$$



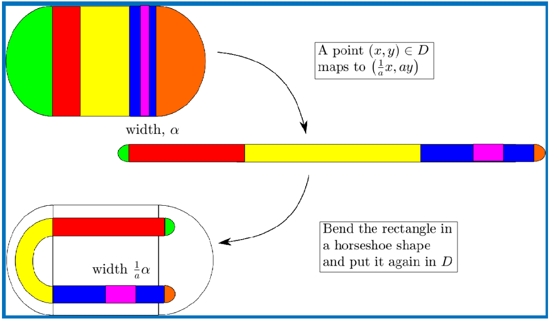
DEFINITION

For any $q \in H_\infty \cap V_\infty \subset \Lambda$, we define $\sigma = \phi(q) \in \Sigma$ as

$$\sigma_n = \begin{cases} 1 & f^n(q) \in P_1 \\ 0 & f^n(q) \in P_0 \end{cases} \quad \Longleftrightarrow \quad f^n(q) \in P_{\sigma_n}.$$

The function ϕ is well defined and $\phi(q) = \{\psi(H_\infty) \cdot \varphi(V_\infty)\}$.

f AND $\tilde{f}$ ACTING ON RECTANGLES



If $V \in P_s$, $s = 0, 1$ is a vertical rectangle of width α then $f(V)$ is rectangle of width $\alpha\frac{1}{a}$ and height a belonging to Q_s
The same for $f^{-1}(H)$ being H an horizontal rectangle.

AN IMPORTANT PROPERTY

Let $q = V_\infty \cap H_\infty \in \Lambda$ and $\sigma = \phi(q)$. Take $N \in \mathbb{N}$ and the square S of $\tilde{\Lambda}^N$ such that $q \in S$. Then $p \in S$ if and only if $f^n(p) \in P_{\sigma_n}$, for $-N + 1 \leq -n \leq N$ (or $-N \leq n \leq N - 1$).

Prove the above result: a piece of exercise 139.

OUTLINE

The conjugacy

Theorem

The map $\phi : \Lambda \to \Sigma$ defined by

$$\sigma = \phi(q) \qquad \Longleftrightarrow \qquad f^n(q) \in P_{\sigma_n}$$

is an homeomorphism satisfying

$$\beta \circ \phi = \phi \circ f_{|\Lambda}.$$

Then $f_{|\Lambda}$ and β are topologically conjugated.

Proof:

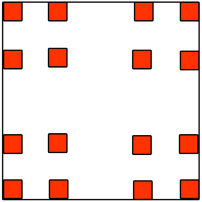
- We first see that $\beta \circ \phi = \phi \circ f_{|\Lambda}$. Indeed, take $q \in \Lambda$ and $\sigma = \phi(q) \in \Sigma$. We have that $f^k(q) \in P_{\sigma_k}$ for all $k \in \mathbb{Z}$. Moreover:

$$\tau = \phi(f(q)) \Longleftrightarrow f^n(f(q)) \in P_{\tau_n}$$

  Therefore, since $f^{n+1}(q) \in P_{\sigma_{n+1}}$ and ϕ is well defined:

$$\sigma_{n+1} = \tau_n \Longleftrightarrow \tau = \beta(\sigma) = \beta \circ \phi(q).$$

CONTINUATION OF THE PROOF



- The set $\tilde{\Lambda}^N = \bigcap_{n=-N+1}^{N} Q^{(n)} = \bigcup_{j=1}^{2^{2N}} S_j^N$ with S_j^N squares of side a^N.
  In the figure for $N = 2$.
- Define $\phi^N : \tilde{\Lambda}^N \to \Sigma^N$ with

$$\Sigma^N = \{(\cdots 0\,\sigma_{-N}\cdots\sigma_{-1}\cdot\sigma_0\cdots\sigma_{N-1}\,0\cdots),\ \ \sigma_n \in \{0,1\}\}$$

  as $\sigma^N = \phi^N(S)$ defined by $p \in S$ if and only if $f^n(p) \in P_{\sigma_n^N}$.
- ϕ^N is injective and cardΣ^N= 2^{2N} . Then ϕ^N is bijective.
- $\Sigma = \bigcup_{N=0}^{\infty} \Sigma^N$ and $\Lambda = \bigcap_{N=0}^{\infty} \tilde{\Lambda}^N$ implies ϕ bijective. Indeed, if $q \in \Lambda$, then $\forall N$, $q \in S^N \subset \tilde{\Lambda}^N$ and $q = \lim_{N\to\infty} S^N$. Let $\sigma^N = \phi^N(S^N)$. Then $\phi(q) = \lim_N \sigma^N$. Now the proof is straightforward.
- If $d(\phi(p), \phi(q)) < 2^{-N}$, then $\phi(p)_n = \phi(q)_n$ for $-N \le n \le N$. This means that $p, q \in S$ the same square in $\tilde{\Lambda}^N$ and then

$$\|p - q\|_\infty \le a^N$$

  and the continuity is proven because $a < 1$.
- The continuity of ϕ^{-1} follows from the fact that Σ is a compact set and ϕ is a bijection.

CHAOS ON THE SMALE'S HORSESHOE

COROLLARY

$f_{|\Lambda}$ is chaotic. In fact $f_{|\Lambda}$ has

- *Countable infinitely many hyperbolic periodic orbits of arbitrarily large period.*
- *Uncountable infinitely many non periodic orbits.*
- *A dense orbit*

- Since $f_{|\Lambda}$ is conjugated to the shift $\beta : \Sigma \to \Sigma$, the result holds *almost* true. It only remains to check that the periodic orbits are hyperbolic.
- Let $f^m(p) = p$, $p \in \Lambda$ a periodic point. It is clear that $p \in P_0 \cup P_1$. Assume for instance that $p \in P_1$. Then, for any $(x, y) \in P_1$ in a neighbourhood of p,

$$f(x, y) = \left(\frac{x}{a}, ay \right).$$

Therefore $Df^m(p) = \operatorname{diag}(1/a^m, a^m)$ which is a hyperbolic matrix.