Poincaré-Melnikov method

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QQMDS, 2022

Outline

SET UP

- To decide if two invariant manifolds intersect is in general a difficult question.
- Even if we are in the easiest case: planar systems.
- However there are some cases where we can perform explicit computations.
- The framework is planar vector fields periodically perturbed:

$$\dot{z} = F(z) + \varepsilon G(z, t, \varepsilon) \tag{1}$$

  where $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $G : U \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$ and

$$G(z, t + T, \varepsilon) = G(z, t, \varepsilon).$$

- When $\varepsilon = 0$, we call (1) unperturbed system.
- We denote the flow by:

$$\varphi(t; t_0, z, \varepsilon).$$

OUTLINE

HYPOTHESES

- The unperturbed system ($\varepsilon = 0$) has a saddle fixed point p_0.
- Assume that

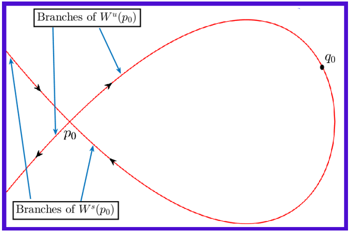
$$W^s(p_0) \cap W^u(p_0) \neq \emptyset.$$

- That means that a branch of the stable manifold coincide with a branch of the unstable one. Indeed, if

$$q_0 \in W^s(p_0) \cap W^u(p_0)$$

then, since $W^s(p_0)$, $W^u(p_0)$ are invariant:

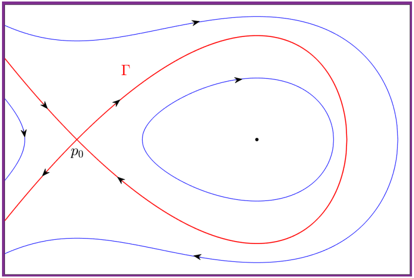
$$\varphi(t; 0, q_0, 0) \subset W^s(p_0) \cap W^u(p_0).$$



Because of $\dim W^{u,s}(p_0) = 1$, the uniqueness of the solutions of the Cauchy problem implies that $W^u(p_0)$ and $W^s(p_0)$ have to have coincident branches. We call one of them Γ.

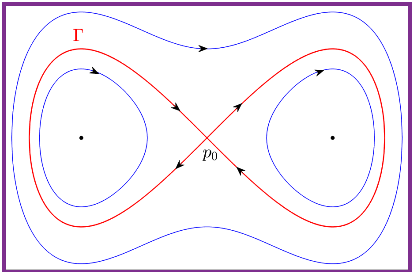
OUTLINE

CLASSICAL EXAMPLES



The fish: $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$.

It has two fixed points $p_0 = (0,0)$ (saddle) and $p_1 = (1,0)$ (center).



Duffing's equation: $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$.

It has three fixed points $p_0 = (0,0)$ (saddle) and $p_\pm = (\pm 1, 0)$ (center).

The stable and unstable manifolds of p_0 are included in the energy level $H(x,y) = 0$:

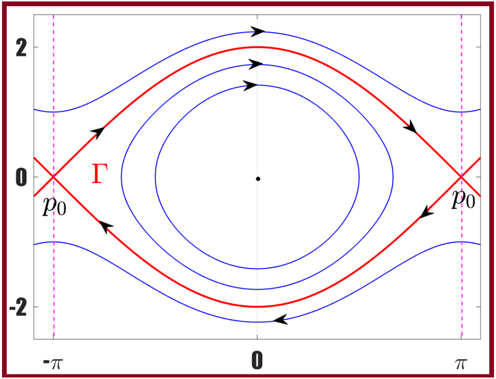
$$y = \pm x\sqrt{1 - \frac{2x}{3}}.$$

The coincident branches are for $x > 0$.

$$y = \pm x\sqrt{1 - \frac{x^2}{2}}.$$

Here we have two coincident branches, one for $x > 0$ and the other for $x < 0$.

MORE EXAMPLES



And finally the pendulum:

$$H(x,y) = \frac{y^2}{2} + 1 - \cos x, \qquad (\text{mod } 1).$$

Defined on $\mathbb{S}^1 \times \mathbb{R}$, it has two fixed points, $p_0 = \pi$ (saddle) and $p_1 = (0,0)$ (center).

Both sides $x = \pi$ are $x = -\pi$ are identify. Recall that the phase space is the cylinder.

The stable and unstable manifolds of p_0 are on the energy level $H(x,y) = 2$. So they are

$$y = \pm\sqrt{2(1+\cos x)}, \qquad x \in (-\pi, \pi).$$

Notice that we have one branch when $+$ sign is considered and the other one with $-$ sign.

HAMILTONIAN SYSTEMS WITH ONE DEGREES OF FREEDOM

- Consider a mechanical Hamiltonian dynamical system:

$$H(x,y) = \frac{y^2}{2} + V(x), \qquad \Longleftrightarrow \dot{x} = y, \;\; \dot{y} = -V'(x).$$

  We call X the associated vector field.
- Assume that it has saddle fixed point $p_0 = (x_0, 0)$, namely $V'(x_0) = 0$ and $V''(x_0) < 0$. Indeed, notice that:

$$DX(p_0) = \begin{pmatrix} 0 & 1 \\ -V''(x_0) & 0 \end{pmatrix}, \qquad \text{has real eigenvalues } \lambda = \pm\sqrt{-V''(x_0)}.$$

- Assume that there exists a non equilibrium point $x_1 \neq x_0$ such that

$$V(x_1) = V(x_0), \qquad V(x) < V(x_0), \text{ for } x \in \overline{x_0, x_1}.$$

- Then the stable and unstable manifolds have at least one coincident branch Γ, belonging to the energy level $H(x,y) = H(p_0)$:

$$\Gamma \subset \{y = \pm\sqrt{2(V(x_0) - V(x))}, \qquad x \in \overline{x_1, x_0}\}.$$

PARAMETERIZATION OF SEPARATRIX

SEPATRIX

We call separatrix to any coincident branch Γ of the stable and unstable invariant manifold.
We emphasize that, in the planar case, the sepatrix is always a solution, for instance $\varphi(t; 0, q_0, 0)$ being $q_0 \in \Gamma$, namely

$$\Gamma = \{\varphi(t; 0, q_0, 0),\ t \in \mathbb{R}\}.$$

We call $\gamma_0(t) := \varphi(t; 0, q_0, 0)$ a parameterization of the separatrix.

- In the general (non hamiltonian) case, we can not provide an explicit formula for $\gamma_0(t)$.
- In the hamiltonian case, we have more information. Indeed, let $q_0 = (x_*, y_*) \in \Gamma$ with $y_* \geq 0$. Then since $y = \dot{x}$, we have that

$$\dot{x} = \sqrt{2(V(x_0) - V(x))} \Longrightarrow \int_0^t ds = \int_{x_*}^{x} \frac{du}{\sqrt{2(V(x_0) - V(u))}}$$

and from this equation *maybe* we can find x as a function of t.

Examples of parameterization

- The parameterization of the pendulum was already computed
$$\gamma(t) = (x_0(t), \dot{x}_0(t)), \qquad x_0(t) = 4\arctan(e^t) - \pi.$$
- The fish. We have to solve
$$\pm t + C = \int \frac{dx}{x\sqrt{1-\frac{2}{3}x}} = \log\left|\frac{\sqrt{1-\frac{2}{3}x}-1}{\sqrt{1-\frac{2}{3}x}+1}\right|$$
- Since the point $(3/2, 0)$ belongs to the separatrix we impose that the equality above is satisfied for $t = 0$ and $x = 3/2$ (why can we do that?) . That implies that $C = 0$.
- Easy computations
$$\left|1 - \sqrt{1-\frac{2}{3}x}\right| = \left|1 + \sqrt{1-\frac{2}{3}x}\right| e^{\pm t}.$$
Since $x > 0$, we can skip the absolute values.
- Again easy computations
$$\sqrt{1-\frac{2}{3}x} = \mp \tanh\left(\frac{t}{2}\right)$$
- Finally
$$x_0(t) = \frac{3}{2}\left[1 - \tanh^2\left(\frac{t}{2}\right)\right] = \frac{3}{2}\frac{1}{\cosh^2\left(\frac{t}{2}\right)}.$$

SUSPENSIONS

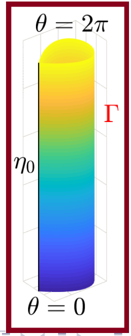
- We consider the suspension:

$$\dot{z} = F(z) + \varepsilon G(z, \theta, \varepsilon), \qquad \dot{\theta} = 1. \tag{2}$$

The flow of (2), $\psi(t; z, \theta, \varepsilon)$, $\psi(0; z, \theta, \varepsilon) = (z, \theta)$ satisfies the relations

$$\psi(t; z, \theta, \varepsilon) = \big(\varphi(t+\theta; \theta, z, \varepsilon), t+\theta\big), \quad \varphi(t; t_0, z, \varepsilon) = \pi_z \psi(t - t_0; z, t_0, \varepsilon).$$

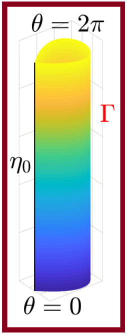
- The phase space for our system is then $\mathbb{R}^2 \times \mathbb{S}^1$.
- When $\varepsilon = 0$, the saddle point p_0 is now the periodic orbit $\eta_0 = \{p_0\} \times \mathbb{S}^1$ and the homoclinic connection Γ is now the *cylinder*, in fact a torus, $\Gamma \times \mathbb{S}^1$.



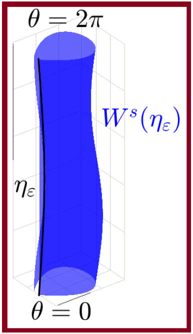
What does happen when $\varepsilon \neq 0$?.

- The fixed point is transformed into a hyperbolic T- periodic orbit $\eta_\varepsilon(t) = \mathcal{O}(\varepsilon)$. This is because the system for $\varepsilon = 0$ is locally structurally stable.
- The $W^s(\eta_\varepsilon)$ and $W^u(\eta_\varepsilon)$ generically have transversal intersections for $\varepsilon \neq 0$:

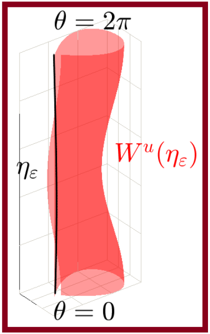
THE POINCARÉ MAP



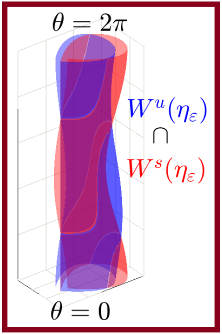
$\varepsilon = 0$



$W^u(\eta_\varepsilon)$ splits



$W^s(\eta_\varepsilon)$ splits



The intersection

POINCARÉ MAP

We can reduce the problem to a planar problem by means of the Poincaré map:

$$P_\varepsilon^{\theta_0}(z) = \pi_z \psi(T; z, \theta_0, \varepsilon) = \varphi(T + \theta_0; \theta_0, z, \varepsilon)$$

defined on

$$P_\varepsilon^{\theta_0} : \Sigma_{\theta_0} \to \Sigma_{\theta_0} = \Sigma_{\theta_0 + T}, \qquad \Sigma_{\theta_0} = \{(z, \theta) \in \mathbb{R}^2 \times \mathbb{R}/(T\mathbb{Z}) \, : \, \theta = \theta_0\}$$

BEHAVIOUR OF THE POINCARÉ MAP

The situation when P_ε^θ is considered:

- It has z_ε^θ a saddle point such that

$$P_\varepsilon^\theta(z_\varepsilon^\theta) = \varphi(T+\theta;\theta, z_\varepsilon^\theta, \varepsilon) = z_\varepsilon^\theta, \qquad \|z_\varepsilon^\theta - p_0\| = \mathcal{O}(\varepsilon).$$

- We have that

$$z_\varepsilon^\theta = \varphi(\theta; 0, z_\varepsilon^0, \varepsilon)$$

  so that, the periodic orbit $\eta_\varepsilon(t) = \varphi(t; 0, z_\varepsilon^0, \varepsilon)$.

- We can always assume, if we need, that $\eta_\varepsilon \equiv 0$ by performing the change of variables

$$v = z - \eta_\varepsilon(t), \qquad \dot{v} = F(v) + \varepsilon\tilde{G}(v,t,\varepsilon), \;\; \tilde{G}(v, t+T, \varepsilon) = \tilde{G}(v,t,\varepsilon).$$

- Notice that

$$(P_\varepsilon^\theta)^n(z) = \varphi(nT+\theta, \theta, z, \varepsilon).$$

  Indeed, it is a consequence from

$$\varphi(t; t_0, z, \varepsilon) = \varphi(t+T; t_0+T, z, \varepsilon)$$

  and

$$\varphi(t; t_1, \varphi(t_1, t_0, z, \varepsilon), \varepsilon) = \varphi(t; t_0, z, \varepsilon).$$

MORE ABOUT THE POINCARÉ MAP

- In this case

$$W^s(\eta_\varepsilon) = \bigcup_{\theta\in\mathbb{R}} W^s(z_\varepsilon^\theta), \qquad W^u(\eta_\varepsilon) = \bigcup_{\theta\in\mathbb{R}} W^u(z_\varepsilon^\theta).$$

Indeed, we assume that $\eta_\varepsilon \equiv 0$. If $q \in W^s(\eta_\varepsilon)$ then

$$0 = \lim_{t\to\infty} \pi^z \psi(t; q, \theta, \varepsilon) = \lim_{t\to\infty} \varphi(t+\theta; \theta, q, \varepsilon).$$

In particular the same happens for $t = nT$. Otherwise, let $q \in W^s(z_\varepsilon^\theta)$ and $t \geq 0$. Let $nT \leq t \leq (n+1)T$. Then, writting

$$\overline{F}(z, t, \varepsilon) = F(z) + \varepsilon G(z, t, \varepsilon)$$

we have that

$$\begin{aligned} \|\varphi(t+\theta;\theta,z,\varepsilon)\| &\leq \|\varphi(nT+\theta;\theta,z,\varepsilon)\| + \int_{nT}^{t} \|\overline{F}(\varphi(s+\theta;\theta,z,\varepsilon))\|\, ds \\ &\leq \|\varphi(nT+\theta;\theta,z,\varepsilon)\| + L\int_{nT}^{t} \|\varphi(s+\theta;\theta,z,\varepsilon)\|\, ds \end{aligned}$$

- Using Gronwall's lemma (Exercise: find the lemma and prove it)

$$\|\varphi(t+\theta;\theta,z,\varepsilon)\| \leq \|\varphi(nT+\theta;\theta,z,\varepsilon)\| e^{L(t-nT)} \leq \|\varphi(nT+\theta;\theta,z,\varepsilon)\| e^{LT}$$

and we are done.

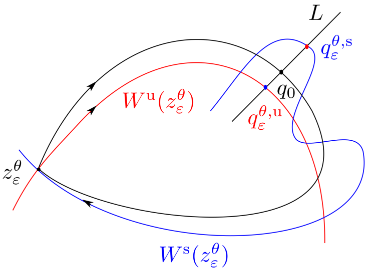
Outline

THE DISTANCE BETWEEN THE INVARIANT MANIFOLDS

- As a consequence,

$$W^s(z_\varepsilon^\theta) = W^s(\eta_\varepsilon) \cap \Sigma_\theta, \qquad W^u(z_\varepsilon^\theta) = W^u(\eta_\varepsilon) \cap \Sigma_\theta.$$

- Therefore, we only need to compute the *distance* between $W^u(z_\varepsilon^\theta)$ and $W^s(z_\varepsilon^\theta)$ on the global section Σ_θ.
- First we have to define what we mean for distance!.
- Take q_0 a point of the separatrix and γ_0 the parameterization such that $\gamma_0(0) = q_0$.



- Let L be the line such that $q_0 \in L$, inside of Σ_θ and ortogonal to the separatrix at q_0:

$$L = q_0 + \langle F(q_0)\rangle^\perp, \qquad \langle F(q_0)\rangle^\perp \subset \Sigma_\theta.$$

- Let $q_\varepsilon^{\theta,s}$, $q_\varepsilon^{\theta,u}$ be the closest points to q_0 belonging to $W^s(z_\varepsilon^\theta) \cap L$ and $W^u(z_\varepsilon^\theta) \cap L$ respectively.
- We want to compute,

$$q_\varepsilon^{\theta,u} - q_\varepsilon^{\theta,s}$$

THE FORMULA FOR THE DISTANCE

- Since

$$q_\varepsilon^{\theta,u}, q_\varepsilon^{\theta,s} \in L = q_0 + \langle F(q_0)\rangle^\perp$$

it is convenient to write

$$q_\varepsilon^{\theta,u} - q_\varepsilon^{\theta,s} = d_\varepsilon(\theta)\frac{1}{\|F(q_0)\|}\big(-F_2(q_0), F_1(q_0)\big).$$

- It is not difficult to check that, denoting $\Omega(u,v) = \det(u,v)$,

$$d_\varepsilon(\theta) = \Omega\left(\frac{F(q_0)}{\|F(q_0)\|}, q_\varepsilon^{\theta,u} - q_\varepsilon^{\theta,s}\right).$$

IMPORTANT REMARKS

- The points $q_\varepsilon^{\theta,u}, q_\varepsilon^{\theta,s}$ are well defined if ε is small enough. This is due to the differentiability of the invariant manifolds with respect to ε. (Why?).
- The function $d_\varepsilon(\theta)$ depends on ε and, obviously, in general cannot be computed.
- However, we know, using Taylor's theorem, that

$$d_\varepsilon(\theta) = \varepsilon\partial_\varepsilon d_\varepsilon(\theta)_{|\varepsilon=0} + \mathcal{O}(\varepsilon^2).$$

- The Melnikov integral, gives a formula for $\partial_\varepsilon d_\varepsilon(\theta)_{|\varepsilon=0}$.

Outline

THE MELNIKOV FUNCTION

PROPOSITION

The distance $d_\varepsilon(\theta)$ between $W^s(z_\varepsilon^\theta)$ and $W^u(z_\varepsilon^\theta)$ is expressed as:

$$d_\varepsilon(\theta) = \varepsilon \frac{M(\theta)}{\|F(q_0)\|} + \mathcal{O}(\varepsilon^2)$$

being $M(\theta)$ the Melnikov function:

$$M(\theta) = \int_{-\infty}^{\infty} exp\left(-\int_0^t trDF(\gamma_0(s))\, ds\right) \Omega(F(\gamma_0(t)), G(\gamma_0(t), t+\theta, 0))\, dt.$$

Remarks:

- The Melnikov function does not depend on ε.
- Remember that γ_0 satisfies $\gamma_0(0) = q_0$.
- When the system is Hamiltonian,

$$M(\theta) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\gamma_0(t), t+\theta, 0)\, dt$$

where $\{H_0, H_1\}$ is the Poisson's bracket:

$$\{H_0, H_1\} = \partial_x H_0 \partial_y H_1 - \partial_y H_0 \partial_x H_1.$$

EXISTENCE OF TRANSVERSAL HOMOCLINIC POINTS

THEOREM

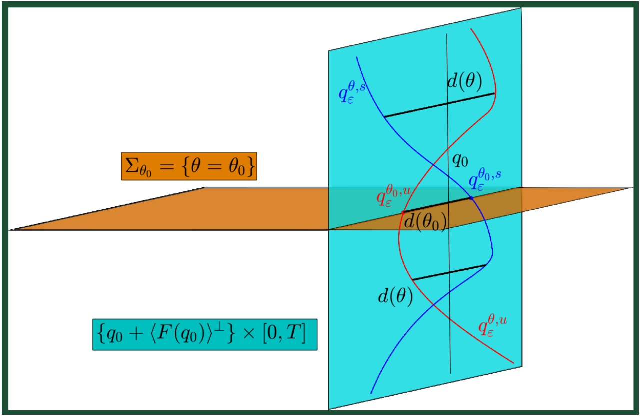
In the previous conditions, let $W^{s,u}(z_\varepsilon^\theta)$ be the stable and unstable manifold of the Poincaré map P_ε^θ. Then

- *If $M(\theta_0) = 0$ and $M'(\theta_0) \neq 0$ (a simple zero), then there exists $\varepsilon_* > 0$ such that for any $|\varepsilon| \leq \varepsilon_*$, $W^s(z_\varepsilon^\theta)$ and $W^u(z_\varepsilon^\theta)$ intersect transversally.*
- *If $M(\theta_0) \neq 0$, then there exists $\varepsilon_* > 0$ such that for any $|\varepsilon| \leq \varepsilon_*$, $W^s(z_\varepsilon^\theta)$ and $W^u(z_\varepsilon^\theta)$ do not intersect transversally close to q_0.*

Remarks

- The proof of this result is straightforward from the previous proposition. Indeed, it is a consequence of the differentiability with respect to parameters and the implicit function theorem.
- Notice that, since all the Poincaré maps are topologically conjugated, if there exists a simple zero of $M(\theta)$, then for every $\theta \in [0, T]$, the Poincaré map P_ε^θ has transversal homoclinic intersections. However they are not always close to q_0.
- As a consequence, $W^s(\eta_\varepsilon)$ and $W^u(\eta_\varepsilon)$ intersect transversally along a homoclinic solution.

SUMMARIZING



This picture shows a transversal homoclinic intersection. In red and blue, the curves

$$\{q_\varepsilon^{\theta,u}\}_{\theta\in[0,T]}, \qquad \{q_\varepsilon^{\theta,s}\}_{\theta\in[0,T]}$$

and in black the straight line

$$\{q_0\} \times [0, T].$$

Outline

THE EXAMPLE

Consider the one and a half degrees of freedom hamiltonian:

$$H(x,y,t)=\frac{y^2}{2}-\frac{x^2}{2}+\frac{x^3}{3}+\varepsilon(\sin t+x\cos t).$$

The homoclinic orbit can be parameterizated by $\gamma_0(t)=(x_0(t),y_0(t))$,

$$x_0(t)=\frac{3}{2}\frac{1}{\cosh^2(t/2)},\qquad y_0(t)=-\frac{3}{2}\frac{\sinh(t/2)}{\cosh^3(t/2)},\qquad \gamma_0(0)=(3/2,0).$$

In this case

$$\begin{aligned}M(\theta)&=-\int_{-\infty}^{\infty}y_0(t)\cos(t+\theta)\,dt=\frac{3}{2}\int_{-\infty}^{\infty}\frac{\sinh(t/2)}{\cosh^3(t/2)}\cos(t+\theta)\,dt\\&=\frac{3}{2}\cos\theta\int_{-\infty}^{\infty}\frac{\sinh(t/2)}{\cosh^3(t/2)}\cos t\,dt-\frac{3}{2}\sin\theta\int_{-\infty}^{\infty}\frac{\sinh(t/2)}{\cosh^3(t/2)}\sin t\,dt\\&=-\frac{3}{2}\sin\theta\int_{-\infty}^{\infty}\frac{\sinh(t/2)}{\cosh^3(t/2)}\sin t\,dt\end{aligned}$$

since $\dfrac{\sinh(t/2)}{\cosh^3(t/2)}\cos t$ is an odd function.

EVALUATING THE MELNIKOV FUNCTION (I)

- We perform the change $t = 2z$ and we obtain

$$I := \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh(t/2)}{\cosh^3(t/2)} \sin t \, dt = \int_{-\infty}^{\infty} \frac{\sinh z}{\cosh^3 z} \sin \omega z \, dz$$

with $\omega = 2$. Notice also that, by parts:

$$I = -\frac{\omega}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh^2 z} \cos \omega z \, dz.$$

- We notice that the function

$$f(z) = \frac{1}{\cosh^2 z} \cos \omega z$$

has poles of order 2 at $z = \pm i\frac{\pi}{2} + 2\pi k i$, $k \in \mathbb{Z}$. Write $a = \pi/2$.

- Recall that, one can compute the Laurent expansion as

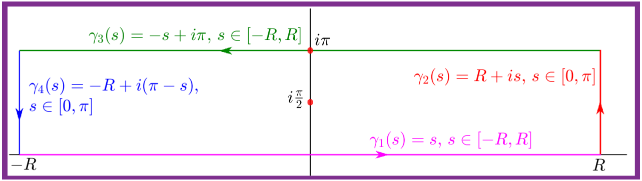
$$f(z) = \frac{a_{-2}}{(z-ia)^2} + \frac{a_{-1}}{(z-ia)} + a_0 + \sum_{k\geq 1} a_k (z-ia)^k, \qquad z \sim ia.$$

- The residue theory states that, if γ is a path having in its interior only one pole, for instance $i\pi/2$:

$$\int_\gamma f(z)\, dz = 2\pi i a_{-1} \Longrightarrow \lim_{R\to\infty} \int_\gamma f(z)\, dz = 2\pi i a_{-1}$$

EVALUATING THE MELNIKOV FUNCTION (II)

- $\gamma = \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$



- Recall that $\cosh iu = \cos u$ and $\sinh iu = i \sin u$ and

$$\cosh(u+v) = \cosh u \cosh v + \sinh u \sin v, \qquad \sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v.$$

- Note that

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz, \qquad \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = \lim_{R\to\infty} \int_{\gamma_4} f(z)\,dz = 0$$

$$\lim_{R\to\infty} \int_{\gamma_3} f(z)\,dz = -\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\cosh^2(-s+i\pi)} \cos\omega(-s+i\pi)\,ds$$

$$= -\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\cosh^2 s} \left[\cos\omega s \cosh\omega\pi + i \sin\omega s \sinh\omega\pi\right] ds$$

$$= -\cosh\omega\pi \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\cosh^2 s} \cos\omega s = -\cosh\omega\pi \int_{-\infty}^{\infty} f(z)\,dz,$$

Evaluating the Melnikov function (III)

- We have then:

$$2\pi i a_{-1} = [1 - \cosh \omega\pi] \int_{-\infty}^{\infty} f(z)\, dz \Longrightarrow \int_{-\infty}^{\infty} \frac{1}{\cosh^2 z} \cos \omega z\, dz = \frac{2\pi i a_{-1}}{1 - \cosh \omega\pi}$$

- It remains to compute a_{-1}, the residue of f at $a = \pi/2$. We have that

$$\cosh z = \cosh\left(i\frac{\pi}{2}\right) + \sinh\left(i\frac{\pi}{2}\right)\left(z - i\frac{\pi}{2}\right) + \cosh\left(i\frac{\pi}{2}\right)\left(z - i\frac{\pi}{2}\right)^2 + \mathcal{O}\left(z - i\frac{\pi}{2}\right)^3$$
$$= i\left(z - i\frac{\pi}{2}\right) + \mathcal{O}\left(z - i\frac{\pi}{2}\right)^3$$

and

$$\cos \omega z = \cos\left(\omega i\frac{\pi}{2}\right) + \omega \sin\left(\omega i\frac{\pi}{2}\right)\left(z - i\frac{\pi}{2}\right) + \mathcal{O}\left(z - i\frac{\pi}{2}\right)^2$$

- Then

$$\frac{\cos \omega z}{\cosh^2 z} = -\frac{\cos\left(\omega i\frac{\pi}{2}\right) + \omega \sin\left(\omega i\frac{\pi}{2}\right)\left(z - i\frac{\pi}{2}\right) + \mathcal{O}\left(z - i\frac{\pi}{2}\right)^2}{\left(z - i\frac{\pi}{2}\right)^2\left(1 + \mathcal{O}\left(z - i\frac{\pi}{2}\right)^2\right)}$$

Therefore

$$a_{-1} = -\omega i \sinh\left(\omega\frac{\pi}{2}\right) \Longrightarrow \int_{-\infty}^{\infty} \frac{1}{\cosh^2 z} \cos \omega z\, dz = 4\pi \frac{\sinh\left(\omega\frac{\pi}{2}\right)}{1 - \cosh \omega\pi} = 2\pi \mathrm{cosech}\left(\omega\frac{\pi}{2}\right)$$

EXISTENCE OF TRANSVERSAL HOMOCLINIC POINTS

- As a consequence

$$M(\theta) = 6\sin\theta \int_{-\infty}^{\infty} \frac{1}{\cosh^2 z} \cos\omega z \, dz = 6\pi \mathrm{cosech}\left(\omega\frac{\pi}{2}\right)\sin\theta$$

- Notice that the Melnikov function, $M(\theta)$ has simple zeroes at $\theta = k\pi$, $k \in \mathbb{Z}$.
- Recall that the distance between the invariant manifolds is given by:

$$d_\varepsilon(\theta) = \varepsilon\frac{M(\theta)}{\|F(q_0)\|} + \mathcal{O}(\varepsilon^2) = \frac{4}{3}\varepsilon M(\theta) + \mathcal{O}(\varepsilon^2).$$

- The implicit function theorem around $\theta_k = k\pi$, says that, there exists ε^k and a $\mathcal{C}^1$ function

$$\Theta^k : (-\varepsilon^k, \varepsilon^k) \to \mathbb{R}, \qquad \Theta^k(0) = \theta_k,$$

  such that,

$$d_\varepsilon(\Theta^k(\varepsilon)) = 0.$$

- That is the system has transversal homoclinic intersections.
- Do exercise 161.

OUTLINE

Idea of the proof (I)

The first thing we need to do is to obtain good expressions for $W^{s,u}(\eta_\varepsilon)$.

Lemma

There exists $\varepsilon_0 > 0$ small enough such that if $|\varepsilon| < \varepsilon_0$, $W^{s,u}(\eta_\varepsilon)$ are

$$W^{s,u}(\eta_\varepsilon) = \{\gamma_\varepsilon^{s,u}(t,t_0)\}$$

with $\gamma_\varepsilon^{s,u}(t,t_0)$ solutions of

$$\dot{z} = F(z) + \varepsilon G(z,t,\varepsilon)$$

and

$$\gamma_\varepsilon^{u}(t,t_0) := \varphi(t;t_0,q_\varepsilon^{t_0,u},\varepsilon) = \gamma_0(t-t_0) + \varepsilon\gamma_1^{u}(t,t_0) + \mathcal{O}(\varepsilon^2), \qquad t \geq t_0$$
$$\gamma_\varepsilon^{s}(t,t_0) := \varphi(t;t_0,q_\varepsilon^{t_0,s},\varepsilon) = \gamma_0(t-t_0) + \varepsilon\gamma_1^{s}(t,t_0) + \mathcal{O}(\varepsilon^2), \qquad t \leq t_0$$

with $\mathcal{O}(\varepsilon^2)$ uniformly bounded with respect to t, t_0.
In addition, $\gamma_1^{s,u}$ are solutions of

$$\dot{z} = DF(\gamma_0(t-t_0))\cdot z + G(\gamma_0(t-t_0),t,0). \tag{3}$$

IDEA OF THE PROOF (II)

- Define the functions

$$\begin{aligned}\Delta^u(t,\theta) &= \Omega\big(F(\gamma_0(t-\theta)), \gamma_1^u(t,\theta)\big),\\ \Delta^s(t,\theta) &= \Omega\big(F(\gamma_0(t-\theta)), \gamma_1^s(t,\theta)\big),\\ \Delta(t,\theta) &= \Delta^u(t,\theta) - \Delta^s(t,\theta).\end{aligned}$$

- We have that

$$q_\varepsilon^{\theta,u} - q_\varepsilon^{\theta,s} = \gamma_\varepsilon^u(\theta,\theta) - \gamma_\varepsilon^s(\theta,\theta) = \varepsilon\big(\gamma_1^u(\theta,\theta) - \gamma_1^s(\theta,\theta)\big) + \mathcal{O}(\varepsilon^2).$$

- Then, since Ω is the determinant:

$$d_\varepsilon(\theta)\|F(q_0)\| = \Omega(F(q_0), q_\varepsilon^{\theta,u} - q_\varepsilon^{\theta,s}) = \varepsilon\Delta(\theta,\theta) + \mathcal{O}(\varepsilon^2).$$

  We have to study $\Delta^{u,s}$.

- Take θ fix and compute $\frac{d}{dt} = \dot{}$ of Δ^s. First

$$\dot{\Delta}^s(t,\theta) = \Omega\left(\frac{d}{dt}(F(\gamma_0(t-\theta))), \gamma_1^s(t,\theta)\right) + \Omega\left(F(\gamma_0(t-\theta)), \frac{d}{dt}(\gamma_1^s(t,\theta))\right).$$

- Use the differential equations that γ_0 and $\gamma_1^{s,u}$ satisfy and the fact that $\Omega(Au,v) + \Omega(u,Av) = \mathrm{tr}A\Omega(u,v)$, for $A \in \mathcal{M}_{2\times 2}$:

IDEA OF THE PROOF (III)

- One can conclude that

$$\dot{\Delta}^s(t,\theta) = \mathrm{tr}\big(DF(\gamma_0(t-\theta))\big)\Delta^s(t,\theta) + \Omega\big(F(\gamma_0(t-\theta)), G(\gamma_0(t-\theta), t, 0)\big)$$

- Now note that, since $F(\gamma_0(t-\theta)) \to 0$ as $t \to \infty$:

$$\lim_{t\to\infty} \Delta^s(t,\theta) = \lim_{t\to\infty} \Omega\big(F(\gamma_0(t-\theta)), \gamma_1^s(t,\theta)\big) = 0.$$

- Assume that $\mathrm{tr}DF(z) \equiv 0$, for instance if we are in the Hamiltonian case. Then,

$$\Delta^s(t,\theta) = \int_\infty^t \Omega\big(F(\gamma_0(s-\theta)), G(\gamma_0(s-\theta), s, 0)\big)\, ds.$$

- A similar computation for $\Delta^u(t,\theta)$ and we obtain that

$$\Delta(\theta,\theta) = \int_{-\infty}^{\infty} \Omega\big(F(\gamma_0(t-\theta)), G(\gamma_0(t-\theta), t, 0)\big)\, dt.$$

In fact $\Delta(t,\theta)$ is constant with respect to t.