Coorbital homoclinic and chaotic dynamics in the Restricted 3-Body Problem

Inmaculada Baldomá[1,4], Mar Giralt*[2], and Marcel Guardia[3,4]

[1]Departament de Matemàtiques & IMTECH, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
[2]IMCCE, CNRS, Observatoire de Paris, Université PSL, Sorbonne Université, 77 Avenue Denfert-Rochereau, 75014 Paris, France
[3]Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain
[4]Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193 Barcelona, Spain

December 22, 2023

Abstract

The description of unstable motions in the Restricted Planar Circular 3-Body Problem, modeling the dynamics of a Sun-Planet-Asteriod system, is one of the fundamental problems in Celestial Mechanics. The goal of this paper is to analyze homoclinic and instability phenomena at coorbital motions, that is when the negligible mass Asteroid is at $1:1$ mean motion resonance with the Planet (i.e. nearly equal periods) and performs close to circular motions. Several bodies in our Solar system belong to such regimes.

In this paper, we obtain the following results. First, we prove that, for a sequence of ratios between the masses of the Planet and the Sun going to 0, there exist a 2-round homoclinic orbit to the Lagrange point L_3, i.e. homoclinic orbits that approach the critical point twice. Second, we construct chaotic motions (hyperbolic sets with symbolic dynamics) as a consequence of the existence of transverse homoclinic orbits to Lyapunov periodic orbits associated to L_3. Finally, we prove that the RPC3BP possesses Newhouse domains by proving that the energy level unfolds generically a quadratic homoclinic tangency to a periodic orbit.

*Corresponding author.
E-mail adresses: immaculada.baldoma@upc.edu (I. Baldomá), mar.giralt@obspm.fr (M. Giralt), guardia@ub.edu (M. Guardia).

Contents

1 Introduction

One of the oldest questions in dynamical systems is to assert whether the Solar System is stable. More precisely, consider the N body problem, that is the motion of N punctual masses under Newtonian gravitational force in the planetary regime (one massive body, the Sun, and $N-1$ light bodies, the planets). For this model, one would like to determine whether the orbits of the planets stay close to ellipses over long time scales or undergo strong deviations due to the mutual gravitational interaction between planets.

The Arnold-Herman-Féjoz Theorem ensures that there is a positive measure set of stable motions lying on quasiperiodic invariant tori, see [Arn63, Rob95, F0́4, CP11]. However, the "gaps" left by the invariant tori in the phase space leave room for instability. M. Herman, in his ICM lecture [Her98], referred to this problem as "the oldest problem in dynamical systems" and, related to it, he posed the following conjecture. Consider the N-body problem in space, with $N \geq 3$ and assume that the center of mass is fixed at the origin and that, on the energy surface of level e, the flow is $\mathcal{C}^\infty$-reparametrized such that the collisions now occur only in infinite time.

Conjecture 1.1. *Is for every e the non-wandering set of the Hamiltonian flow of H_e on $H_e^{-1}(0)$ nowhere dense in $H_e^{-1}(0)$?*

Note that this conjecture is not restricted to the planetary regime but is formulated for any value of the masses of the bodies. This conjecture is nowadays wide open. Even results proving unstable motions in Celestial Mechanics models are rather scarce. Most of these results deal with nearly integrable settings, either the planetary regime or the hierarchical regime (when bodies are increasingly separated) and are tipically of two different types: chaotic motions (i.e. existence of Smale horsehoes) or Arnold diffusion (see Section 1.4 for references).

One of the main sources of instabilities are resonances where, typically, hyperbolic invariant objects with invariant manifolds appear. These invariant manifolds structure the global dynamics and act as "highways" for the unstable motions. Among these resonances, mean motion resonances play a fundamental role in the global dynamics of the Solar System (see, for instance, [Mor02, FGKR16]). They appear when two (or more) bodies have rationally dependent periods.

The aim of this article is to study instability phenomena and how (some) invariant manifolds structure the global dynamics at the 1 : 1 mean motion resonance nearly circular orbits. Such region of the phase space is usually called *coorbital motions*, since two of the bodies, at short time scales, perform approximately the same circular orbit. Many bodies in our Solar System (satellites, asteroids) belong to this region. We focus on the simplest model where such dynamics arise, that is the Restricted Planar Circular 3-Body Problem (RPC3BP).

The Restricted Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform a circular motion. If one also assumes that the massless body moves on the same plane as the primaries one has the RPC3BP.

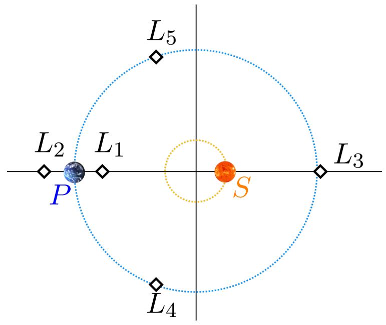


Figure 1: Projection onto the q-plane of the Lagrange equilibrium points for the RPC3BP on rotating coordinates.

Let us name the two primaries S (star) and P (planet) and normalize their masses so that $m_S = 1-\mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$. Choosing a suitable rotating coordinate system, the positions of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$ and then, the position and momenta of the third body, $(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the two degrees of freedom Hamiltonian

$$h(q,p;\mu) = h_0(q,p) + \mu h_1(q;\mu) \tag{1.1}$$

where

$$\begin{aligned} h_0(q,p) &= \frac{||p||^2}{2} - q^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{||q||}, \\ \mu h_1(q;\mu) &= \frac{1}{||q||} - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}. \end{aligned} \tag{1.2}$$

This Hamiltonian is autonomous and the conservation of h corresponds to the preservation of the classical Jacobi constant.

We analyze this model for $\mu > 0$ small enough at coorbital motions. That is, when the orbit of the third body is close to the orbit of the Planet. It is a well known fact that (1.1) has five critical points, usually called Lagrange points, which, for $\mu > 0$ small enough, lie at the coorbital motions region, (see Figure 1). The three collinear Lagrange points, L_1, L_2 and L_3, are of center-saddle type whereas, for small μ, the triangular ones, L_4 and L_5, are of center-center type (see, for instance, [Sze67]).

There is numerical evidence that the invariant manifolds of L_3 play a fundamental role in structuring the global dynamics at coorbital motions. Indeed, its center-stable and center-unstable invariant manifolds act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [GJMS01, SSST13]). The invariant manifolds of L_3 are also relevant in creating transfer orbits from the small primary to L_3 in the RPC3BP (see [HTL07, TFR+10]) or between primaries in the Bicircular 4-Body Problem (see [JN20, JN21]).

Over the past years, one of the main focus of the study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits", first analyzed in [Bro11], which are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5. The interest on these types of orbits arises when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus and near Earth asteroids. Recently, in [NPR20], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus (see also [CH03, BFPC13, CPY19], and [DM81a, DM81b, LO01, BM05, BO06] for numerical studies).

The mentioned results deal with stable coorbital motions. On the contrary, the purpose of this paper is to prove that unstable motions and homoclinic orbits coexist with them. In particular,

1. We construct homoclinic orbits to L_3 for a sequence of mass ratios μ tending to zero. The papers [BGG22, BGG23, BCGG23] prove that the 1-dimensional stable and unstable manifolds of L_3 do not meet the first time they intersect a given transverse section for μ small enough. However, we prove that, for the sequence values of μ, they do meet the second time they hit the section (see Theorem B in Section 1.1 below).

2. We prove the existence of Smale horseshoes, that is of hyperbolic invariant sets with symbolic dynamics. This is a consequence of the existence of transverse homoclinic orbits to certain Lyapunov periodic orbits which lie on the center manifold of L_3. See Theorem C in Section 1.2.

3. We construct Newhouse domains for the RPC3BP by showing that there exist a Lyapunov periodic orbit around L_3 with a quadratic homoclinic tangency which unfolds generically with respect to the energy (see Theorem D). This leads to the existence of hyperbolic sets with Hausdorff dimension arbitrarily close to maximal and to the existence of an infinite number of elliptic islands (see Theorem E in Section 1.3).

A key point to obtain these results is an asymptotic formula for the distance between the 1-dimensional stable and unstable invariant manifolds of the point L_3 (at a first crossing with a suitable transverse section) for $\mu > 0$ small enough which was proven by the authors in [BGG22, BGG23, BCGG23] (see Theorem A below).

Together with the already mentioned KAM results provided in [NPR20], the results presented in this paper show mixed dynamics at coorbital motions. In other words, the coexistence of stable motions (KAM regime) and unstable motions. As far as the authors know this is one of the first papers to build Newhouse domains in Celestial Mechanics (see [GK12]). See Section 1.4 for a brief discussion on related previous results.

1.1 Homoclinic connections to L_3

The critical point L_3 and its eigenvalues satisfy that, as $\mu \to 0$,

$$(q_1, q_2, p_1, p_2) = (d_\mu, 0, 0, d_\mu), \qquad \text{with} \quad d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3) \tag{1.3}$$

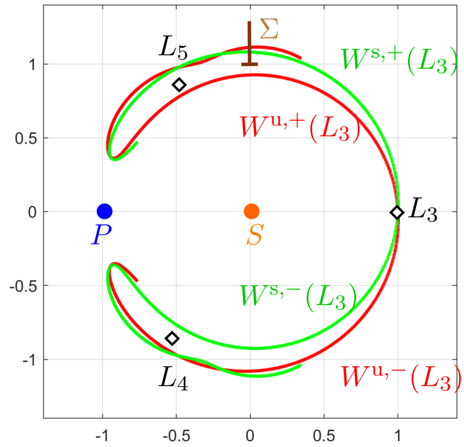


Figure 2: Projection onto the q-plane of the unstable (red) and stable (green) manifolds of L_3, for $\mu = 0.0028$.

and

$$\text{Spec} = \{\pm\sqrt{\mu}\,\rho_{\text{eig}}(\mu), \pm i\,\omega_{\text{eig}}(\mu)\}\,, \quad \text{with} \quad \begin{cases} \rho_{\text{eig}}(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega_{\text{eig}}(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2), \end{cases} \tag{1.4}$$

(see [Sze67]). Therefore, L_3 possesses one-dimensional unstable and stable manifolds, which we denote as $W^{\text{u}}(L_3)$ and $W^{\text{s}}(L_3)$. Notice that, due to the different size in the eigenvalues, the system possesses two time scales which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3.

The manifolds $W^{\text{u}}(L_3)$ and $W^{\text{s}}(L_3)$ have two branches each. One pair, which we denote by $W^{\text{u},+}(L_3)$ and $W^{\text{s},+}(L_3)$ circumvents L_5 whereas the other circumvents L_4 and it is denoted as $W^{\text{u},-}(L_3)$ and $W^{\text{s},-}(L_3)$, see Figure 2. Notice that the Hamiltonian system associated to h in (1.1) is reversible with respect to the involution

$$\Psi(q, p) = (q_1, -q_2, -p_1, p_2). \tag{1.5}$$

Therefore, by (1.3), L_3 belongs to the symmetry axis given by Ψ and the $+$ branches of the invariant manifolds of L_3 are symmetric to the $-$ ones.

In the papers [BGG22, BGG23, BCGG23], the authors provide an asymptotic formula for the distance between the 1-dimensional stable and unstable manifolds of L_3 at a transverse section. To present this formula, we introduce the classical symplectic polar coordinates

$$q = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \qquad p = R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} - \frac{G}{r}\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}, \tag{1.6}$$

where R is the radial linear momentum and G is the angular momentum. We consider as well the 3-dimensional section

$$\Sigma = \left\{(r, \theta, R, G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2 \;:\; r > 1,\, \theta = \frac{\pi}{2}\right\} \tag{1.7}$$

and denote by $(r_*^{\mathrm{u}}, \frac{\pi}{2}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}})$ and $(r_*^{\mathrm{s}}, \frac{\pi}{2}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})$ the first crossing of the invariant manifolds with this section (see Figure 2). The next theorem measures the distance between these points for $0 < \mu \ll 1$.

Theorem A. (Distance between the unstable and stable manifolds of L_3). *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$\|(r_*^{\mathrm{u}}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}}) - (r_*^{\mathrm{s}}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})\| = \sqrt[3]{4}\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$

where the constant $\Theta \in \mathbb{C}$ satisfies

$$\Theta \neq 0 \tag{1.8}$$

and the constant $A > 0$ is given by the real-valued integral

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\, dx \approx 0.177744. \tag{1.9}$$

The asymptotic formula in the theorem is obtained in the papers [BGG22, BGG23]. Then, in [BCGG23], by means of a computer assisted proof, we show that the constant Θ is not zero. The distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$. This is due to the rapidly rotating dynamics of the system (see (1.4)) and it is usually known as a *beyond all orders phenomenon*, since the difference between the manifolds cannot be detected by expanding the manifolds in series of powers of μ. Due to the symmetry in (1.5), an analogous result holds for the opposite branches.

The goal of this section is to analyze the existence of homoclinic orbits to L_3. To this end, let us introduce the following definition.

Definition 1.2. *Let $\Gamma(t)$ be an homoclinic orbit of (1.1) to the critical point L_3 and B_μ a ball centered at L_3 of radius μ. Then, we say that $\Gamma(t)$ is k-round if*

$$\overline{\bigcup_{t\in\mathbb{R}} \Gamma(t)} \setminus B_\mu \qquad \text{has } k \text{ connected components.}$$

Theorem A implies the following corollary.

Corollary A. (1-round homoclinic connections). *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$, the Hamiltonian system associated to (1.1) does not have 1-round homoclinic connections to L_3.*

This corollary does not prevent the existence of multi-round homoclinic orbits. Indeed, E. Barrabés, J.M. Mondelo and M. Ollé in [BMO09] analyzed numerically the existence of multi-round homoclinic connections to L_3 in the RPC3BP and conjectured the existence of 2-round homoclinic orbits for a sequence of mass ratios $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying $\mu_n \to 0$ as $n \to \infty$. The first result of this paper proves this conjecture.

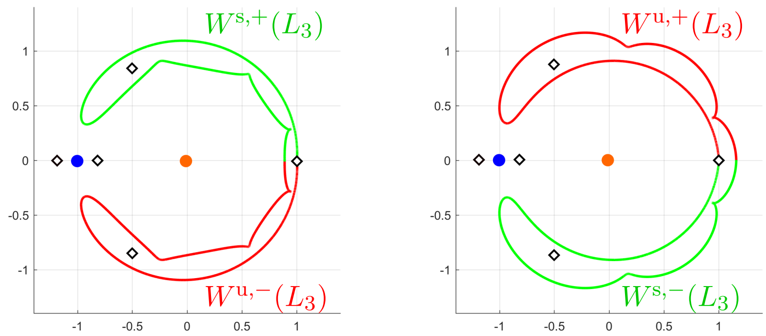


Figure 3: Projection onto the q-plane for examples of 2-round homoclinic connection to L_3. (Left) $\mu = 0.012144$, (right) $\mu = 0.004192$.

Theorem B. (2-round homoclinic connections). *There exists a sequence* $\{\mu_n\}_{n\geq N_0}$ *with* N_0 *big enough, of the form*

$$\mu_n = \frac{A}{n\pi\rho_{\rm eig}(0)}\left(1+\mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \textit{for } n \gg 1,$$

where $\rho_{\rm eig}(\mu)$ *is given in* (1.4) *and* $A > 0$ *is the constant introduced in* (1.9)*, such that the Hamiltonian system* (1.1) *has a* 2*-round homoclinic connection to the equilibrium point* L_3*. These homoclinic orbits coincide with* $W^{\rm u,+}(L_3)$ *and* $W^{\rm s,-}(L_3)$*.*

This theorem is proven in Section 3. Using the same tools, one can obtain an analogous result for the homoclinic connections between $W^{\rm u,-}(L_3)$ and $W^{\rm s,+}(L_3)$ (for a possibly different sequence of mass ratios).

1.2 Coorbital chaotic motions

Next we study the existence of chaotic phenomena associated to L_3 and its invariant manifolds. The Lyapunov Center Theorem (see for instance [MO17]) ensures the existence of a family of periodic orbits emanating from the saddle-center L_3 which, close to the equilibrium point, are hyperbolic. This family can be parametrized by the energy level given by the Hamiltonian h in (1.1).

Proposition 1.3 (Lyapunov periodic orbits to L_3)**.** *There exist* $\mu_0, \varrho_0 > 0$ *small enough such that, for* $\mu \in (0, \mu_0)$*, the Hamiltonian system with Hamiltonian* (1.1) *has a family of hyperbolic periodic orbits*

$$\Pi_3 = \left\{P_{3,\varrho} \textit{ periodic orbit } : \; h(P_{3,\varrho}) = \varrho^2 + h(L_3),\; \varrho \in (0, \varrho_0)\right\},$$

which depend regularly on $\varrho \in (0, \varrho_0)$ *and satisfy that* $\operatorname{dist}(P_{3,\varrho}, L_3) \to 0$ *as* $\varrho \to 0$ *in the sense of Hausdorff distance.*

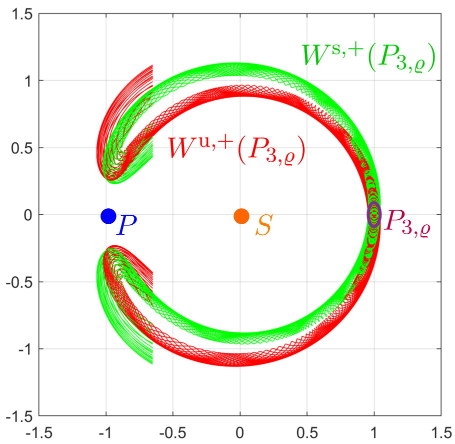


Figure 4: Projection onto the q-plane of the unstable (red) and stable (green) manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$ (blue), for $\mu = 0.003$.

In Proposition 4.1 we state this result in a different set of coordinates and provide estimates for the periodic orbits. Its proof can be found in Appendix A.

We denote by $W^{\mathrm{u}}(P_{3,\varrho})$ and $W^{\mathrm{s}}(P_{3,\varrho})$ the 2-dimensional unstable and stable invariant manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$. Analogously to the L_3 case, the invariant manifolds have two branches each. We denote by $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ the ones that circumvent L_5 and, by $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$, the ones that surround L_4 (see Figure 4). By the Smale-Birkhoff homoclinic Theorem (see [Sma67, KH95]), proving the existence of transverse intersections between $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ implies the existence of chaotic motions on a neighborhood of L_3 and its invariant manifolds. More specifically, we prove the following result, whose proof is deferred to Section 4.

Theorem C. (Chaotic motions). *Let $A > 0$ and $\Theta \neq 0$ be the constants given in Theorem A and ϱ_0 be as in Proposition 1.3. Then, there exist $\mu_0 > 0$ and two functions $\varrho_{\min}, \varrho_{\max} : (0, \mu_0) \to [0, \varrho_0]$ of the form*

$$\varrho_{\min}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[1 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$
$$\varrho_{\max}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[2 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$

such that, for $\mu \in (0, \mu_0)$ and $\varrho \in (\varrho_{\min}(\mu), \varrho_{\max}(\mu)]$, the following statement hold.

1. *The invariant manifolds $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ intersect transversally.*
2. *Consider the section $\widehat{\Sigma}_\varrho = \Sigma \cap \left\{ h = \varrho^2 + h(L_3) \right\}$ with Σ as given in (1.7) and the induced Poincaré map $\mathcal{P} : \widehat{\Sigma}_\varrho \to \widehat{\Sigma}_\varrho$. Then, there exists $M > 0$ such that $\mathcal{P}^M$ has an invariant set $\mathcal{X}$, homeomorphic to $\mathbb{Z}^{\mathbb{N}}$, such that $\mathcal{P}^M|_{\mathcal{X}}$ is topologically conjugated to the shift.*

Due to the symmetry in (1.5), an analogous result holds for the transverse intersections of $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$.

The chaotic motions induced by the Smale's horseshoe maps provided in the previous theorem lie in a tubular neighborhood around the invariant manifolds $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ with the boundary at an energy level of the form $h = h(L_3) + \mathcal{O}(\mu^{\frac{2}{3}} e^{-\frac{2A}{\sqrt{\mu}}})$

To prove Theorem C, we rely on the asymptotic formula given in by Theorem A. Since $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ are exponentially close to each other with respect to $\sqrt{\mu}$, the energy levels where chaotic motions are found are also exponentially close to that of L_3. In addition, by restricting μ one can take $\varrho_{\max}(\mu)$ bigger (see Theorem 4.2 below).

Moreover, following the same ideas behind Theorem C, we prove that the Lyapunov periodic orbit at the energy level $h = \varrho^2_{\min}(\mu) + h(L_3)$ possesses a quadratic homoclinic tangency.

Theorem D. (Homoclinic tangencies). *Denote by f_ϱ the flow of the Hamiltonian system given in (1.1) restricted to the energy level $h = \varrho^2 + h(L_3)$. Let $\varrho_0, \mu_0 > 0$ and $\varrho_{\min}(\mu) : (0, \mu_0) \to [0, \varrho_0]$ be as given in Theorem C. Then, for a fixed $\mu \in (0, \mu_0)$ and ϱ close to $\varrho_{\min}(\mu)$, the flow f_ϱ unfolds generically an homoclinic quadratic tangency between $W^{\mathrm{u},+}(P_{3,\varrho_{\min}(\mu)})$ and $W^{\mathrm{s},+}(P_{3,\varrho_{\min}(\mu)})$.*

To prove this result, we use the definition of generic unfolding of a quadratic homoclinic tangency given in [Dua08] (see also Section 1.3 below). The existence of a quadratic homoclinic quadratic tangency leads to the existence of Newhouse domains for the RPC3BP. This is explained in the next section. As far as the authors know this is one of the first constructions of Newhouse domains in Celestial Mechanics (see also [GK12]).

1.3 Newhouse domains for the RPC3BP at coorbital motions

To describe the dynamics arising from the quadratic homoclinic tangencies provided in Theorem D, we have to introduce several concepts. We follow the approach in [Dua08, Gor12, BFPS22].

Consider a symplectic 2-dimensional manifold M. A hyperbolic basic set for a $\mathcal{C}^r$ diffeomorphism $f \in \mathrm{Diff}^r(M)$, $r \geq 4$, is an invariant compact set Λ which is transitive, hyperbolic and locally maximal (it is the maximal invariant set in one of its neighborhoods U). All the points in Λ have stable and unstable manifolds, which are injectively immersed submanifolds depending continuously on the base point. It is a well known fact that hyperbolic basic sets are robust under $\mathcal{C}^1$ perturbations and the dynamics of the perturbed set is equivalent to that of Λ. We call the perturbed hyperbolic basic set the hyperbolic continuation of Λ.

Given two points $x, y \in \Lambda$, an intersection point of $W^s_x \cap W^u_y$ is called a homoclinic tangency if the corresponding intersection is not transverse.

We say that Λ is a wild basic set over an open set $\mathcal{U} \subset \mathrm{Diff}^r(M)$ containing f if, for all maps $g \in \mathcal{U}$,

1. The hyperbolic continuation Λ_g is a hyperbolic basic set conjugated to Λ.

2. There is at least one orbit of homoclinic tangencies of Λ_g.

The set $\mathcal{U}$ is usually called Newhouse region.

Assume that the symplectic diffeomorphism f has a hyperbolic saddle P. Its homoclinic class $H(P,f)$ is the closure of the union of the transverse homoclinic points to P. It is well known that $H(P,f)$ is a transitive invariant set. Moreover, $H(P,f)$ is the smallest closed invariant set which contains all the basic sets of f containing P.

Close to basic sets there will appear plenty of elliptic periodic points with a particular structure. Let us also describe them. Consider an elliptic periodic point P of period N of f. We say that P is generic if the two eigenvalues λ, λ^{-1} lie in the unit circle and are not resonant up to order 3, that is $|\lambda| = 1$, $\lambda^2 \neq 1$, $\lambda^3 \neq 1$, and the first coefficient of the Birkhoff Normal Form of f^N at P is not zero. By KAM Theory, around such points there exists an invariant set, with full Lebesgue density at P, which is a union of invariant curves for f^N, whose dynamics is conjugated to an irrational rotation of the circle. This structure around P is usually called "elliptic isle".

We want to analyze the Newhouse phenomenon and the existence of wild basic sets for one parameter families of symplectic diffeomorphisms, that is a $\mathcal{C}^r$-function $(\eta, x) \to f_\eta(x)$ defined on $I \times M$ where $I \subset \mathbb{R}$ is an interval, such that $f_\eta \in \mathrm{Diff}^r(M)$ and it is symplectic. We say that the family f_η unfolds generically an orbit of homoclinic quadratic tangencies at $(\eta_0, Q_0) \in I \times M$, associated to some hyperbolic periodic point P if, denoting by P_η its hyperbolic continuation for the map f_η,

1. The stable and unstable manifolds of P for f_{η_0}, $W^s(P, f_{\eta_0})$ and $W^u(P, f_{\eta_0})$, have a quadratic tangency at Q_0.

2. If ℓ is any smooth curve transverse to $W^s(P, f_{\eta_0})$ and $W^u(P, f_{\eta_0})$ at Q_0, then the local intersections of $W^s(P_\eta, f_\eta)$ and $W^u(P_\eta, f_\eta)$ with ℓ cross each other with relative non-zero velocity at (η_0, Q_0).

In [New70, Dua08, Gor12] it is proven the following.

Theorem 1.4. *Fix $0 < \nu \ll 1$. Let f_η be a $\mathcal{C}^r$ one parameter family of symplectic maps in $\mathrm{Diff}^r(M)$, $r \geq 6$. Let O be a hyperbolic periodic orbit and Γ an orbit of homoclinic quadratic tangencies of f_0 which unfolds generically at $\eta = 0$. Denote by O_η the hyperbolic continuation of O and take any small neighborhood U of $O \cup \Gamma$. Then, there is sequence of Newhouse intervals Δ_k converging to $\eta = 0$. Namely, for each $\eta \in \Delta_k$, f_η possesses a wild hyperbolic basic set $\Lambda_{k,\eta}$, which depends continuously on η (with respect to the Hausdorff distance), such that $O_\eta \subset \Lambda_{k,\eta} \subset U$.*

Moreover, for each $k \geq 1$,

- *For every $\eta \in \Delta_k$, the Hausdorff dimension of $\Lambda_{k,\eta}$ satisfies*

$$\dim_H \Lambda_{k,\eta} \geq 2 - \nu.$$

- *Given any periodic point $P_\eta \in \Lambda_{k,\eta}$ (in particular, O_η), there is a dense subset $D_k \subset \Delta_k$ such that for every $\eta \in D_k$, the periodic point P_η has an orbit of homoclinic tangencies.*

- *There is a residual subset $R_k \subset \Delta_k$ such that for every $\eta \in R_k$,*
  1. *The homoclinic class $H(O_\eta, f_\eta)$ is accumulated by generic elliptic periodic points of f_η.*
  2. *The homoclinic class $H(O_\eta, f_\eta)$ contains hyperbolic sets of Hausdorff dimension arbitrarily close to 2. In particular $\dim_H H(P_\eta, f_\eta) = 2$.*

We apply this result to the quadratic homoclinic tangencies of the invariant manifolds of the Lyapunov periodic orbit around L_3 for the RPC3BP obtained in Theorem D. Since the RPC3BP is autonomous, the energy is conserved. Then, it can be seen as a family of 3-dimensional[1] flows depending on two parameters: the mass ratio μ and the energy h. We denote these flows by $\Phi^t_{\mu,h}$. Doing an abuse of language, in the next theorem, we use the concepts defined above (basic set, homoclinic class, generic elliptic orbit, etc) referred to flows instead of maps.

Theorem E. (Newhouse phenomenon for the RPC3BP). *Fix $0 < \nu \ll 1$ and $\mu \in (0, \mu_0)$. Let $P = P_{3,\varrho_{\min}(\mu)}$ be the Lyapunov periodic orbit of $\Phi^t_{\mu,h(\mu)}$ with $h_0(\mu) = \varrho^2_{\min}(\mu) + h(L_3)$ obtained in Theorem D. Let Γ be the associated orbit of quadratic homoclinic tangencies obtained in the same theorem. Take any small neighborhood U of $P \cup \Gamma$. Then,*

- *There exist $h^* > h_0(\mu)$ and a sequence of Newhouse intervals $\Delta_k = \Delta_k(\mu) \subset (h_0(\mu), h_*)$ converging to $h_0(\mu)$. That is for each $h \in \Delta_k$, the flow $\Phi^t_{\mu,h}$ possesses a wild hyperbolic basic set $\Lambda_{k,h}$, which depends continuously on h (with respect to the Hausdorff distance) such that $\Lambda_{k,h} \subset U$ and $\Lambda_{k,h}$ contains P_h, the hyperbolic continuation of P.*
- *For every $h \in \Delta_k$, the Hausdorff dimension of $\Lambda_{k,h}$ satisfies*
$$\dim_H \Lambda_{k,h} \geq 3 - \nu.$$
- *Given any periodic orbit $Q_h \in \Lambda_{k,h}$ (in particular, P_h), there is a dense subset $D_k \subset \Delta_k$ such that for every $h \in D_k$, Q_h has an orbit of homoclinic tangencies.*
- *There is a residual subset $R_k \subset \Delta_k$ such that for every $h \in R_k$,*
  1. *The homoclinic class $H(P_h, \Phi^t_{\mu,h})$ is accumulated by generic elliptic periodic orbits of $\Phi^t_{\mu,h}$.*
  2. *The homoclinic class $H(P_h, \Phi^t_{\mu,h})$ contains hyperbolic sets of Hausdorff dimension arbitrarily close to 3. In particular $\dim_H H(P_h, \Phi^t_{\mu,h}) = 3$.*

This theorem is a consequence of Theorems D and 1.4. Note that one cannot define a global Poincaré map in the energy levels considered. However, the proofs in [New70, Dua08, Gor12] only rely on an induced map close to the periodic orbit. To construct it, it is enough to consider a local transverse section to the periodic orbit and therefore their proofs also apply to our setting.

[1] The energy level is not a manifold at the energy value of L_3, however it defines (locally) a manifold for energy levels close enough to that of the Lyapunov orbit with a quadratic homoclinic tangency.

1.4 State of the art

A fundamental problem in dynamical systems is to prove whether a given system has chaotic dynamics. For many physically relevant models this is usually remarkably difficult. This is the case of many Celestial Mechanics models, where most of the known chaotic motions have been found in nearly integrable regimes where there is an unperturbed problem which already presents some form of "hyperbolicity". This is the case in the vicinity of collision orbits (see for example [Moe89, BM06, Bol06, Moe07]) or close to parabolic orbits (which allows to construct chaotic/oscillatory motions), see [Sit60, Ale76, LS80, Mos01, GMS16, GSMS17, GPSV21, GMPS22]. There are also several results in regimes far from integrable which rely on computer assisted proofs [Ari02, WZ03, Cap12, GZ19].

The problem tackled in this paper is radically different. Indeed, if one takes the limit $\mu \to 0$ in (1.1) one obtains the classical integrable Kepler problem in the elliptic regime, where no hyperbolicity is present. Instead, the (weak) hyperbolicity is created by the $\mathcal{O}(\mu)$ perturbation. The bifurcation scenario we are dealing with is the so called $0^2 i\omega$ resonance or Hamiltonian Hopf-Zero bifurcation. Indeed, for $\mu > 0$ the Hamiltonian system given by h in (1.1) has a saddle-center equilibrium point at L_3. However, for $\mu = 0$, the equilibrium point degenerates and the spectrum of its linear part consists in a pair of purely imaginary and a double 0 eigenvalues, (see (1.4)).

Most of the studies in homoclinc phenomena around a saddle-center equilibrium are focused on the non-degenerate case where all the eigenvalues have comparable size, see [Ler91, MHO92, Rag97a, Rag97b, BRS03]. However, for close to resonance $0^2 i\omega$ cases, to the best of authors knowledge, the results are more rare. The generic unfolding of the reversible $0^2 i\omega$ resonance is considered in [Lom99, Lom00] where the author proves the existence of transverse homoclinic connections for every periodic orbit exponentially close to the origin and the breakdown of homoclinic orbits to the origin itself. In [JBL16], the authors show the existence of homoclinic connections with several loops for every periodic orbit close to the equilibrium point for a generic unfolding of a Hamiltonian $0^2 i\omega$ resonance. Note that the unfolding of the $0^2 i\omega$ resonance in the RPC3BP is highly non-generic due to the strong degeneracies of its Keplerian approximation.

The work here presented shows the existence of homoclinic connections for both the equilibrium point and periodic orbits (exponentially) close to the equilibrium point. In the case of the (non-Hamiltonian) Hopf-zero singularity, we remark the strongly related work [BIS20]. Also, in [GGSZ21], the authors use similar techniques to analyze breather solutions for the nonlinear Klein-Gordon partial differential equation.

Acknowledgements

I Baldomá has been supported by the grant PID-2021- 122954NB-100 funded by the Spanish State Research Agency through the programs MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe".

M. Giralt has been supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101034255. M. Giralt has also been supported by the research project PRIN 2020XB3EFL "Hamiltonian and dispersive PDEs".

M. Guardia has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 757802). M. Guardia is also supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia Prize 2019.

This work is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

2 Scaled Poincaré variables and previous results

Let us notice that, for the unperturbed problem h in (1.1) with $\mu = 0$, the five Lagrange point disappear into a circle of degenerate critical points. For this reason, in [BGG22], we introduced a singular change of coordinates to obtain a new first order Hamiltonian which has a saddle-center equilibrium point (close to L_3) with stable and unstable manifolds that coincide along a separatrix.

First, in Section 2.1, we introduce the main features of this change of coordinates and its relation to L_3. Then, in Section 2.2, we state Theorem 2.4, which is a reformulation of Theorem A in the new set of coordinates.

2.1 A singular perturbation formulation of the problem

Applying a suitable singular change of coordinates, the Hamiltonian h can be written as a perturbation of a pendulum-like Hamiltonian weakly coupled with a fast oscillator. We summarize the most important properties of this set of coordinates, which was studied in detail in [BGG22].

The Hamiltonian h expressed in the classical (rotating) Poincaré coordinates, $\phi^{\text{Poi}} : (\lambda, L, \eta, \xi) \to (q, p)$, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and the Hamiltonian

$$H^{\text{Poi}} = H_0^{\text{Poi}} + \mu H_1^{\text{Poi}}, \tag{2.1}$$

with

$$H_0^{\text{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \qquad \text{and} \qquad H_1^{\text{Poi}} = h_1 \circ \phi_{\text{Poi}}. \tag{2.2}$$

Moreover, the critical point L_3 satisfies

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu) \tag{2.3}$$

and the linearization of the vector field at this point has, at first order, an uncoupled nilpotent and center blocks. Since ϕ^{Poi} is an implicit change of coordinates, there is no explicit expression for H_1^{Poi}. However, since H_1^{Poi} is analytic for $|(L-1,\eta,\xi)| \ll 1$, it is possible to obtain series expansion in powers of $(L-1,\eta,\xi)$ (see [BGG22, Lemma 4.1])

In addition, since the original Hamiltonian h is reversible with respect to the involution Ψ in (1.5), the Hamiltonian H^{Poi} is reversible with respect to the involution

$$\Phi_{\mathrm{Poi}}(\lambda, L, \eta, \xi) = (-\lambda, L, \xi, \eta). \tag{2.4}$$

To capture the slow-fast dynamics of the system, we perform the singular symplectic scaling

$$\phi_\delta : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \qquad L = 1 + \delta^2\Lambda, \quad \eta = \delta x, \quad \xi = \delta y, \tag{2.5}$$

and the time reparametrization $t = \delta^{-2}t'$, where

$$\delta = \mu^{\frac{1}{4}}. \tag{2.6}$$

Defining the potential

$$V(\lambda) = H_1^{\mathrm{Poi}}(\lambda, 1, 0, 0; 0) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \tag{2.7}$$

the Hamiltonian system associated to H^{Poi}, expressed in scaled coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$ and the Hamiltonian

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1, \tag{2.8}$$

where

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad H_{\mathrm{osc}}(x, y; \delta) = \frac{xy}{\delta^2}, \tag{2.9}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\mathrm{Poi}}(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4) - V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2\Lambda), \tag{2.10}$$

and

$$F_{\mathrm{p}}(z) = \left(-\frac{1}{2(1+z)^2} - (1+z)\right) + \frac{3}{2} + \frac{3}{2}z^2 = \mathcal{O}(z^3). \tag{2.11}$$

We introduce a suitable neighborhood where the coordinates (λ, Λ, x, y) are defined. For $c_0 > 0$ we define the domain

$$U_{\mathbb{R}}(c_0, c_1) = \left\{(\lambda, \Lambda, x, \overline{x}) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{C}^2 : |\pi - \lambda| > c_0, |(\Lambda, x)| < c_1\right\}. \tag{2.12}$$

For technical reasons, we consider some of the objects of the system in an analytical extension of the domain $U_{\mathbb{R}}$. In particular we use the domain

$$U_{\mathbb{C}}(c_0, c_1) = \left\{(\lambda, \Lambda, x, y) \in \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}^3 : |\pi - \mathrm{Re}\,\lambda| > c_0, |(\mathrm{Im}\,\lambda, \Lambda, x, y)| < c_1\right\}. \tag{2.13}$$

The next proposition states some properties of the Hamiltonian H.

Proposition 2.1. *Fix $c_0, c_1 > 0$. Then, there exists $\delta_0 = \delta_0(c_0, c_1) > 0$ such that, for $\delta \in (0, \delta_0)$, one has that*

- *The Hamiltonian H in (2.8) is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \delta)$ in the domain $U_{\mathbb{C}}(c_0, c_1)$.*
- *There exists $b_0 > 0$ independent of δ such that, for $(\lambda, \Lambda, x, y) \in U_{\mathbb{C}}(c_0, c_1)$, the second derivatives of the Hamiltonian H_1 given in (2.10) satisfy*

$$\left|\partial_\lambda^2 H_1\right|, \left|\partial_{\lambda x} H_1\right|, \left|\partial_{\lambda y} H_1\right| \leq b_0 \delta, \qquad \left|\partial_{\lambda\Lambda} H_1\right|, \left|\partial_\Lambda^2 H_1\right| \leq b_0 \delta^2,$$
$$\left|\partial_x^2 H_1\right|, \left|\partial_{xy} H_1\right|, \left|\partial_y^2 H_1\right| \leq b_0 \delta^2, \qquad \left|\partial_{\Lambda x} H_1\right| \left|\partial_{\Lambda y} H_1\right| \leq b_0 \delta^3.$$

Moreover[2]*,*

$$\left|\partial_{\alpha_1,\alpha_2,\alpha_3} H_1\right| \leq b_0 \delta, \qquad \text{with} \qquad \alpha_1, \alpha_2, \alpha_3 \in \{\lambda, \Lambda, x, y\}.$$

Proof. The first statement follows from [BGG22, Theorem 2.1]. The second statement is a consequence of [BGG23, Lemma A.3]. □

Remark 2.2. *Consider $M \subseteq \mathbb{C}^4$ a symmetric subset with respect to $\mathbb{R}^4$. We say that a function $\zeta = (\zeta_\lambda, \zeta_\Lambda, \zeta_x, \zeta_y) : M \to U_{\mathbb{C}}(c_0, c_1)$ is real-analytic if, for $m \in M$, $\zeta_\lambda(\overline{m}) = \overline{\zeta_\lambda(m)}$, $\zeta_\Lambda(\overline{m}) = \overline{\zeta_\Lambda(m)}$, $\zeta_x(\overline{m}) = \overline{\zeta_y(m)}$ and $\zeta_y(\overline{m}) = \overline{\zeta_x(m)}$. Notice that, as a consequence, $\zeta(m) \in U_{\mathbb{R}}(c_0)$, for $m \in M \cap \mathbb{R}^4$.*

Notice that, by (2.4), the Hamiltonian H is reversible with respect to the involution

$$\Phi(\lambda, \Lambda, x, y) = (-\lambda, \Lambda, y, x), \tag{2.14}$$

which has symmetry axis

$$\mathcal{S} = \{\lambda = 0,\ x = y\}. \tag{2.15}$$

In the next proposition, proven in [BGG22, Theorem 2.1], we obtain an expression and suitable estimates for the equilibrium point L_3.

Proposition 2.3. *There exist $\delta_0 > 0$ and $b_1 > 0$ such that, for $\delta \in (0, \delta_0)$, the critical point L_3 expressed in coordinates (λ, Λ, x, y) is of the form*

$$\mathfrak{L}(\delta) = \left(0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)\right)^T \in \mathcal{S}, \tag{2.16}$$

with $\left|\mathfrak{L}_\Lambda(\delta)\right|, \left|\mathfrak{L}_x(\delta)\right|, \left|\mathfrak{L}_y(\delta)\right| \leq b_1$ and $\mathcal{S}$ as given in (2.15).

The linearization of $\mathfrak{L}(\delta)$ is given by

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

[2] One can obtain more precise estimates for the third derivatives of H_1. However, these rough estimates are sufficient for the proofs of this paper.

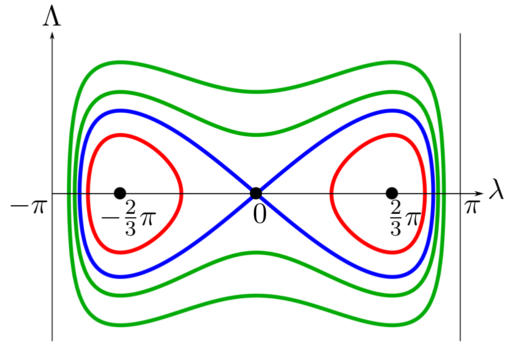


Figure 5: Phase portrait of the system given by Hamiltonian $H_{\mathrm{p}}(\lambda,\Lambda)$ on (2.9). On blue the two separatrices.

This analysis leads us to define a "new" first order for the Hamiltonian H in (2.8) as

$$H_0(\lambda,\Lambda,x,y;\delta) = H_{\mathrm{p}}(\lambda,\Lambda) + H_{\mathrm{osc}}(x,y;\delta), \tag{2.17}$$

and we refer to H_0 as the unperturbed Hamiltonian and to H_1 (see (2.10)) as the perturbation.

Notice that the unperturbed Hamiltonian is uncoupled. In the (x,y)-plane, it displays a fast oscillator of velocity $\frac{1}{\delta^2}$ whereas, in the (λ,Λ)-plane, it has a saddle at $(0,0)$ with two homoclinic connections or separatrices at the energy level $H_{\mathrm{p}}(\lambda,\Lambda) = -\frac{1}{2}$, (see Figure 5). We define

$$\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right), \tag{2.18}$$

which satisfies $H_{\mathrm{p}}(\lambda_0,0) = H_{\mathrm{p}}(0,0) = -\frac{1}{2}$ and corresponds with the crossing point of the right separatrix with the axis $\{\Lambda = 0\}$.

2.2 The invariant manifolds of L_3

The unstable and stable manifolds of the critical point $\mathfrak{L}(\delta)$ for small values of δ, have two branches, which are symmetric with respect to the involution (2.14) (see Figure 2).

For $\delta > 0$, we denote by $\mathcal{W}^{\mathrm{u}}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s}}(\mathfrak{L})$ the 1-dimensional unstable and stable manifolds of $\mathfrak{L}(\delta)$. In addition, as done in Section 1, we consider each branch independently. Let ψ_t be the flow given by the Hamiltonian H and $\mathbf{e_1} = (1,0,0,0)^T$. We denote

$$\mathcal{W}^{\mathrm{u},+}(\mathfrak{L}) = \left\{ z \in \mathcal{W}^{\mathrm{u}}(\mathfrak{L}) \ : \ \lim_{t\to-\infty} \langle \psi_t(z), \mathbf{e_1}\rangle = 0^+ \right\}, \qquad \mathcal{W}^{\mathrm{s},-}(\mathfrak{L}) = \Phi\left(\mathcal{W}^{\mathrm{u},+}\right),$$
$$\mathcal{W}^{\mathrm{s},+}(\mathfrak{L}) = \left\{ z \in \mathcal{W}^{\mathrm{s}}(\mathfrak{L}) \ : \ \lim_{t\to+\infty} \langle \psi_t(z), \mathbf{e_1}\rangle = 0^+ \right\}, \qquad \mathcal{W}^{\mathrm{u},-}(\mathfrak{L}) = \Phi\left(\mathcal{W}^{\mathrm{s},+}\right),$$

the branches of $\mathcal{W}^{\diamond}(\mathfrak{L})$, for $\diamond = \mathrm{u},\mathrm{s}$.

Next result, proven in [BGG23, Theorem 2.2], gives an asymptotic formula for the distance between the first intersection of the one dimensional manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ on a suitable section. In particular, Theorem A is a consequence of this result.

Theorem 2.4. *Fix an interval* $[\lambda_1, \lambda_2] \subset (0, \lambda_0)$ *with* λ_0 *as given in* (2.18)*. There exists* $\delta_0 > 0$ *such that, for* $\delta \in (0, \delta_0)$ *and* $\lambda_* \in [\lambda_1, \lambda_2]$*, the invariant manifolds* $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ *and* $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ *intersect the section* $\{\lambda = \lambda_*, \Lambda > 0\}$*. Denote by* $(\lambda_*, \Lambda^{\mathrm{u}}_\delta, x^{\mathrm{u}}_\delta, y^{\mathrm{u}}_\delta)$ *and* $(\lambda_*, \Lambda^{\mathrm{s}}_\delta, x^{\mathrm{s}}_\delta, y^{\mathrm{s}}_\delta)$ *the first intersection points of the unstable and stable manifolds with this section, respectively. They satisfy*

$$y^{\mathrm{u}}_\delta - y^{\mathrm{s}}_\delta = \sqrt[6]{2}\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\Theta + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], \qquad x^{\mathrm{u}}_\delta - x^{\mathrm{s}}_\delta = \overline{y^{\mathrm{u}}_\delta - y^{\mathrm{s}}_\delta},$$
$$\Lambda^{\mathrm{u}}_\delta - \Lambda^{\mathrm{s}}_\delta = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right),$$

where $A > 0$ *and* $\Theta \in \mathbb{C} \setminus \{0\}$ *are the constants described in Theorem* A*.*

To prove Theorem C and D, it will be convenient to analyze the distance between the invariant manifolds in the "horizontal section"

$$\Sigma_0 = \left\{(\lambda, \Lambda, x, y) \in U_{\mathbb{R}}(c_0, c_1) : \Lambda = \delta^2 \mathfrak{L}_\Lambda(\delta),\ H(\lambda, \Lambda, x, y) = H(\mathfrak{L}(\delta))\right\}, \qquad (2.19)$$

within the energy level of $\mathfrak{L}(\delta)$, where (x, y) define a system of coordinates. The following corollary is a consequence of Theorem 2.4. It is proven in Appendix B.

Corollary 2.5. *There exists* $\delta_0 > 0$ *such that, for every* $\delta \in (0, \delta_0)$*, the invariant manifolds* $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ *and* $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ *intersect the section* Σ_0*. Denote by* $(\lambda^{\mathrm{u}}_\delta, \delta^2\mathfrak{L}_\Lambda, x^{\mathrm{u}}_\delta, y^{\mathrm{u}}_\delta)$ *and* $(\lambda^{\mathrm{s}}_\delta, \delta^2\mathfrak{L}_\Lambda, x^{\mathrm{s}}_\delta, y^{\mathrm{s}}_\delta)$ *the first intersection points of the unstable and stable manifolds, respectively, with the section. Then, they satisfy*

$$|x^{\mathrm{u}}_\delta - x^{\mathrm{s}}_\delta| = |y^{\mathrm{u}}_\delta - y^{\mathrm{s}}_\delta| = \sqrt[6]{2}\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right].$$

3 2-round homoclinic orbits to L_3: Proof of Theorem B

In this section we study the existence of 2-round homoclinic connections to the $\mathfrak{L}(\delta)$ (see (2.16)) for certain values of the parameter δ and we prove Theorem B. We first restate it referred to the Hamiltonian (2.8) (recall that $\delta = \mu^{\frac{1}{4}}$, see (2.6)).

Theorem 3.1. *There exist* $N_0 > 0$ *and a sequence* $\{\delta_n\}_{n \geq N_0}$ *satisfying*

$$\delta_n = \sqrt[8]{\frac{8}{21}} \sqrt[4]{\frac{A}{n\pi}} \left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \text{for } n \geq N_0,$$

such that, for each $n \geq N_0$*, there exist a* 2*-round homoclinic connection to the equilibrium point* $\mathfrak{L}(\delta_n)$ *between* $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ *and* $\mathcal{W}^{\mathrm{s},-}(\mathfrak{L})$*.*

The rest of this section is devoted to prove this theorem.

To prove Theorem 3.1, we take advantage of the fact that the Hamiltonian H is reversible with respect to the axis $\mathcal{S} = \{\lambda = 0,\, x = y\}$ (see (2.15)). Therefore, by symmetry, it is only necessary to see that there exists a sequence of δ such that $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$

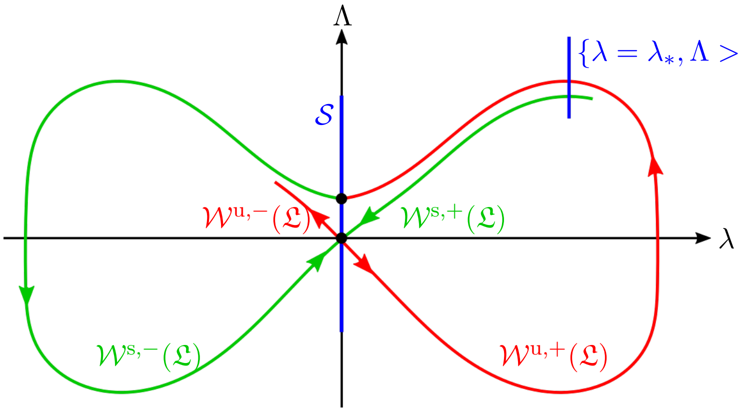


Figure 6: Projection into the (λ, Λ)-plane of the unstable and stable manifolds and its intersections with the symmetry axis and section $\{\lambda = \lambda_*, \Lambda > 0\}$.

intersects the symmetry axis $\mathcal{S}$, see Figure 6. To this end, we extend the manifold $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ from the section $\{\lambda = \lambda_*, \Lambda > 0\}$, studied in Theorem 2.4, to a neighborhood of the critical point $\mathfrak{L}(\delta)$ and look for intersections with $\mathcal{S}$. To study the invariant manifolds near $\mathfrak{L}(\delta)$, we use a normal form result for Hamiltonian systems in a neighborhood of a saddle-center critical point. Note that, the classical normal form result by Moser in [Mos58] is not enough for our purposes. Indeed, we need to control that the radius of convergence of the normal form does not goes to zero when $\delta \to 0$. For that reason, we apply a more quantitative normal form obtained by T. Jézéquel, P. Bernard and E. Lombardi in [JBL16].

3.1 Proof of Theorem 3.1

To prove Theorem 3.1, we first perform a detailed local analysis of the Hamiltonian H in (2.8) close to the equilibrium point $\mathfrak{L}(\delta)$. In the next proposition we introduce the normal form result given by T. Jézéquel, P. Bernard and E. Lombardi in [JBL16] adapted to the Hamiltonian H. Then, in Proposition 3.3, we translate the results in Theorem 2.4 and the symmetry axis $\mathcal{S}$ in (2.15) into the new set of coordinates provided by the normal form.

Proposition 3.2. *There exist $\delta_0, \varrho_0, c_0, c_1 > 0$ and a family of analytic changes of coordinates*

$$\begin{aligned} \mathcal{F}_\delta : B(\varrho_0) = \left\{ \mathbf{z} \in \mathbb{R}^4 : |\mathbf{z}| < \varrho_0 \right\} &\to U_{\mathbb{R}}(c_0, c_1) \\ (v_1, w_1, v_2, w_2) &\mapsto (\lambda, \Lambda, x, y), \end{aligned}$$

defined for $\delta \in (0, \delta_0)$, with the following properties:

1. *It is canonical with respect to the symplectic form $dv_1 \wedge dw_1 + dv_2 \wedge dw_2$.*
2. *$\mathcal{F}_\delta(0) = \mathfrak{L}(\delta)$.*

3. *The Hamiltonian H (see (2.8)) in the new coordinates reads*

$$\begin{aligned}\mathcal{H}(v_1,w_1,v_2,w_2;\delta) &= H(\mathcal{F}_\delta(v_1,w_1,v_2,w_2);\delta) - H(\mathfrak{L}(\delta);\delta)\\ &= v_1 w_1 + \frac{\alpha(\delta)}{2\delta^2}\left(v_2^2+w_2^2\right) + \mathcal{R}(v_1w_1, v_2^2+w_2^2;\delta),\end{aligned}$$

where $\alpha(\delta)$ is a $\mathcal{C}^1$-function satisfying that $\alpha(\delta)=\sqrt{\frac{8}{21}}+\mathcal{O}(\delta^4)$ and $\mathcal{R}$ satisfies

$$\left|\mathcal{R}(v_1w_1, v_2^2+w_2^2;\delta)\right| \leq C|(v_1w_1, v_2^2+w_2^2)|^2,$$

for $(v_1,w_1,v_2,w_2)\in B(\varrho_0)$ and $C>0$ a constant independent of δ.

The proof of this proposition, which is a consequence of the results in [JBL16], is explained in Section 3.2. Observe that the equations associated to the Hamiltonian $\mathcal{H}$ are of the form

$$\begin{aligned}\dot{v_1} &= v_1\left(1+\partial_1\mathcal{R}(v_1w_1,v_2^2+w_2^2;\delta)\right)\\ \dot{w_1} &= -w_1\left(1+\partial_1\mathcal{R}(v_1w_1,v_2^2+w_2^2;\delta)\right)\\ \dot{v_2} &= w_2\left(\frac{\alpha(\delta)}{\delta^2}+2\partial_2\mathcal{R}(v_1w_1,v_2^2+w_2^2;\delta)\right)\\ \dot{w_2} &= -v_2\left(\frac{\alpha(\delta)}{\delta^2}+2\partial_2\mathcal{R}(v_1w_1,v_2^2+w_2^2;\delta)\right).\end{aligned} \tag{3.1}$$

Since this system has two conserved quantities, v_1w_1 and $v_2^2+w_2^2$, its solutions are

$$\begin{aligned}v_1(t) &= v_1(0)e^{\nu_1 t},\\ w_1(t) &= w_1(0)e^{-\nu_1 t},\\ \begin{pmatrix}v_2(t)\\ w_2(t)\end{pmatrix} &= \begin{pmatrix}\cos\nu_2 t & \sin\nu_2 t\\ -\sin\nu_2 t & \cos\nu_2 t\end{pmatrix}\begin{pmatrix}v_2(0)\\ w_2(0)\end{pmatrix},\end{aligned} \tag{3.2}$$

where, for $(v_1(0),w_1(0),v_2(0),w_2(0))\in B(\varrho_0)$,

$$\begin{aligned}\nu_1 = \nu_1(\delta) &= 1+\partial_1\mathcal{R}\left(v_1(0)w_1(0),v_2^2(0)+w_2^2(0);\delta\right)>0,\\ \nu_2=\nu_2(\delta) &= \frac{\alpha(\delta)}{\delta^2}+2\partial_2\mathcal{R}\left(v_1(0)w_1(0),v_2^2(0)+w_2^2(0);\delta\right)>0.\end{aligned} \tag{3.3}$$

Notice that the local unstable and stable manifolds are given by $\{w_1=v_2=w_2=0\}$ and $\{v_1=v_2=w_2=0\}$, respectively.

Proposition 3.3. *Consider the constants ϱ_0,δ_0 given by Proposition 3.2. Then,*

1. *There exists $\lambda_*\in(0,\lambda_0)$ and $\delta_1\in(0,\delta_0)$ such that, for any $\delta\in(0,\delta_1)$, the first intersections $\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*)$ and $\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)$ of the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section $\{\lambda=\lambda_*,\Lambda>0\}$ respectively (see Theorem 2.4), satisfy that*

$$(v_1^{\mathrm{u}},w_1^{\mathrm{u}},v_2^{\mathrm{u}},w_2^{\mathrm{u}}) = \mathcal{F}_\delta\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*(\varrho))\right), \qquad (v_1^{\mathrm{s}},w_1^{\mathrm{s}},v_2^{\mathrm{s}},w_2^{\mathrm{s}}) = \mathcal{F}_\delta\left(\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*(\varrho))\right) \tag{3.4}$$

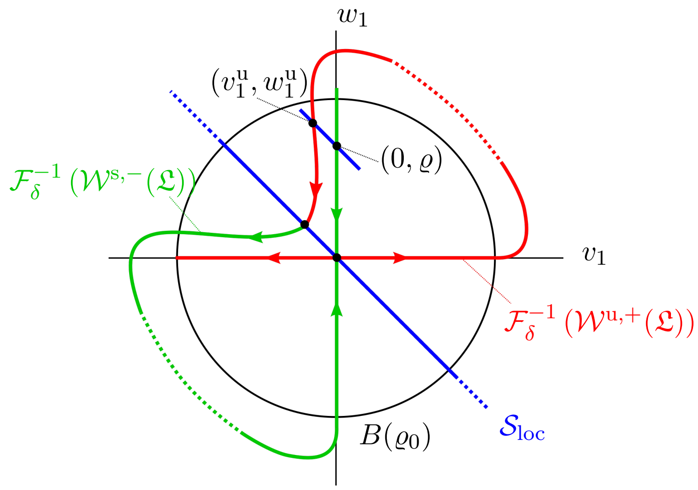


Figure 7: Representation of the unstable and stable manifolds in local coordinates (v_1, w_1, v_2, w_2) given in Propositions 3.2 and 3.3.

belong to the ball $B(\varrho_0)$.

Moreover, there exists $\varrho \in (0, \varrho_0)$ *such that, for* $\delta \in (0, \delta_1)$*, these points can be written as*

$$\begin{aligned}
v_1^{\mathrm{u}} &= -\frac{\sqrt[3]{2}}{\varrho}\delta^{-\frac{4}{3}} e^{-\frac{2A}{\delta^2}} \left[|\Theta|^2 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], & v_1^{\mathrm{s}} &= 0, \\
w_1^{\mathrm{u}} &= \varrho + \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right), & w_1^{\mathrm{s}} &= \varrho, \\
v_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\operatorname{Re}\Theta + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], & v_2^{\mathrm{s}} &= 0, \\
w_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[-\operatorname{Im}\Theta + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], & w_2^{\mathrm{s}} &= 0.
\end{aligned}$$

2. *Let* $\mathcal{S} = \{\lambda = 0, x = y\}$ *be the symmetry axis* (2.15) *of the Hamiltonian* H*. There exist real-analytic functions* $\Psi_1, \Psi_2 : B(\varrho_0) \times (0, \delta_0) \to \mathbb{R}$ *and a constant* $C > 0$ *such that the curve*

$$\mathcal{S}_{\mathrm{loc}} = \left\{v_1 + w_1 = \Psi_1(v_1, w_1, v_2, w_2; \delta),\ w_2 = \Psi_2(v_1, w_1, v_2, w_2; \delta)\right\} \tag{3.5}$$

satisfies that $\mathcal{F}_\delta(\mathcal{S}_{\mathrm{loc}}) \subset \mathcal{S}$ *and, for* $(v_1, w_1, v_2, w_2; \delta) \in B(\varrho_0) \times (0, \delta_0)$,

(a) $|\Psi_1(v_1, w_1, v_2, w_2; \delta)| \leq C\delta\,|(v_1, w_1, v_2, w_2)| + C\,|(v_1, w_1)|^2$,

(b) $|\Psi_2(v_1, w_1, v_2, w_2; \delta)| \leq C\delta\,|(v_1, w_1, v_2, w_2)|$.

This proposition is proven in Section 3.3.

From now on, we work in the set of local coordinates $(v_1, w_1, v_2, w_2) \in B(\varrho_0)$ given in Proposition 3.2. Then, to prove Theorem 3.1, it remains to extend the unstable manifold from the point $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}})$ given in (3.4) and to analyze for which values of $\delta > 0$ it intersects with the symmetry curve $\mathcal{S}_{\mathrm{loc}}$ given in (3.5), (see Figure 7).

To give an intuition of the proof of Theorem 3.1, in the next lemma, we consider the intersection of the unstable manifold with a convenient "first order" of the symmetry axis $\mathcal{S}_{\mathrm{loc}}$. From now on, we denote by C any positive constant independent of δ.

Lemma 3.4. *Let $\Phi^{\mathrm{u}}(t;\delta)$ be the trajectory of the Hamiltonian system given by $\mathcal{H}$ in Proposition 3.2 with initial condition $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}})$ as given in Proposition 3.3. Then, there exist $N_0 > 0$ and sequences $\{\widehat{T}_n\}_{n\geq N_0}$ and $\{\widehat{\delta}_n\}_{n\geq N_0}$ such that, for $n \geq N_0$,*

$$\Phi^{\mathrm{u}}(\widehat{T}_n; \widehat{\delta}_n) \in \{v_1 + w_1 = 0,\ w_2 = 0\}.$$

Moreover,

$$\widehat{\delta}_n = \sqrt[8]{\frac{8}{21}}\sqrt[4]{\frac{A}{n\pi}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \textit{for } n \geq N_0,$$

where $A > 0$ is the constant introduced in Theorem A.

Proof. Let $(v_1(t), w_1(t), v_2(t), w_2(t))$ be a trajectory of the Hamiltonian system given by $\mathcal{H}$. We want to find $\delta > 0$ such that there exists $T_\delta^0 > 0$ satisfying

$$\begin{gathered}
(v_1(0), w_1(0), v_2(0), w_2(0)) = (v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}), \\
\big(v_1(T_\delta^0), w_1(T_\delta^0), v_2(T_\delta^0), w_2(T_\delta^0)\big) \in \{v_1 + w_1 = 0,\ w_2 = 0\}.
\end{gathered}$$

In other words, using (3.2),

$$v_1^{\mathrm{u}} e^{\nu_1 T_\delta^0} + w_1^{\mathrm{u}} e^{-\nu_1 T_\delta^0} = 0, \tag{3.6}$$

$$\cos(\nu_2 T_\delta^0) w_2^{\mathrm{u}} - \sin(\nu_2 T_\delta^0) v_2^{\mathrm{u}} = 0, \tag{3.7}$$

where, by its definition in (3.3) and Proposition 3.3, one has that

$$\nu_1 = \nu_1(\delta) = 1 + \mathcal{O}\left(\delta^{-\frac{8}{3}} e^{-\frac{4A}{\delta^2}}\right), \qquad \nu_2 = \nu_2(\delta) = \frac{1}{\delta^2}\sqrt{\frac{8}{21}} + \mathcal{O}\left(\delta^2\right). \tag{3.8}$$

For any δ small enough, equation (3.6) has the solution

$$\begin{aligned}
T_\delta^0 = -\frac{1}{2\nu_1}\ln\left(-\frac{v_1^{\mathrm{u}}}{w_1^{\mathrm{u}}}\right) &= \frac{A}{\delta^2} + \frac{2}{3}\log\delta - \log\left(\sqrt[6]{2}\,|\Theta|\,\varrho^{-1}\right) + \mathcal{O}\left(\frac{1}{|\log\delta|}\right) \\
&= \frac{A}{\delta^2}\left(1 + \mathcal{O}(\delta^2\,|\log\delta|)\right).
\end{aligned} \tag{3.9}$$

Next, we study equation (3.7). Let us denote $\theta = \arg\Theta$. From Proposition 3.3,

$$\begin{aligned}
v_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\left[|\Theta|\cos\theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right] \\
w_2^{\mathrm{u}} &= -\sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\left[|\Theta|\sin\theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].
\end{aligned}$$

By Theorem A, one has that $\Theta \neq 0$. Then, (3.7) is equivalent to

$$\cos(\nu_2 T_\delta^0)\sin\theta + \sin(\nu_2 T_\delta^0)\cos\theta = \sin\left(\theta + \nu_2 T_\delta^0\right) = g_0(\delta),$$

where $g_0(\delta)$ contains the higher order terms,

$$g_0(\delta) = -\cos(\nu_2 T_\delta^0)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}w_2^{\mathrm{u}} + \sin\theta\right) - \sin(\nu_2 T_\delta^0)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}v_2^{\mathrm{u}} - \cos\theta\right).$$

and satisfies $g_0(\delta) = \mathcal{O}(|\log\delta|^{-1})$. We deduce then that, for $n \in \mathbb{Z}$,

$$\nu_2 T_\delta^0 + \theta = n\pi - \arcsin g_0(\delta).$$

Using the asymptotic expressions of $\nu_2 = \nu_2(\delta)$ and T_δ^0 in (3.8) and (3.9), we have that δ has to satisfy

$$\frac{A}{\delta^4}\sqrt{\frac{8}{21}}\,(1 + g_1(\delta)) = \pi n,$$

where $g_1(\delta) = \mathcal{O}(\delta^2\,|\log\delta|)$. Therefore, there exists $N_0 > 0$ and a sequence $\{\widehat{\delta}_n\}_{n\geq N_0} \subset (0,\delta_1)$ satisfying the previous equation and the asymptotic expression of the lemma. Finally, one has that $\widehat{T}_n = T_\delta^0$ for $\delta = \widehat{\delta}_n$. □

End of the proof of Theorem 3.1. We proceed analogously to the proof of Lemma 3.4. Let us consider the expressions of $(v_1(t), w_1(t), v_2(t), w_2(t))$ given in (3.2) and $T_\delta > 0$, such that

$$\begin{aligned}(v_1(0), w_1(0), v_2(0), w_2(0)) &= (v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}),\\ (v_1(T_\delta), w_1(T_\delta), v_2(T_\delta), w_2(T_\delta)) &\in \mathcal{S}_{\mathrm{loc}},\end{aligned}$$

with $\mathcal{S}_{\mathrm{loc}} = \{v_1 + w_1 = \Psi_1,\ w_2 = \Psi_2\}$ as given in Proposition 3.3.

First, we deal with the equation $v_1 + w_1 = \Psi_1$. Then, T_δ must satisfy

$$v_1(T_\delta) + w_1(T_\delta) = \Psi_1\left(v_1(T_\delta), w_1(T_\delta), v_2(T_\delta), w_2(T_\delta)\right). \tag{3.10}$$

Let us denote $\tau = \tau(\delta) = T_\delta - T_\delta^0$, with T_δ^0 satisfying $v_1(T_\delta^0) + w_1(T_\delta^0) = 0$ (see equations (3.6) and (3.9)). Then, by (3.2), τ has to satisfy

$$\begin{aligned}v_1(T_\delta^0)e^{\nu_1\tau} + w_1(T_\delta^0)e^{-\nu_1\tau} &= w_1(T_\delta^0)(e^{-\nu_1\tau} - e^{\nu_1\tau})\\ &= \Psi_1(v_1(T_\delta^0+\tau), w_1(T_\delta^0+\tau), v_2(T_\delta^0+\tau), w_2(T_\delta^0+\tau)).\end{aligned}$$

Namely, $\tau(\delta) = F[\tau](\delta)$ with

$$F[\tau](\delta) = \frac{e^{-\nu_1\tau} - e^{\nu_1\tau} + 2\tau\nu_1}{2\nu_1} - \frac{\Psi_1(v_1(t), w_1(t), v_2(t), w_2(t))|_{t=T_\delta^0+\tau}}{2\nu_1 w_1(T_\delta^0)}.$$

First we obtain estimates for $F[0](\delta)$. By Proposition 3.3 and (3.8),

$$
\begin{aligned}
|F[0](\delta)| &\leq \frac{\left|\Psi_1(v_1(T_\delta^0), w_1(T_\delta^0), v_2(T_\delta^0), w_2(T_\delta^0))\right|}{2\left|\nu_1 w_1(T_\delta^0)\right|} \\
&\leq C \frac{\delta \left|(v(T_\delta^0), w(T_\delta^0))\right| + \left|(v_1(T_\delta^0), w_1(T_\delta^0))\right|^2}{\left|w_1(T_\delta^0)\right|}.
\end{aligned}
$$

Let us recall that, by (3.9), we have an asymptotic expression for T_δ^0. Then, by (3.2), (3.8) and Proposition 3.3,

$$
\begin{aligned}
w_1(T_\delta^0) &= w_1^{\mathrm{u}} e^{-\nu_1 T_\delta^0} = \sqrt[6]{2}\,|\Theta|\,\delta^{-\frac{2}{3}} e^{-\frac{A}{\delta^2}} \left[1 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], \\
v_2(T_\delta^0) &= \cos(\nu_2 T_\delta^0) v_2^{\mathrm{u}} + \sin(\nu_2 T_\delta^0) w_2^{\mathrm{u}} = \mathcal{O}\left(\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\right), \\
w_2(T_\delta^0) &= -\sin(\nu_2 T_\delta^0) v_2^{\mathrm{u}} + \cos(\nu_2 T_\delta^0) w_2^{\mathrm{u}} = \mathcal{O}\left(\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\right).
\end{aligned}
\tag{3.11}
$$

Since $v_1(T_\delta^0) = -w_1(T_\delta^0)$, one has that $|F[0](\delta)| \leq C\delta$. Next, we study the Lipschitz constant of the operator F. Let us consider continuous functions $\tau_0, \tau_1 : (0, \delta_0) \to \mathbb{R}$ such that $|\tau_0(\delta)|, |\tau_1(\delta)| \leq C\delta$ and the function $\tau_\sigma = \sigma\tau_1 + (1-\sigma)\tau_0$. Then, by the mean value theorem,

$$
\begin{aligned}
&|F[\tau_1](\delta) - F[\tau_0](\delta)| \leq C\,|\tau_1(\delta) - \tau_0(\delta)| \cdot \\
&\qquad \sup_{\sigma\in[0,1]} \left\{ |\tau_\sigma(\delta)|^2 + \delta^{\frac{2}{3}} e^{\frac{A}{\delta^2}} \left| D\Psi_1(v_1, w_1, v_2, w_2) \cdot (\dot{v}_1, \dot{w}_1, \dot{v}_2, \dot{w}_2)^T \right|_{t=T_\delta^0+\tau_\sigma(\delta)} \right\}.
\end{aligned}
$$

Since Ψ_1 is a real-analytic function, by Proposition 3.3, one has $|D\Psi_1| \leq C\delta + C\,|(v_1, w_1)|$. Moreover, using (3.1), one can obtain estimates for the derivatives $(\dot{v}_1, \dot{w}_1, \dot{v}_2, \dot{w}_2)$. Then,

$$
|F[\tau_1](\delta) - F[\tau_0](\delta)| \leq C\delta\,|\tau_1(\delta) - \tau_0(\delta)|\,.
$$

This implies that, taking $\delta > 0$ small enough, F is well defined and contractive. Hence, F has a fixed point $\tau(\delta)$ such that $|\tau(\delta)| \leq C\delta$. Therefore, there exists T_δ satisfying equation (3.10) such that

$$
T_\delta = T_\delta^0 + \tau(\delta) = \frac{A}{\delta^2}(1 + \mathcal{O}(\delta^2\,|\log\delta|)). \tag{3.12}
$$

Next, we study the equation $w_2 = \Psi_2$. One has that $\delta > 0$ must satisfy

$$
w_2(T_\delta) = \Psi_2\left(v(T_\delta), w(T_\delta)\right). \tag{3.13}
$$

Theorem A implies that $\Theta \neq 0$. Then, by (3.2), δ has to satisfy

$$
\sin\left(\theta + \nu_2 T_\delta\right) = \widehat{g_0}(\delta),
$$

where

$$\widehat{g_0}(\delta) = \Psi_2\left(v(T_\delta), w(T_\delta)\right) - \cos(\nu_2 T_\delta)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}w_2^{\mathrm{u}} + \sin\theta\right)$$
$$- \sin(\nu_2 T_\delta)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}v_2^{\mathrm{u}} - \cos\theta\right).$$

Then, we deduce that, for $n \in \mathbb{Z}$,

$$\nu_2 T_\delta + \theta = n\pi - \arcsin\left(\widehat{g_0}(\delta)\right).$$

By Proposition 3.3 and using the asymptotic expressions in (3.11) and (3.12),

$$|\widehat{g_0}(\delta)| \leq C\delta\,|(v(T_\delta), w(T_\delta))| + \frac{C}{|\log\delta|} \leq C\delta\left|(v(T_\delta^0), w(T_\delta^0))\right| + \frac{C}{|\log\delta|} \leq \frac{C}{|\log\delta|}.$$

Therefore, δ has to satisfy

$$\frac{A}{\delta^4}\sqrt{\frac{8}{21}}\left(1 + \widehat{g_1}(\delta)\right) = \pi n,$$

where $\widehat{g_1}(\delta) = \mathcal{O}(\delta^2\,|\log\delta|)$. Then, there exists $N_0 > 0$ and a sequence $\{\delta_n\}_{n \geq N_0} \subset (0, \delta_0)$ satisfying the statement of the Theorem and that

$$\delta_n = \sqrt[8]{\frac{8}{21}}\sqrt[4]{\frac{A}{n\pi}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \text{for } n \geq N_0.$$

□

3.2 A quantitative Moser normal form

To prove Proposition 3.2, we first introduce a series of affine changes of coordinates in order to put the Hamiltonian $H(\lambda, \Lambda, x, y; \delta)$ in (2.8) in the form considered in [JBL16] (see (3.14) below).

Lemma 3.5. *Fix $c_0, c_1 > 0$. There exists $\delta_0, \widehat{\varrho}_0 > 0$ and a family of affine transformations*

$$\widehat{\phi}_\delta : B(\widehat{\varrho}_0) = \left\{\mathbf{z} \in \mathbb{R}^4 : |\mathbf{z}| < \widehat{\varrho}_0\right\} \to U_{\mathbb{R}}(c_0, c_1)$$
$$(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \mapsto (\lambda, \Lambda, x, y),$$

defined for $\delta \in (0, \delta_0)$, with $\mathcal{C}^1$-functions of δ as coefficients such that the Hamiltonian system given by H (see (2.8)) in the new coordinates and after a scaling in time is Hamiltonian with respect to the canonical form and

$$\begin{aligned}\widehat{H}(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta) &= H\left(\widehat{\phi}_\delta(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2); \delta\right) - H(\mathfrak{L}(\delta); \delta)\\ &= \widehat{v}_1\widehat{w}_1 + \frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2 + \widehat{w}_2^2\right) + \widehat{K}(\widehat{v}_1, \widehat{w}_1)\\ &\quad + \delta\widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta),\end{aligned} \tag{3.14}$$

where $\alpha(\delta)$ *is a* $\mathcal{C}^1$*-function in* δ *satisfying* $\alpha(\delta) = \sqrt{\frac{8}{21}} + \mathcal{O}(\delta^4)$ *and, for* $(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \in B(\widehat{\varrho}_0)$*, there exists a constant* $C > 0$ *independent of* δ *such that*

$$|\widehat{K}(\widehat{v}_1, \widehat{w}_1)| \leq C\,|\widehat{v}_1 + \widehat{w}_1|^3, \qquad |\widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta)| \leq C\,|(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)|^3.$$

Moreover, the change of coordinates satisfies that $\widehat{\phi}_\delta(0) = \mathfrak{L}(\delta)$ *and*

$$D\widehat{\phi}_0 = \begin{pmatrix} \frac{2}{\sqrt{7}} & \frac{2}{\sqrt{7}} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \sqrt[4]{\frac{2}{21}} & i\sqrt[4]{\frac{2}{21}} \\ 0 & 0 & \sqrt[4]{\frac{2}{21}} & -i\sqrt[4]{\frac{2}{21}}, \end{pmatrix}, \quad D\widehat{\phi}_0^{-1} = \begin{pmatrix} \frac{\sqrt{7}}{4} & -\sqrt{3} & 0 & 0 \\ \frac{\sqrt{7}}{4} & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt[4]{\frac{21}{32}} & \sqrt[4]{\frac{21}{32}} \\ 0 & 0 & -i\sqrt[4]{\frac{21}{32}} & i\sqrt[4]{\frac{21}{32}} \end{pmatrix}.$$

Proof. The proof of this lemma relies on the approach and techniques of [JBL16]. For technical reasons and to be consistent with [JBL16], we consider the Poincaré Hamiltonian $H^{\mathrm{Poi}}(\lambda, L, \eta, \xi; \mu)$ introduced in (2.1) instead of the scaled version H defined in (2.8). Let us denote the point L_3 in Poincaré coordinates (λ, L, η, ξ) as $L_3^{\mathrm{Poi}} = (\phi^{\mathrm{Poi}})^{-1}(L_3)$. Therefore, L_3^{Poi} is a saddle-center equilibrium point of the system given by H^{Poi} and, by (2.3), it satisfies that

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu) = (1, 0, 0) + \mathcal{O}(\delta^4).$$

We perform several changes of coordinates.

1. Translation of the equilibrium point. Let $\phi^{\mathrm{eq}} : (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi}) \to (\lambda, L, \eta, \xi)$ be the translation such that $\phi^{\mathrm{eq}}(0) = L_3^{\mathrm{Poi}}$. Then, the Hamiltonian system associated to H^{Poi} in the new coordinates defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\widetilde{L} + i d\widetilde{\eta} \wedge d\widetilde{\xi}$ and the Hamiltonian

   $$H^{\mathrm{eq}} = H^{\mathrm{Poi}} \circ \phi^{\mathrm{eq}} - H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; \mu).$$

   Denoting $\widetilde{\mathbf{z}} = (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi})$, $H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu)$ can be written as

   $$H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) = H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) + R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) + R_3^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu),$$

   with

   $$\begin{aligned} H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) &= \frac{1}{2} D^2 H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; 0)[\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}] = -\frac{3}{2}\widetilde{L}^2 + \widetilde{\eta}\widetilde{\xi}, \\ R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) &= \frac{1}{2} D^2 H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; \mu)[\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}] - H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) = \mathcal{O}(\mu\,|\widetilde{\mathbf{z}}|^2), \\ R_3^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) &= (H^{\mathrm{Poi}} \circ \phi^{\mathrm{eq}})(\widetilde{\mathbf{z}}; \mu) - H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) - R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) - H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; \mu) \\ &= \mathcal{O}(\widetilde{L}^3) + \mathcal{O}(\mu\,|\widetilde{\mathbf{z}}|^3), \end{aligned} \tag{3.15}$$

   where we have used that $\widetilde{L} = L - 1 + \mathcal{O}(\mu)$, $\widetilde{\eta} = \eta + \mathcal{O}(\mu)$ and $\widetilde{\xi} = \xi + \mathcal{O}(\mu)$. Notice that as a result, for $\mu > 0$, $\widetilde{\mathbf{z}} = 0$ is a saddle-center point of the system given by the Hamiltonian $H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu)$.

2. Reduction of the terms of order 2. Following the strategy of the proof of [JBL16, Theorem 1.3] in our setting, for $\mu \geq 0$, there exists a family $\phi_\mu^{\text{red}} : \mathbf{x} = (x_\lambda, x_L, x_\eta, x_\xi) \mapsto \widetilde{\mathbf{z}} = (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi})$ of real-analytic linear diffeomorphisms satisfying that $D\phi_0^{\text{red}}(0) = \mathbf{Id}$ and that

$$H^{\text{red}}(\mathbf{x};\mu) = (H^{\text{eq}} \circ \phi_\mu^{\text{red}})(\mathbf{x};\mu) = H_0^{\text{eq}}(\mathbf{x}) + R_2^{\text{red}}(\mathbf{x};\mu) + R_3^{\text{red}}(\mathbf{x};\mu),$$

where $R_2^{\text{red}}(\mathbf{x};\mu)$ is a real polynomial of degree 2 in $\mathbf{x}$ with $\mathcal{C}^1$-functions of μ as coefficients and

$$R_2^{\text{red}}(\mathbf{x};\mu) = \mathcal{O}(\mu\,|\mathbf{x}|^2), \qquad \left\{H_0^{\text{eq}} \circ \mathbf{J}, R_2^{\text{red}}\right\} = 0, \qquad R_3^{\text{red}}(\mathbf{x};\mu) = \mathcal{O}(|\mathbf{x}|^3),$$

where $\mathbf{J}$ is the matrix associated to the symplectic form $dx_\lambda \wedge dx_L + i dx_\eta \wedge dx_\xi$. The fact that $\left\{H_0^{\text{eq}} \circ \mathbf{J}, R_2^{\text{red}}\right\} = 0$ and that R_2^{red} is a homogeneous polynomial of degree 2 and $\mathcal{O}(\mu|\mathbf{x}|^2)$ imply that there exist $\mathcal{C}^1$-functions $\sigma_1(\mu), \sigma_2(\mu) = \mathcal{O}(1)$ such that

$$R_2^{\text{red}}(x_\lambda, x_L, x_\eta, x_\xi;\mu) = \mu\sigma_1(\mu)\frac{x_\lambda^2}{2} + \mu\sigma_2(\mu)x_\eta x_\xi.$$

Since ϕ_μ^{red} is linear and taking into account that $D\phi_0^{\text{red}}(0) = \mathbf{Id}$ and the definition of the potential $V(\lambda)$ in (2.7), one has that

$$\sigma_1(0) = \frac{1}{\mu}\partial_\lambda^2 H^{\text{Poi}}(L_3^{\text{Poi}};\mu)\Big|_{\mu=0} = \partial_\lambda^2 H_1^{\text{Poi}}(0,1,0,0;0) = V''(0) = \frac{7}{8}. \tag{3.16}$$

Therefore, by (3.15), one has that

$$H^{\text{red}}(\mathbf{x};\mu) = -\frac{3}{2}x_L^2 + \mu\sigma_1(\mu)\frac{x_\lambda^2}{2} + (1 + \mu\sigma_2(\mu))x_\eta x_\xi + R_3^{\text{red}}(\mathbf{x};\mu).$$

In addition, since the terms of order 3 and higher of H^{eq} are of the form $\mathcal{O}(\widetilde{L}^3) + \mathcal{O}(\mu\,|\widetilde{\mathbf{x}}|^3)$ (see (3.15)), one has that

$$R_3^{\text{red}}(\mathbf{x};\mu) = \mathcal{O}(x_L^3) + \mathcal{O}(\mu\,|\mathbf{x}|^3).$$

3. Symplectic scaling. We rename the parameter $\delta = \mu^{\frac{1}{4}}$ (see (2.6)) and, similarly to (2.5), we consider $\phi^{\text{sca}} : \mathbf{y} = (y_\lambda, y_L, y_\eta, y_\xi) \mapsto \mathbf{x} = (x_\lambda, x_L, x_\eta, x_\xi)$ such that

$$x_\lambda = \frac{1}{\sqrt{\sigma_1(\delta^4)}}\,y_\lambda, \quad x_L = \frac{\delta^2}{\sqrt{3}}y_L, \quad x_\eta = \frac{\delta}{\sqrt[4]{3\sigma_1(\delta^4)}}y_\eta, \quad x_\xi = \frac{\delta}{\sqrt[4]{3\sigma_1(\delta^4)}}y_\xi,$$

and a scaling in time by a factor of $\delta^2\sqrt{3\sigma_1(\mu)}$. The Hamiltonian system of H^{red} expressed in these coordinates defines a system associated with the form $dy_\lambda \wedge dy_L + i dy_\eta \wedge dy_\xi$ and the Hamiltonian

$$H^{\text{sca}}(y;\delta) = \frac{1}{2}\left(y_\lambda^2 - y_L^2\right) + \alpha(\delta)\frac{y_\eta y_\xi}{\delta^2} + K^{\text{sca}}(y_\lambda) + \delta H_1^{\text{sca}}(\mathbf{y};\delta), \tag{3.17}$$

where

$$\alpha(\delta) = \frac{1+\delta^4\sigma_2(\delta^4)}{\sqrt{3\sigma_1(\delta^4)}} = \sqrt{\frac{8}{21}} + \mathcal{O}(\delta^4),$$

$$K^{\mathrm{sca}}(y_\lambda) = \frac{1}{\delta^4} R_3^{\mathrm{red}}\left(\frac{y_\lambda}{\sqrt{\sigma_1(0)}}, 0,0,0;0\right) = \mathcal{O}(y_\lambda^3),$$

$$\delta H_1^{\mathrm{sca}}(\mathbf{y};\delta) = \frac{1}{\delta^4} R_3^{\mathrm{red}}\left(\phi^{\mathrm{sca}}(\mathbf{y});\delta^4\right) - K^{\mathrm{sca}}(y_\lambda) = \mathcal{O}(\delta\,|\mathbf{y}|^3),$$

where we have used Cauchy estimates to bound DR_3^{red}.

4. Diagonalization. Consider the symplectic change of coordinates $\phi^{\mathrm{diag}} : (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \mapsto \mathbf{y} = (y_\lambda, y_L, y_\eta, y_\xi)$ defined by

$$\begin{pmatrix} y_\lambda \\ y_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \widehat{v}_1 \\ \widehat{w}_1 \end{pmatrix}, \qquad \begin{pmatrix} y_\eta \\ y_\xi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \widehat{v}_2 \\ \widehat{w}_2 \end{pmatrix}.$$

Then, the Hamiltonian system associated to (3.17) expressed in these coordinates defines a Hamiltonian system with respect to the form $d\widehat{v}_1 \wedge d\widehat{w}_1 + d\widehat{v}_2 \wedge d\widehat{w}_2$ and the Hamiltonian

$$\begin{aligned} \widehat{H}(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta) =& \widehat{v}_1\widehat{w}_1 + \frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2 + \widehat{w}_2^2\right) + \widehat{K}(\widehat{v}_1, \widehat{w}_1) \\ &+ \delta \widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta), \end{aligned} \tag{3.18}$$

where

$$\widehat{K}(\widehat{v}_1, \widehat{w}_1) = K^{\mathrm{sca}}\left(\frac{\widehat{v}_1 + \widehat{w}_1}{\sqrt{2}}\right) = \mathcal{O}\left(|\widehat{v}_1 + \widehat{w}_1|^3\right), \qquad \widehat{H}_1 = H_1^{\mathrm{sca}} \circ \phi^{\mathrm{diag}}.$$

□

Next proposition provides a normal form in a neighborhood of the saddle-center equilibrium point. It is a direct consequence of [JBL16, Proposition C.1]. In order to use this result, we introduce the artificial parameter $\nu > 0$ and rewrite $\widehat{H}$ in (3.14) as

$$\begin{aligned} \widehat{H}(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta,\nu) =& \widehat{v}_1\widehat{w}_1 + \frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2 + \widehat{w}_2^2\right) + K(\widehat{v}_1, \widehat{w}_1) \\ &+ \nu \widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta). \end{aligned} \tag{3.19}$$

Note that we are interested in the case $\nu = \delta$.

Proposition 3.6. *There exist $\delta_0, \varrho_0 > 0$ and a family of analytic canonical changes of coordinates, defined for $\nu \in [0, \delta_0)$ and $\delta \in (0, \delta_0)$,*

$$\begin{aligned} \widehat{\mathcal{F}}_{\delta,\nu} = (\varphi_{1,\nu}, \psi_{1,\nu}, \varphi_{2,\nu}, \psi_{2,\nu}) : B(\varrho_0) &\to B(\widehat{\varrho}_0) \subset \mathbb{R}^4 \\ (v_1, w_1, v_2, w_2) &\mapsto (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2), \end{aligned}$$

such that the Hamiltonian $\widehat{H}$ *in* (3.19) *in the new coordinates reads*

$$\begin{aligned}\mathcal{H}(v_1,w_1,v_2,w_2;\delta,\nu) &= \widehat{H}\left(\widehat{\mathcal{F}}_{\delta,\nu}(v_1,w_1,v_2,w_2);\delta,\nu\right)\\ &= v_1w_1 + \frac{\alpha(\delta)}{2\delta^2}\left(v_2^2+w_2^2\right) + \mathcal{R}(v_2w_2, v_2^2+w_2^2)\end{aligned}$$

where $\mathcal{R}$ *satisfies that, for* $(v_1,w_1,v_2,w_2)\in B(\varrho_0)$,

$$\left|\mathcal{R}(v_1w_1, v_2^2+w_2^2;\delta)\right| \leq C|(v_1w_1, v_2^2+w_2^2)|^2,$$

for some $C>0$ *independent of* δ *and* ν.

In addition, for all $(v_1,w_1,v_2,w_2)\in B(\varrho_0)$ *and all* $(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)\in B(\widehat{\varrho}_0)$, *the individual components of the change of coordinates satisfy*

(1) $|\varphi_{1,\nu}(v_1,w_1,v_2,w_2)-v_1| \leq C\left\{|(v_1,w_1)|^2 + \nu\,|(v_1,w_1,v_2,w_2)|^2\right\}$,
$|\varphi_{1,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{v}_1| \leq C\left\{|(\widehat{v}_1,\widehat{w}_1)|^2 + \nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2\right\}$,

(2) $|\psi_{1,\nu}(v_1,w_1,v_2,w_2)-w_1| \leq C\left\{|(v_1,w_1)|^2 + \nu\,|(v_1,w_1,v_2,w_2)|^2\right\}$,
$|\psi_{1,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{w}_1| \leq C\left\{|(\widehat{v}_1,\widehat{w}_1)|^2 + \nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2\right\}$,

(3) $|\varphi_{2,\nu}(v_1,w_1,v_2,w_2)-v_2| \leq C\nu\,|(v_1,w_1,v_2,w_2)|^2$,
$|\varphi_{2,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{v}_2| \leq C\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2$,

(4) $|\psi_{2,\nu}(v_1,w_1,v_2,w_2)-w_2| \leq C\nu\,|(v_1,w_1,v_2,w_2)|^2$,
$|\psi_{2,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{w}_2| \leq C\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2$.

Proposition 3.2 is a direct consequence of Lemma 3.5 and Proposition 3.6.

3.3 The invariant manifolds in normal form variables

To prove Proposition 3.3, we translate the results in Theorem 2.4 (Statement 1) and the axis of symmetry $\mathcal{S}$ (Statement 2) into the set of coordinates (v_1,w_1,v_2,w_2) given in Proposition 3.2. Recall that in the proof of Proposition 3.2, we have used the "intermediate" system of coordinates $(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)$. We translate first the results via the change of coordinates $\widehat{\phi}_\delta:(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)\to(\lambda,\Lambda,x,y)$, given by Lemma 3.5. Then, we apply the second change of coordinates $\widehat{\mathcal{F}}_{\delta,\nu}:(v_1,w_1,v_2,w_2)\to(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)$ with $\nu=\delta$, given by Proposition 3.6.

Statement 1: Let $\lambda_*\in[\lambda_1,\lambda_2]\subset(0,\lambda_0)$ to be chosen later and consider the section $\Sigma(\lambda_*)=\{\lambda=\lambda_*,\Lambda>0\}$. Let $\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*)$ and $\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)$ be the first intersections of the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section $\Sigma(\lambda_*)$, respectively.

Let us recall that, by Proposition 2.3, the critical point $\mathfrak{L}(\delta)$ in (λ,Λ,x,y) coordinates is of the form $\mathfrak{L}(\delta)=\left(0,\delta^2\mathfrak{L}_\Lambda(\delta),\delta^3\mathfrak{L}_x(\delta),\delta^3\mathfrak{L}_y(\delta)\right)^T$, with $\mathfrak{L}_\Lambda,\mathfrak{L}_x,\mathfrak{L}_y=\mathcal{O}(1)$. Then, applying the change of coordinates $\widehat{\phi}_\delta$ given in Lemma 3.5, there exist $\mathcal{C}^1$ functions $\gamma_1, \gamma_2 : (0, \delta_0) \to \mathbb{R}^4$ satisfying $\gamma_1, \gamma_2 = \mathcal{O}(1)$ such that

$$\begin{aligned}\widehat{\Sigma}(\lambda_*, \delta) = \widehat{\phi}_\delta\left(\Sigma(\lambda_*)\right) = \Big\{&\widehat{v}_1 + \widehat{w}_1 + \delta\langle\gamma_1(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle = \tfrac{\sqrt{7}}{2}\lambda_*,\\ &\widehat{w}_1 - \widehat{v}_1 + \delta\langle\gamma_2(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle + \delta^2\sqrt{6}\mathfrak{L}_\Lambda(\delta) > 0\Big\}.\end{aligned} \tag{3.20}$$

Notice that $\widehat{\Sigma}(\lambda_*, 0) = \{\widehat{v}_1 + \widehat{w}_1 = \frac{\sqrt{7}}{2}\lambda_*,\ \widehat{w}_1 > \widehat{v}_1\}$. Moreover, we denote

$$\begin{aligned}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}) &= \widehat{\phi}_\delta^{-1}\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*)\right) \in \widehat{\Sigma}(\lambda_*, \delta),\\ (\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}) &= \widehat{\phi}_\delta^{-1}\left(\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)\right) \in \widehat{\Sigma}(\lambda_*, \delta).\end{aligned}$$

Since $\widehat{\phi}_\delta$ is an affine transformation, by Theorem 2.4 and Lemma 3.5, one has that

$$\begin{aligned}\widehat{v}_1^{\mathrm{u}} - \widehat{v}_1^{\mathrm{s}} &= \left[\left(\frac{\sqrt{7}}{4}, -\sqrt{3}, 0, 0\right) + \mathcal{O}(\delta)\right]\cdot\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*) - \mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\right),\\ \widehat{w}_1^{\mathrm{u}} - \widehat{w}_1^{\mathrm{s}} &= \left[\left(\frac{\sqrt{7}}{4}, \sqrt{3}, 0, 0\right) + \mathcal{O}(\delta)\right]\cdot\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*) - \mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\right),\\ \widehat{v}_2^{\mathrm{u}} - \widehat{v}_2^{\mathrm{s}} &= \left[\left(0, 0, \sqrt[4]{\frac{21}{32}}, \sqrt[4]{\frac{21}{32}}\right) + \mathcal{O}(\delta)\right]\cdot\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*) - \mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)\right)\\ &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left[\operatorname{Re}\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right],\\ \widehat{w}_2^{\mathrm{u}} - \widehat{w}_2^{\mathrm{s}} &= \left[\left(0, 0, -i\sqrt[4]{\frac{21}{32}}, i\sqrt[4]{\frac{21}{32}}\right) + \mathcal{O}(\delta)\right]\cdot\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*) - \mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)\right)\\ &= -\sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left[\operatorname{Im}\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].\end{aligned} \tag{3.21}$$

Next, we consider the change of coordinates $\widehat{\mathcal{F}}_{\delta,\nu}$ with $\nu = \delta$ given in Proposition 3.6. Let us denote

$$(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}) = \widehat{\mathcal{F}}_{\delta,\delta}^{-1}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}), \quad (v_1^{\mathrm{s}}, w_1^{\mathrm{s}}, v_2^{\mathrm{s}}, w_2^{\mathrm{s}}) = \widehat{\mathcal{F}}_{\delta,\delta}^{-1}(\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}). \tag{3.22}$$

Since the local stable manifold is given by $\{v_1 = v_2 = w_2 = 0\}$ (see (3.2)), one has that $v_1^{\mathrm{s}} = v_2^{\mathrm{s}} = w_2^{\mathrm{s}} = 0$ and we call $\varrho = w_1^{\mathrm{s}}$, (see Figure 12). Taking into account that $\widehat{\mathcal{F}}_{\delta,\delta}(0, \varrho, 0, 0) = (\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}) \in \widehat{\Sigma}(\lambda_*, \delta)$ for $\lambda \in [\lambda_1, \lambda_2]$, by (3.20), the value λ_* must satisfy

$$\lambda^* = \sqrt{\frac{7}{4}}\left[\widehat{v}_1^{\mathrm{s}} + \widehat{w}_1^{\mathrm{s}} + \delta^4\langle\gamma_1(\delta), \widehat{\mathcal{F}}_{\delta,\delta}(0, \varrho, 0, 0)\rangle\right].$$

Then, using the notation $\widehat{\mathcal{F}}_{\delta,\delta} = (\varphi_{1,\delta}, \psi_{1,\delta}, \varphi_{2,\delta}, \psi_{2,\delta})$, by Proposition 3.6, one has that for $\varrho \in (0, \varrho_0)$ and $\delta > 0$ small enough,

$$\lambda_* = \sqrt{\frac{7}{4}}\varrho\left(1 + \mathcal{O}(\varrho, \delta)\right).$$

Then, it is clear that taking, for instance, $\varrho = \varrho_0/2$, the corresponding λ_* belongs to a closed interval in $(0, \lambda_0)$ independent of δ.

Next, we consider the difference between $(v_1^{\rm u}, w_1^{\rm u}, v_2^{\rm u}, w_2^{\rm u})$ and $(v_1^{\rm s}, w_1^{\rm s}, v_2^{\rm s}, w_2^{\rm s}) = (0, \varrho, 0, 0)$. By (3.22), one has that

$$\begin{aligned} w_1^{\rm u} - \varrho &= \psi_{1,\delta}^{-1}(\widehat{v}_1^{\rm u}, \widehat{w}_1^{\rm u}, \widehat{v}_2^{\rm u}, \widehat{w}_2^{\rm u}) - \psi_{1,\delta}^{-1}(\widehat{v}_1^{\rm s}, \widehat{w}_1^{\rm s}, \widehat{v}_2^{\rm s}, \widehat{w}_2^{\rm s}), \\ v_2^{\rm u} &= \varphi_{2,\delta}^{-1}(\widehat{v}_1^{\rm u}, \widehat{w}_1^{\rm u}, \widehat{v}_2^{\rm u}, \widehat{w}_2^{\rm u}) - \varphi_{2,\delta}^{-1}(\widehat{v}_1^{\rm s}, \widehat{w}_1^{\rm s}, \widehat{v}_2^{\rm s}, \widehat{w}_2^{\rm s}), \\ w_2^{\rm u} &= \psi_{2,\delta}^{-1}(\widehat{v}_1^{\rm u}, \widehat{w}_1^{\rm u}, \widehat{v}_2^{\rm u}, \widehat{w}_2^{\rm u}) - \psi_{2,\delta}^{-1}(\widehat{v}_1^{\rm s}, \widehat{w}_1^{\rm s}, \widehat{v}_2^{\rm s}, \widehat{w}_2^{\rm s}). \end{aligned}$$

For $w_1^{\rm u} - \varrho$, by the mean value theorem, Proposition 3.6 and (3.21), one obtains

$$|w_1^{\rm u} - \varrho| \leq C\,|\widehat{v}_1^{\rm u} - \widehat{v}_1^{\rm s}| + C\,|\widehat{w}_1^{\rm u} - \widehat{w}_1^{\rm s}| + C\delta\,|\widehat{v}_2^{\rm u} - \widehat{v}_2^{\rm s}| + C\delta\,|\widehat{w}_2^{\rm u} - \widehat{w}_2^{\rm s}| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}.$$

Analogously, for $v_2^{\rm u}$, one has that

$$|v_2^{\rm u} - (\widehat{v}_2^{\rm u} - \widehat{v}_2^{\rm s})| \leq C\delta\,|(\widehat{v}_1^{\rm u} - \widehat{v}_1^{\rm s}, \widehat{w}_1^{\rm u} - \widehat{w}_1^{\rm s}, \widehat{v}_2^{\rm u} - \widehat{v}_2^{\rm s}, \widehat{w}_2^{\rm u} - \widehat{w}_2^{\rm s})| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}$$

and, by (3.21), one obtains the expression of the statement for $v_2^{\rm u}$. An analogous estimate holds for $w_2^{\rm u}$. Lastly, by the expression of Hamiltonian $\mathcal{H}$ in Proposition 3.6, one sees that $\mathcal{H}(v_1^{\rm u}, w_1^{\rm u}, v_2^{\rm u}, w_2^{\rm u}) = \mathcal{H}(0,0,0,0) = 0$ and obtains the expression for $v_1^{\rm u}$.

Statement 2: Let us consider the symmetry axis $\mathcal{S} = \{\lambda = 0, x = y\}$ given in (2.15). Notice that, by Proposition 2.3, one has that $\widehat{\phi}_\delta(0) = \mathfrak{L}(\delta) \in \mathcal{S}$. Then, applying the affine transformation $\widehat{\phi}_\delta$ given in Lemma 3.5, there exist functions $\gamma_3, \gamma_4 : (0, \delta_0) \to \mathbb{R}$ satisfying $\gamma_3, \gamma_4 = \mathcal{O}(1)$ such that

$$\widehat{\phi}_\delta(\mathcal{S}) = \left\{ \widehat{v}_1 + \widehat{w}_1 + \delta\langle\gamma_3(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle = 0,\ \ \widehat{w}_2 + \delta\langle\gamma_4(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle = 0 \right\}.$$

Then, applying the change of coordinates $\widehat{\mathcal{F}}_{\delta,\delta}$, one has that

$$\mathcal{S}_{\rm loc} = \left\{ v_1 + w_1 = \Psi_1(v_1, w_1, v_2, w_2; \delta),\ w_2 = \Psi_2(v_1, w_1, v_2, w_2; \delta) \right\},$$

where

$$\begin{aligned} \Psi_1 &= (\varphi_{1,\delta} - v_1) + (\psi_{1,\delta} - w_1) + \delta\langle\gamma_3(\delta), \mathcal{F}_{\delta,\delta}\rangle, \\ \Psi_2 &= (\psi_{2,\delta} - w_2) + \delta\langle\gamma_4(\delta), \mathcal{F}_{\delta,\delta}\rangle. \end{aligned}$$

Then, Proposition 3.6 implies that for $(v_1, w_1, v_2, w_2) \in B(\varrho_0)$ and $\delta > 0$ small enough,

$$\begin{aligned} |\Psi_1(v_1, w_1, v_2, w_2; \delta)| &\leq C\delta\,|(v_1, w_1, v_2, w_2)| + C\,|(v_1, w_1)|^2, \\ |\Psi_2(v_1, w_1, v_2, w_2; \delta)| &\leq C\delta\,|(v_1, w_1, v_2, w_2)|. \end{aligned}$$

4 The invariant manifolds of the Lyapunov orbits: Proof of Theorems C and D

The goal of this section is to prove Item 1 of Theorem C and Theorem D. First we rephrase these results referred to the Hamiltonian (2.8) (recall that $\delta = \mu^{\frac{1}{4}}$, see (2.6)). We begin with the existence of the Lyapunov periodic orbits given by Proposition 1.3. Note that the Lyapunov Center Theorem (see for instance [MO17]) ensures the existence of a family of periodic orbits emanating from a saddle-center equilibrium point. In our setting, this family corresponds to perturbed orbits of the fast oscillator, centered at $\mathfrak{L}(\delta)$, and therefore the existence of the periodic orbits given by Proposition 1.3 is just a consequence of this classical theorem. However, we need to "reprove" it to have estimates for the periodic orbits.

First, we introduce the following notation. For $d > 0$, we denote

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \qquad \mathbb{T}_d = \{\tau \in \mathbb{C}/2\pi\mathbb{Z} \, : \, |\mathrm{Im}\, \tau| < d\} \, . \tag{4.1}$$

Proposition 4.1. *Let $d, c_0, c_1 > 0$. There exist $\rho_0, \delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, there exists a family of periodic orbits $\{\mathfrak{P}_\rho(\tau;\delta) \, : \, \tau \in \mathbb{T}_d\}_{\rho\in[0,\rho_0]}$, where $\mathfrak{P}_\rho : \mathbb{T}_d \to U_{\mathbb{C}}(c_0, c_1)$ are real-analytic functions satisfying that*

$$H(\mathfrak{P}_\rho(\tau;\delta)) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta)).$$

Furthermore, there exist $\omega_{\rho,\delta} > 0$ and a constant $b_2 > 0$, independent of ρ and δ, such that the parametrization of the periodic orbit satisfies

$$\dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2} \qquad \textit{with} \qquad |\omega_{\rho,\delta} - 1| \le b_2 \delta^4.$$

In addition, the parametrization can be written as

$$\mathfrak{P}_\rho(\tau;\delta) = \mathfrak{L}(\delta) + \rho \cdot \left(0, 0, e^{-i\tau}, e^{i\tau}\right)^T + \delta\rho \cdot \left(\lambda_{\mathfrak{P}}, \Lambda_{\mathfrak{P}}, x_{\mathfrak{P}}, y_{\mathfrak{P}}\right)^T(\tau), \tag{4.2}$$

where $|\lambda_{\mathfrak{P}}(\tau)|, |\Lambda_{\mathfrak{P}}(\tau)| \le b_2$, and $|x_{\mathfrak{P}}(\tau)|, |y_{\mathfrak{P}}(\tau)| \le b_2\delta^3$.

The proof of this proposition can be found in Appendix A.

Let us denote by $\mathcal{W}^{\mathrm{u}}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s}}(\mathfrak{P}_\rho)$ the 2-dimensional unstable and stable manifolds of the periodic orbit $\mathfrak{P}_\rho(\cdot, \delta)$. Analogously to the invariant manifolds of $\mathfrak{L}(\delta)$, we denote each branch as $\mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\diamond,-}(\mathfrak{P}_\rho)$ for $\diamond \in \{\mathrm{u}, \mathrm{s}\}$ (see Figure 4). To prove Theorem C and D, we focus on the study of the "+" invariant manifolds. By symmetry, there exist analogous results for the "−" invariant manifolds.

We look for intersections between $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ in the 2-dimensional section

$$\Sigma_\rho = \left\{(\lambda, \Lambda, x, y) \in U_{\mathbb{R}}(c_0, c_1) : \Lambda = \delta^2 \mathfrak{L}_\Lambda(\delta), \, H(\lambda, \Lambda, x, y) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta))\right\}, \tag{4.3}$$

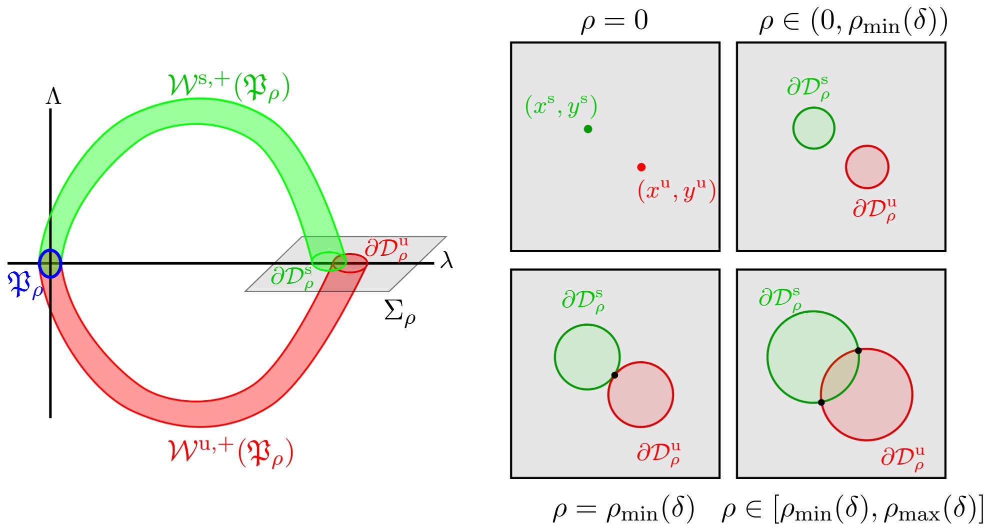


Figure 8: Intersection of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ with section Σ_ρ. The pictures in the right show the different possibilities given in Corollary 2.5 and Theorem 4.2.

where $\mathfrak{L} = (0, \delta^2\mathfrak{L}_\Lambda, \delta^3\mathfrak{L}_x, \delta^3\mathfrak{L}_y)^T$ as given in Proposition 2.3 and $U_{\mathbb{R}}(c_0, c_1)$ is the domain introduced in (2.12). Note that this definition is consistent with that of Σ_0 in (2.19) and that, by Proposition 4.1, the periodic orbit $\mathfrak{P}_\rho$ belongs to the energy level $H = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta))$ where Σ_ρ is included.

In the next result, we see that the 2-dimensional invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ intersect in the section Σ_ρ for certain values of ρ (see Figure 8). Note that the intersection of the invariant manifolds for $\rho = 0$ has been analyzed in Corollary 2.5. Both Item 1 of Theorem C and Theorem D are a consequence of the following result.

Theorem 4.2. *Let ρ_0 and $\mathfrak{P}_\rho$, for $\rho \in [0, \rho_0]$, be as given in Proposition 4.1. Then, the following is satisfied.*

- *There exists $\delta_0 > 0$ such that, for every $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$, the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ intersect the section Σ_ρ. The first intersection is given by closed curves, which we denote by $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$.*

- *Let $R > 1$. There exists $\delta_R > 0$, satisfying $\lim_{R\to\infty} \delta_R = 0$, and functions $\rho_{\min}, \rho_{\max} : (0, \delta_R) \to [0, \rho_0]$ such that, for $\delta \in (0, \delta_R)$ and $\rho \in [\rho_{\min}(\delta), \rho_{\max}(\delta)]$, the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ intersect. Moreover,*

$$\rho_{\min}(\delta) = \frac{\sqrt[6]{2}}{2}|\Theta|\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[1 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right],$$

$$\rho_{\max}(\delta) = \frac{\sqrt[6]{2}}{2}|\Theta|\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[R + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].$$

- *For $\rho \in (\rho_{\min}(\delta), \rho_{\max}(\delta)]$, the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ intersect transversally at least twice.*

- *For $\rho = \rho_{\min}(\delta)$, the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ have at least one quadratic tangency at a point $Q_0 \in \partial\mathcal{D}^{\mathrm{u}}_\rho \cap \partial\mathcal{D}^{\mathrm{s}}_\rho$.*
- *Fix $\delta \in (0, \delta_0)$ and let ζ be any smooth curve transverse to $\partial\mathcal{D}^{\mathrm{u}}_{\rho_{\min}}$ and $\partial\mathcal{D}^{\mathrm{s}}_{\rho_{\min}}$ within $\Sigma_{\rho_{\min}}$ at Q_0. Then, for ρ close to $\rho_{\min}$, the local intersections of $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ with the curve ζ cross each other with relative non-zero velocity at $(Q_0, \rho_{\min})$.*

Theorem 4.2 implies in particular that, for small values of δ, there exist transverse intersections between some unstable and stable manifolds of Lyapunov periodic orbits of $\mathfrak{L}(\delta)$. By symmetry, an analogous result holds for $\mathcal{W}^{\mathrm{u},-}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},-}(\mathfrak{L})$. This proves Item 1 of Theorem C.

Moreover, the last two statements of Theorem 4.2 imply the existence of a generic unfolding of a quadratic tangency between $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ (we follow the definition of generic unfolding given in [Dua08]). Indeed, denoting by f_ϱ to the flow of h in (1.1) restricted to the energy level $h = \varrho + h(L_3)$, for $\delta \in (0, \delta_0)$, one has that f_ϱ unfolds generically an homoclinic quadratic tangency. Finally, noticing that the energy level $H(\lambda, \Lambda, x, y; \delta) = \frac{\varrho^2}{\delta^2} + H(\mathfrak{L})$ corresponds to $h(q, p; \mu) = \sqrt{\mu}\rho^2 + h(L_3)$ (see (2.6) and (2.5)), one proves Theorem D.

The rest of this section is devoted to prove Theorem 4.2. First, in Section 4.1, we sum up the results concerning the unperturbed separatrix of the Hamiltonian H_{p} in (2.9) presented in [BGG22]. Next, in Section 4.2, we obtain and analyze parametrizations of the unstable and stable manifolds of the Lyapunov periodic orbits given in Proposition 4.1. Last, in Section 4.3, we analyze the intersections between these manifolds to complete the proof of Theorem 4.2.

Throughout this section and the following ones, we denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3, f_4)^T$, unless stated otherwise. In addition, we denote the canonical basis of $\mathbb{C}^4$ by $\{\mathbf{e_j}\}_{j=1..4}$.

4.1 The unperturbed separatrix

Let us consider the unperturbed Hamiltonian H_0 as given in (2.17). Notice that the plane $\{x = y = 0\}$ is invariant for H_0 and the dynamics on it is described by

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad V(\lambda) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}},$$

(see (2.9)). The origin $(\lambda, \Lambda) = (0, 0)$ is a saddle with two separatrices associated to it (see Figure 5). In [BGG22], we studied their real-analytic time-parametrizations. The following result summarizes Theorem 2.2 and Corollary 2.4 in [BGG22] and it establishes a suitable domain for these parametrizations, which we denote as

$$\sigma_{\mathrm{p}}(u) = (\lambda_{\mathrm{p}}(u), \Lambda_{\mathrm{p}}(u), 0, 0)^T. \tag{4.4}$$

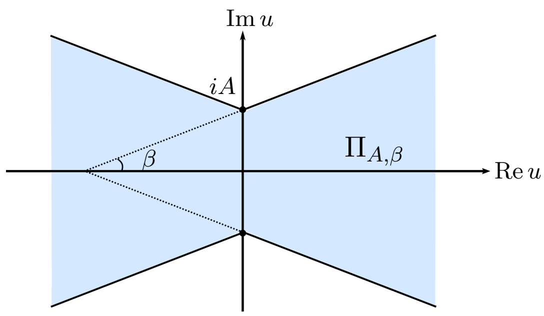


Figure 9: Representation of the domain $\Pi_{A,\beta}$ in (4.5).

Proposition 4.3. *Let $\lambda_0 > 0$ be as given in (2.18). There exists $0 < \beta < \frac{\pi}{2}$ such that the time-parametrization $(\lambda_{\mathrm{p}}(u), \Lambda_{\mathrm{p}}(u))$ of the right separatrix (i.e, $\lambda_{\mathrm{p}}(u) \in (0, \pi)$) of H_{p} with $(\lambda_{\mathrm{p}}(0), \Lambda_{\mathrm{p}}(0)) = (\lambda_0, 0)$ extends analytically to*

$$\begin{aligned} \Pi_{A,\beta} = &\{u \in \mathbb{C} \,:\, |\mathrm{Im}\, u| < \tan\beta\, \mathrm{Re}\, u + A\} \cup \\ &\{u \in \mathbb{C} \,:\, |\mathrm{Im}\, u| < -\tan\beta\, \mathrm{Re}\, u + A\}, \end{aligned} \tag{4.5}$$

with $A > 0$ as given in (1.9), (see Figure 9). Moreover,

- *There exists $C > 0$ such that, for $|\mathrm{Re}\, u| \gg 1$, $|\lambda_{\mathrm{p}}(u)|, |\Lambda_{\mathrm{p}}(u)| \leq Ce^{-\sqrt{\frac{21}{8}}|\mathrm{Re}\, u|}$.*
- *For $u \in \overline{\Pi_{A,\beta}}$, $\lambda_{\mathrm{p}}(u) = \pi$ if and only if $u = \pm iA$.*
- *For $u \in \overline{\Pi_{A,\beta}}$, $\Lambda_{\mathrm{p}}(u) = 0$ if and only if $u = 0$.*

4.2 Existence of the perturbed invariant manifolds

We devote this section to obtain and analyze parametrizations of the 2-dimensional branches of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$, where $\{\mathfrak{P}_\rho\}_{\rho\in(0,\rho_0]}$ is the family of periodic orbits given in Proposition 4.1. We find these parametrizations through a Perron-like method. In particular, following the ideas in [BFGS12], we write the perturbed manifolds as functions of (u, τ), where u parametrizes the unperturbed homoclinic orbit $\sigma_{\mathrm{p}}(u)$ (see (4.4)) and τ parametrizes the Lyapunov periodic orbit $\mathfrak{P}_\rho(\tau; \delta)$.

Let us define the following complex domains (see Figure 10),

$$D^{\mathrm{u}} = \left\{u \in \mathbb{C} \,:\, |\mathrm{Im}\, u| < \tfrac{A}{2} - \tan\beta\, \mathrm{Re}\, u\right\}, \qquad D^{\mathrm{s}} = \{u \in \mathbb{C} \,:\, -u \in D^{\mathrm{u}}\}. \tag{4.6}$$

Then, for $\diamond \in \{\mathrm{u}, \mathrm{s}\}$, we consider the parametrizations $Z^\diamond(u, \tau)$ satisfying that

$$\{Z^\diamond(u, \tau) \,:\, (u, \tau) \in D^\diamond \times \mathbb{T}_d\} \subseteq \mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho).$$

Notice that, for the unperturbed problem, since $\sigma_{\mathrm{p}}(u)$ is a time-parametrization it satisfies $\dot{u} = 1$. In addition, by Proposition 4.1, the dynamics in $\mathfrak{P}_\rho(\tau; \delta)$ satisfy $\dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}$.

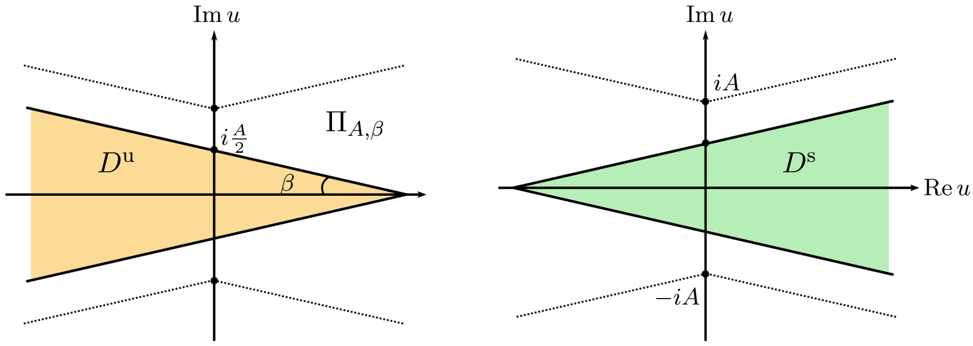


Figure 10: Representation of the domains D^{u} and D^{s} in (4.6).

Therefore, we impose that the dynamics on the perturbed parametrizations $Z^\diamond$ are given by

$$\dot{u} = 1, \qquad \dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}.$$

Hence, the parametrizations satisfy

$$\partial_u Z^\diamond(u,\tau) + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau Z^\diamond(u,\tau) = \begin{pmatrix} \mathbf{J} & 0 \\ 0 & i\mathbf{J} \end{pmatrix} DH(Z^\diamond(u,\tau);\delta) \quad \text{with } \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.7}$$

and the asymptotic conditions

$$\lim_{\operatorname{Re} u \to -\infty} Z^{\mathrm{u}}(u,\tau) = \lim_{\operatorname{Re} u \to +\infty} Z^{\mathrm{s}}(u,\tau) = \mathfrak{P}_\rho(\tau;\delta), \quad \text{for all} \quad \tau \in \mathbb{T}_d. \tag{4.8}$$

To prove their existence and behavior, we consider the decomposition

$$Z^\diamond(u,\tau) = \mathfrak{P}_\rho(\tau;\delta) + \sigma_{\mathrm{p}}(u) + Z_1^\diamond(u,\tau), \tag{4.9}$$

with σ_{p} as given in (4.4). The proof of the following result is deferred to Section 4.4.

Proposition 4.4. *Fix $d > 0$ and $\diamond \in \{\mathrm{u},\mathrm{s}\}$. Let $\rho_0 > 0$ be the constant given in Proposition 4.1. There exist $c_0, c_1, \delta_0, b_3 > 0$ such that, for $\rho \in [0,\rho_0]$ and $\delta \in (0,\delta_0)$, equation (4.7) together with the condition (4.8) has a unique real-analytic solution $Z^\diamond : D^\diamond \times \mathbb{T}_d \to U_{\mathbb{C}}(c_0,c_1)$ that can be decomposed as in (4.9) and satisfies*

$$\langle Z_1^\diamond(0,\tau), \mathbf{e_2}\rangle = 0, \qquad \textit{for all } \tau \in \mathbb{T}_d.$$

In addition, for $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,

$$|Z_1^\diamond(u,\tau)| \le b_3 \delta e^{-\nu|\operatorname{Re} u|}, \qquad \textit{for } (u,\tau) \in D^\diamond \times \mathbb{T}_d.$$

Notice that, by Proposition 4.1, when $\rho = 0$, $\mathfrak{P}_0(\tau;\delta) \equiv \mathfrak{L}(\delta)$ is a fixed point and that, $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ are 1-dimensional invariant manifolds. Then, for $\diamond \in \{\mathrm{u},\mathrm{s}\}$, Proposition 4.4 provides parametrizations $z_1^\diamond$ independent of τ satisfying

$$\{z^\diamond(u) \ : \ u \in D^\diamond\} \subseteq \mathcal{W}^{\diamond,+}(\mathfrak{L}),$$

that can be decomposed as

$$z^\diamond(u) = \mathfrak{L} + \sigma_{\rm p}(u) + z_1^\diamond(u). \tag{4.10}$$

Corollary 4.5. *Let $\diamond \in \{{\rm u}, {\rm s}\}$. There exist $c_0, c_1, \delta_0, b_3 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\rho = 0$, equation (4.7) together with the conditions (4.8) has a unique real-analytic solution $z^\diamond : D^\diamond \to U_{\mathbb{C}}(c_0, c_1)$ that can be decomposed as in (4.10) and satisfies $\langle z_1^\diamond(0), \mathbf{e_2}\rangle = 0$. In addition, for $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,*

$$|z_1^\diamond(u)| \leq b_3 \delta e^{-\nu|\operatorname{Re} u|}, \qquad \textit{for } u \in D^\diamond.$$

Finally, for $\diamond \in \{{\rm u}, {\rm s}\}$, we can measure how accurately the 1-dimensional manifolds $\mathcal{W}^{\diamond,+}(\mathfrak{L})$ approximate the 2-dimensional manifolds $\mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho)$.

Proposition 4.6. *Fix $d > 0$ and $\diamond \in \{{\rm u}, {\rm s}\}$. Let ρ_0 be the constant in Proposition 4.1 and $Z_1^\diamond$ and $z_1^\diamond$ be the parametrizations given in Proposition 4.4 and Corollary 4.5, respectively. Then, there exists $\delta_0 > 0$ and a constant $b_4 > 0$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$,*

$$|Z_1^\diamond(u, \tau) - z_1^\diamond(u)| \leq b_4 \delta \rho, \qquad \textit{for} \quad (u, \tau) \in D^\diamond \times \mathbb{T}_d.$$

The proof of this proposition is postponed to Section 4.5.

4.3 End of the proof of Theorem 4.2

To prove the first statement of Theorem 4.2, in the next lemma we study the intersections between the section Σ_ρ (see (4.3)) and the unstable and stable manifolds of $\mathfrak{P}_\rho$ parametrized by $Z^{\rm u}$ and $Z^{\rm s}$, respectively.

Lemma 4.7. *Fix $\diamond \in \{{\rm u}, {\rm s}\}$. Let ρ_0 and $\mathfrak{P}_\rho$ be as given in Proposition 4.1, $Z^\diamond$ be the parametrization given in (4.9) and Proposition 4.4 and Σ_ρ the section given in (4.3). Then, there exists $\delta_0 > 0$ and a real-analytic function $\mathcal{U}_\rho^\diamond : \mathbb{T}_d \to D^\diamond$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$,*

$$Z^\diamond(\mathcal{U}_\rho^\diamond(\tau), \tau) \in \Sigma_\rho, \qquad \textit{for } \tau \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Moreover, there exists $C > 0$ independent of ρ and δ such that, for $\tau \in \mathbb{T}_d$,

$$\mathcal{U}_0^\diamond \equiv 0, \qquad |\mathcal{U}_\rho^\diamond(\tau)| \leq C\delta\rho.$$

Proof. Since the parametrization $Z^\diamond$ is real-analytic (see Remark 2.2), one has that

$$Z^\diamond(u, \tau) \in U_{\mathbb{R}}(c_0, c_1) \qquad \text{for } (u, \tau) \in (D^\diamond \cap \mathbb{R}) \times \mathbb{T}.$$

In addition, by Propositions 4.1 and 4.4, one has that

$$H(Z^\diamond(u, \tau)) = H(\mathfrak{P}_\rho(\tau; \delta)) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta)), \qquad \text{for } (u, \tau) \in (D^\diamond \cap \mathbb{R}) \times \mathbb{T}.$$

Therefore, it is only necessary to find a function $\mathcal{U}_\rho^\diamond(\tau)$ satisfying that $\langle Z^\diamond(\mathcal{U}_\rho^\diamond(\tau),\tau),\mathbf{e}_2\rangle = \delta^2\mathfrak{L}_\Lambda(\delta)$ for all $\tau\in\mathbb{T}$. Then, by the decomposition (4.9) of $Z^\diamond$ and Proposition 4.1,

$$\delta\rho\Lambda_{\mathfrak{P}}(\tau)+\Lambda_{\mathrm{p}}(\mathcal{U}_\rho^\diamond(\tau))+\langle Z_1^\diamond(\mathcal{U}_\rho^\diamond(\tau),\tau),\mathbf{e}_2\rangle=0.$$

By Proposition 4.3, one has that $\Lambda_{\mathrm{p}}(u)=\dot{\Lambda}_{\mathrm{p}}(0)u+\mathcal{O}(u^2)$ with $\dot{\Lambda}_{\mathrm{p}}(0)=-V'(\lambda_0)\neq 0$. Then, $\mathcal{U}_\rho^\diamond$ is a solution of the fixed point equation given by the operator

$$F[\mathcal{U}_\rho^\diamond](\tau)=-\frac{1}{\dot{\Lambda}_{\mathrm{p}}(0)}\left[\delta\rho\Lambda_{\mathfrak{P}}(\tau)+\left(\Lambda_{\mathrm{p}}(\mathcal{U}_\rho^\diamond(\tau))-\dot{\Lambda}_{\mathrm{p}}(0)\mathcal{U}_\rho^\diamond\right)+\langle Z_1^\diamond(\mathcal{U}_\rho^\diamond(\tau),\tau),\mathbf{e}_2\rangle\right].$$

Notice that, by Propositions 4.1 and 4.4,

$$|F[0](\tau)|=\delta\rho\frac{|\Lambda_{\mathfrak{P}}(\tau)|}{|\dot{\Lambda}_{\mathrm{p}}(0)|}\leq C\delta\rho.$$

Moreover, for real-analytic functions $\mathcal{U},\mathcal{V}:\mathbb{T}_d\to D^\diamond$ satisfying that $|\mathcal{U}|,|\mathcal{V}|\leq C\delta\rho$ and applying the mean value theorem and Proposition 4.4, one can see that the operator F satisfies that, if δ small enough,

$$\begin{aligned}|F[\mathcal{U}]-F[\mathcal{V}]|&\leq C|\mathcal{U}^2-\mathcal{V}^2|+|\mathcal{U}-\mathcal{V}|\sup_{s\in[0,1]}|\langle\partial_u Z_1^\diamond(s\mathcal{U}+(1-s)\mathcal{V},\tau),\mathbf{e}_2\rangle|\\&\leq C\delta\rho\,|\mathcal{U}-\mathcal{V}|\leq\frac{1}{2}\,|\mathcal{U}-\mathcal{V}|,\end{aligned}$$

where we have used that $\langle Z_1^\diamond(0,\tau),\mathbf{e_2}\rangle=0$. Hence, F has a fixed point $\mathcal{U}_\rho^\diamond$ satisfying that $\left|\mathcal{U}_\rho^\diamond(\tau)\right|\leq C\delta\rho$, for $\tau\in\mathbb{T}_d$. ◻

The first statement of Theorem 4.2 is a direct consequence of Lemma 4.7. We denote by $\partial\mathcal{D}_\rho^{\mathrm{u}}$ and $\partial\mathcal{D}_\rho^{\mathrm{s}}$ the first intersection of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ with the section Σ_ρ, respectively, that can be parametrized as

$$\partial\mathcal{D}_\rho^\diamond=\left\{Z^\diamond(\mathcal{U}_\rho^\diamond(\tau^\diamond),\tau^\diamond)\;:\;\tau^\diamond\in\mathbb{T}\right\}\subset\Sigma_\rho\cap\mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho),\quad\diamond\in\{\mathrm{u},\mathrm{s}\}.\tag{4.11}$$

In particular, the first intersection of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section Σ_0 corresponds to the points $\partial\mathcal{D}_0^{\mathrm{u}}=\{z^{\mathrm{u}}(0)\}$ and $\partial\mathcal{D}_0^{\mathrm{s}}=\{z^{\mathrm{s}}(0)\}$.

To prove the rest of the statements, we study the difference between the parametrizations of the curves considered in (4.11). Since $\Sigma_\rho\subset U_{\mathbb{R}}(c_0,c_1)$ (see (4.3)), for $\tau^{\mathrm{u}},\tau^{\mathrm{s}}\in\mathbb{T}$ one has that

$$\begin{aligned}&\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_2}\rangle=0,\\&\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_4}\rangle=\overline{\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau),\tau^{\mathrm{s}}),\mathbf{e_3}\rangle},\end{aligned}$$

and $\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_1}\rangle$ can be recovered by the conservation of energy $H=\frac{\rho^2}{\delta^2}+H(\mathfrak{L})$. Therefore, to analyze the intersections between $\partial\mathcal{D}_\rho^{\mathrm{s}}$ and $\partial\mathcal{D}_\rho^{\mathrm{u}}$, it suffices to study the zeroes of the complex function

$$\Delta(\tau^{\mathrm{u}},\tau^{\mathrm{s}},\rho,\delta):=\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_4}\rangle.$$

Let us recall that, by Proposition 4.6, the difference $\Delta(\tau^{\mathrm{u}}, \tau^{\mathrm{s}})$ is given at first order, by the difference $z^{\mathrm{u}} - z^{\mathrm{s}}$. Therefore, using the decompositions (4.9) and (4.10), for $\diamond \in \{\mathrm{u}, \mathrm{s}\}$, we write

$$Z^{\diamond}(\mathcal{U}^{\diamond}(\tau), \tau) = \mathfrak{P}_{\rho}(\tau) + \sigma_{\mathrm{p}}(\mathcal{U}^{\diamond}(\tau)) + z_1^{\diamond}(\mathcal{U}^{\diamond}(\tau)) + \left(Z_1^{\diamond}(\mathcal{U}^{\diamond}(\tau), \tau) - z_1^{\diamond}(\mathcal{U}^{\diamond}(\tau))\right),$$

where $Z_1^{\diamond}$ and $z_1^{\diamond}$ are given in Proposition 4.4 and Corollary 4.5, respectively. Recall that, $\sigma_{\mathrm{p}} = (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)$ (see (4.4)) and, by Proposition 4.1, $\mathfrak{P}_{\rho} = \mathfrak{L} + \rho(0, 0, e^{i\tau}, e^{-i\tau}) + \delta\rho(\lambda_{\mathfrak{P}}, \Lambda_{\mathfrak{P}}, x_{\mathfrak{P}}, y_{\mathfrak{P}})$. Therefore, for δ small enough, we look for $(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \rho)$ such that

$$\Delta(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \rho, \delta) = 0, \tag{4.12}$$

where

$$\Delta(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \rho, \delta) = \rho(e^{-i\tau^{\mathrm{u}}} - e^{-i\tau^{\mathrm{s}}}) + \sqrt[6]{2}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \,|\Theta|\, e^{i\theta} + M(\delta) + R(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \rho),$$

with $\theta = \arg \langle z_1^{\mathrm{u}}(0) - z_1^{\mathrm{s}}(0), \mathbf{e_4}\rangle$ and

$$\begin{aligned}
M(\delta) &= \langle z_1^{\mathrm{u}}(0) - z_1^{\mathrm{s}}(0), \mathbf{e_4}\rangle - \sqrt[6]{2}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \,|\Theta|\, e^{i\theta}, \\
R(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \rho) &= \langle Z_1^{\mathrm{u}}(\mathcal{U}_{\rho}^{\mathrm{u}}(\tau^{\mathrm{u}}), \tau^{\mathrm{u}}) - z_1^{\mathrm{u}}(\mathcal{U}_{\rho}^{\mathrm{u}}(\tau^{\mathrm{u}})), \mathbf{e_4}\rangle - \langle Z_1^{\mathrm{s}}(\mathcal{U}_{\rho}^{\mathrm{s}}(\tau^{\mathrm{s}}), \tau^{\mathrm{s}}) - z_1^{\mathrm{s}}(\mathcal{U}_{\rho}^{\mathrm{s}}(\tau^{\mathrm{s}})), \mathbf{e_4}\rangle \\
&\quad + \langle z_1^{\mathrm{u}}(\mathcal{U}_{\rho}^{\mathrm{u}}(\tau^{\mathrm{u}})) - z_1^{\mathrm{u}}(0), \mathbf{e_4}\rangle - \langle z_1^{\mathrm{s}}(\mathcal{U}_{\rho}^{\mathrm{s}}(\tau^{\mathrm{s}})) - z_1^{\mathrm{s}}(0), \mathbf{e_4}\rangle \\
&\quad + \delta\rho(y_{\mathfrak{P}}(\tau^{\mathrm{u}}) - y_{\mathfrak{P}}(\tau^{\mathrm{s}})).
\end{aligned}$$

Notice that, by Corollary 2.5, Propositions 4.1 and 4.6 and Lemma 4.7,

$$M(\delta) = \mathcal{O}\left(\frac{\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}}{|\log \delta|}\right), \qquad R(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \rho) = \mathcal{O}(\delta\rho).$$

Since, by Theorem A, $\Theta \neq 0$, we can consider the auxiliary parameter $r \in (0, r_0]$,

$$r = \frac{2e^{\frac{A}{\delta^2}}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\rho, \qquad \text{and} \qquad r_0 = \frac{2e^{\frac{A}{\delta^2}}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\rho_0. \tag{4.13}$$

Then, equation (4.12) is equivalent to

$$r(e^{-i(\tau^{\mathrm{u}}+\theta)} - e^{-i(\tau^{\mathrm{s}}+\theta)}) + 2 + g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) = 0, \tag{4.14}$$

where

$$\begin{aligned}
g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) &= \frac{2e^{\frac{A}{\delta^2}} e^{-i\theta}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\left(M(\delta) + R\left(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}}|\Theta| e^{-\frac{A}{\delta^2}} r\right)\right) \\
&= \mathcal{O}\left(\frac{1}{|\log \delta|}\right) + \mathcal{O}(\delta r).
\end{aligned}$$

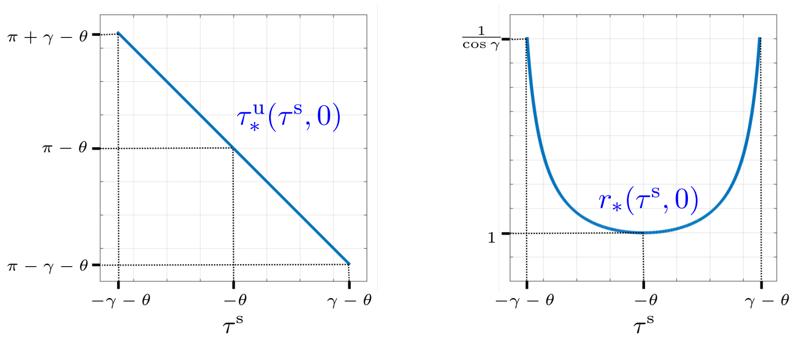


Figure 11: Plot in $\tau^{\rm s}$ of functions $\tau^{\rm u}_*(\tau^{\rm s},0)$ and $r_*(\tau^{\rm s},0)$ as given in Lemma 4.8.

By introducing $G=(G_1,G_2):\mathbb{T}^2\times[0,r_0]\times[0,\delta_0)\to\mathbb{R}^2$, as

$$\begin{aligned} G_1(\tau^{\rm u},\tau^{\rm s},r,\delta)&=r\left(\cos(\tau^{\rm u}+\theta)-\cos(\tau^{\rm s}+\theta)\right)+2+\operatorname{Re}g(\tau^{\rm u},\tau^{\rm s},r,\delta),\\ G_2(\tau^{\rm u},\tau^{\rm s},r,\delta)&=r\left(\sin(\tau^{\rm u}+\theta)-\sin(\tau^{\rm s}+\theta)\right)+\operatorname{Im}g(\tau^{\rm u},\tau^{\rm s},r,\delta),\end{aligned} \tag{4.15}$$

equation (4.14) is equivalent to $G(\tau^{\rm u},\tau^{\rm s},r,\delta)=(0,0)$.

Next result characterizes the solutions of this equation (see also Figure 11). Note that it would be reasonable to look for the zeros of G for a fixed r. This would give the intersections between the invariant manifolds of a given periodic orbit. Instead, we parameterize the zeros writing $(\tau^{\rm u},r)$ as functions of $\tau^{\rm s}$. This makes the application of the implicit function theorem easier and allows us to analyze at the same time transverse intersections and quadratic tangencies.

Lemma 4.8. *Fix $\gamma\in(0,\frac{\pi}{2})$ and consider $I_\gamma=[-\theta-\gamma,-\theta+\gamma]$. There exists δ_γ satisfying $\lim_{\gamma\to\pi/2}\delta_\gamma=0$ and functions $(\tau^{\rm u}_*,r_*):I_\gamma\times(0,\delta_\gamma)\to\mathbb{T}\times\mathbb{R}$, such that $G(\tau^{\rm u}_*(\tau^{\rm s},\delta),\tau^{\rm s},r_*(\tau^{\rm s},\delta),\delta)=(0,0)$ and*

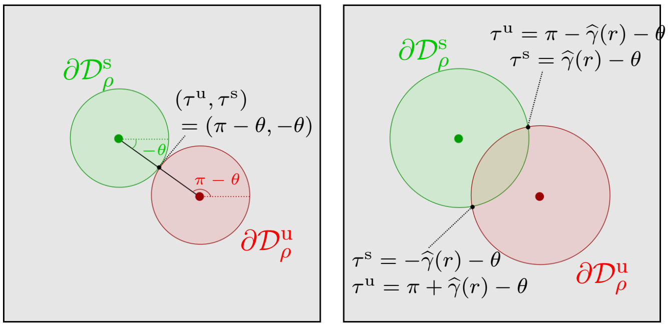
$$\begin{aligned}\tau^{\rm u}_*(\tau^{\rm s},\delta)&=\pi-\tau^{\rm s}-2\theta+\mathcal{O}\left(\frac{1}{|\log\delta|}\right),\\ r_*(\tau^{\rm s},\delta)&=\frac{1}{\cos(\tau^{\rm s}+\theta)}+\mathcal{O}\left(\frac{1}{|\log\delta|}\right).\end{aligned}$$

Proof. For $r\geq 1$ and $\delta=0$, the equation $G(\tau^{\rm u},\tau^{\rm s},r,0)=(0,0)$ has a family of solutions given by

$$S_\alpha=\left\{(\tau^{\rm u},\tau^{\rm s},r,0)=\left(\pi-\alpha-\theta,\alpha-\theta,\frac{1}{\cos\alpha},0\right)\right\},\quad \text{with } \alpha\in[-\gamma,\gamma]\subset\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Therefore, for $\delta>0$, it only remains to find zeroes of the function G using the implicit function theorem around every solution of this family. □

The second statement of Theorem 4.2 is a consequence of this lemma. Indeed, take $R>1$ and $\gamma=\arccos(\frac{1}{R})\in(0,\frac{\pi}{2})$. Then, Lemma 4.8 implies that the equation



(a) $r=1$

(b) $r\in(1,R]$
$\widehat{\gamma}(r) = \arccos(\frac{1}{r})$

Figure 12: Representation of solutions of the equation (4.15) in function of the coordinate r.

$G(\tau^{\rm u},\tau^{\rm s},r,\delta)=(0,0)$ has at least one solution for $r\in[r_{\min}(\delta),r_{\max}(\delta)]$ and $\delta\in(0,\delta_\gamma)$, with

$$r_{\min}(\delta)=1+\mathcal{O}\left(\frac{1}{|\log\delta|}\right),\qquad r_{\max}(\delta)=R+\mathcal{O}\left(\frac{1}{|\log\delta|}\right).$$

Taking into account (4.13), we define

$$\rho_{\min}(\delta)=\frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}|\Theta|r_{\min}(\delta),\qquad \rho_{\max}(\delta)=\frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}|\Theta|r_{\max}(\delta),$$

and assume $\delta>0$ small enough such that $\rho_{\max}(\delta)<\rho_0$. Then, for $\rho\in[\rho_{\min}(\delta),\rho_{\max}(\delta)]$, the closed curves $\partial\mathcal{D}^{\rm u}_\rho$ and $\partial\mathcal{D}^{\rm s}_\rho$ (see (4.11)) intersect at least once. See Figure 12 for a representation of the case $\delta=0$.

Finally, we prove the third and fourth statement of Theorem 4.2. Let us denote the solutions of equation $G(\tau^{\rm u},\tau^{\rm s},r,\delta)=(0,0)$ given in Lemma 4.8 as

$$P(\tau^{\rm s},\delta)=(\tau^{\rm u}_*(\tau^{\rm s},\delta),\tau^{\rm s},r_*(\tau^{\rm s},\delta),\delta)$$

and consider the function

$$\widetilde{G}(\tau^{\rm s},\delta)=\det\left(\frac{\partial G}{\partial(\tau^{\rm u},\tau^{\rm s})}(P(\tau^{\rm s},\delta))\right),\qquad (\tau^{\rm s},\delta)\in I_\gamma\times(0,\delta_\gamma).$$

Then, the values such that $\widetilde{G}\neq0$ correspond to transverse intersections of the closed curves $\partial\mathcal{D}^{\rm u}_\rho$ and $\partial\mathcal{D}^{\rm s}_\rho$. Likewise, the values such that $\widetilde{G}=0$ and $\partial_{\tau^{\rm s}}\widetilde{G}\neq0$ correspond to quadratic tangencies.

To characterize the transverse intersections and the quadratic tangencies, we define $\tau^{\rm s}_{\min}(\delta)$, the value of $\tau^{\rm s}$ where r_* reaches its minimum value $r_{\min}$. Note that this corresponds to a critical point, which, by Lemma 4.8, satisfies

$$\tau^{\rm s}_{\min}(\delta)=-\theta+\mathcal{O}\left(\frac{1}{|\log\delta|}\right),\quad r_{\min}(\delta)=r_*(\tau^{\rm s}_{\min}(\delta),\delta),\quad \partial_{\tau^{\rm s}}r_*(\tau^{\rm s}_{\min}(\delta),\delta)=0. \quad (4.16)$$

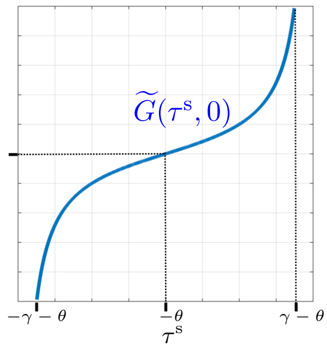


Figure 13: Plot in $\tau^{\rm s}$ of function $\widetilde{G}(\tau^{\rm s},0)$ as given in (4.15).

Now we prove that that $(\tau^{\rm s}_{\rm min}(\delta),\delta)$ is a simple zero of $\widetilde{G}$ and otherwise $\widetilde{G}\neq 0$, for $\tau^{\rm s}\neq\tau^{\rm s}_{\rm min}(\delta)$. By the definition of function G in (4.15) and Lemma 4.8, for $(\tau^{\rm s},\delta)\in I_\gamma\times(0,\delta_\gamma)$, one has that

$$\widetilde{G}(\tau^{\rm s},\delta)=2\tan(\tau^{\rm s}+\theta)+\mathcal{O}\left(\frac{1}{|\log\delta|}\right),\quad \partial_{\tau^{\rm s}}\widetilde{G}(\tau^{\rm s},\delta)=\frac{2}{\cos^2(\tau^{\rm s}+\theta)}+\mathcal{O}\left(\frac{1}{|\log\delta|}\right),$$

(see Figure 13). Notice that, for δ small enough,

$$\partial_{\tau^{\rm s}}\widetilde{G}(\tau^{\rm s},\delta)\geq 2+\mathcal{O}\left(\frac{1}{|\log\delta|}\right)>0.$$

Therefore, $\widetilde{G}$ is a strictly increasing function in $\tau^{\rm s}$ and can only have one simple zero. Moreover, this zero corresponds to $\tau^{\rm s}=\tau^{\rm s}_{\rm min}(\delta)$. Indeed, since $G(P(\tau^{\rm s}_{\rm min}(\delta),\delta))=(0,0)$ and $\partial_{\tau^{\rm s}}r_*(\tau^{\rm s}_{\rm min}(\delta),\delta)=0$ (see (4.16)), taking the derivatives one has that

$$\partial_{\tau^{\rm s}}G(P(\tau^{\rm s}_{\rm min}(\delta),\delta))+\partial_{\tau^{\rm u}}G(P(\tau^{\rm s}_{\rm min}(\delta),\delta))\partial_{\tau^{\rm s}}\tau^{\rm u}_*(\tau^{\rm s}_{\rm min}(\delta),\delta)=(0,0),$$

and, as a result, the vectors $\partial_{\tau^{\rm s}}G$ and $\partial_{\tau^{\rm u}}G$ at $P(\tau^{\rm s}_{\rm min}(\delta),\delta)$ are linearly dependent and, therefore, $\widetilde{G}(\tau^{\rm s}_{\rm min},\delta)=0$. Hence, there exists at least one quadratic tangency at $r=r_{\rm min}(\delta)$ and at least two transverse intersection for each $r\in(r_{\rm min}(\delta),r_{\rm max}(\delta)]$.

4.4 Proof of Proposition 4.4

From now on, we consider a fixed $d>0$ and the corresponding complex torus $\mathbb{T}_d$ (see (4.1)). We also set ρ_0 satisfying the conditions in Proposition 4.1 and $\rho\in[0,\rho_0]$. To avoid cumbersome notations, throughout the rest of the section, we omit the dependence on the parameter δ unless necessary and denote by C any positive constant independent of δ and ρ to state estimates. We only prove the results for the unstable manifold, the proof for the stable manifold is analogous.

We look for parametrizations of the invariant manifold $\mathcal{W}^{u,+}(\mathfrak{P}_\rho)$ of the form

$$Z^{\rm u}(u,\tau)=\mathfrak{P}_\rho(\tau)+\sigma_{\rm p}(u)+Z^{\rm u}_1(u,\tau),\qquad (u,\tau)\in D^{\rm u}\times\mathbb{T}_d,$$

(see (4.9)) satisfying the equation (4.7) and the asymptotic condition given in (4.8).

Let us recall that we split the Hamiltonian H as $H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1$ (see (2.8)). Since $\sigma_{\mathrm{p}} = (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)$ is a solution of the unperturbed system $H_{\mathrm{p}} + H_{\mathrm{osc}}$, it satisfies the invariance equation (4.7) for the unperturbed Hamiltonian (see Proposition 4.3). By Proposition 4.1, $\mathfrak{P}_\rho$ also satisfies (4.7) (for the full Hamiltonian H). Then, the parametrization $Z_1^\diamond$ satisfies

$$\mathcal{L}_\rho Z_1^{\mathrm{u}} = \mathcal{R}_\rho[Z_1^{\mathrm{u}}], \tag{4.17}$$

where

$$\mathcal{L}_\rho \zeta = \left(\partial_u + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau - \mathcal{A}(u)\right)\zeta, \quad \mathcal{A} = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_{\mathrm{p}}(u)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} \tag{4.18}$$

and

$$\mathcal{R}_\rho[\zeta] = \begin{pmatrix} \partial_\Lambda H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) - \partial_\Lambda H_1(\mathfrak{P}_\rho) \\ -T_\rho[\zeta_1] - \partial_\lambda H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) + \partial_\lambda H_1(\mathfrak{P}_\rho) \\ i\partial_y H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) - i\partial_y H_1(\mathfrak{P}_\rho) \\ -i\partial_x H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) + i\partial_x H_1(\mathfrak{P}_\rho) \end{pmatrix}, \tag{4.19}$$

with

$$T_\rho[\zeta_1] = V'(\lambda_{\mathrm{p}} + \mathfrak{P}_{\rho,1} + \zeta_1) - V'(\lambda_{\mathrm{p}}) - V'(\mathfrak{P}_{\rho,1}) - V''(\lambda_{\mathrm{p}})\zeta_1. \tag{4.20}$$

We solve equation (4.17) by means of a fixed point scheme on a suitable Banach space. For $\alpha \geq 0$, we consider the Banach space

$$\mathcal{Y}_\alpha = \left\{\zeta : D^{\mathrm{u}} \times \mathbb{T}_d \to \mathbb{C} \ : \ \zeta \text{ real-analytic, } \|\zeta\|_\alpha := \sup_{(u,\tau)\in D^{\mathrm{u}}\times\mathbb{T}_d} \left|e^{-\alpha u}\zeta(u,\tau)\right| < +\infty\right\},$$

where D^{u} is the domain introduced in (4.6). We also consider the product Banach space $\mathcal{Y}_\alpha^4 = \mathcal{Y}_\alpha \times \dots \times \mathcal{Y}_\alpha$ endowed with the norm

$$\|\zeta\|_\alpha^\times = \sum_{j=1}^4 \|\zeta_j\|_\alpha.$$

In the next lemma, we state some properties of these Banach spaces. We will use them throughout the section.

Lemma 4.9. *The following statements hold.*

1. *If $\alpha \geq \beta \geq 0$, then $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$. Moreover, for $\zeta \in \mathcal{Y}_\alpha$, $\|\zeta\|_\beta \leq C\|\zeta\|_\alpha$.*
2. *If $\zeta \in \mathcal{Y}_\alpha$ and $\eta \in \mathcal{Y}_\beta$, then $\zeta\eta \in \mathcal{Y}_{\alpha+\beta}$ and $\|\zeta\eta\|_{\alpha+\beta} \leq \|\zeta\|_\alpha\|\eta\|_\beta$.*

Next, we obtain and analyze a suitable right-inverse of the operator $\mathcal{L}_\rho$ introduced in (4.18). The first step is to construct a fundamental matrix for $\dot{\zeta} = \mathcal{A}(u)\zeta$.

Lemma 4.10. *Fix $u_0 \in \mathbb{R} \setminus \{0\}$ and consider the linear differential equation $\dot{\zeta} = \mathcal{A}(u)\zeta$, with $\mathcal{A}$ as given in (4.18). Then, a real-analytic fundamental matrix of this equation is*

$$\Phi(u) = \begin{pmatrix} 3f_\Phi(u) & 3g_\Phi(u) & 0 & 0 \\ -\dot{f}_\Phi(u) & -\dot{g}_\Phi(u) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}u} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}u} \end{pmatrix},$$

with

$$f_\Phi(u) = \frac{1}{3\xi(0)}\left(\xi(u) - \frac{\dot{\xi}(0)}{\dot{\Lambda}_\mathrm{p}(0)}\dot{\Lambda}_\mathrm{p}(u)\right), \quad g_\Phi(u) = -\frac{\Lambda_\mathrm{p}(u)}{\dot{\Lambda}_\mathrm{p}(0)}, \quad \xi(u) = \Lambda_\mathrm{p}(u)\int_{u_0}^{u}\frac{dv}{\Lambda_\mathrm{p}^2(v)},$$

where, in the last integral, we consider an integration path in D^u given by the straight line if $u \in \mathbb{C} \setminus \mathbb{R}$ and by a path avoiding $u = 0$ when $u \in \mathbb{R}$.

Moreover, $\Phi(u)$ satisfies that $\det \Phi(u) = 1$, $\Phi(0) = \mathbf{Id}$ *and that there exists a constant $C > 0$ such that, denoting $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,*

$$\|g_\Phi\|_{2\nu} \leq C, \quad \|\dot{g}_\Phi\|_{2\nu} \leq C, \quad \|f_\Phi\|_{-2\nu} \leq C, \quad \|\dot{f}_\Phi\|_{-2\nu} \leq C.$$

Proof. Let us recall that, by Proposition 4.3, the time-parametrization of the separatrix satisfies that $\dot{\lambda}_\mathrm{p}(u) = -3\Lambda_\mathrm{p}(u)$ and $\dot{\Lambda}_\mathrm{p}(u) = -V'(\lambda_\mathrm{p}(u))$, for $u \in \Pi_{A,\beta}$. Then, a fundamental matrix of the equation $\dot{\zeta} = \mathcal{A}(u)\zeta$ is given by

$$\phi(u) = \begin{pmatrix} 3\xi(u) & 3\Lambda_\mathrm{p}(u) & 0 & 0 \\ -\dot{\xi}(u) & -\dot{\Lambda}_\mathrm{p}(u) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}u} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}u} \end{pmatrix}.$$

We stress that ξ is real-analytic in $D^\mathrm{u} \subset \Pi_{A,\beta}$. Indeed, one has that $u = 0$ is the only zero of $\Lambda_\mathrm{p}(u)$ (see Proposition 4.3), that $\dot{\Lambda}_\mathrm{p}(0) = -V'(\lambda_\mathrm{p}(0)) \neq 0$ and $\ddot{\Lambda}_\mathrm{p}(0) = 0$. Thus, $\Lambda_\mathrm{p}(u) = \dot{\Lambda}_\mathrm{p}(0)u + \mathcal{O}(u^3)$. That implies that the integral appearing on ξ does not depend on the path of integration since its residue is zero. As a consequence, $\xi(u) \in \mathbb{R}$ for $u \in \mathbb{R}$. In addition, since $\xi(0) = -\dot{\Lambda}_\mathrm{p}^{-1}(0) \neq 0$, we can perform a linear transformation to $\phi(u)$ to obtain the fundamental matrix $\Phi(u)$ satisfying $\Phi(0) = \mathrm{Id}$ and $\det \Phi(u) = 1$. Lastly, recalling that, by Proposition 4.3, $\|\lambda_\mathrm{p}\|_{2\nu} \leq C$ and $\|\Lambda_\mathrm{p}\|_{2\nu} \leq C$, we obtain the corresponding estimates for f_Φ and g_Φ. □

We use this matrix Φ to construct a right-inverse of the operator $\mathcal{L}_\rho$ in (4.18). For $\zeta \in \mathcal{Y}_\nu^4$, we consider the operator

$$\mathcal{G}_\rho[\zeta](u,\tau) = \sum_{j=1}^{4}\mathcal{G}_{\rho,j}[\zeta](u,\tau)\mathbf{e_j}, \tag{4.21}$$

given by

$$\begin{pmatrix} \mathcal{G}_{\rho,1}[\zeta](u,\tau) \\ \mathcal{G}_{\rho,2}[\zeta](u,\tau) \end{pmatrix} = \begin{pmatrix} 3f_\Phi(u) & 3g_\Phi(u) \\ -\dot{f}_\Phi(u) & -\dot{g}_\Phi(u) \end{pmatrix} \begin{pmatrix} \displaystyle\int_{-\infty}^0 \mathcal{I}_1[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt \\ \displaystyle\int_{-u}^0 \mathcal{I}_2[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt \end{pmatrix}$$

and

$$\mathcal{G}_{\rho,3}[\zeta](u,\tau) = \int_{-\infty}^0 e^{-\frac{i}{\delta^2}t}\zeta_3\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt,$$
$$\mathcal{G}_{\rho,4}[\zeta](u,\tau) = \int_{-\infty}^0 e^{\frac{i}{\delta^2}t}\zeta_4\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt,$$

where

$$\mathcal{I}_1[\zeta_1,\zeta_2](u,\tau) = -\dot{g}_\Phi(u)\zeta_1(u,\tau) - 3g_\Phi(u)\zeta_2(u,\tau),$$
$$\mathcal{I}_2[\zeta_1,\zeta_2](u,\tau) = \dot{f}_\Phi(u)\zeta_1(u,\tau) + 3f_\Phi(u)\zeta_2(u,\tau).$$

Lemma 4.11. *For $\rho \in [0,\rho_0]$ and $\delta \in (0,1)$, the operator $\mathcal{G}_\rho : \mathcal{Y}_\nu^4 \to \mathcal{Y}_\nu^4$ is well defined and is a right-inverse of the operator $\mathcal{L}_\rho$ given in (4.18). Moreover, $\mathcal{G}_{\rho,2}[\zeta](0,\cdot) \equiv 0$ and there exists a constant $C>0$ independent of ρ and δ such that*

$$\|\mathcal{G}_\rho[\zeta]\|_\nu^\times \le C\|\zeta\|_\nu^\times.$$

In addition, if $\partial_\tau\zeta \equiv 0$, one has that $\mathcal{G}_\rho[\zeta] = \mathcal{G}_{\tilde\rho}[\zeta]$ for $\rho,\tilde\rho \in [0,\rho_0]$.

Proof. The fact that $\mathcal{G}_\rho$ is a right inverse of $\mathcal{L}_\rho$ is straightforward. We show how to obtain estimates for $\mathcal{G}_{\rho,1}$. The estimates for $\mathcal{G}_{\rho,2}$, $\mathcal{G}_{\rho,3}$ and $\mathcal{G}_{\rho,4}$ are analogous.

Let $\zeta_1,\zeta_2 \in \mathcal{Y}_\nu$. By the estimates in Lemma 4.10, for $(u,\tau) \in D^{\mathrm{u}} \times \mathbb{T}_d$ one has

$$|\mathcal{I}_1[\zeta_1,\zeta_2](u,\tau)| \le C|e^{3\nu u}|\left(\|\zeta_1\|_\nu + \|\zeta_2\|_\nu\right),$$
$$|\mathcal{I}_2[\zeta_1,\zeta_2](u,\tau)| \le C|e^{-\nu u}|\left(\|\zeta_1\|_\nu + \|\zeta_2\|_\nu\right).$$

Then,

$$\begin{aligned}\left|\mathcal{G}_{\rho,1}(u,\tau)e^{-\nu u}\right| &\le C\left|e^{-3\nu u}\right|\left|\int_{-\infty}^0 \mathcal{I}_1[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt\right| \\ &\quad + C\left|e^{\nu u}\right|\left|\int_{-u}^0 \mathcal{I}_2[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt\right| \\ &\le C\left(\|\zeta_1\|_\nu + \|\zeta_2\|_\nu\right).\end{aligned}$$

□

We introduce the fixed point operator

$$\mathcal{F}_\rho = \mathcal{G}_\rho \circ \mathcal{R}_\rho, \tag{4.22}$$

with $\mathcal{R}_\rho$ and $\mathcal{G}_\rho$ as given in (4.19) and (4.21), respectively. Then, equation (4.17) can be expressed as $Z_1^{\mathrm{u}} = \mathcal{F}_\rho[Z_1^{\mathrm{u}}]$.

Proving Proposition 4.4 is equivalent to prove the following result.

Proposition 4.12. *Let $\rho_0 > 0$ be the constant given in Proposition 4.1. There exist $\delta_0 > 0$ and $b_3 > 0$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$, the equation $Z_1^{\mathrm{u}} = \mathcal{F}[Z_1^{\mathrm{u}}]$ has a unique solution $Z_1^{\mathrm{u}} \in \mathcal{Y}_\nu^4$ satisfying*

$$\|Z_1^{\mathrm{u}}\|_\nu^\times \leq b_3 \delta.$$

Proof. For $\varsigma > 0$, let us consider $B(\varsigma) = \left\{\zeta \in \mathcal{Y}_\nu^4 : \|\zeta\|_\nu^\times \leq \varsigma\right\}$. We will check that $\mathcal{F}_\rho : B(\varsigma) \to B(\varsigma)$ is a contraction for a suitable ς.

We first claim that there exist $\delta_0 > 0$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$,

$$\|\mathcal{R}_\rho[\zeta]\|_\nu^\times \leq C\delta, \qquad \|\partial_j \mathcal{R}_\rho[\zeta]\|_0^\times \leq C\delta, \tag{4.23}$$

for $\zeta \in B(\varsigma\delta)$ and $j = 1, .., 4$. Indeed, we obtain the estimates for $\mathcal{R}_{\rho,2}[\zeta]$, the other cases are proven analogously. For the derivatives it is enough to apply Cauchy estimates.

We recall the definitions

$$\begin{aligned}
\sigma_{\mathrm{p}} &= (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)^T, \\
\mathfrak{P}_\rho &= (0, \delta^2 \mathfrak{L}_\Lambda, \delta^3 \mathfrak{L}_x, \delta^3 \mathfrak{L}_y)^T + \rho(0, 0, e^{i\tau}, e^{-i\tau})^T + \delta\rho\big(\lambda_{\mathfrak{P}}, \Lambda_{\mathfrak{P}}, x_{\mathfrak{P}}, y_{\mathfrak{P}}\big)^T, \\
\mathcal{R}_{\rho,2}[\zeta] &= -\partial_\lambda H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) + \partial_\lambda H_1(\mathfrak{P}_\rho) - T_\rho[\zeta_1], \\
T_\rho[\zeta_1] &= V'(\lambda_{\mathrm{p}} + \delta\rho\lambda_{\mathfrak{P}} + \zeta_1) - V'(\lambda_{\mathrm{p}}) - V'(\delta\rho\lambda_{\mathfrak{P}}) - V''(\lambda_{\mathrm{p}})\zeta_1,
\end{aligned} \tag{4.24}$$

where V is the potential given in (2.7). Then, by the mean value theorem,

$$\begin{aligned}
\mathcal{R}_{\rho,2}[\zeta](u,\tau) = &- \int_0^1 D\partial_\lambda H_1(s\sigma_{\mathrm{p}}(u) + s\zeta(u,\tau) + \mathfrak{P}_\rho(\tau)) ds \, (\sigma_{\mathrm{p}}(u) + \zeta(u,\tau)) \\
&- \zeta_1(u,\tau) \left[V''(\lambda_{\mathrm{p}}(u) + \delta\rho\lambda_{\mathfrak{P}}(\tau)) - V'''(\lambda_{\mathrm{p}}(u))\right] + \mathcal{O}\left(\zeta_1(u,\tau)\right)^2 \\
&- \delta\rho\lambda_{\mathfrak{P}}(\tau)\lambda_{\mathrm{p}}(u)V'''(0) + \mathcal{O}\left(\delta\rho\lambda_{\mathfrak{P}}(\tau)\lambda_{\mathrm{p}}(u)\right)^2.
\end{aligned}$$

From Proposition 4.1 and Proposition 4.3, one easily checks that

$$\|s\sigma_{\mathrm{p}} + s\zeta + \mathfrak{P}_\rho\|_0^\times \leq C, \qquad \text{for } s \in [0,1].$$

Thus, applying the estimates in Proposition 2.1 and using that $\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}} \in \mathcal{Y}_{2\nu}$,

$$\begin{aligned}
\|\mathcal{R}_{\rho,2}[\zeta]\|_\nu \leq\, & C\delta\|\lambda_{\mathrm{p}} + \zeta_1\|_\nu + C\delta^2\|\Lambda_{\mathrm{p}} + \zeta_2\|_\nu + C\delta\|\zeta_3\|_\nu + C\delta\|\zeta_4\|_\nu \\
&+ C\|\zeta_1\|_\nu + C\delta\rho\|\lambda_{\mathrm{p}}\|_\nu \leq C\delta,
\end{aligned}$$

which proves (4.23).

As a consequence of (4.23) and using Lemma 4.11, there exists a constant $b_3 > 0$ such that

$$\|\mathcal{F}_\rho[0]\|_\nu^\times \leq C\|\mathcal{R}_\rho[0]\|_\nu^\times \leq \frac{1}{2}b_3\delta. \tag{4.25}$$

In addition, for $\zeta, \tilde{\zeta} \in B(b_3\delta)$ and by the mean value theorem,

$$\mathcal{R}_\rho[\zeta] - \mathcal{R}_\rho[\tilde{\zeta}] = \left[\int_0^1 D\mathcal{R}_\rho[s\zeta + (1-s)\tilde{\zeta}]ds\right](\zeta - \tilde{\zeta}).$$

Then, from Lemma 4.11 and the estimates in (4.23), we deduce that

$$\begin{aligned}\|\mathcal{F}_\rho[\zeta] - \mathcal{F}_\rho[\tilde{\zeta}]\|_\nu^\times &\leq C\|\mathcal{R}_\rho[\zeta] - \mathcal{R}_\rho[\tilde{\zeta}]\|_\nu^\times \\ &\leq \sup_{s\in[0,1]}\sum_{k=1}^4 \|\partial_k\mathcal{R}_\rho[s\zeta + (1-s)\tilde{\zeta}]\|_0\|\zeta_k - \tilde{\zeta}_k\|_\nu \leq C\delta\|\zeta - \tilde{\zeta}\|_\nu^\times.\end{aligned} \tag{4.26}$$

This implies that, taking δ small enough, $\|\mathcal{F}_\rho[\zeta] - \mathcal{F}_\rho[\tilde{\zeta}]\|_\nu^\times \leq \frac{1}{2}\|\zeta - \tilde{\zeta}\|_\nu^\times$ and, therefore, $\mathcal{F}_\rho : B(b_3\delta) \to B(b_3\delta)$ is well defined and contractive. Hence, $\mathcal{F}_\rho$ has a fixed point $Z_1^{\mathrm{u}} \in B(b_3\delta)$. □

Proposition 4.12 completes the proof of Proposition 4.4. Note that, since $\mathcal{G}_{\rho,2}[\zeta](0,\cdot) \equiv 0$ (see (4.21)) and $\mathcal{F}_\rho = \mathcal{G}_\rho \circ \mathcal{R}_\rho$, the solution obtained in Proposition 4.12 satisfies

$$\langle Z_1^\diamond(0,\tau), \mathbf{e_2}\rangle = 0, \qquad \text{for all } \tau \in \mathbb{T}_d.$$

4.5 Proof of Proposition 4.6

To prove Proposition 4.6, let us consider the parametrizations $Z_1^{\mathrm{u}}(u,\tau)$ and $z_1^{\mathrm{u}}(u)$ given in Proposition 4.4 and Corollary 4.5, respectively.

Let us recall that, by Proposition 4.12, Z_1^{u} satisfies $Z_1^{\mathrm{u}} = (\mathcal{G}_\rho \circ \mathcal{R}_\rho)[Z_1^{\mathrm{u}}]$ and, as a result, $z_1^{\mathrm{u}} = (\mathcal{G}_0 \circ \mathcal{R}_0)[z_1^{\mathrm{u}}]$. By Lemma 4.11, since z_1^{u} does not depend on τ, one has that

$$z_1^{\mathrm{u}} = \mathcal{G}_\rho \circ \mathcal{R}_0[z_1^{\mathrm{u}}], \qquad \text{for any } \rho \in [0,\rho_0]. \tag{4.27}$$

Then, by Proposition 4.12,

$$\begin{aligned}Z_1^{\mathrm{u}} - z_1^{\mathrm{u}} &= \mathcal{F}_\rho[Z_1^{\mathrm{u}}] - \mathcal{G}_\rho \circ \mathcal{R}_0[z_1^{\mathrm{u}}] \\ &= \mathcal{F}_\rho[Z_1^{\mathrm{u}}] - \mathcal{F}_\rho[z_1^{\mathrm{u}}] + \mathcal{G}_\rho\left(\mathcal{R}_\rho[z_1^{\mathrm{u}}] - \mathcal{R}_0[z_1^{\mathrm{u}}]\right),\end{aligned} \tag{4.28}$$

where we recall that $\mathcal{F}_\rho = \mathcal{G}_\rho \circ \mathcal{R}_\rho$ (see (4.22)).

Let us consider the constant b_3 as given in Proposition 4.4. It is clear that,

$$Z_1^{\mathrm{u}}, z_1^{\mathrm{u}} \in B(b_3\delta) := \left\{\zeta \in \mathcal{Y}_\nu^4 \,:\, \|\zeta\|_\nu^\times \leq b_3\delta\right\}.$$

Since $\mathcal{F}_\rho$ is contractive with Lipschitz constant $\mathrm{Lip}(\mathcal{F}_\rho) \leq C\delta$ (see (4.26)), for δ small enough, one has that

$$\|\mathcal{F}_\rho[Z_1^{\mathrm{u}}] - \mathcal{F}_\rho[z_1^{\mathrm{u}}]\|_\nu^\times \leq C\delta \|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times \leq \frac{1}{2}\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times.$$

Thus, by (4.28) and Lemma 4.11,

$$\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times \leq \frac{1}{2}\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times + C\|\mathcal{R}_\rho[z_1^{\mathrm{u}}] - \mathcal{R}_0[z_1^{\mathrm{u}}]\|_\nu^\times, \tag{4.29}$$

We claim that, for $\rho \in [0, \rho_0]$ and $\delta > 0$ small enough,

$$\|\mathcal{R}_\rho[z_1^{\mathrm{u}}] - \mathcal{R}_0[z_1^{\mathrm{u}}]\|_\nu^\times \leq C\delta\rho. \tag{4.30}$$

Indeed, first we consider estimates for $\mathcal{R}_{\rho,1}$ as given in (4.19). One has that

$$\begin{aligned}\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}] &= \big(\partial_\Lambda H_1(\sigma_{\mathrm{p}} + \mathfrak{P}_\rho + z_1^{\mathrm{u}}) - \partial_\Lambda H_1(\mathfrak{P}_\rho)\big) \\ &\quad - \big(\partial_\Lambda H_1(\sigma_{\mathrm{p}} + \mathfrak{P}_0 + z_1^{\mathrm{u}}) - \partial_\Lambda H_1(\mathfrak{P}_0)\big).\end{aligned}$$

Denoting $\mathfrak{P}^s = (1-s)\mathfrak{P}_0 + s\mathfrak{P}_\rho$, by the mean value theorem,

$$\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}] = (\mathfrak{P}_\rho - \mathfrak{P}_0)^T \left[\int_{[0,1]^2} D^2\partial_\Lambda H_1(r(\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}) + \mathfrak{P}^s) dr ds\right](\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}).$$

Then, using Lemma 4.9 and for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\lambda, \Lambda, x, y)$, one sees that

$$\begin{aligned}\|\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}]\|_\nu \leq \sum_{j=1}^{4}\sum_{k=1}^{4} \sup_{s\in[0,1]} \sup_{r\in[0,1]} \|\partial_{\alpha_j\alpha_k\Lambda} H_1(r(\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}) + \mathfrak{P}^s)\|_0 \\ \cdot \|\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}\|_\nu^\times \|\mathfrak{P}_\rho - \mathfrak{P}_0\|_0^\times.\end{aligned}$$

Notice that, Proposition 4.1 implies that $\|\mathfrak{P}_\rho - \mathfrak{P}_0\|_0^\times \leq C\rho$ and Proposition 4.3 and Corollary 4.5 imply that $\|\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}\|_\nu^\times \leq C$. These estimates and those of Proposition 2.1, which bound $\|\partial_{\alpha_j\alpha_k\Lambda} H_1\|_0$, imply that

$$\|\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho.$$

Analogously, it can be seen that

$$\begin{aligned}&\|\mathcal{R}_{\rho,2}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,2}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho + \|T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]\|_\nu, \\ &\|\mathcal{R}_{\rho,3}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,3}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho, \\ &\|\mathcal{R}_{\rho,4}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,4}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho,\end{aligned}$$

with T_ρ defined in (4.20). Therefore, it only remains to analyze $T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]$. Indeed, applying the mean value theorem one sees that

$$\begin{aligned}T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}] &= V'(\lambda_{\mathrm{p}} + \mathfrak{P}_{\rho,1} + z_1^{\mathrm{u}}) - V'(\mathfrak{P}_{\rho,1}) - V'(\lambda_{\mathrm{p}} + z_1^{\mathrm{u}}) + V'(0) \\ &= \mathfrak{P}_{\rho,1}(\lambda_{\mathrm{p}} + z_1^{\mathrm{u}}) \int_{[0,1]^2} V'''(s\lambda_{\mathrm{p}} + r\mathfrak{P}_{\rho,1} + s z_1^{\mathrm{u}}) dr ds.\end{aligned}$$

Then, since $\lambda_{\mathrm{p}} \in \mathcal{Y}_{2\nu}$ and taking into account that $\mathfrak{P}_{\rho,1}(\tau) = \delta\rho\lambda_{\mathfrak{P}}(\tau)$ with $\|\lambda_{\mathfrak{P}}\|_0 \leq C$ (see Proposition 4.1), one has that $\|T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho$. This proves (4.30) and, by (4.29), Proposition 4.6 holds.

A Lyapunov periodic orbits

In this appendix we prove Proposition 4.1. Let us recall that, by Proposition 2.3, the equilibrium point L_3 in the coordinates (λ, Λ, x, y) (see (2.5)), is given by

$$\mathfrak{L}(\delta) = \left(0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)\right)^T,$$

with $|\mathfrak{L}_\Lambda(\delta)|, |\mathfrak{L}_x(\delta)|\, |\mathfrak{L}_y(\delta)| \leq b_1$ for $\delta > 0$ small enough. Using that one can write H as $H = H_0 + H_1$, we have that

$$\begin{aligned} \partial_\lambda H_1(\mathfrak{L}(\delta); \delta) &= 0, & \partial_\Lambda H_1(\mathfrak{L}(\delta); \delta) &= 3\delta^2 \mathfrak{L}_\Lambda(\delta), \\ \partial_x H_1(\mathfrak{L}(\delta); \delta) &= -\delta \mathfrak{L}_y(\delta), & \partial_y H_1(\mathfrak{L}(\delta); \delta) &= -\delta \mathfrak{L}_x(\delta). \end{aligned} \tag{A.1}$$

In addition, one can easily check that

$$H(\mathfrak{L}(\delta); \delta) = -\frac{1}{2} - \frac{3}{2}\delta^4 \mathfrak{L}_\Lambda^2(\delta) + \delta^4 \mathfrak{L}_x(\delta)\mathfrak{L}_y(\delta) + H_1(\mathfrak{L}(\delta); \delta). \tag{A.2}$$

For $\rho > 0$, we consider a polar symplectic change of coordinates $\phi_{\mathrm{Lya}} : (\lambda, J, \varphi, I) \to (\lambda, \Lambda, x, y)$ given by

$$\Lambda = J + \delta^2 \mathfrak{L}_\Lambda(\delta), \quad x = \sqrt{\rho^2 + I} e^{-i\varphi} + \delta^3 \mathfrak{L}_x(\delta), \quad y = \sqrt{\rho^2 + I} e^{i\varphi} + \delta^3 \mathfrak{L}_y(\delta). \tag{A.3}$$

The Hamiltonian H expressed in the coordinates (λ, J, φ, I) becomes $H^{\mathrm{Lya}} = H \circ \phi_{\mathrm{Lya}}$, given by

$$\begin{aligned} H^{\mathrm{Lya}}(\lambda, J, \varphi, I; \rho, \delta) = &-\frac{3}{2}J^2 + V(\lambda) + \frac{\rho^2 + I}{\delta^2} + H_1(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I); \delta) - 3\delta^2 J \mathfrak{L}_\Lambda \\ &+ \delta\sqrt{\rho^2 + I}\left(e^{-i\varphi}\mathfrak{L}_y + e^{i\varphi}\mathfrak{L}_x\right) - \frac{3}{2}\delta^4 \mathfrak{L}_\Lambda + \delta^4 \mathfrak{L}_x \mathfrak{L}_y, \end{aligned}$$

which, using (A.1) and (A.2), can be rewritten as

$$\begin{aligned} H^{\mathrm{Lya}}(\lambda, J, \varphi, I; \rho, \delta) = &-\frac{3}{2}J^2 + V(\lambda) + \frac{1}{2} + \frac{I}{\delta^2} + H_1(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I); \delta) \\ &- H_1(\mathfrak{L}; \delta) - DH_1(\mathfrak{L}; \delta) \cdot (\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I) - \mathfrak{L})^T \\ &+ \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta). \end{aligned} \tag{A.4}$$

We are interested in proving the existence of a periodic orbit in the energy level $H^{\mathrm{Lya}} = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta)$. To this end, in the following lemma, we first obtain an expression of I in terms of the other coordinates. Let us denote by $B(\varsigma) = \{z \in \mathbb{C} : |z| < \varsigma\}$, the open ball of radius ς.

Lemma A.1. *Fix $d, \varsigma_\lambda, \varsigma_J, \rho_0 > 0$. There exists $\delta_0 > 0$ such that, for all $\rho \in (0, \rho_0]$ and $\delta \in (0, \delta_0)$, there exists a function*

$$\widehat{I}_{\rho,\delta} : B(\delta\rho\varsigma_\lambda) \times B(\delta\rho\varsigma_J) \times \mathbb{T}_d \to \mathbb{C},$$

such that $H^{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}(\lambda, J, \varphi); \rho, \delta) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta)$.

Moreover, there exists a constant $C > 0$ *independent of* ρ *and* δ *such that*

$$|\widehat{I}_{\rho,\delta}(\lambda, J, \varphi; \delta)| \le C\delta^4\rho^2, \qquad |\partial_\lambda \widehat{I}_{\rho,\delta}(\lambda, J, \varphi; \delta)| \le C\delta^3\rho,$$
$$|\partial_J \widehat{I}_{\rho,\delta}(\lambda, J, \varphi; \delta)| \le C\delta^3\rho, \qquad |\partial_\varphi \widehat{I}_{\rho,\delta}(\lambda, J, \varphi; \delta)| \le C\delta^4\rho^2.$$

Proof. One has that the function $\widehat{I}_{\rho,\delta}$ must satisfy the equation $\widehat{I}_{\rho,\delta} = F[\widehat{I}_{\rho,\delta}]$ with

$$\begin{aligned} F[I](\lambda, J, \varphi) &= \delta^2 H^{\mathrm{Lya}}(\lambda, J, \varphi, I; \rho, \delta) - I - \rho^2 - \delta^2 H(\mathfrak{L}; \delta) \\ &= \delta^2 \bigg[\frac{3}{2}J^2 + V(\lambda) + \frac{1}{2} + H_1(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I); \delta) \\ &\qquad - H_1(\mathfrak{L}; \delta) - DH_1(\mathfrak{L}; \delta) \cdot (\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I) - \mathfrak{L})^T\bigg]. \end{aligned}$$

Let $(\lambda, J, \varphi) \in B(\delta\rho\varsigma_\lambda) \times B(\delta\rho\varsigma_J) \times \mathbb{T}_d$. Then, using the estimates of D^2H_1 in Proposition 2.1, one has that

$$|F[0](\lambda, J, \varphi)| \le C\delta^2\rho^2.$$

In addition, for functions $\iota_1, \iota_2 : B(\delta\rho\varsigma_\lambda) \times B(\delta\rho\varsigma_J) \times \mathbb{T}_d \to \mathbb{C}$ such that $|\iota_1(\lambda, J, \varphi)|, |\iota_2(\lambda, J, \varphi)| \le C\delta^2\rho^2$, by the estimates of the third derivatives of H_1 in Proposition 2.1 and the mean value theorem, one has that

$$|F[\iota_1](\lambda, J, \varphi) - F[\iota_2](\lambda, J, \varphi)| \le C\delta^3\rho\, |\iota_1(\lambda, J, \varphi) - \iota_2(\lambda, J, \varphi)| \le C\delta_0^3\rho_0\, |\iota_1(\lambda, J, \varphi) - \iota_2(\lambda, J, \varphi)|.$$

Then, taking δ_0 small enough and applying the fixed point theorem, one obtains the existence of the function $\widehat{I}_{\rho,\delta}$ and its corresponding bounds. The bounds for the derivatives of $\widehat{I}_{\rho,\delta}$ are a direct consequence of Cauchy estimates. ☐

By Lemma A.1, the Hamiltonian system on the energy level $H^{\mathrm{Lya}} = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta)$ is of the form

$$\dot{\lambda} = -3J + f_1(\lambda, J, \varphi), \qquad \dot{J} = -\frac{7}{8}\lambda + f_2(\lambda, J, \varphi), \qquad \dot{\varphi} = \frac{1}{\delta^2} + g(\lambda, J, \varphi), \tag{A.5}$$

where, denoting $\widehat{I}_{\rho,\delta} = \widehat{I}_{\rho,\delta}(\lambda, J, \varphi)$ and using the expression of H^{Lya} in (A.4) and that $V''(0) = -\frac{7}{8}$,

$$\begin{aligned} f_1(\lambda, J, \varphi) &= \partial_\Lambda H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_\Lambda H_1(\mathfrak{L}(\delta); \delta), \\ f_2(\lambda, J, \varphi) &= -V'(\lambda) + V''(0)\lambda - \partial_\lambda H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) + \partial_\lambda H_1(\mathfrak{L}(\delta); \delta), \\ g(\lambda, J, \varphi) &= \frac{e^{-i\varphi}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left(\partial_x H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_x H_1(\mathfrak{L}(\delta); \delta)\right) \\ &\quad + \frac{e^{i\varphi}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left(\partial_y H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_y H_1(\mathfrak{L}(\delta); \delta)\right). \end{aligned} \tag{A.6}$$

We look for the periodic orbit of the system (A.5) as a graph over φ provided $\dot{\varphi} \neq 0$ (which will be true on the periodic orbits). In other words, we look for periodic functions

$$w = (w_\lambda, w_J) : \mathbb{T}_d \to \mathbb{C}^2, \qquad w = w(\varphi),$$

satisfying the invariance equation $\mathcal{L}w = \mathcal{R}[w]$, with

$$\begin{aligned} \mathcal{L}w &= (\partial_\varphi - \delta^2 \mathcal{A})w, \qquad \mathcal{A} = \begin{pmatrix} 0 & -3 \\ -\frac{7}{8} & 0 \end{pmatrix}, \\ \mathcal{R}[w](\varphi) &= \delta^2 \left(\frac{\mathcal{A}w + f(w_\lambda(\varphi), w_J(\varphi), \varphi)}{1 + \delta^2 g(w_\lambda(\varphi), w_J(\varphi), \varphi)} - \mathcal{A}w \right) \quad \text{where} \quad f = (f_1, f_2). \end{aligned} \tag{A.7}$$

Let us consider the Banach space

$$\mathcal{Z} = \left\{ h : \mathbb{T}_d \to \mathbb{C} \;:\; h \text{ analytic}, \; \|h\| := \sup_{\varphi \in \mathbb{T}_d} |h(\varphi)| < +\infty \right\},$$

and the space $\mathcal{Z}^2$ endowed with the product norm $\|h\|^\times = \|h_1\| + \|h_2\|$.

Proposition A.2. *There exist $\rho_0, \delta_0, b_6 > 0$ such that, for $\rho \in (0, \rho_0]$ and $\delta \in (0, \delta_0)$, there exists a solution of $\mathcal{L}w = \mathcal{R}[w]$ belonging to $\mathcal{Z}^2$ and satisfying*

$$\|w\|^\times \leq b_6 \delta \rho.$$

To prove Proposition A.2 we first study the right-inverse of the operator $\mathcal{L} = \partial_\varphi - \delta^2 \mathcal{A}$ in $\mathcal{Z}^2$. First, notice that

$$\mathcal{A} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1} \quad \text{where} \quad \mathcal{D} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 3 & 3 \\ -\nu & \nu \end{pmatrix}, \quad \nu = \sqrt{\frac{21}{8}}.$$

Lemma A.3. *The operator $\mathcal{G} : \mathcal{Z}^2 \to \mathcal{Z}^2$ defined as*

$$\begin{aligned} \mathcal{G}[h](\varphi) = &\mathcal{P} e^{\varphi \delta^2 \mathcal{D}} (e^{-2\pi\delta^2 \mathcal{D}} - \mathrm{Id})^{-1} \int_0^{2\pi} e^{-\theta\delta^2\mathcal{D}} \mathcal{P}^{-1} h(\theta) d\theta \\ &+ \mathcal{P} e^{\varphi\delta^2\mathcal{D}} \int_0^\varphi e^{-\theta\delta^2\mathcal{D}} \mathcal{P}^{-1} h(\theta) d\theta, \end{aligned} \tag{A.8}$$

is a right-inverse of the operator $\mathcal{L}$ given in (A.7). In addition, there exists $C > 0$ such that, for $\delta \in (0, 1)$,

$$\|\mathcal{G}[h]\|^\times \leq \frac{C}{\delta^2} \|h\|^\times, \qquad \textit{for } h \in \mathcal{Z}^2.$$

Proof. If w is a solution of $\mathcal{L}[w] = h$, it must exist $K_0 \in \mathbb{R}^2$ such that

$$w(\varphi) = \mathcal{P} e^{\varphi\delta^2\mathcal{D}} \left[K_0 + \int_0^\varphi e^{-\theta\delta^2\mathcal{D}} \mathcal{P}^{-1} h(\theta) d\theta \right].$$

Then, imposing that w is 2π-periodic, one obtains (A.8). The estimates for the operator are straightforward taking into account that

$$\|(e^{-2\pi\delta^2\mathcal{D}} - \mathrm{Id})^{-1}\| \leq (1 - \|e^{-2\pi\delta^2\mathcal{D}} - \mathrm{Id}\|)^{-1} \leq \frac{C}{\delta^2}.$$

□

For $\varsigma > 0$, we denote $\mathcal{B}(\varsigma) = \left\{ h \in \mathcal{Z}^2 \,:\, \|h\|^\times \leq \varsigma \right\}$.

Lemma A.4. *Fix constants $\rho_0, \varsigma > 0$. Then, there exist $\delta_0, C > 0$ such that, for $\rho \in (0, \rho_0]$, $\delta \in (0, \delta_0)$ and $h \in \mathcal{B}(\varsigma\delta\rho)$, the function $\mathcal{R}$ in (A.7) satisfies*

$$\|\mathcal{R}_1[h]\| \leq C\delta^5\rho, \qquad \|\mathcal{R}_2[h]\| \leq C\delta^3\rho$$

and

$$\|\partial_1\mathcal{R}_1[h]\| \leq C\delta^4, \quad \|\partial_2\mathcal{R}_1[h]\| \leq C\delta^4, \quad \|\partial_1\mathcal{R}_2[h]\| \leq C\delta^3, \quad \|\partial_2\mathcal{R}_2[h]\| \leq C\delta^4.$$

Proof. Let $h = (h_1, h_2) \in \mathcal{B}(\varsigma\delta\rho)$ and $\varphi \in \mathbb{T}_d$. For $s \in [0,1]$, we denote

$$z^s(\varphi) = s\,\phi_{\mathrm{Lya}}\left(h_1(\varphi), h_2(\varphi), \varphi, \widehat{I}_{\rho,\delta}(h(\varphi))\right) + (1-s)\mathfrak{L}(\delta).$$

We notice that, by the definition in (A.3) of ϕ_{Lya},

$$z^1(\varphi) - z^0(\varphi) = \left(h_1(\varphi), h_2(\varphi), \sqrt{\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))}e^{-i\varphi}, \sqrt{\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))}e^{i\varphi}\right)^T.$$

We recall that $f_1 = \partial_\Lambda H_1(\phi_{\mathrm{Lya}}) - \partial_\Lambda H_1(\mathfrak{L})$ (see (A.6)) and then, by the mean value theorem and the estimates in Proposition 2.1 and Lemma A.1,

$$\begin{aligned}|f_1(h(\varphi),\varphi)| \leq \sup_{s\in[0,1]} \Big\{ &|\partial_{\Lambda\lambda}H_1\left(z^s(\varphi)\right)|\,|h_1(\varphi)| + \left|\partial^2_\Lambda H_1\left(z^s(\varphi)\right)\right|\,|h_2(\varphi)| \\ &+ \big(|\partial_{\Lambda x}H_1\left(z^s(\varphi)\right)| + |\partial_{\Lambda y}H_1\left(z^s(\varphi)\right)| \big)\,|\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))|^{\frac{1}{2}} \Big\} \leq C\delta^3\rho.\end{aligned} \tag{A.9}$$

Analogously,

$$|f_2(h(\varphi),\varphi)| \leq C\delta\rho, \qquad |g(h(\varphi),\varphi)| \leq C\delta^2. \tag{A.10}$$

To obtain estimates for the derivatives of f_1, f_2 and g, note that

$$\partial_\lambda f_1(h(\varphi),\varphi) = \partial_{\lambda\Lambda}H_1\left(z^1(\varphi)\right) + \frac{\partial_\lambda \widehat{I}_{\rho,\delta}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left[e^{-i\varphi}\partial_{\Lambda x}H_1\left(z^1(\varphi)\right) + e^{i\varphi}\partial_{\Lambda y}H_1\left(z^1(\varphi)\right)\right],$$

$$\partial_J f_1(h(\varphi),\varphi) = \partial^2_\Lambda H_1\left(z^1(\varphi)\right) + \frac{\partial_J \widehat{I}_{\rho,\delta}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left[e^{-i\varphi}\partial_{\Lambda x}H_1\left(z^1(\varphi)\right) + e^{i\varphi}\partial_{\Lambda y}H_1\left(z^1(\varphi)\right)\right],$$

where $\widehat{I}_{\rho,\delta} = \widehat{I}_{\rho,\delta}(h(\varphi))$. Then, using the estimates in Proposition 2.1 and Lemma A.1,

$$|\partial_\lambda f_1(h(\varphi),\varphi)| \leq C\delta^2, \qquad |\partial_J f_1(h(\varphi),\varphi)| \leq C\delta^2. \tag{A.11}$$

Analogously,

$$\begin{aligned}|\partial_\lambda f_2(h(\varphi),\varphi)| &\leq C\delta, \qquad |\partial_J f_2(h(\varphi),\varphi)| \leq C\delta^2, \\ |\partial_\lambda g(h(\varphi),\varphi)| &\leq \frac{C\delta}{\rho}, \qquad |\partial_J g(h(\varphi),\varphi)| \leq \frac{C\delta^3}{\rho}.\end{aligned} \tag{A.12}$$

Finally, joining the just obtained bounds with the definition of the operator $\mathcal{R}$ in (A.7), we obtain the statement of the lemma. □

Proof of Proposition A.2. A fixed point of $w = \mathcal{F}[w]$ with $\mathcal{F} = \mathcal{G} \circ \mathcal{R}$ is a periodic solution of $\mathcal{L}w = \mathcal{R}[w]$. By Lemmas A.3 and A.4, there exists $b_6 > 0$ such that

$$\|\mathcal{F}[0]\|^{\times} \leq \frac{C}{\delta^2}\left(\|\mathcal{R}_1[0]\| + \|\mathcal{R}_2[0]\|\right) \leq \frac{b_6}{2}\delta\rho. \tag{A.13}$$

Moreover, for $h, \hat{h} \in \mathcal{B}(b_6\delta\rho)$, by the mean value theorem,

$$\|\mathcal{R}[h] - \mathcal{R}[\hat{h}]\|^{\times} \leq \sup_{s\in[0,1]} \left[\|D\mathcal{R}[(1-s)h + s\hat{h}](h-\hat{h})\|^{\times}\right].$$

Thus, by Lemmas A.3 and A.4,

$$\|\mathcal{F}[h] - \mathcal{F}[\hat{h}]\|^{\times} \leq \frac{C}{\delta^2}\|\mathcal{R}[h] - \mathcal{R}[\hat{h}]\|^{\times} \leq C\delta\|h - \hat{h}\|^{\times}. \tag{A.14}$$

Then, if δ is small enough, the operator $\mathcal{F} : \mathcal{B}(b_6\delta\rho) \to \mathcal{B}(b_6\delta\rho)$ is well defined and contractive and, as a consequence, it has a fixed point $w \in \mathcal{B}(b_6\delta\rho)$. □

End of the proof of Proposition 4.1. Let $w(\varphi) = (w_\lambda(\varphi), w_J(\varphi))$ be the solution of $\mathcal{L}w = \mathcal{R}[w]$ given by Proposition A.2 and introduce $w_I(\varphi) = \widehat{I}_{\rho,\delta}(w(\varphi), \varphi)$ as given in Lemma A.1. Then, the curve $(w_\lambda(\varphi), w_J(\varphi), \varphi, w_I(\varphi))$ is a graph parametrization of the Lyapunov periodic solution in the energy level $H^{\text{Lya}} = \frac{\rho^2}{\delta^2} + H(\mathfrak{L})$. However, $\dot{\varphi} = \partial_t\varphi = \frac{1}{\delta^2} + g(w(\varphi), \varphi)$. Then, to complete the proof of Proposition 4.1, we look for a reparametrization $\varphi = \widehat{\varphi}(\tau)$ and a constant $\omega_{\rho,\delta}$ such that $\dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}$. Moreover, we impose $\varphi(t)|_{t=0} = 0$ and therefore $\widehat{\varphi}(2\pi) = 2\pi$. Then, $\widehat{\varphi}$ must satisfy that

$$\partial_\tau\widehat{\varphi} = \frac{1 + \delta^2 g(w(\widehat{\varphi}), \widehat{\varphi})}{\omega_{\rho,\delta}} \qquad \text{and} \qquad \widehat{\varphi}(2\pi) = 2\pi.$$

Notice that, by (A.10) and for δ small enough, one has that $\partial_\tau\widehat{\varphi} \neq 0$. Then, its inverse $\tau \equiv \widehat{\tau}(\varphi)$ satisfies that

$$\partial_\varphi\widehat{\tau} = \frac{\omega_{\rho,\delta}}{1 + \delta^2 g(w(\varphi), \varphi)} \qquad \text{and} \qquad \widehat{\tau}(2\pi) = 2\pi.$$

These conditions give definitions for the function $\widehat{\tau}(\varphi)$ and the constant $\omega_{\rho,\delta}$,

$$\widehat{\tau}(\varphi) = \omega_{\rho,\delta}\int_0^{\varphi} \frac{d\eta}{1 + \delta^2 g(w(\eta), \eta)} \qquad \text{and} \qquad \omega_{\rho,\delta} = \frac{2\pi}{\int_0^{2\pi} \frac{d\varphi}{1+\delta^2 g(w(\varphi),\varphi)}}.$$

We notice that $\widehat{\tau}(\varphi + 2\pi) = 2\pi + \widehat{\tau}(\varphi)$. By the estimate for g in (A.10), we obtain

$$|\omega_{\rho,\delta} - 1| \leq C\delta^4, \qquad |\widehat{\tau}(\varphi) - \varphi| \leq C\delta^4, \qquad |\widehat{\varphi}(\tau) - \tau| \leq C\delta^4. \tag{A.15}$$

Then, for $\tau \in \mathbb{T}_d$, the curve

$$\mathfrak{P}_\rho(\tau;\delta) = \phi_{\text{Lya}}\big(w_\lambda(\widehat{\varphi}(\tau)), w_J(\widehat{\varphi}(\tau)), \widehat{\varphi}(\tau), w_I(\widehat{\varphi}(\tau))\big)$$

is a real-analytic and 2π-periodic solution of the Hamiltonian system given by the Hamiltonian H in (2.8) and it belongs to the energy level $H = \frac{\rho^2}{\delta^2} + H(\mathfrak{L})$. In addition, the functions in (4.2) are given by

$$\lambda_{\mathfrak{P}}(\tau) = \frac{w_\lambda(\widehat{\varphi}(\tau))}{\delta\rho}, \qquad x_{\mathfrak{P}}(\tau) = \frac{\sqrt{\rho^2 + w_I(\widehat{\varphi}(\tau))}e^{-i\widehat{\varphi}(\tau)} - \rho e^{-i\tau}}{\delta\rho},$$
$$\Lambda_{\mathfrak{P}}(\tau) = \frac{w_J(\widehat{\varphi}(\tau))}{\delta\rho}, \qquad y_{\mathfrak{P}}(\tau) = \frac{\sqrt{\rho^2 + w_I(\widehat{\varphi}(\tau))}e^{i\widehat{\varphi}(\tau)} - \rho e^{i\tau}}{\delta\rho},$$

and, by Lemma A.1, Proposition A.2 and (A.15), satisfy that $|\lambda_{\mathfrak{P}}(\tau)|, |\Lambda_{\mathfrak{P}}(\tau)| \leq C$ and $|x_{\mathfrak{P}}(\tau)|, |y_{\mathfrak{P}}(\tau)| \leq C\delta^3$. □

B Difference between the invariant manifolds of L_3 on Σ_0

In this appendix we prove Corollary 2.5, relying on the results in Sections 4.2 and 4.4. Let us consider the real-analytic time parametrizations z^{u} and z^{s} of the unstable and stable manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ defined in Corollary 4.5. Notice that, for $u \in D^{\mathrm{u}} \cap D^{\mathrm{s}}$ (see (4.6)), they satisfy

$$\left|z^{\mathrm{u}}(u) - \sigma_{\mathrm{p}}(u) - \delta^2\mathfrak{L}_\Lambda\right| \leq C\delta, \qquad \left|z^{\mathrm{s}}(u) - \sigma_{\mathrm{p}}(u) - \delta^2\mathfrak{L}_\Lambda\right| \leq C\delta, \tag{B.1}$$

where $\sigma_{\mathrm{p}} = (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)^T$ is given in (4.4). Moreover, $z^{\mathrm{u}}(0), z^{\mathrm{s}}(0) \in \left\{\Lambda = \delta^2\mathfrak{L}_\Lambda\right\}$ and, since z^{u} and z^{s} satisfy equation (4.7) and are independent of τ, for $z^\diamond = (\lambda^\diamond, \Lambda^\diamond, x^\diamond, y^\diamond)$, $\diamond = \mathrm{u}, \mathrm{s}$, one has that

$$\begin{aligned}
\frac{d\lambda^\diamond}{du} &= -3\Lambda^\diamond + \partial_\Lambda H_1(z^{\mathrm{u}};\delta), & \frac{dx^\diamond}{du} &= \frac{i}{\delta^2}x^\diamond + i\partial_y H_1(z^{\mathrm{u}};\delta)\\
\frac{d\Lambda^\diamond}{du} &= -V'(\lambda^\diamond) - \partial_\lambda H_1(z^{\mathrm{u}};\delta), & \frac{dy^\diamond}{du} &= -\frac{i}{\delta^2}y^\diamond - i\partial_x H_1(z^{\mathrm{u}};\delta).
\end{aligned} \tag{B.2}$$

Fix $\lambda_* \in \left(\frac{2\pi}{3}, \lambda_0\right)$, (see (2.18)). By Proposition 4.3, there exists $u_* > 0$ such that $\lambda_* = \lambda_{\mathrm{p}}(u_*)$. Therefore, by (B.1) and for $\delta > 0$ small enough, there exist $T^{\mathrm{u}}, T^{\mathrm{s}} = u_* + \mathcal{O}(\delta)$ such that $z^{\mathrm{u}}(T^{\mathrm{u}}), z^{\mathrm{s}}(T^{\mathrm{s}}) \in \{\lambda = \lambda_*, \Lambda > 0\}$. Moreover, by Theorem 2.4,

$$z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}}) = \sqrt[6]{2}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left[\left(0, 0, \overline{\Theta}, \Theta\right)^T + \mathcal{O}_\delta\right], \tag{B.3}$$

where $\mathcal{O}_\delta = (0, \mathcal{O}(\delta), \mathcal{O}(|\log\delta|^{-1}), \mathcal{O}(|\log\delta|^{-1}))^T$.

To prove Corollary 2.5, we deduce the difference $z^{\mathrm{u}}(0) - z^{\mathrm{s}}(0)$ from (B.3). To this end, we define $\Delta(u) = z^{\mathrm{u}}(u) - z^{\mathrm{s}}(u)$, for $u \in [0, T^{\mathrm{u}}]$. It is clear that, by (B.2), the function $\Delta(u)$ satisfies the linear equation

$$\frac{d}{du}\Delta(u) = (M_0(u) + M_1(u))\Delta(u),$$

with

$$M_0(u) = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_{\mathrm{p}}(u)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix},$$

$$M_1(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ m(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \int_0^1 \mathbf{J} D^2 H_1 \left(\varsigma z^{\mathrm{u}}(u) + (1-\varsigma) z^{\mathrm{s}}(u)\right) d\varsigma,$$

$$m(u) = V''(\lambda_{\mathrm{p}}(u)) - \int_0^1 V''\left(\varsigma \lambda^{\mathrm{u}}(u) + (1-\varsigma)\lambda^{\mathrm{s}}(u)\right) d\varsigma,$$

where $\mathbf{J}$ is the symplectic matrix associated with the form $d\lambda \wedge d\lambda + i dx \wedge dy$. Moreover, from Proposition 2.1 and Corollary 4.5, we deduce that $|M_1(u)| \leq C\delta$, for $u \in [0, T^{\mathrm{u}}]$. Let $\Phi(u)$ be the fundamental matrix of the differential equation $\frac{d}{du}\Phi(u) = M_0(u)\Phi(u)$ given in Lemma 4.10, which satisfies $\Phi(0) = \mathrm{Id}$. Then,

$$\Delta(u) = \Phi(u) \left[\Phi^{-1}(T^{\mathrm{u}})\Delta(T^{\mathrm{u}}) + \int_{T^{\mathrm{u}}}^{u} \Phi^{-1}(\sigma) M_1(\sigma) \Delta(\sigma) d\sigma \right].$$

On one hand, using Gronwall's Lemma, one has that $|\Delta(u)| \leq C\,|\Delta(T^{\mathrm{u}})|$ for $u \in [0, T^{\mathrm{u}}]$ and, on the other hand

$$\left|\Delta(0) - \Phi^{-1}(T^{\mathrm{u}})\Delta(T^{\mathrm{u}})\right| \leq C\delta T^{\mathrm{u}}\,|\Delta(T^{\mathrm{u}})|. \tag{B.4}$$

Thus, to obtain an asymptotic formula for $\Delta(0)$, we need to compute $\Delta(T^{\mathrm{u}})$. We write

$$\Delta(T^{\mathrm{u}}) = z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}}) + z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}}). \tag{B.5}$$

Since the difference $z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}})$ is given by (B.3), we only need to analyze the term $z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}})$. To do so, we first bound $T^u - T^s$. Since $z^{\mathrm{s}} = (\lambda^{\mathrm{s}}, \Lambda^{\mathrm{s}}, x^{\mathrm{s}}, y^{\mathrm{s}})$ satisfies equation (B.2) and using the mean value theorem, we obtain that

$$T^{\mathrm{u}} - T^{\mathrm{s}} = \frac{\Lambda^{\mathrm{u}}(T^{\mathrm{s}}) - \Lambda^{\mathrm{u}}(T^{\mathrm{u}})}{V'(\lambda_{\mathrm{p}}(u_*)) + \beta(T^{\mathrm{u}}, T^{\mathrm{s}})},$$

where, denoting $T(r) = rT^{\mathrm{u}} + (1-r)T^{\mathrm{s}}$, the function β is given by

$$\beta(T^{\mathrm{u}}, T^{\mathrm{s}}) = \int_0^1 \left[V'(\lambda^{\mathrm{u}}(T(r))) - V'(\lambda_{\mathrm{p}}(u_*))\right] dr + \int_0^1 \partial_\lambda H_1(z^{\mathrm{u}}(T(r))) dr.$$

Notice that $V'(\lambda_{\mathrm{p}}(u_*)) = V'(\lambda_*) \neq 0$ (see (2.7)). Moreover, since $T^{\mathrm{u}}, T^{\mathrm{s}} = u_* + \mathcal{O}(\delta)$, by (B.1) and the estimates in Proposition 2.1, one can see that $|\beta(T^{\mathrm{u}}, T^{\mathrm{s}})| \leq C\delta$. Therefore, one has that $|T^{\mathrm{u}} - T^{\mathrm{s}}| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}$. Then, since

$$z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}}) = (T^{\mathrm{s}} - T^{\mathrm{u}}) \int_0^1 \partial_u z^{\mathrm{s}}(rT^{\mathrm{u}} + (1-r)T^{\mathrm{s}}) dr,$$

we have $z^{\rm s}(T^{\rm s}) - z^{\rm s}(T^{\rm u}) = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right)$. Therefore, by (B.3) and (B.5)

$$\Delta(T^{\rm u}) = \sqrt[6]{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\left(0, 0, \overline{\Theta}, \Theta\right)^T + \widetilde{\mathcal{O}_\delta} \right], \tag{B.6}$$

where $\widetilde{\mathcal{O}_\delta} = (\mathcal{O}(\delta), \mathcal{O}(\delta), \mathcal{O}(|\log\delta|^{-1}), \mathcal{O}(|\log\delta|^{-1}))^T$. Lastly, joining the results in (B.4) and (B.6), we obtain

$$|\Delta(0)| = \sqrt[6]{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Phi^{-1}(T^{\rm u}) \left(0, 0, \overline{\Theta}, \Theta\right)^T| + \widetilde{\mathcal{O}_\delta} \right].$$

Then, applying the expression of the fundamental matrix Φ given in Lemma 4.10, we obtain the statement of the corollary.

References

[Ale76] V. M. Alekseev. Quasi-random oscillations and qualitative problems of celestial mechanics. *STIA*, 77:212–341, 1976.

[Ari02] G. Arioli. Periodic orbits, symbolic dynamics and topological entropy for the restricted 3-body problem. *Comm. Math. Phys.*, 231(1):1–24, 2002.

[Arn63] V. I. Arnol'd. Small denominators and problems of stability of motion in classical and Celestial Mechanics. *Uspehi Mat. Nauk*, 18(6 (114)):91–192, 1963.

[BCGG23] I. Baldomá, M. Capiński, M. Giralt, and M. Guardia. Breakdown of homoclinic orbits to L_3: Nonvanishing of the Stokes constant. *Preprint arXiv:2312.13138*, 2023.

[BFGS12] I. Baldomá, E. Fontich, M. Guardia, and T. M. Seara. Exponentially small splitting of separatrices beyond Melnikov analysis: rigorous results. *J. Differential Equations*, 253(12):3304–3439, 2012.

[BFPC13] A. Bengochea, M. Falconi, and E. Pérez-Chavela. Horseshoe periodic orbits with one symmetry in the general planar three-body problem. *Discrete & Continuous Dynamical Systems*, 33(3):987, 2013.

[BFPS22] P. Berger, A. Florio, and D. Peralta-Salas. Steady euler flows on $\mathbb{R}^3$ with wild and universal dynamics. *Preprint arXiv:2202.02848*, 2022.

[BGG22] I. Baldomá, M. Giralt, and M. Guardia. Breakdown of homoclinic orbits to L3 in the RPC3BP (I). Complex singularities and the inner equation. *Advances in Mathematics*, 408:108562 (64), 2022.

[BGG23] I. Baldomá, M. Giralt, and M. Guardia. Breakdown of homoclinic orbits to L_3 in the RPC3BP (II). An asymptotic formula. *Adv. Math.*, 430:109218(72), 2023.

[BIS20] I. Baldomá, S. Ibáñez, and T. M. Seara. Hopf-Zero singularities truly unfold chaos. *Communications in Nonlinear Science and Numerical Simulation*, 84:105162, 2020.

[BM05] E. Barrabés and S. Mikkola. Families of periodic horseshoe orbits in the restricted three-body problem. *Astronomy & Astrophysics*, 432(3):1115–1129, 2005.

[BM06] S. V. Bolotin and R. S. MacKay. Nonplanar Second Species Periodic and Chaotic Trajectories for the Circular Restricted Three-Body Problem. *Celestial Mech. Dynam. Astronom.*, 94(4):433–449, 2006.

[BMO09] E. Barrabés, J. M. Mondelo, and M. Ollé. Dynamical aspects of multi-round horseshoe-shaped homoclinic orbits in the RTBP. *Celestial Mech. Dynam. Astronom.*, 105(1-3):197–210, 2009.

[BO06] E. Barrabés and M. Ollé. Invariant manifolds of L3 and horseshoe motion in the restricted three-body problem. *Nonlinearity*, 19:2065–2089, 2006.

[Bol06] S. Bolotin. Symbolic dynamics of almost collision orbits and skew products of symplectic maps. *Nonlinearity*, 19(9):2041–2063, 2006.

[Bro11] E. W. Brown. Orbits, Periodic, On a new family of periodic orbits in the problem of three bodies. *Monthly Notices of the Royal Astronomical Society*, 71:438–454, 1911.

[BRS03] P. Bernard, C. G. Ragazzo, and P. Salamao. Homoclinic orbits near saddle-center fixed points of Hamiltonian systems with two degrees of freedom. *Astérisque*, 286:151–165, 2003.

[Cap12] M. J. Capiński. Computer Assisted Existence Proofs of Lyapunov Orbits at L2 and Transversal Intersections of Invariant Manifolds in the Jupiter–Sun PCR3BP. *SIAM Journal on Applied Dynamical Systems*, 11(4):1723–1753, 2012.

[CH03] J. M. Cors and G. R. Hall. Coorbital periodic orbits in the three body problem. *SIAM Journal on Applied Dynamical Systems*, 2(2):219–237, 2003.

[CP11] L. Chierchia and G. Pinzari. The planetary N-body problem: symplectic foliation, reductions and invariant tori. *Inventiones mathematicae*, 186(1):1–77, 2011.

[CPY19] J. M. Cors, J. F. Palacián, and P. Yanguas. On co-orbital quasi-periodic motion in the three-body problem. *SIAM J. Appl. Dyn. Syst.*, 18(1):334–353, 2019.

[DM81a] S. F. Dermott and C. D. Murray. The dynamics of tadpole and horseshoe orbits. I - Theory. *Icarus*, 48(1):1–11, 1981.

[DM81b] S. F. Dermott and C. D. Murray. The dynamics of tadpole and horseshoe orbits. II - The coorbital satellites of Saturn. *Icarus*, 48(1):12–22, 1981.

[Dua08] P. Duarte. Elliptic isles in families of area-preserving maps. *Ergodic Theory and Dynamical Systems*, 28(6):1781–1813, 2008.

[F0́4] J. Féjoz. Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman). *Ergodic Theory Dynam. Systems*, 24(5):1521–1582, 2004.

[FGKR16] J. Féjoz, M. Guardia, V. Kaloshin, and P. Roldan. Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem. *Journal of the European Mathematical Society*, 18(10):2313–2401, 2016.

[GGSZ21] O. M. L. Gomide, M. Guardia, T. M. Seara, and C. Zeng. On small breathers of nonlinear Klein-Gordon equations via exponentially small homoclinic splitting. *arXiv preprint arXiv:2107.14566*, 2021.

[GJMS01] G. Gómez, A. Jorba, J. J. Masdemont, and C. Simó. *Dynamics And Mission Design Near Libration Points - Volume 4: Advanced Methods For Triangular Points*, volume 5. World Scientific, 2001.

[GK12] A. Gorodetski and V. Kaloshin. Hausdorff dimension of oscillatory motions for restricted three body problems. Preprint, 2012.

[GMPS22] M. Guardia, P. Martín, J. Paradela, and T. M. Seara. Hyperbolic dynamics and oscillatory motions in the 3 body problem. Preprint arXiv:2207.14351, 2022.

[GMS16] M. Guardia, P. Martín, and T. M. Seara. Oscillatory motions for the restricted planar circular three body problem. *Inventiones mathematicae*, 203(2):417–492, 2016.

[Gor12] A. Gorodetski. On stochastic sea of the standard map. *Comm. Math. Phys.*, 309(1):155–192, 2012.

[GPSV21] M. Guardia, J. Paradela, T. M. Seara, and C. Vidal. Symbolic dynamics in the restricted elliptic isosceles three body problem. *Journal of Differential Equations*, 294:143–177, 2021.

[GSMS17] M. Guardia, T. M. Seara, Pau Martín, and L. Sabbagh. Oscillatory orbits in the restricted elliptic planar three body problem. *Discrete & Continuous Dynamical Systems - A*, 37(1):229, 2017.

[GZ19] A. Gierzkiewicz and P. Zgliczyński. A computer-assisted proof of symbolic dynamics in Hyperion's rotation. *Celestial Mech. Dynam. Astronom.*, 131(7):1–17, 2019.

[Her98] M. Herman. Some open problems in dynamical systems. In *Proceedings of the International Congress of Mathematicians*, volume 2, pages 797–808. Berlin, 1998.

[HTL07] X. Hou, J. Tang, and L. Liu. Transfer to the collinear libration point L3 in the Sun–Earth+Moon system. *Nasa Technical Report*, 2007.

[JBL16] T. Jézéquel, P. Bernard, and E. Lombardi. Homoclinic connections with many loops near a $0^2 iw$ resonant fixed point for Hamiltonian systems. *Discrete and Continuous Dynamical Systems*, 36(6):3153–3225, 2016.

[JN20] A. Jorba and B. Nicolás. Transport and invariant manifolds near L3 in the Earth-Moon Bicircular model. *Commun. Nonlinear Sci. Numer. Simul.*, 89:19pp, 2020.

[JN21] À. Jorba and B. Nicolás. Using invariant manifolds to capture an asteroid near the L3 point of the Earth-Moon Bicircular model. *Commun. Nonlinear Sci. Numer. Simul.*, 102:23pp, 2021.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[Ler91] L. M. Lerman. Hamiltonian systems with loops of a separatrix of a saddle-center. *Selecta Math. Sov*, 10(1991):297–306, 1991.

[LO01] J. Llibre and M. Ollé. The motion of Saturn coorbital satellites in the restricted three-body problem. *Astronomy & astrophysics*, 378(3):1087–1099, 2001.

[Lom99] E Lombardi. Non-persistence of homoclinic connections for perturbed integrable reversible systems. *J. Dynam. Differential Equations*, 11(1):129–208, 1999.

[Lom00] E. Lombardi. *Oscillatory Integrals and Phenomena Beyond all Algebraic Orders: with Applications to Homoclinic Orbits in Reversible Systems*. Springer, 2000.

[LS80] J. Llibre and C. Simó. Oscillatory solutions in the planar restricted three-body problem. *Mathematische Annalen*, 248(2):153–184, 1980.

[MHO92] A. Mielke, P. Holmes, and O. O'Reilly. Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle-center. *J. Dynam. Differential Equations*, 4(1):95–126, 1992.

[MO17] K. R. Meyer and D. C. Offin. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, volume 90. Springer Science+Business Media, 2017.

[Moe89] R. Moeckel. Chaotic dynamics near triple collision. *Archive for Rational Mechanics and Analysis*, 107(1):37–69, 1989.

[Moe07] R. Moeckel. Symbolic dynamics in the planar three-body problem. *Regular and Chaotic Dynamics*, 12(5):449–475, 2007.

[Mor02] A. Morbidelli. *Modern celestial mechanics. Aspects of solar system dynamics*. CRC Press, 2002.

[Mos58] J. Moser. On the generalization of a theorem of A. Liapounoff. *Communications on Pure and Applied Mathematics*, 11(2):257–271, 1958.

[Mos01] J. Moser. *Stable and random motions in dynamical systems : with special emphasis on celestial mechanics*. Princeton University Press, 2001.

[New70] S. E. Newhouse. Nondensity of axiom A(a) on S^2. In *Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968)*, volume XIV-XVI of *Proc. Sympos. Pure Math.*, pages 191–202. Amer. Math. Soc., Providence, RI, 1970.

[NPR20] L. Niederman, A. Pousse, and P. Robutel. On the Co-orbital Motion in the Three-Body Problem: Existence of Quasi-periodic Horseshoe-Shaped Orbits. *Communications in Mathematical Physics*, 377(1):551–612, 2020.

[Rag97a] C. G. Ragazzo. Irregular dynamics and homoclinic orbits to Hamiltonian saddle centers. *Communications on Pure and Applied Mathematics*, 50(2):105–147, 1997.

[Rag97b] C. G. Ragazzo. On the stability of double homoclinic loops. *Communications in* *mathematical physics*, 184(2):251–272, 1997.

[Rob95] P. Robutel. Stability of the planetary three-body problem. II. KAM theory and existence of quasiperiodic motions. *Celestial Mech. Dynam. Astronom.*, 62(3):219–261, 1995.

[Sit60] K. Sitnikov. The existence of oscillatory motions in the three-body problem. In *Dokl. Akad. Nauk SSSR*, volume 133, pages 303–306, 1960.

[Sma67] S. Smale. Differentiable Dynamical Systems. *Bull. of the AMS*, 73(6):747–817, 1967.

[SSST13] C. Simó, P. Sousa-Silva, and M. Terra. Practical Stability Domains Near L4,5 in the Restricted Three-Body Problem: Some Preliminary Facts. In *Progress and Challenges in Dynamical Systems*, pages 367–382. Springer, 2013.

[Sze67] V. G. Szebehely. *Theory of orbits : the restricted problem of three bodies.* Academic Press, New York [etc.], 1967.

[TFR+10] M. Tantardini, E. Fantino, Y. Ren, P. Pergola, G. Gómez, and J. J. Masdemont. Spacecraft trajectories to the L3 point of the Sun–Earth three-body problem. *Celestial Mech. Dynam. Astronom.*, 108(3):215–232, 2010.

[WZ03] D. Wilczak and P. Zgliczyński. Heteroclinic connections between periodic orbits in planar restricted circular three-body problem – a computer assisted proof. *Comm. Math. Phys.*, 234(1):37–75, 2003.