LOCAL BIFURCATIONS

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Outline

1. General concepts
   - Preliminary definitions
   - Elementary bifurcations in real vector fields
   - Further definitions
2. Local bifurcations for planar vector fields
   - The Saddle-Node singularity
   - The Hopf bifurcation
   - Cusp bifurcation or Bogdanov Takens bifurcation

Outline

Families of dynamical systems

Families of vector fields

A family of dynamical systems are either a vector field or a diffeomorphism depending on parameters. Namely, $X, f : U \times \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, with $U \times \Lambda$ an open set, belonging to $\mathcal{C}^r(U \times \Lambda)$.
We call $\mu \in \Lambda$ the parameter, which has m components.

- The goal of the bifurcation theory is to study how the qualitative behaviour changes with respect to the parameters.
- As when we study the structural stable property, we can focus on either the global behaviour or the local behaviour around some invariant object.
- Notice that, in the previous lesson, we have encountered different behaviours by means of the splitting of separatrices in one degrees of freedom Hamiltonians.

FAMILIES WE HAVE ALREADY STUDIED

The Lorenz equation

$$\begin{cases} \dot{x} &= 10(y-x) \\ \dot{y} &= \rho x - y - \mu x z \\ \dot{z} &= -\frac{8}{3} z + \mu x y. \end{cases}$$

- When $\rho \in \mathbb{R}$, $\mu = 0$, the system is linear.
- When $\rho = 28$, $\mu = 1$, the system has a chaotic attractor.
- When $\rho < 24.74$, $\mu = 1$ the system has three fixed points, two of them attractors.

The splitting of separatrices of hamiltonian systems:

$$H(x,y,t) = H_0(x,y) + \mu H_1(x,y,t,\mu).$$

- A lot of examples such that when $\mu = 0$, the system has a homoclinic separatrix.
- When $\mu \neq 0$ the separatrix splits and appear transversal homoclinic points.
- As a consequence, the system for $\mu \neq 0$ is chaotic meanwhile for $\mu = 0$ the system is integrable

POINCARÉ SAYS

Bifurcations like torches enlighten the way from simple systems to complicated ones.

What does mean bifurcations?

Bifurcations of dynamical systems

Bifurcations

We say that the family $X(x,\mu)$ (or $f(x,\mu)$) has a bifurcation at $\mu = \mu_*$ if for any $V \subset \Lambda$ neighborhood of μ_* there exists $\mu \in V$ such that $X(x,\mu)$ (or $f(x,\mu)$) exhibits a different qualitative behaviour as $X(x,\mu_*)$. That is:

- Vector fields: $X(x,\mu)$ and $X(x,\mu_*)$ are not topologically equivalent;
- Diffeomorphism: $f(x,\mu)$ and $f(x,\mu_*)$ are not topologically conjugated.

Notice that

- A family can not have a bifurcation at $\mu = \mu_*$ if the system when $\mu = \mu_*$ is structurally stable.
- We focus on the local behaviour and moreover only in the simplest scenario: around a fixed point, which has to be non hyperbolic.
- Remember that if a fixed point is hyperbolic, the system is locally structurally stable.

Bifurcations associated to fixed points

- For $\mu = \mu_*$, assume that the system has a non hyperbolic fixed point x_*.
- We are interested in studying the local behaviour of the family. That is, the behaviour for (x, μ) as close as we want of (x_*, μ_*).
- The bifurcation parameter and the fixed point can be assumed to be $(x_*, \mu_*) = (0, 0)$.
- The concept of local family is then introduced as a family defined in a neighbourhood of $(x, \mu) = (0, 0)$.
- The concept of local bifurcation is introduced as well: it is a bifurcation of a local family. Namely, the systems exhibit different local qualitative behaviours.

From now on ...

We only consider the family defined in a neighbourhood N of $(x, \mu) \sim (0, 0)$. For instance,

$$\dot{x} = \mu x - x^2 - \mu x^3,$$

has three fixed points

$$x_1 = 0, \qquad x_2 = \frac{-1 + \sqrt{1 + 4\mu^2}}{2\mu}, \qquad x_3 = \frac{-1 - \sqrt{1 + 4\mu^2}}{2\mu}$$

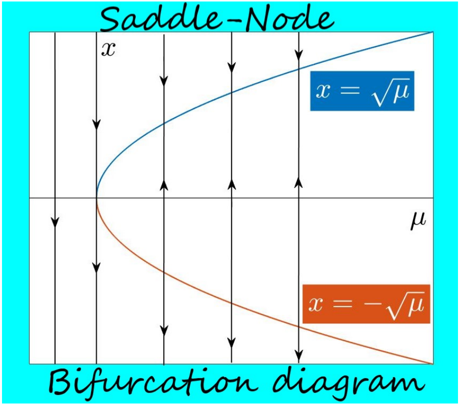
but only x_1, x_2 are close to 0. Therefore x_3 would be discarded of our analysis.

OUTLINE

Saddle-node bifurcation

Consider the following family around $\mu, x \sim 0$

$$\dot{x} = X(x,\mu), \qquad X(x,\mu) = \mu - x^2, \qquad \mu, x \in \mathbb{R}$$

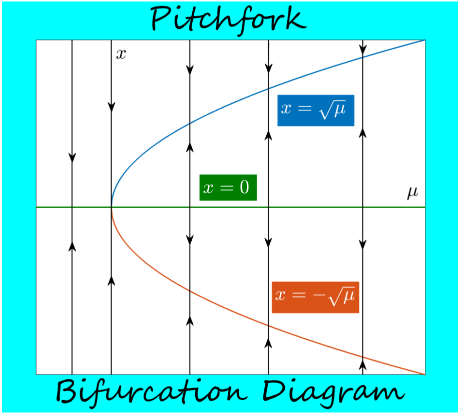


- If $\mu < 0$, there are not fixed points.
- If $\mu = 0$, $x = 0$ is the unique fixed point. It is non hyperbolic and is neither attractor nor repeller.
- If $\mu > 0$, there are two fixed points $x_{\pm} = \pm\sqrt{\mu}$. In addition, x_+ is an attractor and x_- is a repeller.
- For any μ the phase space (for x) is $\mathbb{R}$.
- In the figure, is represented the phase portrait.
- We say that we have a bifurcation at $\mu = 0$.

Pitchfork bifurcation

Consider the family

$$\dot{x} = \mu x - x^3, \qquad \mu, x \sim 0$$

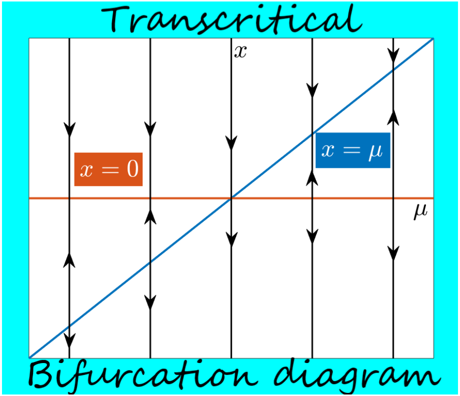


- The point $x = 0$ is always a fixed point.
- If $\mu < 0$, $x = 0$ is the unique fixed point and it is an attractor.
- If $\mu = 0$ $x = 0$ is the unique fixed point and it is an attractor.
- If $\mu > 0$, $x = 0$ is a repeller.
- If $\mu > 0$, there are two more fixed points $x_{\pm} = \pm\sqrt{\mu}$. Both are attractor.
- For any μ the phase space (for x) is $\mathbb{R}$.
- In the figure, is represented the phase portrait.
- We say that we have a bifurcation at $\mu = 0$.

Transcritical bifurcation

Consider the family

$$\dot{x} = \mu x - x^2, \qquad \mu, x \sim 0$$



- The points $x = 0$ and $x = \mu$ are always fixed points.
- If $\mu < 0$, $x = 0$ is an attractor and $x = \mu$ is a repeller.
- If $\mu = 0$ $x = 0$ is the unique fixed point and it is neither attractor nor repeller.
- If $\mu > 0$, $x = 0$ is a repeller and $x = \mu$ is an attractor.
- For any μ the phase space (for x) is $\mathbb{R}$.
- In the figure, is represented the phase portrait.
- We say that we have a bifurcation at $\mu = 0$.

Outline

Unfoldings

- Let $X_0(x)$ (or $f_0(x)$) be a dynamical system having a non-hyperbolic singularity at $x = 0$. We say that X_0 (or f_0) has a singularity at $x = 0$ or (shorter) we say that X_0 (or $f_0(x)$) is a singularity. For instance take $X_0(x) = -x^2$.

Unfoldings

An unfolding of X_0 (or f_0) is a local family $X, f : N \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ such that $X(x, 0) = X_0(x)$ (or $f(x, 0) = f_0(x)$) and it has a bifurcation at $(x, \mu) = (0, 0)$. Sometimes we will write $X_\mu(x) = X(x, \mu)$ ($f_\mu(x) = f(x, \mu)$).

- For instance if $X_0(x) = -x^2$, $X_\mu(x) = (-1 + \mu)x^2$ is not an unfolding. However, $X_\mu(x) = \mu - x^2$ is an unfolding as well as $Y_\mu(x) = \mu x - x^2$.

Equivalent and induced families

- Let $Y, g : N \subset \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $X, f : N \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be unfoldings of X_0, f_0 respectively.

Equivalent families

X, Y (or f, g) are said to be equivalent if $m = k$ and for any μ small enough, X_μ, Y_μ (f_μ, g_μ) are topologically equivalent (topologically conjugated) by means of a continuous map $h(x, \mu)$.

- For instance the families $X(x, \mu) = \mu x - x^2$ and $Y(y, \mu) = \frac{\mu^2}{4} - y^2$ are equivalent.

  Indeed, consider $y = h(x, \mu) = x - \frac{\mu}{2}$ then

$$\dot{y} = \dot{x} = \mu x - x^2 = \mu\left(y + \frac{\mu}{2}\right) - \left(y + \frac{\mu}{2}\right)^2 = \frac{\mu^2}{4} - y^2.$$

Induced families

We say that X (f) is induced by Y (g) if $X(x, \mu) = Y(x, \varphi(\mu))$ $(f(x, \mu) = g(x, \varphi(\mu)))$ with $\varphi : V \subset \mathbb{R}^m \to \mathbb{R}^k$ a continuous map.

- The family $Y(y, \mu) = \frac{\mu^2}{4} - y^2$ is induced by $Z(z, \nu) = \nu - z^2$ by the map $\varphi(\nu) = \nu^2/4$.

VERSAL UNFOLDINGS

VERSAL UNFOLDINGS

We say that the family $X(x,\mu)$ ($f(x,\mu)$) is a versal unfolding of the singularity X_0 (f_0) if every unfolding of $Y(y,\nu)$ ($g(x,\mu)$) of X_0 (f_0) is equivalent to an induced by X (f) family. That is, there exists φ such that $X(x,\varphi(\nu))$ ($f(x,\varphi(\nu))$) is equivalent to $Y(y,\nu)$ ($g(x,\nu)$).

- $X(x,\nu) = \nu - x^2$ is a versal unfolding of $X_0(x) = -x^2$.

We can prove the result above by using the Malgrange preparation theorem:

THEOREM

Let $U \times \Lambda \subset \mathbb{R} \times \mathbb{R}^m$ be an open neighbourhood of the origin and $F : U \times \Lambda \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Assume that

$$F(x,0) = x^k g(x), \qquad g(0) \neq 0, \text{ with } \qquad g \in \mathcal{C}^\infty(U).$$

Then there exists $q(x,\mu)$ a $\mathcal{C}^\infty$ function at $(0,0)$ and functions $s_0(\mu), \cdots, s_{k-1}(\mu)$ which are C^∞ at $\mu = 0$ such that

$$q(x,\mu)F(x,\mu) = x^k + \sum_{i=0}^{k-1} s_i(\mu)x^i, \qquad s_i(0) = 0.$$

$X(x,\nu)=\nu-x^2$, versal unfolding of $X_0(x)=-x^2$

- Take $Y(y,\mu)$, $\mu\in\mathbb{R}^m$ an unfolding of X_0 ($Y(y,0)=-y^2$). By Malgrange preparation theorem:

$$Y(y,\mu)=\frac{1}{q(y,\mu)}(y^2+s_0(\mu)+s_1(\mu)y)\Longrightarrow -q(y,0)y^2=y^2+s_0(0)+s_1(0)y$$

- Since $s_0(0)=s_1(0)=0$ and q is $\mathcal{C}^\infty$, $q(0,0)=-1$ and therefore, $q(y,\mu)<0$.
- As a consequence the family $Y(x,\mu)$ is topologically equivalent (and the homeomorphism is the identitly) to

$$\widetilde{Y}(y,\mu)=-y^2-s_0(\mu)-s_1(\mu)y.$$

- Since

$$\widetilde{Y}(y,\mu)=-s_0(\mu)+\frac{s_1(\mu)^2}{4}-\left(y+\frac{s_1(\mu)}{2}\right)^2,$$

taking

$$x=y+\frac{s_1(\mu)}{2},\qquad \varphi(\mu)=-s_0(\mu)+\frac{s_1(\mu)^2}{4},$$

we conclude that $\widetilde{Y}(y,\mu)$ is equivalent to $X(x,\varphi(\mu))$ and as a consequence, $X(x,\nu)$ is a versal unfolding.
- Notice that also $\hat{X}(x,\eta)=\eta_0+\eta_1x-x^2$ is a versal unfolding of X_0, but X has less parameters!

Do exercise 164,165

What about unfoldings of $X_0(x) = -x^k$?

- Let $Y(y,\mu)$ $\mu \in \mathbb{R}^m$ an unfolding of $X_0(x) = -x^k$. Using the Malgrange preparation theorem

$$Y(y,\mu) = \frac{1}{q(y,\mu)}(y^k + s_0(\mu) + \cdots + s_{k-1}(\mu)y^{k-1}) \Longrightarrow q(0,0) = -1.$$

- Therefore, Y is equivalent to $\overline{Y}(y,\mu) = -y^k - s_0(\mu) - \cdots - s_{k-1}(\mu)y^{k-1}$, which is induced by

$$\widetilde{Y}(y,\eta) = -y^k + \eta_0 + \eta_1 y + \cdots + \eta_{k-1}y^{k-1}, \qquad \eta_i = -s_i(\mu).$$

- Performing the change of variables

$$y = x + \frac{\eta_{k-1}}{k}$$

we obtain that $\widetilde{Y}$ is equivalent to

$$\hat{X}(x,\eta) = -\, x^k + \eta_0 + f_0(\eta_1,\cdots,\eta_{k-1}) + x[\eta_1 + f_0(\eta_2,\cdots,\eta_{k-1})] + \cdots$$
$$+\, x^{k-2}[\eta_{k-2} + f_{k-2}(\eta_{k-1})]$$

for some $\mathcal{C}^\infty$ functions f_i (which can be explicitly computed).

- Take $\nu_i = \eta_i + f_i(\eta_{i+1},\cdots,\eta_{k-1})$.
- We conclude that $X(x,\nu) = -x^k + \nu_0 + \cdots + \nu_{k-2}x^{k-2}$ is a versal unfolding of X_0.

The unfoldings $X(x,\eta) = \eta_1 x + \eta_2 x^2 - x^3$

$X(x,\eta)$

The local family $X(x,\eta)$ is a versal unfolding of the singularity $X_0(x) = -x^3$.

- We need to check that if $Y(y,\mu)$ is a versal unfolding of X_0, it is equivalent to an induced by X family.
- As we have seen, Y is equivalent to a family induced by

$$\hat{Y}(y,\nu) = -y^3 + \nu_0 + \nu_1 y.$$

- Write $\nu = (\nu_0, \nu_1)$. Since $\hat{Y}$ is a odd polynomial, it has at least one real zero $\alpha(\nu)$ which depends continuously on ν at $\nu = 0$. Then

$$\hat{Y}(y,\nu) = -(y - \alpha(\nu))(y^2 + \alpha(\nu)y - \nu_1 + \alpha^2(\nu))$$

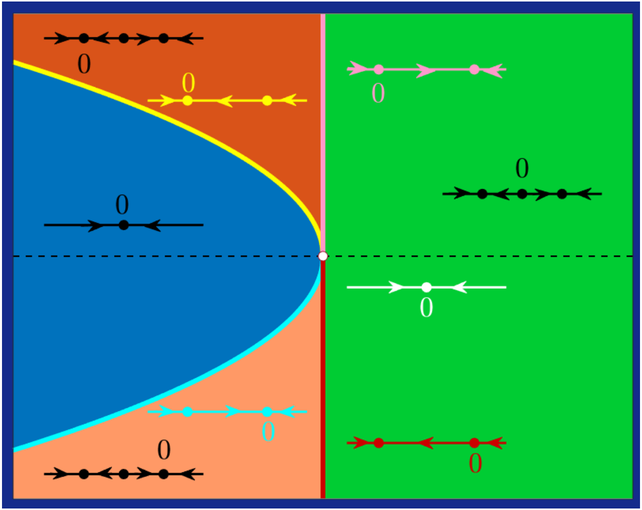
and $x = y - \alpha(\nu)$ gives

$$\hat{X}(x,\nu) = -x(x^2 + 3\alpha(\nu)x + 3\alpha^2(\nu) - \nu_1)$$

- The family $\hat{X}$ is induced by $X(x,\eta)$ taking

$$\eta_1 = -3\alpha(\nu)^2 + \nu_1, \qquad \eta_2 = -3\alpha(\nu)$$

Bifurcation diagram of $X(x,\eta) = \eta_1 x + \eta_2 x^2 - x^3$



- Four regions in the parameter space (η_1, η_2).
- Their phase portrait in black located at each region.
- Curves in yellow and blue are

$$\eta_2 = \sqrt{-4\eta_1},$$
$$\eta_2 = -\sqrt{-4\eta_1}.$$

- The phase portrait in the boundaries are the ones of the same color.

Mini versal unfoldings (I)

Definition

If a versal unfolding has the minimum number of parameters we say that it is a mini versal unfolding.

- A mini versal unfolding of $X_0(x) = -x^k$ has to have $k-1$ parameters.
- In fact, $X(x,\nu) = -x^k + \nu_0 + \cdots + \nu_{k-2}x^{k-2}$ is a mini versal unfolding of X_0.
- Indeed, by Malgrange preparation theorem, if $Y(y,\mu)$, $\mu \in \mathbb{R}^m$ is a versal unfolding of X_0 is equivalent to

$$Y(y,\mu) = -y^k + s_{k-1}(\mu)y^{k-1} + \cdots + s_1(\mu)y + s_0(\mu)$$

and, in fact, it is equivalent to

$$\hat{X}(z,\mu) = -z^k + \hat{s}_{k-2}(\mu)z^{k-2} + \cdots + \hat{s}_1(\mu)z + \hat{s}_0(\mu), \qquad y = z + \frac{s_{k-1}(\mu)}{k}.$$

- We need to assure that

$$\forall(\nu_0, \cdots, \nu_{k-2}) \sim 0,\ \exists\mu \text{ such that } (\nu_0, \cdots, \nu_{k-2}) = (\hat{s}_0(\mu), \hat{s}_1(\mu), \cdots, \hat{s}_{k-2}(\mu)).$$

For that reason, we need $\mu \in \mathbb{R}^{k-1}$.

INTRODUCTION

- We study the versal unfoldings of the planar singularities X_0 such that

$$X_0(0) = 0, \qquad \det DX_0(0) = 0.$$

- Assume that $DX_0(0)$ is in Jordan form.
- The first case is that, for $\lambda \neq 0$:

$$DX_0(0) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{Saddle-Node singularity.}$$

- The second is two conjugated complex eigenvalues:

$$DX_0(0) = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \qquad \text{Hopf singularity.}$$

- The third is two eigenvalues 0 but $DX_0(0) \neq 0$:

$$DX_0(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{cusp singularity.}$$

- The last (non studied) one is $DX_0(0) = 0$.

CODIMENSION NOTION

We say that the singularity has codimension ℓ if a mini versal unfolding of it has ℓ independent parameters.

OUTLINE

THE SADDLE-NODE SINGULARITY

Let $X_0(x,y)$ be a singularity

$$X_0(x,y) = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} + \mathcal{O}(\|(x,y)\|^2).$$

- The normal form is (we use the same notation)

$$X_0(x,y) = \begin{pmatrix} x(\lambda + ay) \\ by^2 \end{pmatrix} + \mathcal{O}(\|(x,y)\|^3).$$

- The general case is when $b \neq 0$. That is, this is the less degenerated case.

PROPOSITION

If $b \neq 0$, the local family

$$X(x,y,\nu) = \begin{pmatrix} x(\lambda + ay) \\ \nu + by^2 \end{pmatrix} + \mathcal{O}(\|(x,y)\|^3)$$

is a versal unfolding of the saddle-node singularity X_0. The $\mathcal{O}(\|(x,y)\|^3)$ terms does not depend on ν.
As a consequence, the saddle-node singularity has codimension 1.

VERSAL UNFOLDINGS OF THE SADDLE-NODE SINGULARITY (I)

Sketch of the proof

- Let $Y(z,\mu)$, $z \in \mathbb{R}^2$ be an unfolding of the saddle-node singularity. Consider the vector field

$$\dot{z} = Y(z,\mu), \qquad \dot{\mu} = 0.$$

- After normal form procedure, we have that $Y(z,0) = (z_1(\lambda + az_2), bz_2^2) + \mathcal{O}(\|z\|^3)$. Clearly, $(z,\mu) = (0,0)$ is a non-hyperbolic fixed point.
- The central manifold of $(z,\mu) = (0,0)$ is two dimensional and can be expressed as the graph of $z_1 = h(z_2,\mu)$.
- Write $Y(z,\mu) = (Y_1(z,\mu), Y_2(z,\mu))$. The central manifold theorem assures that it is topologically equivalent to

$$\dot{z}_h = \lambda z_h, \qquad \dot{z}_c = Z(\zeta,\mu) := Y_2(h(z_c,\mu), z_c, \mu), \qquad \dot{\mu} = 0.$$

- $Z(\zeta,\mu)$ is an unfolding of $\dot{\zeta} = b\zeta^2$. In addition $\dot{\zeta} = \eta + b\zeta^2$ is a versal unfolding of the singularity $\dot{\zeta} = b\zeta^2$ (the proof is the same as the one for $\dot{\zeta} = -\zeta^2$).
- We then conclude that $Y(z,\mu)$ is equivalent to an induced by

$$\hat{X}(x,y,\eta) = (\lambda x, \eta + by^2), \qquad \eta = \varphi(\mu) \in \mathbb{R}$$

family.

VERSAL UNFOLDINGS OF THE SADDLE-NODE SINGULARITY (II)

- Let $X(x,y,\nu)$ be the family
$$X(x,y,\nu) = X_0(x,y) + (0,\nu)^\top = \left(X_0^1(x,y), X_0^2(x,y)\right) + (0,\nu)^\top.$$
- We also have that $X(x,y,\nu)$ is equivalent to an induced by $\hat{X}(x,y,\eta)$ family. Let $\hat{\varphi}(\nu)=\eta$.
- Recall that the topological equivalent is a transitive equivalence relation, but to prove the result we need to check that $\hat{\varphi}$ is invertible and then $\nu = \hat{\varphi}^{-1}(\varphi(\mu))$ will be the transformation between the parameters we need.
- There is a topological equivalence between X and
$$\tilde{X}(x,y,\nu) = \left(\lambda x, \nu + X_0^2(h(y,\nu),\nu)\right).$$
- After we apply the results for the one dimensional case to assure that $\nu + X_0^2(h(y,\nu),\nu)$ is equivalent to an induced by $\eta + b\xi^2$ family. To do so, using the Malgrange preparation theorem
$$\nu + X_0^2(h(y,\nu),\nu) = q(y,\nu)(y^2 + s_0(\nu) + s_1(\nu)y), \qquad q(y,\nu) = b + \nu\mathcal{O}(\|(y,\nu)\|),\ s_1(0)=0.$$
- Evaluating at $y=0$, $\nu + X_0^2(h(0,\nu),\nu) = q(0,\nu)s_0(\nu)$ and we conclude that
$$s_0(\nu) = b^{-1}\nu + \mathcal{O}(\nu^2).$$
- Finally recall that $\hat{\varphi}(\nu) = -s_0(\nu) + \mathcal{O}(s_1(\nu)^2) = -b^{-1}\nu + \mathcal{O}(\nu^2)$.

THE BIFURCATION DIAGRAM. ANALYSIS

Assume that $b < 0$ and (renaming $-b$ by b), consider the unfolding $X(x, y, \nu)$ defined by

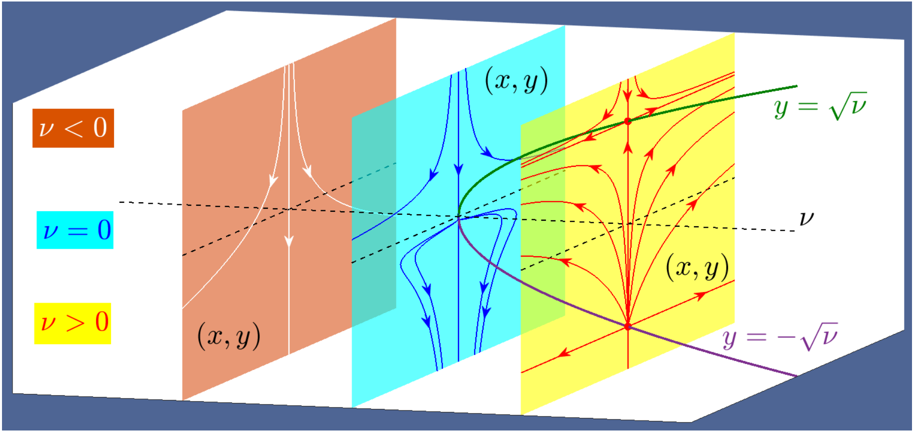
$$\dot{x} = x\lambda + axy, \qquad \dot{y} = \nu - by^2, \qquad b > 0.$$

- For any ν, $x = 0$ is invariant and the dynamics on it is given by the (known) vector field $\dot{y} = \nu - by^2$ (see the saddle-node bifurcation diagram for unidimensional vector fields).
- When $\nu < 0$, the system has no fixed points.
- When $\nu = 0$, the system has only one fixed point at $(0, 0)$.
- When $\nu > 0$, the system has only two fixed points at $p_- = \left(0, -\sqrt{\frac{\nu}{b}}\right)$, $p_+ = \left(0, \sqrt{\frac{\nu}{b}}\right)$.
- We have that

$$DX(p_-) = \begin{pmatrix} \lambda - a\sqrt{\frac{\nu}{b}} & 0 \\ 0 & 2b\sqrt{\frac{\nu}{b}} \end{pmatrix}, \qquad DX(p_+) = \begin{pmatrix} \lambda + a\sqrt{\frac{\nu}{b}} & 0 \\ 0 & -2b\sqrt{\frac{\nu}{b}} \end{pmatrix}.$$

- When $\lambda > 0$, p_- is a repeller node and p_+ is a saddle. Conversely, when $\lambda < 0$, p_- is a saddle and p_+ is an attractor node.
- For any ν, $x = 0$ is the stable (unstable) manifold of p_+ (p_-) when $\lambda > 0$ ($\lambda < 0$).

THE BIFURCATION DIAGRAM. DRAWING



- To do this diagram, we have taken $\lambda > 0$.
- There is no qualitative difference between this diagram and the one corresponding to a local family with $\mathcal{O}(\|(x, y, \mu)\|^3)$.

Unfolding a saddle-node singularity

- The Taylor expansion of $X(x, y, \mu)$ is (after translation and linear change of coordinates):

$$X(x, y, \mu) = \begin{pmatrix} \lambda x + \sum_{j=1}^{m} a_j \mu_j + \sum_{j=1}^{m} \mu_j (b_j x + c_j y) + d_1 x^2 + d_2 xy + d_3 y^2 \\ \sum_{j=1}^{m} \alpha_j \mu_j + \sum_{j=1}^{m} \mu_j (\beta_j x + \gamma_j y) + \delta_1 x^2 + \delta_2 xy + \delta_3 y^2 \end{pmatrix} + R(x, y, \mu),$$

  with $R(x, y, \mu) = \mathcal{O}(\|\mu\|^2) + \mathcal{O}(\|(\mu, x, y)\|^3)$.
- We call $X_2(x, y, \mu)$ the up to order 2 terms of the local family $X(x, y, \mu)$

Proposition

If $\delta_3 \neq 0$ and $\alpha_j \neq 0$ for some $j = 1, \cdots, m$, the local family $X(x, y, \mu)$ is equivalent to an induced by

$$\hat{X}(x, y, \nu) = \begin{pmatrix} x(\lambda + ay) \\ \nu + by^2 \end{pmatrix}$$

family with $a = 0$.
In addition, the local family $X_2(x, y, \mu)$ is differentiably conjugated to a one induced by the $\hat{X}$ family, by allowing $a \neq 0$.

Do exercise 175 for the proof

OUTLINE

THE HOPF SINGULARITY

Let $X_0(x,y)$ be a singularity

$$X_0(x,y) = \begin{pmatrix} -\beta y \\ \beta x \end{pmatrix} + \mathcal{O}(\|(x,y)\|^2).$$

- The normal form is (we use the same notation)

$$X_0(x,y) = \begin{pmatrix} -\beta y \\ \beta x \end{pmatrix} + (x^2+y^2)\left\{ a \begin{pmatrix} x \\ y \end{pmatrix} + b \begin{pmatrix} -y \\ x \end{pmatrix} \right\} + \mathcal{O}(\|(x,y)\|^5).$$

- The general case is when $\beta > 0$ and $a \neq 0$.

PROPOSITION

If $\beta, a \neq 0$, the local family

$$X(x,y,\nu) = \begin{pmatrix} \nu x - \beta y \\ \nu y + \beta x \end{pmatrix} + (x^2+y^2)\left\{ a \begin{pmatrix} x \\ y \end{pmatrix} + b \begin{pmatrix} -y \\ x \end{pmatrix} \right\} + \mathcal{O}(\|(x,y)\|^5).$$

is a versal unfolding of the Hopf singularity. The $\mathcal{O}(\|(x,y)\|^5)$ terms does not depend on ν. As a consequence, the Hopf singularity has codimension 1.

The proof is difficult!

THE BIFURCATION DIAGRAM. ANALYSIS

Assume that $a < 0$ and consider the unfolding $X(x, y, \nu)$ defined by

$$\dot{x} = \nu x - \beta y + (x^2 + y^2)(ax - by), \qquad \dot{y} = \nu y + \beta x + (x^2 + y^2)(ay + bx), \qquad a < 0$$

which in polar coordinates is

$$\dot{r} = r(\nu + ar^2), \qquad \dot{\theta} = \beta + br^2.$$

- For any ν, the system has one fixed point at $(0, 0)$ (recall that $r \sim 0$). In addition

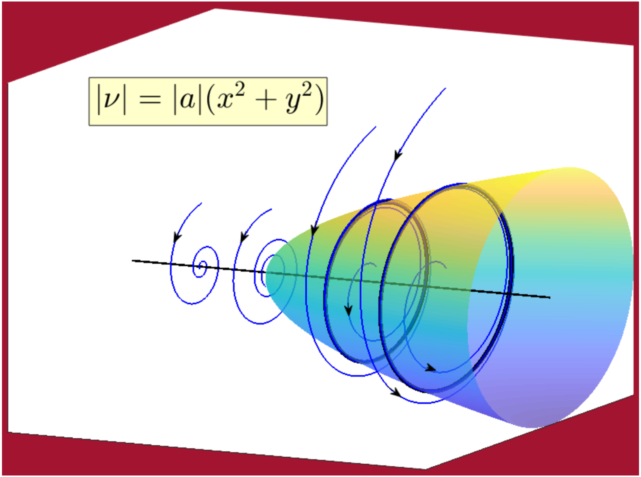
$$DX(0, 0, \nu) = \begin{pmatrix} \nu & -\beta \\ \beta & \nu \end{pmatrix}.$$

- Then, if $\nu > 0$, the origin is a repeller focus and if $\nu < 0$, the origin is an attractor focus.
- When $\nu = 0$, $\dot{r} = ar^3 < 0$, $\dot{\theta} = \beta + br^2 > 0$. Then the origin is an attractor degenerated focus, in particular a non hyperbolic fixed point.
- When $\nu > 0$, the local family has a periodic orbit placed at the circumference of radius $r_\nu = \sqrt{\nu/|a|}$. If $\nu \le 0$ there is no periodic orbits.
- Again take $\nu > 0$. Notice that $\dot{r} = r(\nu + ar^2)$ satisfies

$$\dot{r} > 0 \text{ if } 0 \le r < \sqrt{\frac{\nu}{|a|}}, \qquad \dot{r} < 0 \text{ if } r > \sqrt{\frac{\nu}{|a|}}$$

Then we conclude that the periodic orbit is attracting.

THE BIFURCATION DIAGRAM



UNFOLDING A HOPF SINGULARITY

THEOREM

Let $Y(x,\mu)$, $x \in \mathbb{R}^2$, $\mu \in \mathbb{R}^1$. Assume that $Y(0,\mu) = 0$ for all μ and that the eigenvalues $\lambda_1(\mu), \lambda_2(\mu)$ of $DY(0,\mu)$ are pure imaginary for some value of $\mu = \mu_*$. Assume in addition that

- $\dfrac{d}{d\mu}\mathrm{Re}\,\lambda_1(\mu)_{|\mu=\mu_*} > 0.$
- *The origin ($x = 0$) is an asymptotically stable fixed point when $\mu = \mu_*$.*

Then

1. $\mu = \mu_*$ *is a bifurcation point.*
2. *The origin is a stable focus when $\mu < \mu_*$.*
3. *The origin is a unstable focus surrounded by a stable limit cycle when $\mu > \mu_*$.*

Remarks

- Since we are working with local families, the values of μ are close to μ_*.
- If the local family $Z(x,\eta)$ has as parameter $\eta \in \mathbb{R}^m$, we can consider

$$Y(x,\mu) = Z(x, \eta_1^*, \cdots, \eta_i^*, \mu, \eta_{i+2}^*, \cdots, \eta_m^*), \qquad \eta_j^* \text{ given.}$$

MORE REMARKS

- The fact that $\frac{d}{d\mu}\mathrm{Re}\,\lambda_1(\mu)_{|\mu=\mu_*} > 0$ assures the change of stability of the origin.
- A sufficient condition for the origin to be asymptotically stable is $a < 0$, but it turns out to be non-necessary. For instance, the family written in polar coordinates as:

$$\dot{r} = r(\nu + \hat{a}r^4), \qquad \dot{\theta} = \beta + br^2$$

  satisfies the conditions but the corresponding $a = 0$ (the coefficient of r^2 in $\dot{r}$).
- To prove that the origin is asymptotically stable, is the most difficult hypothesis. One can use either Lyapunov functions or perform the normal form procedure to compute a and check if $a < 0$ or not.
- This coefficient a can be computed by means of the third derivatives of the local families at $(x, \mu) = (0, \mu_*)$ (see the course book or Guckenheimer and Holmes *Nonlinear Oscillations, Dynamical Systems and Bifurcations of vector fields*, page 152...)
- To prove the result, use the normal form theorem and the central manifold theorem.

A DIFFERENT POINT OF VIEW

THEOREM

Let $\dot{x} = X(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^1$ has an equilibrium at $(0, 0)$. Assume that

- *The central part of $DX(0, 0)$ is a simple pair of pure imaginary eigenvalues.*

Let $x(\mu)$ be the equilibrium $X(x(\mu), \mu)$ arising from 0 by using the implicit function theorem. Denote by $\lambda(\mu), \overline{\lambda(\mu)}$ the imaginary eigenvalues of $DX(x(\mu), \mu)$. Assume

- $\dfrac{d}{d\mu}\mathrm{Re}\,\lambda(\mu)_{|\mu=0} = d \neq 0.$

Then, by using differentiable changes of variables, the Taylor expansion of order 3 of $X(x, \mu)$ is given by (in polar coordinates)

$$\dot{r} = r(d\mu + ar^2), \qquad \dot{\theta} = \beta + c\mu + br^2.$$

If $a \neq 0$, there is a surface of periodic solutions in the center manifold. If $a < 0$ these periodic orbits are stable, while if $a > 0$, the periodic orbits are repelling.

Hint of the proof. Use the normal form theorem and the central manifold theorem.

OUTLINE

THE CUSP SINGULARITY

Let $X_0(x, y)$ be a singularity of the form

$$X_0(x, y) = \begin{pmatrix} y \\ 0 \end{pmatrix} + \mathcal{O}(\|(x, y)\|^2).$$

- The normal form is (we use the same notation)

$$X_0(x, y) = \begin{pmatrix} y + ax^2 \\ bx^2 \end{pmatrix} + \mathcal{O}(\|(x, y)\|^3).$$

- The general case is when $a, b \neq 0$. That is, this is the less degenerated case.

PROPOSITION

If $a, b \neq 0$, the local family

$$X(x, y, \nu) = \begin{pmatrix} y + \nu_2 x + ax^2 \\ \nu_1 + bx^2 \end{pmatrix} + \mathcal{O}(\|(x, y)\|^3)$$

is a versal unfolding of the cusp singularity.
As a consequence, the cusp singularity has codimension 2.

The proof is difficult!

SOME REMARKS

- Recall that the normal form of the singularity $X_0(x, y)$ is not unique.
- In fact, the most part of the work with this singularity is due to Bogdanov who use another alternative normal form.
- He prove that any two-parameter unfolding of a cusp singularity is equivalent to an induced by:

$$\dot{x} = y, \qquad \dot{y} = \eta_1 + \eta_2 x + x^2 \pm xy.$$

  unfolding.
- However we will do the analysis of the cusp bifurcation by taking the family

$$X_2(x, y, \nu) = \begin{pmatrix} y + \nu_2 x + ax^2 \\ \nu_1 + bx^2 \end{pmatrix}.$$

  That is the terms up to order two of X.
- In addition we take $a < 0$ and $b > 0$.
- Note that scaling variables $(u, v) = (bx, by)$ and renaming the parameter $\eta_1 = b\nu_1$ and the constant $\alpha = -a/b > 0$, the system becomes

$$X_2(u, v, \eta) = \begin{pmatrix} v + \eta_2 u - \alpha u^2 \\ \eta_1 + u^2 \end{pmatrix}.$$

  That is $b = 1$.
- As usual rename u, v, α and η by x, y, a and ν.

THE BIFURCATION DIAGRAM (I)

- There are no fixed point if $\nu_1 > 0$.
- When $\nu_1 = 0$, there is only one fixed point $(0,0)$.
- When $\nu_1 < 0$, there are two fixed points

$$p_\pm = (x_\pm, (-\nu_2 + a x_\pm) x_\pm), \qquad x_\pm = \pm\sqrt{|\nu_1|}.$$

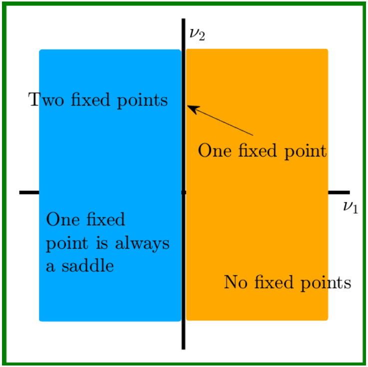
- For $\nu_1 < 0$, the linearized system

$$A_\pm := DX_2(p_\pm, \nu) = \begin{pmatrix} \nu_2 - 2a x_\pm & 1 \\ 2x_\pm & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1 = \frac{\operatorname{tr} A_\pm + \sqrt{(\operatorname{tr} A_\pm)^2 - 4 \det A_\pm}}{2}, \qquad \lambda_2 = \frac{\operatorname{tr} A_\pm - \sqrt{(\operatorname{tr} A_\pm)^2 - 4 \det A_\pm}}{2}.$$

- Since $\det A_+ = -2x_+ < 0$, x_+ is always a saddle.



The bifurcation diagram (II)

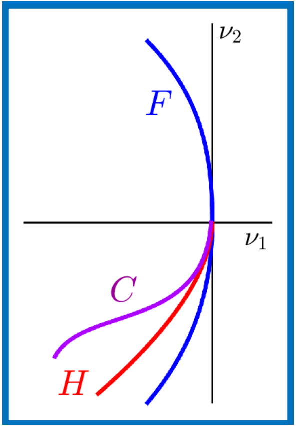
- The character of x_- changes. The eigenvalues are

$$\lambda_{1,2} = \frac{\nu_2 + 2a\sqrt{|\nu_1|} \pm \sqrt{(\nu_2 + 2a\sqrt{|\nu_1|})^2 - 8\sqrt{|\nu_1|}}}{2}$$

- Consider the curves

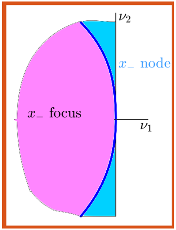
$$F = \{(\nu_2 + 2a\sqrt{|\nu_1|})^2 = 8\sqrt{|\nu_1|}\},$$
$$H = \{\nu_2 + 2a\sqrt{|\nu_1|} = 0\}$$

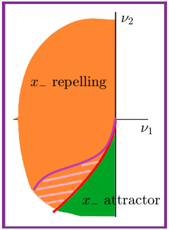
- Then if $(\nu_2 + 2a\sqrt{|\nu_1|})^2 \geq 8\sqrt{|\nu_1|}$, x_- is an attractor or repelling node depending on the sign of $\nu_2 + 2a\sqrt{|\nu_1|}$.



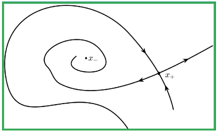
- For any $\nu_2 \neq 0$ constant, the system has a saddle-node bifurcation at $(x, y, \nu_1) = (0, 0, 0)$.
- If $(\nu_2 + 2a\sqrt{|\nu_1|})^2 < 8\sqrt{|\nu_1|}$, x_- is a focus.
- If $(\nu_1, \nu_2) \in H$, the eigenvalues are $\lambda_{1,2} = \pm i\sqrt{2}|\nu_1|^{1/4}$.
- One can check that crossing transversally H one has a Hopf bifurcation.

THE BIFURCATION DIAGRAM (III)

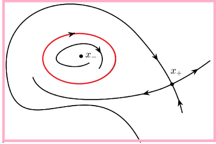




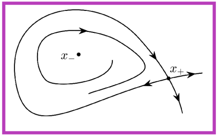
- Taking values of ν in different regions we get different qualitative behaviours.
- The blue line is not a bifurcation line!
- Look the homoclinic connection that appears when the curve C is crossing!



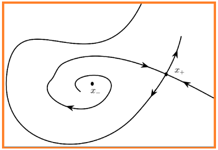
x_- attracting focus



crossing the Hopf bifurcation



on the homoclinic connection



x_- repelling focus