CHAOTIC PHENOMENA AROUND L_3 IN THE RPC3BP

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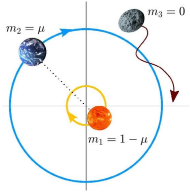
MFO Workshop 2128 Dynamische Systeme

OUTLINE

RESTRICTEDPLANARCIRCULAR3BP

We consider:

- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.
- Changing unities: $m_1 = 1 - \mu$, $m_2 = \mu$ and $T = 2\pi$.



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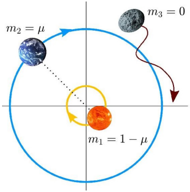

- In rotating (synodic) coordinates, the primaries are located at $(\mu, 0)$ and $(\mu - 1, 0)$ and the massless body follows a 2 degrees of freedom **autonomous** hamiltonian system.

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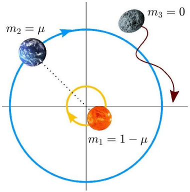

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The massless body follows the hamiltonian

$$\frac{\|p\|^2}{2} - q^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1-\mu}{\|q - (\mu, 0)\|} - \frac{\mu}{\|q - (\mu - 1, 0)\|}.$$

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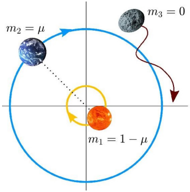

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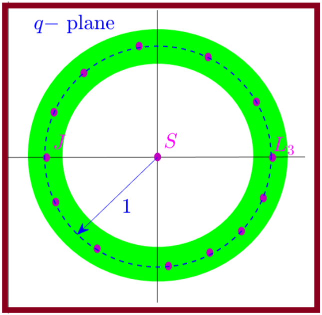
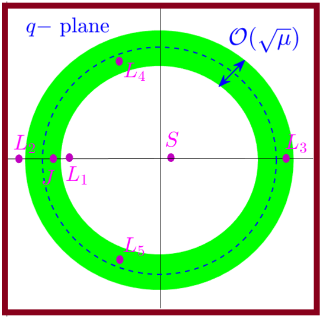
- We assume a perturbative setting, $0 < \mu \ll 1$.
- Notice that when $\mu = 0$, the third body follows a two body problem

μ AS A SINGULAR PARAMETER



$\mu = 0$. A cercle of equilibrium points



$\mu > 0$. $L_1, \cdots, L_5$ equilibrium points.

- We focus on the Lagrangian point L_3 which belongs to the mean motion resonance $1 : 1$.

MEAN MOTION RESONANCE

The mean motion resonance $1 : 1$ is a region of the phase space close to the motions of the third body having the same period τ, as the primaries. That is, $\tau = 2\pi$ (major axis $a = 1$).
The green zone in the figure corresponds to $a = 1$, $e = 0$.

MAIN RESULT

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no intersection (In the figure is the projection of $W^{u,s}$ on the q-plane).
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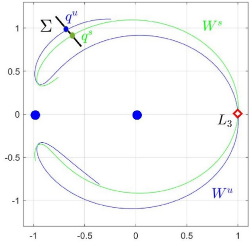

THEOREM

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

$$\|q^u - q^s\| + \|p^u - p^s\| \sim \boxed{K}\,\mu^{\frac{1}{3}} e^{-\frac{\boxed{A}}{\sqrt{\mu}}}.$$

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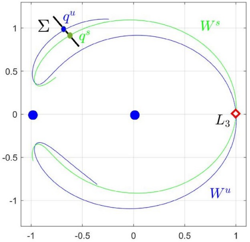

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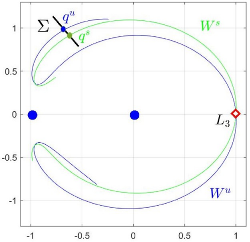

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Stokes constant

Known constant

THE CONSTANTS

The constants A, K have a different nature:

- The constant

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}dx \sim 0.177744.$$

  Is the height of the analyticity strip of a suitable homoclinic connection.
- K corresponds a Stokes constant, depending on the full jet of the hamiltonian. Can be numerically computed by means of the so called *inner equation*. We obtain $K \sim 1.63$. In fact, by means of a computer assisted proof, we expect to prove that $K \neq 0$.
- We can not use a *Melnikov-like* theory.

DYNAMICS AROUND L_3 AND ITS MANIFOLDS

- The motion takes place far from collision.
- Mean motion resonances can lead to inestabilities, see for instance

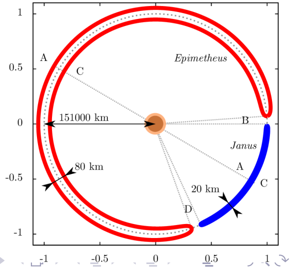
  J. Féjoz, M. Guardia, V. Kaloshin, and P. Roldan.
  Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem

- The center-stable and center-unstable invariant manifold act as boundaries of effective stability of the stability domains around L_4 and L_5.

  C. Simó, P. Sousa-Silva, and M. Terra
  Practical Stability Domains Near $L_{4,5}$ in the Restricted Three-Body Problem: Some Preliminary Facts

- **Horseshoe-shaped orbits**: quasi-periodic orbits encompassing L_3, L_4 and L_5. These orbits can model the motion of **co-orbital satellites** (Janus and Epimetheus, for example).

  L. Niederman, A. Pousse and P. Robutel. 2020.
  On the co-orbital motion in the 3*-BP: Existence of quasi-periodic horseshoe-shaped orbits.*



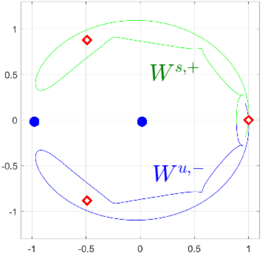
COMMENTS

- One should expect that our result implies that there exist Lyapunov periodic orbits exponentially close to L_3 whose stable and unstable invariant manifolds intersect transversally.

- We have not primary homoclinic connection, but in

  E. Barrabés, J. M. Mondelo, and M. Ollé
  Dynamical aspects of multi-round horseshoe-shaped homoclinic orbits in the RTBP

  is conjectured the existence of multiround homoclinic orbits to L_3 for $\{\mu_k\}$, with $\mu_k \to 0$.



- A more difficult problem is to consider the elliptic case (the primaries move in an elliptic motion) and try to prove, for small excentricities, the existence of diffussing orbits.

DIFFERENT SCALES

- We use Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y)=H_0(\lambda,\Lambda,x,y)+o(1)$$

with

$$H_0(\lambda,\Lambda,x,y)=i\frac{xy}{\sqrt{\mu}}-\frac{3}{2}\Lambda^2+1-\cos\lambda-\frac{1}{\sqrt{2+2\cos\lambda}}$$

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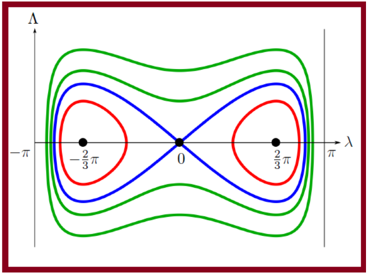
Fast variables

Slow variables

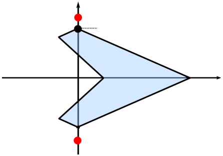
- The slow system is a pendulum-like hamiltonian system with homoclinic connections



- L_3 corresponds to the origin.
- The homoclinic connection is parameterizated by $(\lambda_0(t), \Lambda_0(t))$
- $\lambda_0(t)$ is analytic in some complex strip.
- We have no explicit expression for $\lambda_0(t)$

EXPONENTIALLY SMALL BOUND

- We prove the existence of analytic parameterizations of the invariant manifolds in a common domain. The black point is iB.



- The one dimensional stable and unstable manifold are solutions of the same equation.
- The difference between the invariant manifold is a solution of a linear homogeneous system satisfying

$$\dot{\Delta x} \sim \frac{i}{\sqrt{\mu}} \Delta x, \qquad \Delta x(t) \sim e^{i\frac{t}{\sqrt{\mu}}} C.$$

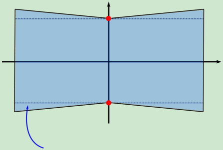
- Then $\Delta x(iB) \sim e^{\frac{B}{\sqrt{\mu}}} C$ implies $C \sim e^{-\frac{B}{\sqrt{\mu}}}$.

Bounded in complex domain implies exponentially small in real domain.

- Bigger B better bound. Since we expect that the homoclinic connection $\lambda_0(t)$ be a *good* approximation of the invariant manifolds, we need to analyze its complex singularities.
- However, to capture the first order we will need a better approximation of the invariant manifolds than the homoclinic orbit, $\sqrt{\mu}$-close to the singularities.

COMPLEX SINGULARITIES OF $\lambda_0(t)$

RESULT



We prove that the only singularities of $\lambda_0(t)$ in the complex domain are $\pm iA$.

The homoclinic connection satisfies

$$-\frac{3}{2}\Lambda_0(t)^2 + 1 - \cos\lambda_0(t) - \frac{1}{\sqrt{2 + 2\cos\lambda_0(t)}} = -\frac{1}{2}$$

From this relation, we have that

$$t = \int_{\lambda_+}^{\lambda_0(t)} \frac{1}{\sqrt{V(s)}} ds.$$

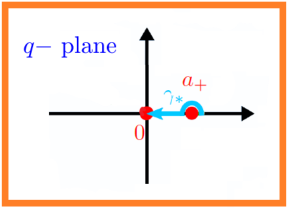
COMPLEX SINGULARITIES (II)

- Take $q = \cos(\lambda/2)$ and $a_\pm = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. We have the identity:

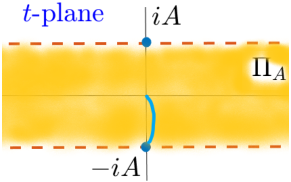
$$t = \mathcal{F}(q) = \int_{a_+}^{q} f(s) = \int_{a_+}^{q} \frac{1}{s-1}\sqrt{\frac{s}{3(s+1)(s-a_+)(s-a_-)}}ds.$$

WHERE THE SINGULARITIES ARE?

$\mathcal{F}$ is analytic in the Riemann surface of f. Then, if the inverse function theorem can be applied to q_h, $q_0(t)$ will not have singularities at $t_h = \mathcal{F}(q_h)$.
Then we have to study $\mathcal{F}(q_*)$ for $q_* = -1, a_-, 0, a_+, 1, |q| \to \infty$ using different complex paths.



$$\int_{a_+}^{0} f(\gamma_*)d\gamma_* = -iA$$



FINAL COMMENTS ON THE PROOF

- To prove the analytic extension of the invariant manifolds we use fixed point theorem arguments.
- After the changes of variables, the hamiltonian we deal with has no explicit expression but Despite of this, we can perform Taylor expansion
- We have not proven a bound for the distance, but an asymptotic expression. For that we have had to deal with a better approximation for the invariant manifolds than the homoclinic connection around the singularities $\pm iA$.
- This approximation comes from special solutions $Z^{u,s}$ of the *inner equation* which is explicit:

$$\begin{aligned}\mathcal{H}(U,W,X,Y) =& 1+\frac{4}{9}U^{-\frac{2}{3}}W^2-\frac{16}{27}U^{-\frac{4}{3}}W+\frac{16}{81}U^{-2}+\frac{4i}{3}U^{-\frac{2}{3}}(X-Y)\\ &-\frac{4}{9}U^{-1}W(X+Y)+\frac{8}{27}U^{-\frac{5}{3}}(X+Y)-\frac{1}{3}U^{-\frac{4}{3}}(X^2+Y^2)\\ &+\frac{10}{9}U^{-\frac{4}{3}}XY.\end{aligned}$$

- By using *matching complex* techniques we relate $Z^{u,s}$ with the parameterization of the invariant manifolds we already had and prove the result.