Chaotic phenomena around L_3 in the RPC3BP

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OUTLINE

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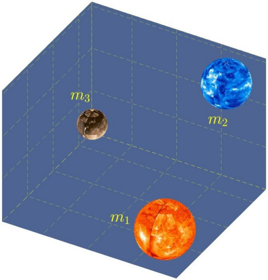
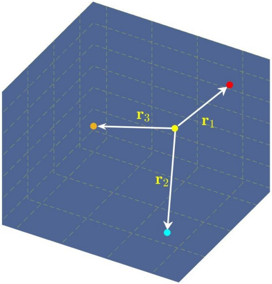
1. RPC3BP
   - Formulation of the problem
   - Mean motion resonance 1:1
   - The Lagrangian point L_3
2. MAIN RESULT
3. HEURISTICS OF THE PROOF
   - Good variables
   - Singularities analysis
   - Exponentially small phenomenon
4. REFERENCES

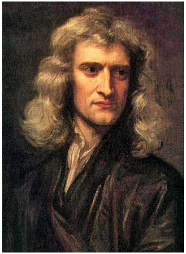
3BODYPROBLEM

Consider three bodies of mass m_1, m_2, m_3 under their mutual attraction law.



Assuming that the bodies are particles:





Newton's law states that, if $r_{ij} = \|\mathbf{r}_i - \mathbf{r}_j\|$ is the norm of the difference vector

$$
\begin{aligned}
m_1\ddot{\mathbf{r}}_1 &= \frac{\mathcal{G} m_1 m_2}{r_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + \frac{\mathcal{G} m_1 m_3}{r_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) \\
m_2\ddot{\mathbf{r}}_2 &= \frac{\mathcal{G} m_1 m_2}{r_{12}^3}(\mathbf{r}_1 - \mathbf{r}_2) + \frac{\mathcal{G} m_2 m_3}{r_{23}^3}(\mathbf{r}_3 - \mathbf{r}_2) \\
m_3\ddot{\mathbf{r}}_3 &= \frac{\mathcal{G} m_1 m_3}{r_{13}^3}(\mathbf{r}_1 - \mathbf{r}_3) + \frac{\mathcal{G} m_2 m_3}{r_{23}^3}(\mathbf{r}_2 - \mathbf{r}_3).
\end{aligned}
$$

RESTRICTEDPLANARCIRCULAR3BP

Make some assumptions:

- **Planar**: the motion takes place into a plane, $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{R}^2$.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two masses bodies (primaries) moves in a circular motion of period T. As a consequence, we only need to pay attention to the massless body: $(\mathbf{r}_3, \dot{\mathbf{r}}_3) \in \mathbb{R}^4$.

Moreover,

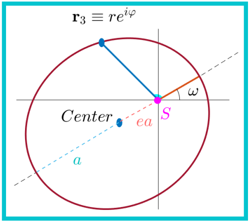
- Changing unities we obtain: $\mathcal{G} = 1$, $m_1 = 1 - \mu$, $m_2 = \mu$, $T = 2\pi$.
- Primaries position: $\mathbf{r}_1(t) = (\mu \cos t, \mu \sin t)$, $\mathbf{r}_2(t) = ((\mu - 1)\cos t, (\mu - 1)\sin t)$.

- When $\mu = 0$, $\mathbf{r}_1 = (0, 0)$, $\mathbf{r}_2(t) = (-\cos t, -\sin t)$ and the third body follows a two body problem:

$$\ddot{\mathbf{r}}_3 = -\frac{1}{\|q\|^3} \quad \rightarrow \quad \frac{r_3}{\|r_3\|^3}$$

- Then calling $r = \|\mathbf{r}_3\|$ we have that:

$$r(\varphi) = \frac{a(1 - e^2)}{1 + e\cos(\varphi - \omega)}.$$



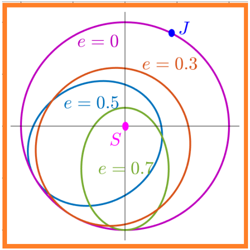
Outline

MEAN MOTION RESONANCE $1:1$

DEFINITION

The mean motion resonance $1:1$ is a region of the phase space close to the motions of the third body having the same period τ, as the primaries. That is, $\tau = 2\pi$.

- When $\mu = 0$, by Kepler's third law $\frac{a^3}{\tau^2} = \frac{1}{4\pi^2}$, being a the major axis of the ellipse.
- $\tau = 2\pi$ is equivalent to $a = 1$.
- The figure is on the $\mathbf{r}_3-$ plane.
- We will study the motion around $a = 1$ and $e = 0$.



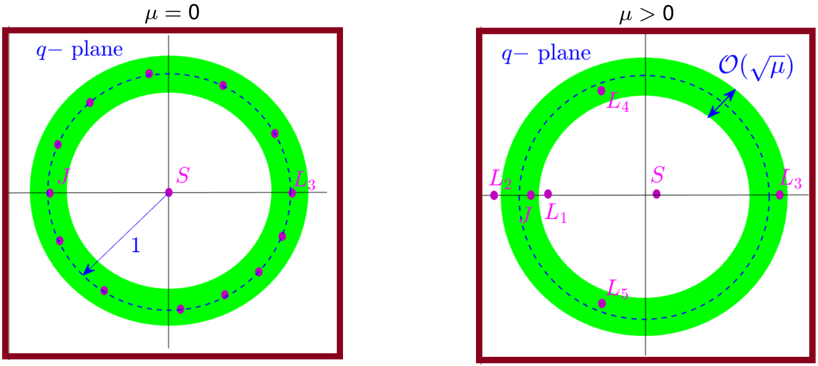
- In rotating (synodic) coordinates:

$$\mathbf{r}_3(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} q(t), \quad p(t) = \dot{q} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q(t),$$

the primaries are at $(\mu, 0)$ and $(\mu - 1, 0)$. The massless body follows the hamiltonian

$$\frac{\|p\|^2}{2} - q^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1-\mu}{\|q - (\mu, 0)\|} - \frac{\mu}{\|q - (\mu - 1, 0)\|}.$$

μ AS A SINGULAR PARAMETER



- In these coordinates, $e = 0$ and $a = 1$ correspond to a circle of degenerated equilibrium points.
- The massless primary is located at $(-1, 0)$.

- There are 5 equilibrium points. The Euler-Lagrange fixed points.
- We study a region (the resonance zone) in the phase space $\mathcal{O}(\sqrt{\mu})-$ close to this circle.

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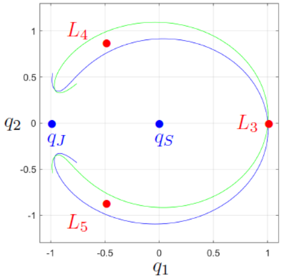
L_3 Lagrangian point

- L_3 is an equilibrium point of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$



- L_3 has one dimensional stable and unstable manifolds, $W^{u,s}$ which will lie in the resonance zone.
- The figure is the projection of $W^{u,s}$ on the q-plane. W^s in green and W^u in blue.
- Remember that the phase space is $\mathbb{R}^4$. We have not drawn the momenta.
- Of course the manifolds do not intersect!

Our goal

To measure the distance between the one dimensional stable and unstable invariant manifolds of L_3 for small values of μ. **This is a singular perturbation problem**.

MAIN RESULT

THEOREM

Take the section $\Sigma = \{(q,p) \in \mathbb{R}^4 \; : \; q_1 = 0, q_2 > 0\}$ *and let* $(q^{u,s}, p^{u,s})$ *be the intersection of* $W^{u,s}(L_3)$ *with* Σ*. Then there exist constants* A, C *such that for* μ *small enough:*

$$\|q^u - q^s\| + \|p^u - p^s\| \sim C\mu^{-\frac{1}{12}} e^{-A/\sqrt{\mu}}.$$

- The constant

$$A = \int_0^{a_+} \frac{1}{1-x}\sqrt{\frac{x}{3(x+1)(a_+ - x)(x - a_-)}}dx \sim 0.177744,$$

  with $a_\pm = -\frac{1}{2} \pm \sqrt{\frac{\sqrt{2}}{2}}$.

- C corresponds a Stokes constant, depending on the full jet of the hamiltonian. Can be numerically computed by means of the so called *inner equation*.

COMMENTS

- Consider a close to integrable one
$$h_0(I)+\varepsilon f(I,\varphi), \qquad I\in\mathbb{R}^2.$$
with a resonant frequency $\partial_I h_0(0)=(0,\omega)$.
- Under some assumptions, the normal form of the hamiltonian can be expressed
$$\frac{\omega}{\sqrt{\varepsilon}}\tilde{I}+\frac{p^2}{2}+V(q)+\frac{1}{2}\tilde{I}^2+\mathcal{O}(\sqrt{\varepsilon}).$$
- The *unperturbed system* $\frac{p^2}{2}+V(q)$ has typically saddle points with homoclinic connections.
- Notice that $\dot{\varphi}=\frac{\omega}{\sqrt{\varepsilon}}+\cdots$ is a *fast* variable. We have a singular perturbation setting.

SKELETON OF THE PROOF

- Use first **Delaunay**'s variables, which provide a non explicit action-angles change of variables.
- However, these variables are not defined for our resonance zone: we have to consider **Poincaré** variables. **The changes of variables are not explicit**.
- We describe the **complex singularities** of the unperturbed separatrix.
- We prove the **exponentially small** splitting of separatrices.

OUTLINE

DELAUNAY VARIABLES

Analytic point of view

- Polar symplectic change of variables:

$$\frac{1}{2}\left(R^2+\frac{G^2}{r^2}\right)-\frac{1}{r}-G+\mu\mathcal{H}_1.$$

- Take $\alpha = L[L-(L^2-G^2)]^{1/2}$ and

$$\mathcal{S}(r,\theta,L,G)=\theta G+\int_\alpha^r\left\{-\frac{G^2}{\xi^2}+\frac{2}{\xi}-\frac{1}{L^2}\right\}d\xi.$$

- The Delaunay variables (ℓ, g, L, G), for $|L| \neq |G|$, are implicitly defined by

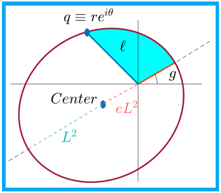
$$R=\partial_r\mathcal{S},\qquad \ell=\partial_L\mathcal{S},\qquad g=\partial_G\mathcal{S}$$

- In Delaunay variables the hamiltonian is

$$\mathcal{H}(\ell,g,L,G)=-\frac{1}{2L^2}-G+\mathcal{H}_1.$$

Geometric point of view

- Take L, G, g, ℓ and $e(L,G)>0$:



- Notice that

$$r(\theta)=\frac{L^2(1-e^2)}{1+e\cos(\theta-g)}.$$

- When $\mu = 0$ is a two body problem and then

$$e=e(L,G)=\sqrt{1-\frac{G^2}{L^2}}.$$

POINCARÉ VARIABLES

- The polar coordinates (r, θ, R, G) are

$$q = (r\cos\theta, r, \sin\theta), \qquad p = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} R \\ \dfrac{G}{r} \end{pmatrix}$$

The circle of equilibrium points is given by $(1, \theta, 0, 1)$ and $L_3 \sim (1, 0, 0, 1)$

- The region we want to study is $\mathcal{O}(\sqrt{\mu})$- close to this circle. Then, $r \sim 1$, $G \sim 1$ (and $e \sim 0$).

DELAUNAY COORDINATES FAIL!

The Delaunay coordinates are not defined for $e = 0$, that is for $L = G = 1$.

- We use Poincaré coordinates instead, namely (λ, η, L, ξ):

$$\lambda = \ell + g, \qquad \eta = \sqrt{L - G}e^{ig}, \qquad \xi = \sqrt{L - G}e^{-ig}.$$

- The change has the symplectic form $d\lambda \wedge dL + id\eta \wedge d\xi$.
- It is analytic when $e = 0$, i.e. $\xi = \eta = 0$ (hard to check!). We also have that $\lambda \to \theta$ when $e \to 0$.
- In Poincaré variables $L_3 = (\lambda, \eta, L, \xi) = (0, 0, 1, 0)$ and the resonance zone is $\xi, \eta \sim 0$.
- The hamiltonian in this variables is $\mathcal{H}(\lambda, \eta, L, \xi) = -\dfrac{1}{2L^2} - L + \xi\eta + \mu\mathcal{H}_1$.

SCALINGS

- Take the usual scaling $L = 1 + \sqrt{\mu}\Lambda$.
- In order to make the change symplectic, $\xi = \sqrt[4]{\mu}x$, $\eta = \sqrt[4]{\mu}$ and $t = \mu^{-1}\tau$.

WHICH ARE NOW THE RELEVANT TERMS?

The hamiltonian (up to constant terms) is now

$$\mu^{-1}\left(-\frac{1}{2L^2} - L + \xi\eta + \mathcal{O}(\mu)\right) \equiv -\frac{3}{2}\Lambda^2 + \frac{xy}{\sqrt{\mu}} + \mathcal{H}_1.$$

- Recall that $\theta \sim \lambda$ and $r \sim 1$ when $e \sim 0$, namely when $\mu \sim 0$. Then $\mu\mathcal{H}_1$ is

$$\begin{aligned}
&\frac{1}{r} - \frac{1-\mu}{(r^2 - 2\mu r\cos\theta + \mu^2)^{1/2}} - \frac{\mu}{(r^2 - 2(1-\mu)r\cos\theta + (1-\mu)^2)^2} \\
=&\frac{1}{r} - \frac{(1-\mu)}{r}\left(1 + \frac{\mu\cos\theta}{r} + o(1)\right) - \frac{\mu}{(2 - 2\cos\lambda + o(1))^{1/2}} \\
=&\mu(1-\cos\lambda) + \frac{\mu}{(2-2\cos\lambda)^{1/2}} + o(1).
\end{aligned}$$

- The hamiltonian is then

$$-\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}} + \frac{xy}{\sqrt{\mu}} + o(1).$$

Terms $o(1)$ are not explicit

OUTLINE

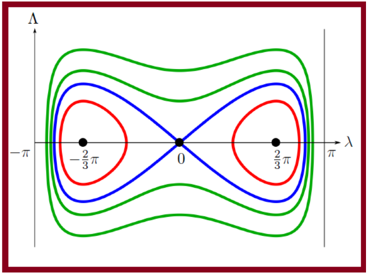
THE HOMOCLINIC CONNECTION

The hamiltonian

$$H_0(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}$$

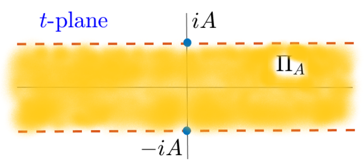
has two homoclinic connection

- Let $(\lambda_0(t), \Lambda_0(t))$ be the parameterization of the right separatrix with $\Lambda_0(0) = 0$.
- It is well known that $\lambda_0(t)$ is analytic in some complex strip.



GOAL

To prove that, for some $A > 0$, the only singularities of $\lambda_0(t)$ in $\overline{\Pi_A}$ are $\pm iA$.



However, there is no explicit parameterization with respect to t of this connection. We only know the relation:

$$t = \int_{\lambda_+}^{\lambda} \frac{1}{\tilde{V}(s)} ds.$$

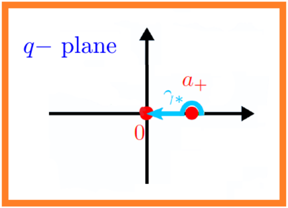
COMPLEX SINGULARITIES

- Take $q = \cos(\lambda/2)$ and $a_\pm = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. We have the identity:

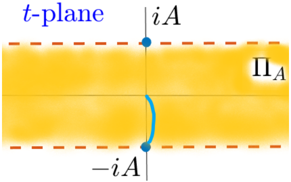
$$t = \mathcal{F}(q) = \int_{a_+}^{q} f(s) = \int_{a_+}^{q} \frac{1}{s-1}\sqrt{\frac{s}{3(s+1)(s-a_+)(s-a_-)}}\,ds.$$

WHERE THE SINGULARITIES ARE?

$\mathcal{F}$ is analytic in the Riemann surface of f. Then, if the inverse function theorem can be applied to q_h, $q_0(t)$ will not have singularities at $t_h = \mathcal{F}(q_h)$.
Then we have to study $\mathcal{F}(q_*)$ for $q_* = -1, a_-, 0, a_+, 1, |q| \to \infty$ using different complex paths.

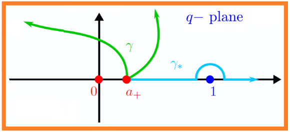


$$\int_{a_+}^{0} f(\gamma_*)d\gamma_* = -iA$$

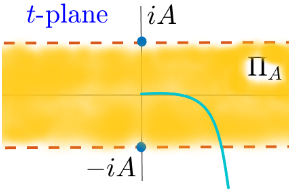


STRATEGY

We also have



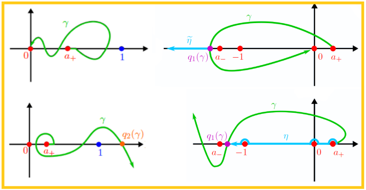
$$\mathrm{Im}\left(\int_{a_+}^{\infty} f(\gamma_*)d\gamma*\right) < -A$$



VISIBLE SINGULARITY

t_* is visible if we can encounter a path γ_* in the $q-$ complex plane such that $\mathrm{int}(\mathcal{F}(\gamma_*(\sigma))) \in \Pi_A$ for all σ. Notice that $\mathcal{F}(\gamma_*)$ is a path in the t-complex plane.
This definition allows us to search singularities in the first sheet of the Riemann surface.

- We encounter singularities with real part, but having bigger imaginary part.
- We study all the homotopic paths and we conclude that the associated singularity is either $\pm iA$ or not visible.



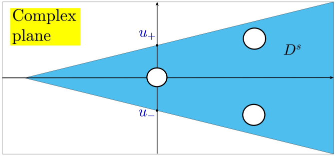
OUTLINE

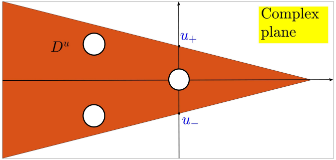
SPLITTING STUFF

- Recall that the hamiltonian is

$$-\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{2+2\cos\lambda} + \frac{xy}{\sqrt{\mu}} + o(1).$$

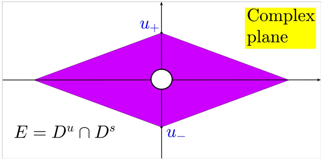
- We perform the symplectic change of variables $\lambda = \lambda_0(u)$, $\Lambda = \Lambda_0(u) - \frac{w}{3\Lambda_0(u)}$.
- We can parameterize the one dimensional invariant manifolds by $\zeta^{s,u}(u)$ in domains





- For $u \in E$, the function $\Delta\zeta = \pi_{x,y}(\zeta^u - \zeta^s)$ satisfies an equation of the form

$$\partial_u \Delta\zeta = \frac{i}{\sqrt{\mu}}(M + \cdots)\Delta\zeta$$



- Since $\Delta\zeta$ is bounded in the domain E it has to be exponentially small for real values of u.

FINAL COMMENTS

- As far as we know, this is the first time that the complex singularities of the homoclinic are analyzed without an explicit formula for $\lambda_0(t)$.
- To prove the existence of $\zeta^{s,u}$ we use fixed point theorem arguments.
- The hamiltonian we deal with is not explicit! Despite of this, we can perform all the computations.
- We have not proven a bound for the distance, but an asymptotic expression. For that we have had to deal with a better approximation for $\zeta^{s,u}$ than the homoclinic connection around the singularities $\pm iA$.
- This approximation comes from special solutions $Z^{u,s}$ of the *inner equation* which is explicit:

$$\begin{aligned}\mathcal{H}(U,W,X,Y) =& 1 + \frac{4}{9}U^{-\frac{2}{3}}W^2 - \frac{16}{27}U^{-\frac{4}{3}}W + \frac{16}{81}U^{-2} + \frac{4i}{3}U^{-\frac{2}{3}}(X-Y) \\ & - \frac{4}{9}U^{-1}W(X+Y) + \frac{8}{27}U^{-\frac{5}{3}}(X+Y) - \frac{1}{3}U^{-\frac{4}{3}}(X^2+Y^2) \\ & + \frac{10}{9}U^{-\frac{4}{3}}XY.\end{aligned}$$

- By using *matching complex* techniques we relate $\zeta^{s,u}$ with $Z^{u,s}$ and prove the result.
- It remains a lot of work to do: Smale's horseshoes, Lyapunov orbits, ...

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Thanks!


THANK YOU FOR YOUR ATTENTION
YESSSS, FINALLY OVER!

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Have a nice
WEEKEND!
MEGAPIXL


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