Hyperbolic non-linear fixed points

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OUTLINE

Outline

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INTRODUCTION

- Dynamical systems are defined on manifolds M.
- We are now interested in what happen around an equilibrium point, namely a point $x_* \in M$ such that

$$x_* = f(x_*), \qquad X(x_*) = 0$$

either if we are dealing with maps, f, or vector fields, X.

- Since we can use only one chart to study the behaviour of the system in a (*small*) neighbourhood of the equilibrium point, we can assume that:

$$f, X : U \subset \mathbb{R}^n \to \mathbb{R}^n.$$

The question is

When the linear part is dominant?

The linear approximation

Around an equilibrium point x_*, by Taylor's theorem

$$f(x) \sim f(x_*) + Df(x_*)(x - x_*) = x_* + Df(x_*)(x - x_*).$$

Then,

$$f(x) - x_* \sim \left[Df(x*)\right](x - x*).$$

It is close to linear homogeneous. Linear systems are *easy*.

Recall that $x_{n+1} = Ax_n$ has explicit solutions

$$x_n = A^n x_0$$

and the same happen for flows $\dot{x} = Ax$:

$$x(t) = e^{At} x_0.$$

LINEARISED PART

We define the **linearised part** around an isolated equilibrium point x_* of a dynamical system as:

If we have $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, a map

$$\bar{x} = x_* + Df(x*)(x - x_*).$$

If we have $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$, a vector field

$$\dot{x} = DX(x_*)(x - x_*).$$

THE QUESTION IS:

When can we relate the qualitative behaviour of the dynamical systems

$$\bar{x} = f(x) \qquad \text{or} \qquad \dot{x} = X(x)$$

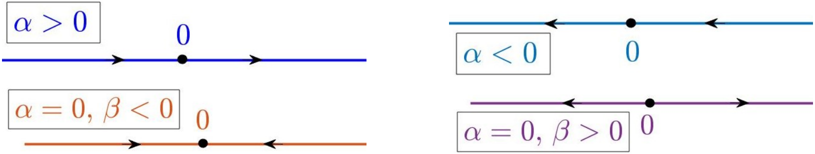
with their **linearised parts** around an equilibrium point x_*.?

Recall that:

$$Df(x_*) = \left(\partial_{x_j} f_i(x_*)\right)_{i,j}, \qquad x = (x_1, \cdots, x_n),\ f = (f_1, \cdots, f_n).$$

NON DOMINANT LINEAR PART

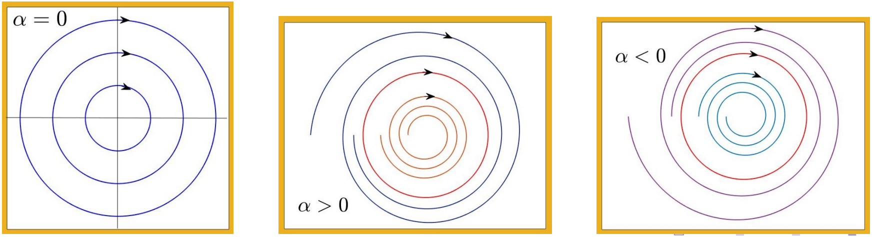
Eigenvalue 0. Take $\dot{x} = \alpha x^2 + \beta x^3$. The linear part $\dot{x} = 0$ is not dominant:



Eigenvalues $\pm i$. Consider the system

$$\dot{x} = y + \alpha x(x^2 + y^2)(1 - (x^2 + y^2)), \qquad \dot{y} = -x + \alpha y(x^2 + y^2)(1 - (x^2 + y^2))$$

which in polar coordinates is $\dot{r} = \alpha r^3(1 - r^2)$, $\dot{\theta} = -1$. Linear part $\alpha = 0$.



HYPERBOLIC EQUILIBRIUM POINTS

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field.

- An equilibrium point $x_* = f(x_*)$ of f is hyperbolic if

$$Df(x_*) \text{ has no eigenvalue with modulus 1.}$$

- An equilibrium point $X(x_*) = 0$ of X is hyperbolic if

$$DX(x_*) \text{ has no eigenvalue with real part equal to 0.}$$

- The stable, E^s, and unstable, E^u, linear subsets of f are the maximal invariant by $Df(x_*)$ subspaces such that

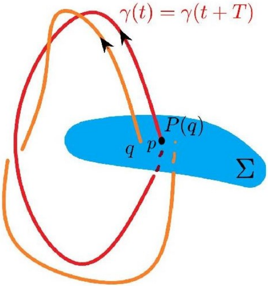
$$\mathrm{Spec}(Df(x_*)_{|E^s}) \subset \{\lambda \in \mathbb{C} \; : \; |\lambda| < 1\},$$
$$\mathrm{Spec}(Df(x_*)_{|E^u}) \subset \{\lambda \in \mathbb{C} \; : \; |\lambda| > 1\}.$$

- The stable, E^s, and unstable, E^u, linear subsets of X are the maximal invariant by $DX(x_*)$ subspaces such that

$$\mathrm{Spec}(DX(x_*)_{|E^s}) \subset \{\lambda \in \mathbb{C} \; : \; \mathrm{re}\,\lambda < 0\},$$
$$\mathrm{Spec}(DX(x_*)_{|E^u}) \subset \{\lambda \in \mathbb{C} \; : \; \mathrm{re}\,\lambda > 0\}.$$

CLOSED HYPERBOLIC CURVES

- Let γ be a closed orbit of a vector field.
  The Poincaré map $P : \Sigma \to \Sigma$ has $p = \gamma \cap \Sigma$ as a fixed point.
  We say that γ is hyperbolic if p is a fixed hyperbolic point of P.



- If γ is a closed curve of a map.... we will see the definition another day!

Recalling Conjugacy

Definition

We say that two maps $f : W \subset \mathbb{R}^n \to U$, $g : V \subset \mathbb{R}^n \to \mathbb{R}^n$ are topologically conjugate if there exists an homeomorphism $h : U \to V$ such that

$$h \circ f = g \circ h.$$

We say that two flows $\varphi_t : W \subset \mathbb{R}^n \to U$, $\psi_t : V \subset \mathbb{R}^n \to \mathbb{R}^n$ are topologically conjugate if there exists an homeomorphism $h : U \to V$ such that

$$h \circ \varphi_t = \psi_t \circ h, \qquad \forall t \in \mathbb{R}.$$

The flows φ_t, ψ_t are topologically equivalent if $h(\varphi_t(x)) = \psi_{\tau(t,x)}(h(x))$.

In particular

- If x_* is a fixed point of f (resp. φ_t), $h(x_*)$ is a fixed point of g (resp. ψ_t).
- The image of the periodic orbits of f (resp. φ_t) by h are also periodic orbits of g (resp. ψ_t).

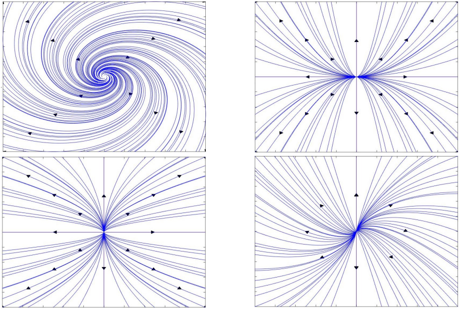
Conjugacy of linear maps

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $X(x) = Ax$. We define the stability index as

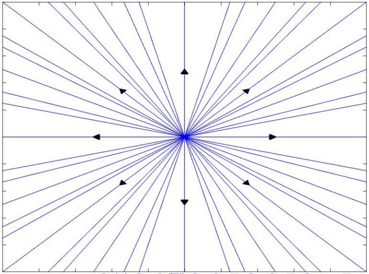
$$n_s = \mathrm{card}\{\lambda \,:\, \text{eigenvalues of } A \ \text{ with re} \, \lambda < 0\}, \quad n_u = n - n_s.$$

On a neighbourhood of the origin, the flow of X is topollogically conjugated to the flow of

$$\dot{\xi}_s = -\xi_s, \qquad \dot{\xi}_u = \xi_u, \qquad \xi_s \in \mathbb{R}^{n_s}, \, \xi_u \in \mathbb{R}^{n_u}.$$



Are topologically conjugated to:



OUTLINE

Normed spaces

Definition (Normed space)

A normed space, E, is a vector space on $K = \mathbb{R}, \mathbb{C}$, equipped with a norm. A norm is a map $\|\cdot\| : E \to [0, +\infty)$ with the properties

- $\|x\| = 0$ if and only if $x = 0$,
- for all $\lambda \in K$, $\|\lambda x\| = |\lambda| \|x\|$,
- for all $x, y \in E$, $\|x + y\| \leq \|x\| + \|y\|$

For instance

- $E = \mathbb{R}^n$, with norms $\|x\|_\infty = \sup_i |x_i|$, $\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.
- The space of the continuous funtions:

$$E = C^0([a, b], \mathbb{R}^n) = \{f : [a, b] \to \mathbb{R}^n \ : \ f \text{ is continuous}\}$$

with the norm $\|f\|_\infty = \sup_{x \in [a,b]} \|f(x)\|$.

BANACH SPACES

Let $(E, \|\cdot\|)$ be a normed space and $(x_n)_{n\in\mathbb{N}} \subset E$ a sequence.
We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is a Cauchy's sequence if

$$\forall\varepsilon > 0,\ \exists n_0,\ \text{such that if } \ n, m \geq n_0,\ \|x_n - x_m\| < \varepsilon.$$

We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is convergent if there exists $x_* \in E$ such that

$$\forall\varepsilon > 0,\ \exists n_0,\ \text{such that if } \ n \geq n_0,\ \|x_n - x_*\| < \varepsilon.$$

DEFINITION (BANACH SPACE)

We say that a normed space, $(E, \|\cdot\|)$ is a Banach space if it is complete, that is if every Cauchy's sequence is convergent.

The normed space $C^0([a, b], \mathbb{R}^n)$ with the norm $\|\cdot\|_\infty$ is a Banach space.

THE FIXED POINT THEOREM

This result try to answer to the question:

Are there solutions of $x = F(x)$?.

THEOREM

Let $(E, \|\cdot\|)$ be a Banach space, $X \subset E$ a closed subset and $F : X \to X$ satisfying that there exists a constant $L \in [0, 1)$ such that

$$\forall x, y \in X, \quad \|F(x) - F(y)\| \leq L\|x - y\|.$$

Then there exists a unique $x_* \in X$ satisfying

- *$x_* = F(x_*)$. (We say that x_* is a fixed point)*
- *For all $x \in X$:*

$$\|F^n(x) - x_*\| \leq \frac{L^n}{1-L}\|F(x) - x\| \implies \lim_{n\to\infty} F^n(x) = x_*.$$

PROOF OF THE FIXED POINT THEOREM

If $x_*^1, x_*^2 \in X$ are two different fixed points of F, then

$$\|x_*^1 - x_*^2\| = \|F(x_*^1) - F(x_*^2)\| \leq L\|x_*^1 - x_*^2\| < \|x_*^1 - x_*^2\|$$

which is a contradiction. The uniqueness is proven.
Let $x \in X$. Consider $x_n = F^n(x) = F(x_{n-1})$ with $x_0 = x$. On the one hand, for $n \geq m$

$$\|x_n - x_m\| = \|F(x_{n-1}) - F(x_n)\| \leq L\|x_{n-1} - x_{m-1}\| \leq L^m\|x_{n-m} - x_0\|.$$

On the other hand

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq L^{n-1}\|x_1 - x_0\| + L^{n-2}\|x_1 - x_0\| + \cdots + L^m\|x_1 - x_0\| \leq \frac{L^m}{1 - L}\|x_1 - x_0\|.\end{aligned}$$

Then $(x_n) \subset X$ is a Cauchy's sequence and, since E is complete, is convergent to $x_* \in E$. we deduce that $x_* \in X$, because X is a closed subset.

- Since $x_n = F(x_{n-1})$ with F continuous, taking $n \to \infty$ we prove that x_* is a fixed point.
- In addition, taking $n \to \infty$ from the above inequality, we get the second statement.

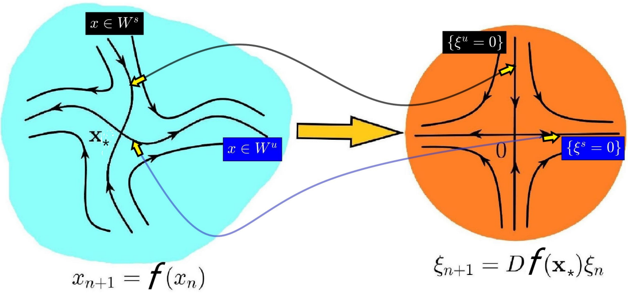
OUTLINE

HARTMAN'S THEOREM FOR MAPS

THEOREM

Let $U \subset \mathbb{R}^n$ be an open set. Consider $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism having a hyperbolic fixed point x_*.
There exists a neighbourhood $N \subset U$ of x_* and a neighbourhood $N' \subset \mathbb{R}^n$ containing the origin such that

$$f_{|N} \text{ is topologically conjugate to } Df(x_*)_{|N'}$$



HARTMAN'S THEOREM FOR FLOWS

THEOREM

Let $U \subset \mathbb{R}^n$ be open. Consider a C^1 vector field $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ having x_* a hyperbolic fixed point. Let $\varphi_t : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be its flow. There exists a neighbourhood $N \subset U$ of x_* on which

$$\varphi_t \text{ is topologically conjugate to the linear flow } \exp(DX(x_*)t)x$$

As a consequence we have that:

CLASSIFICATION OF HYPERBOLIC POINTS

On the same conditions of Hartman's theorem, let

$$n_s = \text{card}\{\lambda : \text{eigenvalues of } DX(x_*) \text{ with re}\,\lambda < 0\}, \quad n_u = n - n_s.$$

On a neighbourhood of x_*, φ_t is topologically conjugated to the flow of

$$\dot{\xi}_s = -\xi_s, \qquad \dot{\xi}_u = \xi_u, \qquad \xi_s \in \mathbb{R}^{n_s}, \xi_u \in \mathbb{R}^{n_u}.$$

OUTLINE

HARTMAN'S THEOREM, GLOBAL VERSION

THEOREM

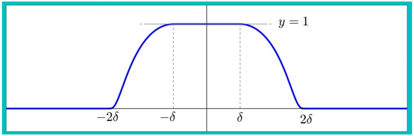
Let L be a linear hyperbolic map $L : \mathbb{R}^n \to \mathbb{R}^n$. Then, there exists $\varepsilon > 0$ such that for any $\Delta f, \Delta g : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz functions with

$$lip\,\Delta f,\, lip\,\Delta g < \varepsilon$$

the maps $L + \Delta f$ and $L + \Delta g$ are topologically conjugated on $\mathbb{R}^n$.
The conjugation homeomorphism h satisfies that $h - Id$ is bounded on $\mathbb{R}^n$

We can prove Hartmans's theorem from this one by:

- Take $L = Df(x_*)$ and perform the change $y = x - x_*$. We get $\tilde{f}(y) = Ly + \tilde{f}_1(y)$ with $\tilde{f}_1(y) = o(y)$.
- For any $\varepsilon > 0$, there exists δ such that if $\|y\| < 2\delta$, then lip $\tilde{f}_1 < \varepsilon$
- We can extend $\tilde{f}$ into $\mathbb{R}^n$ by means of a regular bump function φ.



$\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\hat{f}(y) = \begin{cases} \tilde{f}(y)\varphi(y), & |y| < 2\delta \\ 0 & |y| \geq 2\delta \end{cases}$$

IDEA OF THE PROOF OF THE GLOBAL VERSION (I)

- Write the conjugation $h = \mathrm{Id} + u$. It has to satisfy

$$(L + \Delta f) \circ (\mathrm{Id} + u) = (\mathrm{Id} + u) \circ (L + \Delta g).$$

This equation is equivalent to

$$\mathcal{F}(u) := Lu - u \circ (L + \Delta g) = -\Delta f \circ (\mathrm{Id} + u) + \Delta g. \tag{1}$$

- If ε is small enough, $L + \Delta g$ is a homeomorphism. Using this fact, prove that $\mathcal{F}$ is a linear homeomorphism. Let $\mathcal{F}^{-1}$ its inverse
- Write equation (1) as a fixed point equation :

$$u = \mathcal{G}(u) := \mathcal{F}^{-1}(-\Delta f \circ (\mathrm{Id} + u) + \Delta g).$$

- Prove that

$$E = \left\{ f : \mathbb{R}^n \to \mathbb{R}^n \ : \ \text{continuous and } \|f\|_\infty = \sup_{x \in \mathbb{R}^n} \|f(x)\| < \infty \right\}.$$

is a Banach space

IDEA OF THE PROOF OF THE GLOBAL VERSION (II)

- If $\varepsilon > 0$ is small enough, $\mathcal{G} : E \to E$ satisfies the hypotheses of the fixed point theorem. Indeed:

$$\|\mathcal{G}(u_1) - \mathcal{G}(u_2)\| \leq \|\mathcal{F}^{-1}\|\|\Delta f \circ (\mathrm{Id} + u_1) - \Delta f \circ (\mathrm{Id} + u_2)\| \leq \|\mathcal{F}^{-1}\|\mathrm{lip}\, \Delta f\|u_1 - u_2\|.$$

  Since $\|\mathcal{F}^{-1}\|\mathrm{lip}\, \Delta f < \|\mathcal{F}^{-1}\|\varepsilon < 1$ we are done.
- Then there exists an unique $u \in E$ such that $h = \mathrm{Id} + u$ satisfies

$$(L + \Delta f) \circ h = h \circ (L + \Delta g).$$

- It remains to see that h is an homeomorphism. Changing Δf and Δg in the above equation, there exists $\bar{h} = Id + \bar{u}$ such that

$$(L + \Delta g) \circ \bar{h} = \bar{h} \circ (L + \Delta f).$$

  Then

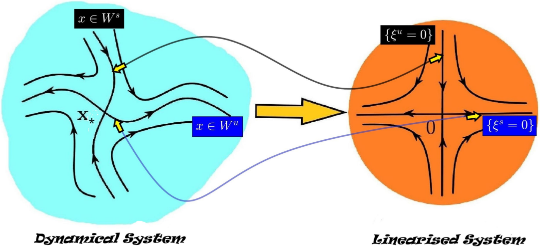
$$h \circ \bar{h} \circ (L + \Delta f) = h \circ (L + \Delta g) \circ \bar{h} = (L + \Delta f) \circ h \circ \bar{h}.$$

  That is, $h \circ \bar{h}$ is a conjugation between $L + \Delta f$ itself. By uniqueness, $h \circ \bar{h} = \mathrm{Id}$. Using similar arguments, we prove $\bar{h} \circ h = Id$.

Outline

PRELIMINARIES

Recall that by Hartman's theorem we have that:



FIRST APPROACH

The invariant sets W^u, W^s are the corresponding ones to E^u, E^s. How can we characterize W^u, W^s without using Hartman's theorem?

For linear maps, by means of a linear change of coordinates, $f(\xi^s, \xi^u) = (A^s\xi^s, A^u\xi^u)$ with

$$\|A^s\| < 1, \qquad \|(A^u)^{-1}\| < 1$$

then $E^s = \{\xi^u = 0\}$, $E^u = \{\xi^s = 0\}$. Notice that,

$$\lim_{m\to\infty} f^m(\xi^s, 0) = 0, \qquad \lim_{m\to-\infty} f^m(0, \xi^u) = 0.$$

Although we can also characterize $E^{s,u}$ as, given B a neighbourhood of the origin:

$$E^s = \{x \,:\, f^m(x) \in B,\ \forall m \geq 0\}, \qquad E^u = \{x \,:\, f^m(x) \in B,\ \forall m \leq 0\}.$$

DEFINITION

INVARIANT MANIFOLDS FOR GENERAL MAPS

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with a hyperbolic fixed point x_*. For any neighborhood $N \subset U$ of x_* we define the local stable invariant set

$$W^s_N(x_*) = \{x \in \mathbb{R}^n : f^m(x) \in N, \ \forall m \geq 0\} = \bigcap_{m \in \mathbb{N}} f^{-m}(N)$$

and the local unstable invariant set

$$W^u_N(x_*) = \{x \in \mathbb{R}^n : f^{-m}(x) \in N, \ \forall m \geq 0\} = \bigcap_{m \in \mathbb{N}} f^{m}(N).$$

OUTLINE

INVARIANT MANIFOLDS FOR HYPERBOLIC POINTS

THEOREM

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism C^r, $r \geq 1$, with an hyperbolic fixed point at $x_* \in U$. Let E^s, E^u be the stable and unstable subspaces of $Df(x_*)$. Then, on a sufficiently small ball $N \subset U$ of x_*, $W_N^{s,u}(x_*)$ are C^r invariant manifolds satisfying that

$$dim\, W_N^s(x_*) = dim\, E^s, \qquad dim\, W_N^u(x_*) = dim\, E^u$$

and they are tangent to the linear subspaces $E^{s,u}$ respectively.

In fact,

- There exist $\gamma^s : N^s \to E^u$, $\gamma^u : N^u \to E^s$, C^r such that

$$\text{graph}\, \gamma^{s,u} = W_N^{s,u}(x_*).$$

- γ^s is tangent at x_* to E^s and γ^u is tangent at x_* to E^u.

REMARKS (I)

FIRST REMARK

$$x \in W^s_N(x_*) \Longrightarrow \lim_{m\to\infty} f^m(x) = x_*, \qquad x \in W^u_N(x_*) \Longrightarrow \lim_{m\to-\infty} f^m(x) = x_*.$$

Indeed, by Hartman's theorem, if $L\xi = Df(x*)\xi$

$$h \circ L = f \circ h, \Longrightarrow f^m = h \circ L^m \circ h^{-1}, \quad m \in \mathbb{Z}.$$

Let $B = h^{-1}(N)$ be a neighbourhood of the origin. We have that

$$E^s_B = \bigcap_{m\in\mathbb{N}} L^{-m}(B) = \{\xi_u = 0,\ \xi \in B\}.$$

Moreover,

$$W^s_N(x_*) = \bigcap_{n\in\mathbb{N}} f^{-m}(N) = \bigcap_{m\in\mathbb{N}} h \circ L^m(h^{-1}(N)) = h(E^s_B)$$

and recall that

$$\xi \in E^s_B \Longleftrightarrow \lim_{m\to\infty} L^m\xi = 0.$$

Take now $x \in W^s_N(x_*)$, and let ξ be such that $h(\xi) = x$. Then

$$\lim_{m\to\infty} f^m(x) = \lim_{m\to\infty} f^m(h(\xi)) = \lim_{m\to\infty} h(L^m\xi) = h(0) = x_*.$$

The analogous argument for the unstable manifold.

REMARKS (II)

UNIQUENESS RESULT (AGAIN AS A CONSEQUENCE OF HARTMAN'S THEOREM)

If $x \in N$ sufficiently small neighbourhood of x_*, then $W_N^s(x_*) = \text{graph}\,\gamma^s$.
With the Hartman theorem we can only prove that γ^s is continuous.

Indeed, let $h = Id + u$ be the homeomorphism such that $h \circ L = f \circ h$, with $L = Df(x_*)$.
We first notice that, changing coordinates if necessary, we can write $x = (x^s, x^u)$, $\xi = (\xi^s, \xi^u)$, $h = (h^s, h^u)$, $u = (u^s, u^u)$ with

$$E^s = \{\xi^u = 0\}, \qquad E^u = \{\xi^s = 0\}, \qquad L = \begin{pmatrix} L^s & 0 \\ 0 & L^u \end{pmatrix}.$$

We define the homeomorphism

$$\bar{h}^s(\xi^s) := h^s(\xi^s, 0) = \xi^s + u^s(\xi^s, 0)$$

and we recall that $W_N^s(x_*) = h(E_B^s) = \{h(\xi^s, 0)\}_{\xi \in B}$. Then, if $x = (x^s, x^u) \in W_N^s(x_*)$, it satisfies $x = h(\xi_s, 0)$ and

$$x^s = h^s(\xi^s, 0) = \bar{h}^s(\xi^s) \Longrightarrow \xi^s = (\bar{h}^s)^{-1}(x_s).$$

Therefore

$$x^u = h^u(\xi^s, 0) = h^u((\bar{h}^s)^{-1}(x_s), 0) =: \gamma^s(x_s).$$

OUTLINE

STRATEGY TO FIND STABLE INVARIANT MANIFOLDS

P.M. WAS CREATED BY CABRÉ, DE LA LLAVE AND FONTICH

The graph transform method, which is the classical one, is a particular case. It consists on searching a function γ^s such that, if $\pi_{u,s}$ is the projection on $x_{u,s}$:

$$\pi_u f(x, \gamma^s(x_s)) = \gamma^s(\pi_s f(x_s, \gamma^s(x_s)).$$

The Parameterization Method (P.M.), search the invariant manifold parameterized instead as a graph. That is to say,

$$W^s_N(x_*) = \{K(t)\}_t$$

Then, the invariance condition is that $f(K(t)) = K(t')$ or written in a better way

$$f(K(t)) = K(R(t)).$$

HOW DOES THE PARAMETERIZATION METHOD WORK?

- Decompose $\mathbb{R}^n = \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$ being $x \in \mathbb{R}^{n_s}$ the *stable* directions and $y \in \mathbb{R}^{n_u}$ the *unstable* directions.
- Perform a change of variables to ensure that $x_* = 0$ and the stable invariant manifold is tangent to $y = 0$.
- Find K, R solving the invariance condition $f \circ K = K \circ R$, where

$$K : V \subset \mathbb{R}^{n_s} \to \mathbb{R}^n, \qquad R : V \subset \mathbb{R}^{n_s} \to N,$$

and $0 \in N$. To do so,

  - *A posteriori result*. Assuming

$$f \circ K^{\leq}(x) - K^{\leq} \circ R(x) = \mathcal{O}(\|x\|^{\ell}), \;\; \ell >> 1.$$

  and using the fixed point theorem, it is proven the existence of $K^{>}$ belonging to an appropriate Banach space and satisfying

$$f \circ (K^{\leq} + K^{>}) - (K^{\leq} + K^{>}) \circ R = 0.$$

  - *An approximation result*. An algorithm to compute $K^{\leq}$ and R is provided.

Outline

DEFINITION

INVARIANT MANIFOLDS FOR FLOWS

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with an equilibrium point x_*. We call its flow φ_t. For any neighborhood $N \subset U$ of x_* we define the local stable invariant set

$$W_N^s(x_*) = \{x \in \mathbb{R}^m : \varphi_t(x) \in N, \ \ \forall t \geq 0\} = \bigcap_{t \geq 0} \varphi_{-t}(N)$$

and the local unstable invariant set

$$W_N^u(x_*) = \{x \in \mathbb{R}^m : \varphi_{-t}(x) \in N, \ \ \forall t \geq 0\} = \bigcap_{t \geq 0} \varphi_t(N).$$

As before we have that

$$x \in W_N^s(x_*) \Longrightarrow \lim_{t \to +\infty} \varphi_t(x) = x_*, \qquad x \in W_N^u(x_*) \Longrightarrow \lim_{t \to -\infty} \varphi_t(x) = x_*.$$

OUTLINE

THE RESULT (THE SAME AS FOR MAPS)

THEOREM

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r, $r \geq 1$, vector field having a hyperbolic equilibrium point at $x_* \in U$. Let E^s, E^u be the stable and unstable subspaces of $DX(x_*)$.
Then, on a sufficiently small ball $N \subset U$ of x_*, $W_N^{s,u}(x_*)$ are C^r invariant manifolds satisfying that

$$dim\, W_N^s(x_*) = dim\, E^s, \qquad dim\, W_N^u(x_*) = dim\, E^u$$

and that they are tangent to the linear subspaces $E^{s,u}$ respectively.

In fact, the same for maps happens:

- There exist $\gamma^s : N^s \to E^u$, $\gamma^u : N^u \to E^s$, C^r such that

$$\text{graph}\, \gamma^{s,u} = W_N^{s,u}(x_*).$$

- γ^s is tangent at x_* to E^s and γ^u is tangent at x_* to E^u.

THE PARAMETERIZATION METHOD FOR FLOWS

The idea is the same as for maps. We search the invariant set as the image of a suitable parameterization:

$$W_N^s = \{K(s)\}_s.$$

To find it we have to solve what is called the invariance equation. Let us to explain how we can obtain it:

- Since we ask $K(s)$ to be invariant:

$$\varphi_t(K(s)) = K(s'), \qquad \text{for some } s'.$$

- It is possible to find a new flow ψ such that $s' = \psi_t(s)$?. In this case

$$\varphi_t(K(s)) = K(\psi_t(s)). \tag{2}$$

- Instead to use this invariance equation we try to work with vector fields which are qualitatively easier to find than flows. We call Y the vector field associated to ψ_t.
- Differentiating with respect to t equation (2):

$$X(\varphi_t(K(s)) = DK(\psi_t(s))Y(\psi_t(s)).$$

- And evaluating to $t = 0$ we obtain the infinitesimal version

$$X(K(s)) = DK(s)Y(s).$$

OUTLINE

THE APPROXIMATED MANIFOLD (I)

We clarify these ideas with a simple example.

- Consider the map

$$f(x,y) = \begin{pmatrix} \lambda x + x^2 + y^2 \\ \mu y + x^2 \end{pmatrix}, \qquad (x,y) \in \mathbb{R}^2, \ \ |\lambda| < 1, |\mu| > 1.$$

  Since the linear part is

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

  it is clear that $E^s = \{y = 0\}$ and $E^u = \{x = 0\}$.

- The idea is to find $K^{\leq}$ and R two polynomials satisfying

$$f \circ K^{\leq}(t) - K^{\leq}(R(t)) = \mathcal{O}(t^N).$$

- On the one hand, the origin is a fixed point, then $K(0) = (0,0)$. On the other hand, the stable manifold is tangent to E^s, then $\partial_t K(0) = (1,0)$.
- Write

$$K^{\leq}(t) = \left(\sum_{k=1}^{N} a_k t^k, \sum_{k=2}^{N} b_k t^k \right), \qquad R(t) = \sum_{l=1}^{N} r_l t^l, \qquad a_1 = 1.$$

THE APPROXIMATED MANIFOLD (II)

- What we need is:

$$\lambda \sum_{k=1}^{N} a_k t^k + \left(\sum_{k=1}^{N} a_k t^k\right)^2 + \left(\sum_{k=2}^{N} b_k t^k\right)^2 = \sum_{k=1}^{N} a_k \left(\sum_{l=1}^{N} r_l t^l\right)^k$$

$$\mu \sum_{k=1}^{N} b_k t^k + \left(\sum_{k=1}^{N} a_k t^k\right)^2 = \sum_{k=2}^{N} b_k \left(\sum_{l=1}^{N} r_l t^l\right)^k .$$

- Let do it only the first terms. For instance, if we look at the terms of order $\mathcal{O}(t)$:

$$\lambda a_1 = a_1 r_1 \implies r_1 = \lambda \quad (\text{recall } a_1 = 1).$$

The terms of order $\mathcal{O}(t^2)$ are

$$\lambda a_2 + a_1^2 = a_1 r_2 + a_2 r_1^2$$
$$\mu b_2 + a_1^2 = b_2 r_1^2 .$$

We have freedom. We can choose, for instance $r_2 = 0$ and

$$(\mu - \lambda^2) b_2 = -1, \qquad (\lambda - \lambda^2) a_2 = -1.$$

- Iteratively, we can encounter $r_l = 0$ and

$$(\mu - \lambda^2) b_k = \text{known} , \qquad (\lambda - \lambda^2) a_k = \text{known} .$$

OUTLINE

THE PROBLEM AS A FIXED POINT EQUATION

- We have now an approximated solution

$$f \circ K^{\leq} - K^{\leq} \circ R = \mathcal{O}(t^N), \qquad R(t) = \lambda t. \tag{3}$$

- We look for δK such that $K = K^{\leq} + \delta K$ satisfies

$$f \circ (K^{\leq} + \delta K) = (K^{\leq} + \delta K) \circ R.$$

  Call $\tilde{f} = f - A = (x^2 + y^2, x^2)$.
- We have then

$$AK^{\leq} + A\delta K + \tilde{f}(K^{\leq} + \delta K) = K^{\leq} \circ R + \delta K \circ R.$$

  Reorganizing and using (3)

$$A\delta K - \delta K \circ R = \mathcal{O}(t^N) - \tilde{f}(K^{\leq} + \delta K) + \tilde{f}(K^{\leq}).$$

- The following linear operator is invertible at some Banach space:

$$\mathcal{L}\delta K = A\delta K - \delta K \circ R$$

- The fixed point equation we have to deal with is

$$\delta K = \mathcal{F}\delta K = \mathcal{L}^{-1}\left(\mathcal{O}(t^N) - \tilde{f}(K^{\leq} + \delta K) + \tilde{f}(K^{\leq})\right).$$

  It is checked that the conditions of the fixed point theorem are satisfied.

EXISTENCE RESULT FOR FLOWS

THEOREM

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r, $r \geq 1$ vector field. Assume that X has an equilibrium point x_*. We call $A = DX(x_*)$, $E^{c,u,s}$ the linear subspaces satisfying

$$Spec\, A_{|E^c} \subset \{Re\,\lambda = 0\}, \qquad Spec\, A_{|E^u} \subset \{Re\,\lambda > 0\}, \qquad Spec\, A_{|E^s} \subset \{Re\,\lambda < 0\}$$

and $n_{c,s,u} = dim(E^{c,u,s})$.
Then, there exists a sufficiently small ball $N \subset U$ of x_*, such that

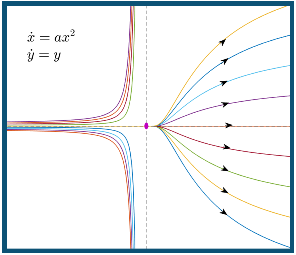
- *There exists a locally invariant C^r manifold W^c_N such that $x_* \in W^c_N$, $\dim W^c_N = \dim E^c$ and it is tangent to E^c at x_*.*
- *There exist unique locally invariant C^r manifolds $W^{u,s}_N$ such that $x_* \in W^{u,s}_N$, $\dim W^{u,s}_N = \dim E^{u,s}$ and it is tangent to $E^{u,s}$ at x_*.*
- *If $r = \infty$, W^c_N is C^k for all k and $W^{u,s}_N$ are C^∞.*

In fact it happens that there exists $\gamma^c : N^c \to E^u \times E^s$ such that

$$\text{graph}\, \gamma^c = W^c_N.$$

Analogously for $W^{u,s}_N$.

COMMENTS (I)



- Center manifold is not unique. Indeed, take $\dot{x} = ax^2$, $\dot{y} = y$. Then $y = e^{-\frac{1}{ax}}C$ and $(0,0)$ is an equilibrium point with $DX(0) = \text{diag}\{0,1\}$ and $E^c = [(1,0)]$.
  Any solution with initial point $x_0 > 0$ and $y = 0$ are tangent to E^c at 0 so that $W^c = \{x > 0\} \cup \{y = 0\}$.
- Dynamics on the W^c_N is unknown without extra hypotheses. It can be attracting, repelling, as a center and almost any behaviour. Indeed, consider

$$\dot{x} = x^2 - \mu^2, \qquad \dot{y} = y + x^2 - \mu^2, \qquad \dot{\mu} = 0.$$

  The center manifold of $(0,0,0)$, W^c, is tangent to $\{y = 0\}$. For $\mu \neq 0$, the equilibrium points $P^{\pm}_{\mu_0} = (\pm\mu_0, 0, \mu_0)$ are not hyperbolic being $E^c = [(1,0,\pm 1)]$.
  Restricting the dynamics to $\mu = \mu_0$, $P^{+}_{\mu_0}$ is an unstable node and $P^{-}_{\mu_0}$ is a saddle when $\mu_0 > 0$.
- We can only guarantee finite differentiability. That is, when the vector field is C^∞, the center manifold is C^k for any order k, but the differentiability domain can (and usually does) depend on k.

SYSTEM $\dot{x} = x^3$, $\dot{y} = 2y - 2x^2$

- The origin is a non hyperbolic equilibrium point with linear part

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

  The center manifolds is tangent to $E^c = [(1,0)]$ at $(0,0)$.
- If the center manifold was analytic, then it would have a convergent Taylor expansion.
- Write the center manifold $y = h(x) = \sum_{i \geq 2} a_i x^i$.
- The invariance condition, $\dot{y} = h'(x)\dot{x}$ is $2h(x) - 2x^2 = h'(x)x^3$:

$$2\sum_{i\geq 2} a_i x^i - 2x^2 = x^3 \sum_{i \geq 2} i a_i x^{i-1} = \sum_{i \geq 4} (i-2) a_{i-2} x^i$$

- Equating same order terms, $a_2 = 1$, $a_3 = 0$ and for $j \geq 4$:

$$2a_j = (j-2)a_{j-2} \Longleftrightarrow a_{2j} = j a_{2j-2}, \quad a_{2j+1} = 0.$$

- However, $a_{2j} = j!$ which does not give a convergent series.
- As a conclusion the center manifold is not analytic, but C^∞.

THE SYSTEM $\dot{x} = \mu x - x^3$, $\dot{y} = y + x^4$, $\dot{\mu} = 0$

- The $(0,0,0)$ has an associated center manifold with *formal expansion*:

$$y = h(x,\mu) = \sum_{i\geq 2, j\geq 0} b_{i,j} x^i \mu^j = \sum_{i\geq 2} a_i(\mu) x^i.$$

- The invariance equation is $(\mu x - x^3)\partial_x h(x,\mu) = h(x,\mu) + x^4$:

$$(\mu x - x^3) \sum_{i\geq 2} i a_i(\mu) x^{i-1} = \sum_{i\geq 2} a_i(\mu) x^i + x^4$$

  or in other words:

$$2\mu a_2(\mu)x^2 + 3\mu a_3(\mu)x^3 + \sum_{j\geq 4}(j\mu a_j(\mu) - (j-2)a_{j-2}(\mu))x^j = \sum_{i\geq 2} a_i(\mu)x^i + x^4.$$

- Equating terms of the same order in x^j, we have that $a_2(\mu) = a_3(\mu) = 0$

$$4\mu a_4(\mu) = a_4(\mu) + 1, \quad a_5(\mu) = 0, \quad j\mu a_j(\mu) - (j-2)a_{j-2}(\mu) = a_j(\mu).$$

  That is:

$$a_4(\mu) = -\frac{1}{1-4\mu}, \quad a_{2j}(\mu) = -(2j-2)\frac{a_{2j-2}(\mu)}{1-2j\mu}, \quad a_{2j+1}(\mu) = 0.$$

- As a conclusion, the center manifold is C^5 if $\mu < 1/4$, C^7 if $\mu < 1/6$ and, in general, C^{2j+1} if $\mu < 1/(2j)$.

AGAIN SYSTEM $\dot{x} = x^2 - \mu^2$, $\dot{y} = y + x^2 - \mu^2$, $\dot{\mu} = 0$

- The set $\{x = \mu,\ y = 0\} \subset W^c = \text{graph} h$, since it is invariant and tangent to $\{y = 0\}$.
- Consider the Taylor expansion of h at $x = \mu$:

$$y(x) = g(x) = \sum_{j=1}^{k} a_j(\mu)(x-\mu)^j + o((x-\mu)^k), \qquad g(\mu) = 0.$$

- We have that $\dot{y} = g'(x)\dot{x}$, that is:

$$\sum_{j=1}^{k} a_j(\mu)(x-\mu)^j + x^2 - \mu^2 = (x^2-\mu^2)\left(\sum_{j=1}^{k} ja_j(\mu_0)(x-\mu)^{j-1}\right) + o((x-\mu)^k).$$

- Skip the dependence of a_j on μ and write $x^2 - \mu^2 = (x-\mu)(2\mu + x - \mu)$:

$$\sum_{j=1}^{k} a_j(x-\mu)^j + (x-\mu)^2 + 2\mu(x-\mu) = \sum_{j=1}^{k} 2\mu ja_j(x-\mu_0)^j + \sum_{j=2}^{k}(j-1)a_{j-1}(x-\mu)^j.$$

- Same order terms are equal, so that $a_1 + 2\mu = 2\mu a_1$, $a_2 + 1 = 4\mu a_2 + a_1$ and for $j \geq 3$:

$$a_j = 2\mu ja_j + (j-1)a_{j-1} \iff a_j = \frac{(j-1)a_{j-1}}{1-2\mu j}.$$

- When $\mu = \dfrac{1}{2m}$, W^c is C^{m-1} but it is not C^m.

EQUIVALENCE RESULT

RESTRICTED DYNAMICS IN AN INVARIANT SET

Take $X(x, y)$ a vector field, $\dot{x} = X_1(x, y)$, $\dot{y} = X_2(x, y)$. On any invariant set $\{y = f(x)\}$, one can consider the *restricted dynamics*:

$$\dot{x} = X_1(x, f(x)), \qquad X = (X_1, X_2).$$

We emphasize that the y-variable is induced by the dynamics on x: $\dot{y} = f'(x)\dot{x}$.

THEOREM

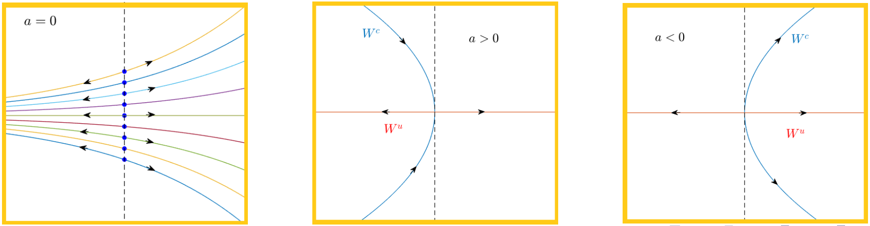
Let X be a C^r, $r \geq 1$ vector field with $W_N^{c,u,s}$ the local invariant manifolds associated to a fixed point with $E^{c,u,s}$ the corresponding subspaces. We write $x = (x_c, x_s, x_u)$ with $(x_c, 0, 0) \in E^c$, $(0, x_s, 0) \in E^s$ and $(0, 0, x_u) \in E^u$. We call $\tilde{X}_c = X_{|W_N^c}$, the restriction to X to W_N^c. Then X is topologically conjugated to

$$\dot{x}_c = \tilde{X}_c(x_c), \qquad \dot{x}_s = -x_s, \qquad \dot{x}_u = x_u.$$

- Since $E^c \oplus E^s \oplus E^u = \mathbb{R}^n$, we indeed can decompose $x = (x_c, x_s, x_u)$.
- W_N^c can be expressed as the graph of $(x_s, x_u) = \gamma^c(x_c)$.
- With these coordinates, $X = (X_c, X_s, X_u)$ and then $\tilde{X}_c(x_c) = X_c(x_c, \gamma^c(x_c))$.
- This result allows to classify non hyperbolic equilibrium points.

SYSTEM $\dot{x} = x + ay^2$, $\dot{y} = xy$

- The origin is an equilibrium point with $E^c = [(0, 1)]$.
- The case $a = 0$ gives $x = 0$ as an equilibrium points line and $y = Ke^x$ the general solution. We assume then $a \neq 0$.
- We compute the Taylor expansion of the center manifold $x = h(y) = cy^2 + \cdots$. Recall that we already know that it is C^k for any k.
- The invariance equation is $cy^2 + ay^2 + \cdots = y(cy^2 + \cdots)(2cy + \cdots)$ and then $c_2 + a = 0$.
- As a consequence $W^c = \{x = -ay^2 + \cdots\}$ and the restricted dynamics is $\dot{y} = -ay^3 + \cdots$
- Then if $a < 0$ the origin is a degenerated node and when $a > 0$, the origin is a degenerated saddle.



THE RESULT FOR MAPS

THEOREM

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r, $r \geq 1$ diffeomorphism. Assume that f has an equilibrium point x_* which can be assumed to be 0. We call $A = Df(0)$, $E^{c,u,s}$ the linear subspaces satisfying

$$Spec\, A_{|E^c} \subset \{|\lambda| = 1\}, \qquad Spec\, A_{|E^u} \subset \{|\lambda| > 1\}, \qquad Spec\, A_{|E^s} \subset \{|\lambda| < 1\}$$

and $n_{c,s,u} = dim(E^{c,u,s})$.
Then there exists a C^r funcion, defined in a neighbourhood $V \subset \mathbb{R}^{n_c}$ of 0, namely, $\gamma^c : V \to \mathbb{R}^{n_u} \times \mathbb{R}^{n_s}$ satisfying:

- *$\gamma^c(0) = 0$ and graph γ^c is tangent to E^c at 0.*
- *graph $\gamma^c \cap U$ is locally invariant, i.e. if $(x_c, \gamma^c(x_c)), f(x_c, \gamma^c(x_c)) \in U$, then $f(x_c, \gamma^c(x_c)) \in$ graph γ^c.*

graph γ^c is called W^c_{loc}, the center manifold.
We also have $W^{u,s}_{loc}$ satisfying the same properties as the ones enunciated in the stable and unstable invariant manifolds theorem.

- Same comments as for the flow case.

POLAR BLOW UP

- Perform *singular* change of variables which expand (make bigger) the non hyperbolic fixed point into a curve with a number of singularities.
- Analize the singularities by using the Harman's theorem.
- Then we go back to the original variables to interpret the analysis.

POLAR BLOW UP

Assume that $X : U \subset \mathbb{R}^2$ can be written in polar coordinates as

$$\dot{r} = r^{k+1} R(r,\theta), \qquad \dot{\theta} = r^k \Theta(r,\theta).$$

The phase curves of the above system as the same as the ones in

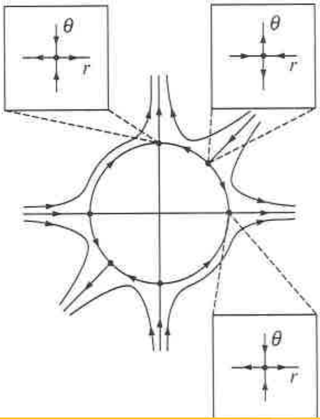
$$\dot{r} = rR(r,\theta), \qquad \dot{\theta} = \Theta(r,\theta)$$

Notice that the origin $(x, y) = 0$ goes to $r = 0$ and that $(r,\theta) \in [0,\infty) \times \mathbb{S}^1$.

AN EXAMPLE

- Consider $\dot{x} = x^2 - 2xy, \qquad \dot{y} = y^2 - 2xy.$
- In polar coordinates $\dot{r} = r^2 R(r,\theta), \qquad \dot{\theta} = r\Theta(r,\theta).$
- Consider $\dot{r} = rR(r,\theta)$, $\dot{\theta} = \Theta(r,\theta)$. This system has singularities at $r = 0$ and $\theta = 0, \pi/4, \pi/2, 3\pi/2, 5\pi/4$.

- We have that:



- Contracting to $r = 0$