HOMOCLINIC POINTS

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Outline

1. Definition
2. Homoclinic points in the Smale's horseshoe
3. Smale-Birkhoff Homoclinic theorem
4. Some examples of chaotic dynamics
5. Hyperbolic sets

GLOBALIZING THE INVARIANT MANIFOLD

- Assume that f is a diffeomorphism f defined on a manifold M (which could be $\mathbb{R}^n$) having a saddle fixed point.
- p has associated two local invariant manifolds, $W_N^{s,u}(p)$ contained in N, a neighbourhood of the fixed point. Namely:

$$W_N^s(p) = \{x \in N \,:\, f^n(x) \in N,\; n \geq 0\}, \qquad W_N^u(p) = \{x \in N \,:\, f^{-n}(x) \in N,\; n \geq 0\}.$$

  In fact one can prove that

$$W_N^s(p) = \{x \in N \,:\, \lim_{n\to\infty} f^n(x) = p\}, \qquad W_N^u(p) = \{x \in N \,:\, \lim_{n\to\infty} f^{-n}(x) = p\}.$$

- We define now the stable and unstable set, $W^{s,u}(p)$ (sometimes we call them global stable and unstable manifold):

$$W^s(p) = \{x \in M \,:\, \lim_{n\to\infty} f^n(x) = p\}, \qquad W^u(p) = \{x \in M \,:\, \lim_{n\to\infty} f^{-n}(x) = p\}.$$

- Besides these sets are

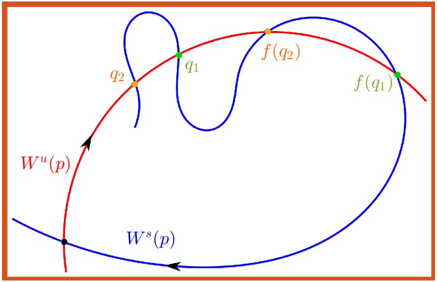
$$W^s(p) = \bigcup_{n\geq 0} f^{-n}(W_N^s(p)), \qquad W^u(p) = \bigcup_{n\geq 0} f^{n}(W_N^u(p)).$$

- When the global stable and unstable manifold are considered one can encountered really crazy behaviours. One of them is produced by the homoclinic transversal points.

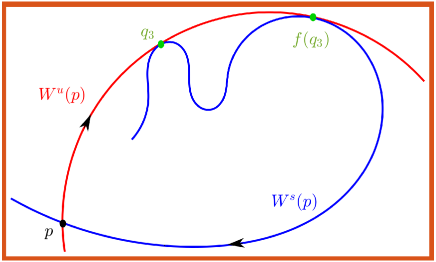
HOMOCLINIC TRANSVERSAL INTERSECTION

HOMOCLINIC POINTS

Let f be a diffeomorphism having a fixed point p of saddle type. Consider $W^{u,s}(p)$ the global unstable and stable sets.
Every point q such that $q \in W^u(p) \cap W^s(p)$ is said to be homoclinic. If the intersection is transversal, we say that q is a transversal homoclinic point.



In the figure, q_1, q_2 are transversal homoclinic points. Notice that $f^m(q_i)$ are homoclinic points.



In the figure, $q_3, f(q_3), \cdots$ are homoclinic tangencies.

HOMOCLINIC POINTS IN THE SMALE'S HORSESHOE

PROPOSITION

The Smale's horseshoe has two hyperbolic fixed points of saddle type $p_0 \in P_0$ and $p_1 \in P_1$. They have stable and unstable sets (not local):

$$W^u(p_i) = \{x \,:\, \lim_{n\to\infty} \|f^{-n}(x) - p_i\| = 0\}, \qquad W^s(p_i) = \{x \,:\, \lim_{n\to\infty} \|f^n(x) - p_i\| = 0\}.$$

If $p_i = V^i_\infty \cap H^i_\infty$, then

$$W^s(p_i) = \bigcup_{n\geq 0} \tilde{f}^n(V^i_\infty), \qquad W^u(p_i) = \bigcup_{n\geq 0} f^n(V^i_\infty).$$

In addition $W^s(p_i)$ and $W^u(p_i)$ have infinitely many transversal intersection on Q.

Remark:

- We will think that the Smale's we are dealing with is in fact the extension g to $\mathbb{S}^2$ which is an invertible map.
- Recall that, besides the sterographic projection, $g_{|D}$ is f and $g^{-1}_{|D'}$ is $\tilde{f}$.
- So when we write f^{-1} we really means $\tilde{f}$.

PROOF OF TRANSVERSAL HOMOCLINIC INTERSECTIONS

- Let $\sigma^0 = \{\cdots 0 \cdot 0 \cdots\}$ and $\sigma^1 = \{\cdots 1 \cdot 1 \cdots\}$. Then

$$p_0 = \phi^{-1}(\sigma^0), \qquad p_1 = \phi^{-1}(\sigma^1)$$

  are fixed points. Notice that $f^n(p_0) \in P_0$ and $f^n(p_1) \in P_1$ for all $n \in \mathbb{Z}$.
- In addition, both are saddle type provided $Df(p_i)$ has eigenvalues $1/a, a$ with $a < 1$.
- We focus in $p_1 = \phi^{-1}(\sigma^1)$. We first look for $W^{s,u}(p_1) \cap \Lambda$. Clearly $W^s(p_1) \cap \Lambda = \phi^{-1}(W^s(\sigma^1))$ and

$$W^s(\sigma^1) = \{\sigma \, : \, \lim_{n\to\infty} d(\beta^n(\sigma), \sigma^1) = 0\}.$$

- Compute

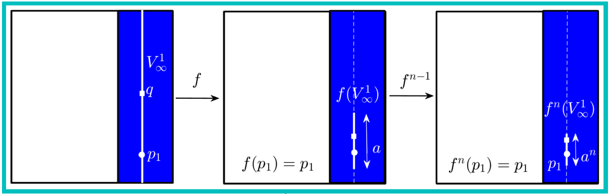
$$d_n(\sigma) = d(\beta^n(\sigma), \sigma^1) = \sum_{k=-\infty}^{\infty} \frac{|\sigma_{n+k} - 1|}{2^{|k|}} = \sum_{k=-\infty}^{\infty} \frac{|\sigma_k - 1|}{2^{|k-n|}}.$$

  We have that $d_n(\sigma) \to 0$ as $n \to \infty$ if and only if $\sigma_k = 1$ for $k \geq n_0$. Then

$$W^s(p_1) \cap \Lambda = \phi^{-1}(W^s(\sigma^1)) = \phi^{-1}\left(\{\sigma \in \Sigma \, : \, \exists n_0 \in \mathbb{Z} \text{ such that } \sigma_n = 1, \ \forall n \geq n_0\}\right).$$

CONTINUATION OF THE PROOF (I)

- We claim that $f^n(q) \in P_1,\ \forall n \geq 0 \iff q \in V^1_\infty \implies q \in W^s(p_1)$.



Where

$$p_1 = V^1_\infty \cap H^1_\infty.$$

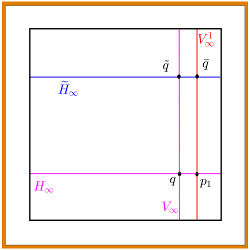
This proves the $\Longleftarrow$ implication.

- We first observe that, $\sigma^1 = \phi(p_1)$ satisfies

$$\sigma^1_n = 1,\ n \geq 0 \iff \varphi(V^1_\infty)_n = 1,\ n \geq 0 \iff f^n(V^1_\infty) \in P_1.$$

- Assume now that $f^n(q) \in P_1$ for all $n \geq 0$. Let V_∞ be such that $q \in V_\infty$. That means that $\sigma = \phi(q)$ satisfies

$$\sigma_n = 1,\ n \geq 0 \iff \varphi(V_\infty)_n = 1,\ n \geq 0 \iff f^n(V_\infty) \in P_1.$$



- Take $\tilde{q} = V_\infty \cap \widetilde{H}_\infty \in V_\infty \cap \Lambda$ and $\bar{q} = V^1_\infty \cap \widetilde{H}_\infty \subset \Lambda$.
- Since $\phi(\tilde{q}) = \phi(\bar{q})$, $\bar{q}, \tilde{q} \in \Lambda$ and $\phi_{|\Lambda}$ is an homeomorphism $\tilde{q} = \bar{q}$.

CONTINUATION OF THE PROOF (II)

- Let now q be such that $f^n(q) \in P_1$ for $n \geq n_0 \geq 0$. Then $q_0 = f^{n_0}(q) \in V^1_\infty$ satisfies $f^n(q_0) \in P_1$ for all $n \geq 0$ and thus $q_0 \in W^s(p_1)$ which implies that $q \in W^s(p_1)$:

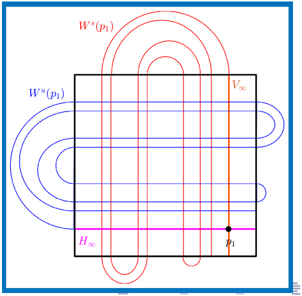
$$\lim_{n\to\infty} \|f^n(q) - p_1\| = \lim_{n\to\infty} \|f^{n-n_0}(q_0) - p_1\| = 0.$$

- Then $W^s(p_1) = \bigcup_{n\geq 0} \tilde{f}^n(V^1_\infty)$. The same for $W^u(p_1)$ and $W^{s,u}(p_0)$.
- $W^s(p_1) \cap W^u(p_1) \cap Q$ contains the corresponding points to

$$W^s(\sigma_1) \cap W^u(\sigma_1) = \{\sigma \in \Sigma \, : \, \exists n_\pm \text{ such that } \sigma_n = 1 \text{ if } n \leq n_-, \; n \geq n_+\}$$

which is obviously a countable infinite set.

- Since $\tilde{f}^n(V^1_\infty) \cap Q$ and $f^n(H^1_\infty) \cap Q$ are 2^n vertical, respectively horizontal, segments, the intersection has to be transversal.
- Both $\tilde{f}(V^1_\infty)$, $f(H^1_\infty)$ has a horseshoe shape. Therefore what we have is, in fact, the figure:



THE λ-LEMMA. A CLASSICAL RESULT

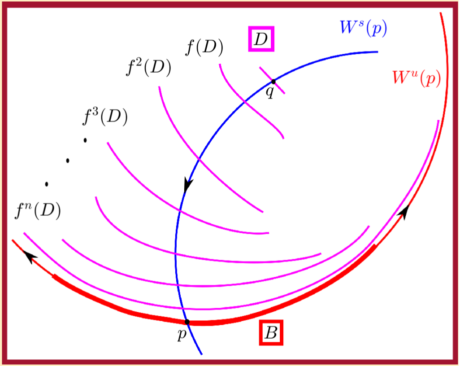
THEOREM

Let $f : M \to M$ be a diffeomorphism with a hyperbolic fixed point p. Take $q \in W^s(p)$ and $n^u = \dim E^u(p)$. Assume that B, D are two $\mathcal{C}^1$ embedded discs of dimension n^u in M such that

- $B \subset W^u(p)$
- $q \in D$ *and* $D \cap W^s(p)$ *is transversal:* $T_qD + T_qW^s(p) = T_qM$.

Then for any $\varepsilon > 0$, $\exists n_0 \geq 0$ such that for all $n \geq n_0$,

$$\|B_n - B\|_{\mathcal{C}_1} < \varepsilon, \qquad B_n \subset f^n(D).$$



Kupka-Smale diffeomorphisms

Kupka-Smale diffeomorphisms

Let M be a compact two dimensional manifold. The subset $\mathcal{K}(M)$ of $\mathrm{Diff}^1(M)$ having all their fixed points hyperbolic and having all intersections between stable and unstable transversal is residual.
Every diffeomorphism belonging to $\mathcal{K}(M)$ is refereed as a Kupka-Smale diffeomorphism.

The key idea is that

$$f \in \mathcal{K}(M) \text{ having a transversal homoclinic point} \implies f_{|\Lambda} \text{ is conjugated to } \beta.$$

Recall that:

- Residual is the countable intersection of open and dense sets.
- $\beta : \Sigma \to \Sigma$ is the right Bernoulli's shift:

$$\beta(\sigma) = \{\cdots \sigma_{-2} \cdot \sigma_{-1}\sigma_0\sigma_1 \cdots\}.$$

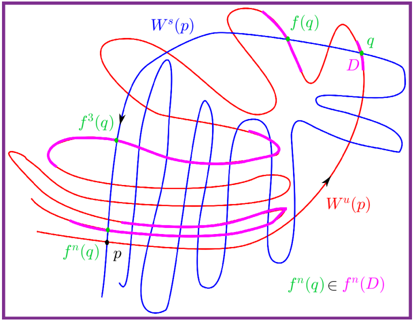
- Λ is a Cantor set.

SMALE-BIRKHOFF THEOREM

THEOREM

Let $f \in \mathcal{K}(M)$ be having a transverse homoclinic point q of a periodic hyperbolic point p of f, namely a fixed point of f^m.
Then there exists $\Lambda = \overline{\Lambda} \subset \Omega(f)$ a Cantor set such that

$$f^m(\Lambda) = \Lambda, \qquad f^m_{|\Lambda} \text{ is topollogically conjugated to } \beta : \Sigma \to \Sigma.$$

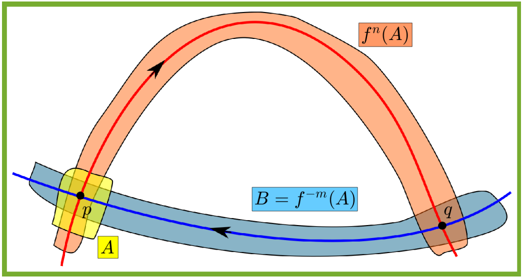


- $q, f(q), \cdots, f^n(q)$ are transversal homoclinic points.
- $\lim_{n\to\infty} f^n(q) = p$ and $\lim_{n\to\infty} f^{-n}(q) = p$.
- The disc D included in $W^u(p)$ is transversal to $W^s(p)$. Therefore by λ-lemma has to accumulate to $W^u(p)$.
- We obtain then the homoclinic tangle in the figure which recall the structure of the transversal intersection in the Smale's horseshoe.

ANOTHER ARGUMENT

Another way to prove the existence of Horseshoes when an homoclinic transversal point occurs is

- Let p be a saddle point and $q \in W^s(p) \cap W^u(p)$ be an homoclinic transversal point.
- Take A a neighbourhood of the saddle point p.



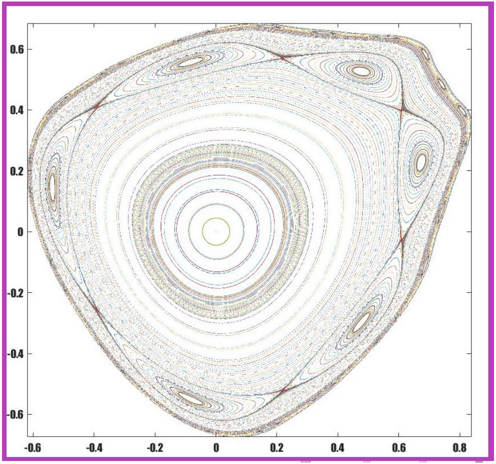
- Let $n, m \geq 0$ be such that $q \in f^n(A)$ and $q \in f^{-m}(A)$.
- Take $\bar{f} = f^{n+m}$ a diffeomorphism. We have that $\bar{f}$ is a Smale's horseshoe type map taking $B = f^{-m}(A)$ as the *square* Q.

The Henon map

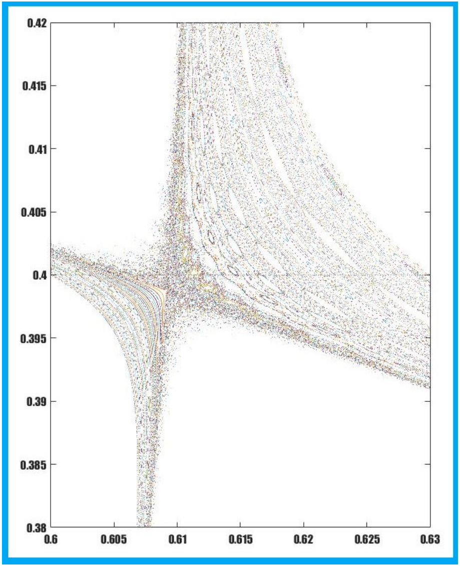
We consider the Henon map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

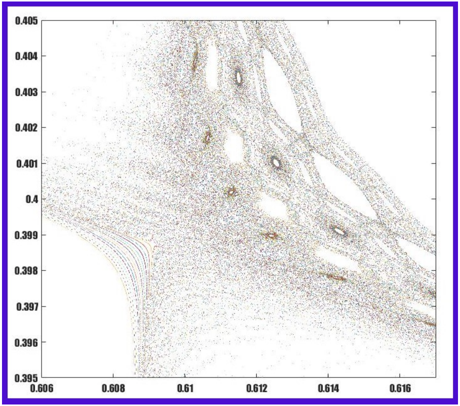
$$f(x,y) = \begin{pmatrix} x\cos\alpha - y\sin\alpha + x^2\sin\alpha \\ x\sin\alpha + y\cos\alpha - x^2\cos\alpha \end{pmatrix}$$

- We draw the phase portrait: the curves resulting of applying the Henon map several times. There is no dynamical sense in them.
- We have taken $\alpha = 0.4$.
- Observe that we have an evident chain of periodic orbits of period 6.
- But also have other chains of large period.
- Every island is surrounded by something similar to a heteroclinic connection. However....



CHAOS IN THE HENON MAP





When we magnify the pictures we encounter

- There are homoclinic transversal points.
- Islands of all the periods.
- Summarizing chaos.

Definition

- Since the maps with transversal homoclinic points, have associated horseshoes, they posses Cantor invariant sets.
- Can we provide some structure to these invariant sets?.

Hyperbolic sets

Let $f : U \subset \mathbb{R}^n \to U$ be a diffeomorphism having an invariant set $\mathcal{S}$. We say that $\mathcal{S}$ is hyperbolic if for any $x \in \mathcal{S}$, there exists a decomposition of the form

$$T_x\mathcal{S} \oplus E^s_x \oplus E^u_x = \mathbb{R}^n$$

where $T_x\mathcal{S}$ is the tangent space of $\mathcal{S}$ at x, satisfying that

- if $v \in E^s_x$, then $Df^n(x)v \in E^s_{f^n(x)}$ and
$$\|Df^n(x)v\| \leq C\mu^n\|v\|,$$
- if $v \in E^u_x$, then $Df^{-n}(x)v \in E^u_{f^{-n}(x)}$ and
$$\|Df^n(x)v\| \geq c\mu^{-n}\|v\|,$$
- the subspaces E^s_x, E^u_x depend on x continuously.

REMARKS

- The definition of hyperbolic set can be extended to diffeomorphism defined on M a compact manifold.
- To do so, it is necessary to use adequate charts to extend the definition of differential.
- The invariant set Λ of the horseshoe map is an hyperbolic set.
- In the cat map $f : \mathbb{T}^2 \to \mathbb{T}^2$, in the torus, with lift $\overline{f}(x) = Ax$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

  the full $\mathbb{T}^2$ has also a hyperbolic structure. In fact in this case, the stable and unstable sets are constants.

Non wandering hyperbolic sets

Theorem

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism having a hyperbolic non-wandering compact set Ω. If the periodic orbits are dense in Ω, then

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_n, \qquad \Omega_i, \ \textit{basic sets}, \qquad \Omega_i \cap \Omega_j = \emptyset$$

with $\Omega_1, \cdots, \Omega_n$ closed, invariant and containing a dense orbit.

A comment

If $f : M \to M$ with M a compact manifold with boundary, the same is true and moreover,

$$M = \bigcup_{i=1}^{n} \mathrm{in}(\Omega_i) \qquad \text{with } \mathrm{in}(\Omega_i) = \left\{ x \in U \ : \ \lim_{m \to \infty} \mathrm{dist}\left(f^m(x), \Omega_i\right) = 0 \right\}$$

- The diffeomorphisms having a hyperbolic non-wandering set with the periodic orbits dense in Ω, are called Axiom-A diffeomorphisms.
- If a non-wandering set contains only finitely many hyperbolic fixed points or periodic orbits, is Axiom-A.
- On the contrary, when a non-wandering set has a dense orbit and the periodic orbits are dense in it, is called a chaotic set.

STRANGE ATTRACTORS

A DEFINITION

There is a lot of discussion about the definition of strange attractor.
One possibility is to say that a strange attractor is an attracting chaotic set.
Notice that such a definition can be applied in both, vector fields and diffeomorphisms.

One of the most popular are:

Consider the Lorenz equation

$$\begin{cases} \dot{x} &= 10(y-x) \\ \dot{y} &= x(28-z)-y \\ \dot{z} &= xy - \frac{8}{3}z. \end{cases}$$

It comes from a model for fluid flow of the atmosphere.

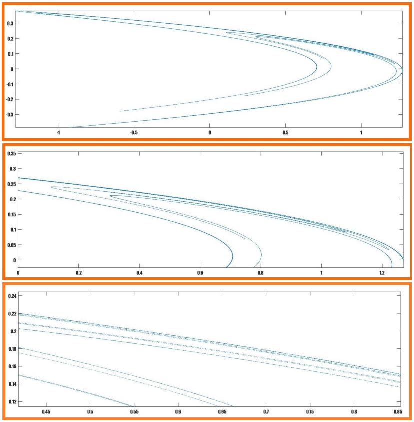
The Henon map

$$f(x,y) = \begin{pmatrix} y - ax^2 + 1 \\ bx \end{pmatrix}$$

for some values of a, b. Concretely we take the classical ones $a = 1.4$, $b = 0.3$.
The Henon map is an approximation of a Poincaré map of the Lorenz equation.

THE PICTURES

Atractor for the Henon map



Lorenz's attractor