Some instances where we can encounter a beyond all order phenomenon

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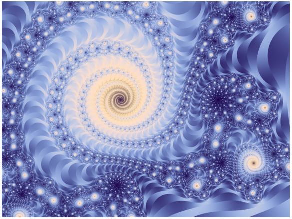
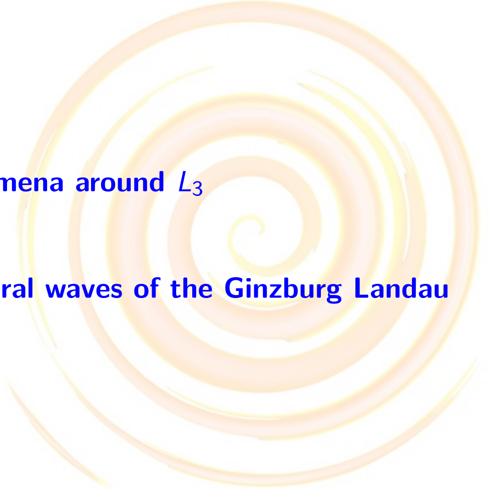


Outline



Beyond all orders phenomenon

Chaotic and homoclinic phenomena around L_3

Asymptotic wavenumber of spiral waves of the Ginzburg Landau equation

Beyond all orders phenomenon

Definition

In a family $\dot{x} = X(x, \varepsilon)$ $(\varepsilon \sim 0)$ if a phenomenon can be described by a flat function $\psi(\varepsilon)$ we say that it is a *beyond all orders phenomenon (BOP)*. Namely $\psi(\varepsilon) = \mathcal{O}(|\varepsilon|^m)$ for all $m \geq 0$. The regular perturbation theory does not work.

- They appear in singular perturbed systems
$$\frac{dx}{dt} = f(x, y, \varepsilon), \ \frac{dy}{dt} = \varepsilon g(x, y, \varepsilon), \quad \text{or} \quad \varepsilon \frac{dx}{d\tau} = f(x, y, \varepsilon), \ \frac{dy}{d\tau} = g(x, y, \varepsilon),$$
with $\tau = \varepsilon t$.
- See that as $\varepsilon = 0$ we get
$$\dot{x} = f(x, y, 0), \ \dot{y} = 0, \quad \text{not equivalent to} \quad 0 = f(x, y, 0), \ y' = g(x, y, 0).$$
- Plethora of models with this phenomena: crystal growth, fluid mechanics (see [Segur, Tarveer, Levine, 1991]), biological problems of several nature (see [Geertje Hek, 2010]), unfoldings of singularities, rapidly forced hamiltonian systems, etc.
- People with results in this area: M. Aguareles, F. Batelli, H. Broer, O. Castejón, S.J. Chapman, A. Delshams, E. Fontich, G. Gallavotti, G. Gentile, V. Gelfreich, M. Giralt, M. Guardia, P. Gutiérrez, V. Hakim, P. Holmes, A. Jorba, M. Kruskal, T. Lázaro, V. Lazutkin, E. Lombardi, P. Loschak, K. Mallic, J.P. Marco, P. Martín, J. Marsden, Mastropietro, A. Neishtad, C. Olivé, J. Paradela, R. Ramírez, M. Rudnev, D. Sauzin, T.M. Seara, H. Segur, J. Sheurle, C. Simó, D. Treshev, Vegter, S. Wiggins and many others

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Singular perturbation. Naive examples (I)

First naive example

Consider $\varepsilon y' + y = f(\varepsilon)$, $y(0) = 1$.

- If $\varepsilon = 0$ there is solution only when $f(0) = 1$.
- If $\varepsilon \neq 0$, then $y(x;\varepsilon) = f(\varepsilon) + e^{-\frac{x}{\varepsilon}}(1 - f(\varepsilon))$ is a solution of our problem.
- If we consider $y(x;\varepsilon) = \sum_{n\geq 0} \varepsilon^n y_n(x)$ and $f(\varepsilon) = \sum_{n\geq 0} f_n \varepsilon^n$ then we have that

$$y_0 \equiv f_0 = 1, \qquad y'_{n-1} + y_n = f_n \Longrightarrow y_n \equiv f_n.$$

The series $\sum_{n\geq 0} \varepsilon^n y_n(x) = f(\varepsilon)$ is convergent but does not describes the solution.

However

Changing $x = \varepsilon u$, $\dot{y} + y = f(\varepsilon)$ is a totally regular system. Expanding in power series

$$y(x;\varepsilon) = \sum_{n\geq 0} \varepsilon^n (f_n + e^{-x}(1 - f_n)) = f(\varepsilon) + e^{-u}(1 - f(\varepsilon))$$

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Singular perturbation. Naive examples (II)

A exponentially small selection

Consider now $\varepsilon y' + y = c$ and we look for solutions $y(0) = y_0$ and $y(1) = y_1$. Of course it has to exists a selection mechanism for the constant c.

- The solutions satisfying $y(0;\varepsilon) = y_0$ are

$$y(x;\varepsilon) = e^{-\frac{x}{\varepsilon}} \left[y_0 + c(e^{\frac{x}{\varepsilon}} - 1) \right]$$

- Imposing $y(1;\varepsilon) = y_1$, we have that

$$c = c(\varepsilon) = \frac{y_1 - y_0 e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} = y_1 + \mathcal{O}(e^{-\frac{1}{\varepsilon}}).$$

- If a classical perturbation expansion of $y(x;\varepsilon)$ and $c(\varepsilon)$ is performed does not provide a solution for c.

However

Thinking in $x - 1 = \varepsilon v$, we have that $\hat{y}(v;\varepsilon) = e^{-v}[y_1 + c(e^{v} - 1)]$. Notice that $\varepsilon \to 0$ implies $v \to -\infty$. Thus $\hat{y}$ bounded, implies $c = y_1 + o(1)$.

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Singular perturbation. Naive examples (III)

An example of divergence

Consider now, $x > 0$, $\varepsilon > 0$, $\varepsilon y' + y = \frac{1}{x}$.

- Clearly for $\varepsilon = 0$ we have only one solution $y(x;\varepsilon) = \frac{1}{x}$.
- For $\varepsilon \neq 0$ we have that all the solutions are

$$y(x;\varepsilon) = e^{-\frac{x}{\varepsilon}} \left[y_0 + \int_0^x e^{\frac{s}{\varepsilon}} \frac{1}{\varepsilon s}\, ds \right].$$

- If we look for $y(x;\varepsilon) = \sum_{n\geq 0} \varepsilon^n y_n(x)$ we obtain a divergent series $\sum_{n\geq 0} (-1)^n n! \varepsilon^n x^{-n-1}$.
- All the solutions have the same divergent expansion and the difference between two of them is $Ce^{-\frac{x}{\varepsilon}}$ for some constant C.
- When $x > 0$ we can not distinguish between two solutions up to any order in ε.

However

If $x \sim 0$, change $x = u\varepsilon$. The difference between two solutions is Ce^{-u} and any term of the divergence series is of the same order $\mathcal{O}(\varepsilon^{-1})$.
The change $\eta(u) = \varepsilon^{-1} y(u\varepsilon)$ leads to the free parameter equation

$$\dot{\eta} + \eta = \frac{1}{u}.$$

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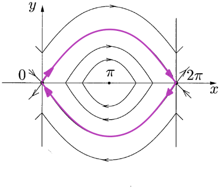
The rapidly perturbed pendulum

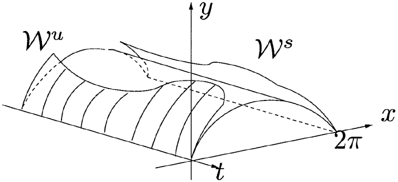
Consider the pendulum perturbed periodically

$$\frac{y^2}{2}+\cos x-1+\mu H_1(x,y,t/\varepsilon), \qquad \langle H_1(x,y,\cdot)\rangle=0,\ |\mu|\ll 1, 0<\varepsilon\ll 1.$$

for $\mu=0$, $(x_0(u),y_0(u))$ the homoclinic connection $(0,0)$ and $(2\pi,0)$. Notice that $y_0(u)=2\cosh^{-1}(u))$ has poles at $u=\pm i\frac{\pi}{2}$.

- The generic situation is that the homoclinic connection is destroyed for $\mu\neq 0$.





- The question is, can we measure the distance between $\mathcal{W}^u$ and $\mathcal{W}^s$ when $\mu\neq 0$?
- $\mathcal{W}^{u,s}$ can be expressed as graphs $y=\partial_x S^{u,s}(x,\tau)$ with $S^{u,s}$ satisfying $\partial_x S^u(0,\tau)=\partial_x S^s(2\pi,\tau)=0$ and the Hamilton-Jacobi equation

$$\varepsilon^{-1}\partial_\tau S^{u,s}+H_0(x,\partial_x S^{u,s})+\mu H_1(x,\partial_x S^{u,s},\tau)=0$$

- For $\mu=0$ $S_0(x)=4(1-\cos(x/2))$.

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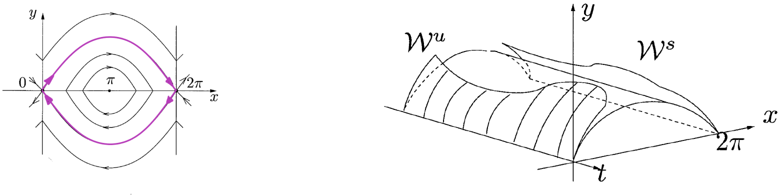

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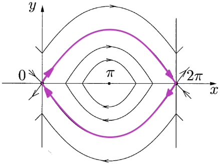

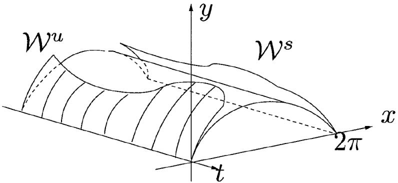

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Exponentially small splitting

- Take
$$T^{u,s}(u,\tau) = S^{u,s}(x_0(u),\tau) - S_0(x_0(u)),$$
that are real analytic functions in $u \in [-\rho,\rho]$ satisfying
$$\varepsilon^{-1}\partial_\tau T^{u,s}(u,\tau) + \partial_u T^{u,s}(u,\tau) = \mu\mathcal{F}(\partial_u T^{u,s}, u, \tau)$$

- Substracting the equations for $T^{u,s}$, the difference $\Delta = T^u - T^s$ satisfies a linear homogeneous equation which is close to
$$\varepsilon^{-1}\partial_\tau\Delta + \partial_u\Delta = 0 \Longrightarrow \Delta(u,\tau) = \Upsilon\left(\tau - \frac{u}{\varepsilon}\right).$$

- Since $\Delta(u,\tau+2\pi) = \Delta(u,\tau)$, $\Upsilon(z+2\pi) = \Upsilon(z)$. Then
$$\Delta(u,\tau) = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{ik\left(\tau - \frac{u}{\varepsilon}\right)} = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{-ik\frac{u}{\varepsilon}} e^{ik\tau},$$
that is
$$\Upsilon^{[k]} e^{-ik\frac{u}{\varepsilon}} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\tau}\Delta(u,\tau)\,d\tau$$

- $\Delta(u,\tau)$ is real analytic for $|\mathrm{Im}\,u| \le b$ bounded by $C|\mu|$. Taking $u = -ib$ if $k < 0$ and $u = ib$ if $k > 0$
$$|\Upsilon^{[k]}| \le C|\mu|e^{-\frac{|k|b}{\varepsilon}} \Longrightarrow \sup_{u\in[-\rho,\rho]} |\Delta(u,\tau) - \langle\Delta(u,\cdot)\rangle| \le C|\mu|e^{-\frac{b}{\varepsilon}}$$

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Exponentially small splitting

- Take

$$T^{u,s}(u,\tau) = S^{u,s}(x_0(u),\tau) - S_0(x_0(u)),$$

  that are real analytic functions in $u \in [-\rho, \rho]$ satisfying

$$\varepsilon^{-1}\partial_\tau T^{u,s}(u,\tau) + \partial_u T^{u,s}(u,\tau) = \mu\mathcal{F}(\partial_u T^{u,s}, u, \tau)$$

- Substracting the equations for $T^{u,s}$, the difference $\Delta = T^u - T^s$ satisfies a linear homogeneous equation which is close to

$$\varepsilon^{-1}\partial_\tau \Delta + \partial_u \Delta = 0 \Longrightarrow \Delta(u,\tau) = \Upsilon\left(\tau - \frac{u}{\varepsilon}\right).$$

- Since $\Delta(u,\tau + 2\pi) = \Delta(u,\tau)$, $\Upsilon(z + 2\pi) = \Upsilon(z)$. Then

$$\Delta(u,\tau) = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{ik\left(\tau - \frac{u}{\varepsilon}\right)} = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{-ik\frac{u}{\varepsilon}} e^{ik\tau},$$

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Comments

- The difference between $\mathcal{W}^{u,s}$ is measured by $|\partial_u T^u - \partial_u T^s| \leq |\partial_u \Delta(u,\tau)| \leq C|\mu| e^{-\frac{b}{\varepsilon}}$. The bigger b is, a sharper bound we obtain.
- Under some conditions, $b = \pi/2 - \mathcal{O}(\varepsilon)$.
- What is happen here is that

$$T^{u,s}(u,\tau) \sim \sum_{n \geq 0} \varepsilon^n F_n(u,\tau), \qquad \text{is a divergent series}$$

Chaotic motions

The presence of transversal homoclinic intersections, leads to chaos by means of the conjugation with the Smale's horseshoe. Then

- To prove that $\mathcal{W}^{u,s}$ intersect transversally, we need to provide a *known and computable (at least numerically)* first order $\Delta_0(u,\tau)$ of $\Delta(u,\tau)$.
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- The procedure can be generalized for mechanic unperturbed hamiltonian [I.B., E. Fontich, M. Guardia, T.M. Seara, 2012]

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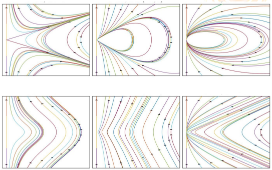
Hopf-zero singularity truly unfolds chaos



- The Hopf-zero singularity is a vector field $X_{0,0}$ with linear part having eigenvalues $\pm i\alpha$, 0.
- A family $X_{\mu,\nu}$ such that $X_{0,0}$ is a Hopf-zero singularity, it is called an unfolding. The eigenvalues are $\mathcal{O}(\mu)$ and $\pm i\alpha + \mathcal{O}(\mu)$.

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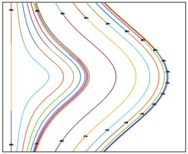

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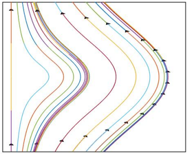

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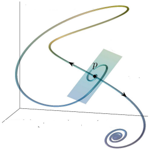

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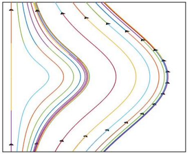

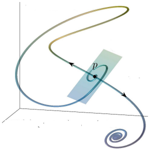

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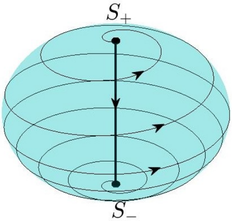
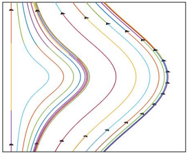


Normal form up to any order

Hopf-zero singularity truly unfolds chaos



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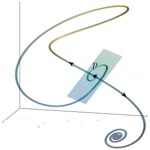

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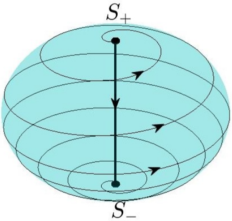

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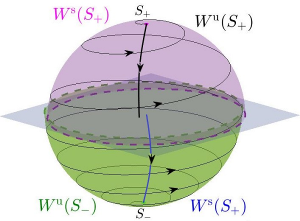

Distance exponentially small

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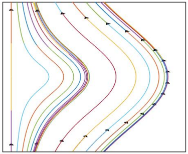

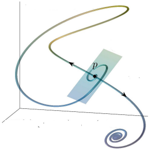

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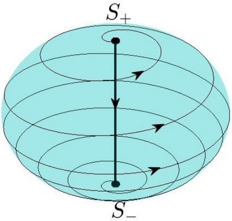

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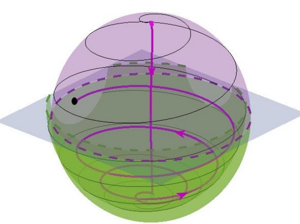

Global $W^u(S_+)$

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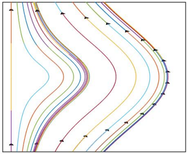

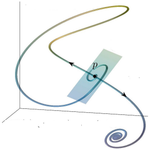

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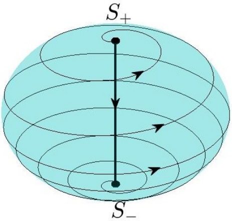

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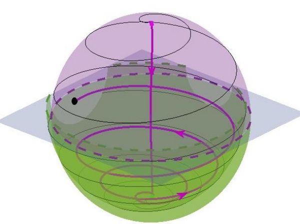

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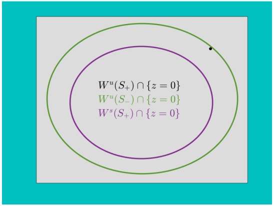

Bolzano and fast oscillation

Comments

- These last two examples can be enclosed in a fast oscillation set up by means of a result due to [Neishtadt, 1984] for systems having fast oscillations

$$\dot{x} = \varepsilon f(x, \varphi, \varepsilon), \qquad \dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon),$$

  with $(x, \varphi) \in \mathbb{R}^n \times \mathbb{S}^1$. He proved that the system can be decoupled up to terms of order $e^{-\frac{c_1}{\varepsilon}}$ with $c_1 > 0$.
- However, as we said, we have not deal with upper bounds but with asymptotic expansions in order to decide if weather a system possesses chaotic dynamics via topological conjugation with the Smale's horseshoe.
- The methodology developed can be implemented by means of computed assisted proofs.
- It is important to mention that there are also results for maps providing exponentially small splitting. [Lazutkin, 84] in his celebrated paper, provides (without a complete proof) the first asymptotic formula for the splitting in the standard. Later, in [Fontich-Simó, 90] provide a sharp bound for the splitting of the invariant manifolds of the origin for diffeomorphisms close to the identity and planar rapidly forced systems. Gelfreich, P. Martin, D. Sauzin, T.M. Seara also have dealt with asymptotic expressions for the splitting.
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Comments

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  $$\dot{x} = \varepsilon f(x, \varphi, \varepsilon), \qquad \dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon),$$

  with $(x, \varphi) \in \mathbb{R}^n \times \mathbb{S}^1$. He proved that the system can be decoupled up to terms of order $e^{-\frac{c_1}{\varepsilon}}$ with $c_1 > 0$.
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The last two examples of beyond all order phenomena

- Homoclinic phenomena around L_3. In the RCP3BP, we exploit the fast oscillations with respect to the small mass parameter. This is a joint work with M. Giralt and M. Guardia.
- Spiral waves in Ginzburg-Landau equation with exponentially small asymptotic wavenumber. Roughly speaking, we reduce the problem to a boundary value problem depending on two parameters

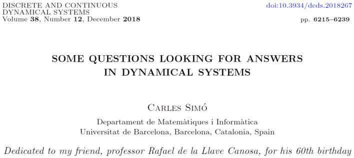
  $$\varepsilon x'' + \varepsilon x' = f(x, y, \varepsilon, \lambda), \qquad y' = g(x, y, \varepsilon, \lambda)$$

  with $x(0) = x'(0) = y(0) = 0$, $y(\infty) = \lambda$, $x, y > 0$ and bounded for $r > 0$. This boundary problem has too much conditions and this will imply a selection mechanism for $\lambda = \lambda(\varepsilon)$. It turns out that

  $$\lambda(\varepsilon) \sim Ae^{-\frac{B}{\varepsilon}}, \qquad B > 0.$$

  This context has no fast oscillations.
  It is a joint work with M. Aguareles and T.M. Seara.

A Carles Simó's problem



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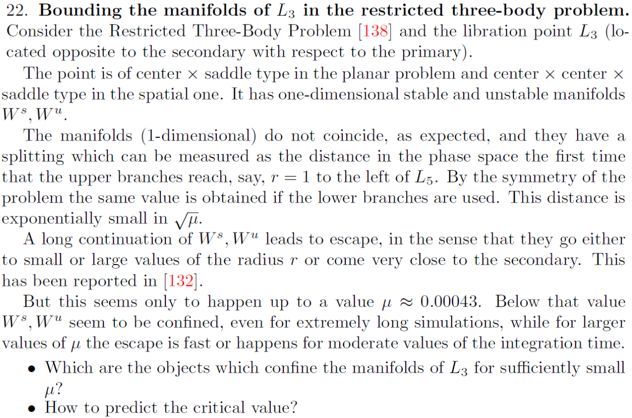
SOME QUESTIONS LOOKING FOR ANSWERS
IN DYNAMICAL SYSTEMS

CARLES SIMÓ
Departament de Matemàtiques i Informàtica
Universitat de Barcelona, Barcelona, Catalonia, Spain

Dedicated to my friend, professor Rafael de la Llave Canosa, for his 60th birthday

It appears 22 problems



22. **Bounding the manifolds of L_3 in the restricted three-body problem.** Consider the Restricted Three-Body Problem [138] and the libration point L_3 (located opposite to the secondary with respect to the primary).

The point is of center × saddle type in the planar problem and center × center × saddle type in the spatial one. It has one-dimensional stable and unstable manifolds W^s, W^u.

The manifolds (1-dimensional) do not coincide, as expected, and they have a splitting which can be measured as the distance in the phase space the first time that the upper branches reach, say, $r = 1$ to the left of L_5. By the symmetry of the problem the same value is obtained if the lower branches are used. This distance is exponentially small in $\sqrt{\mu}$.

A long continuation of W^s, W^u leads to escape, in the sense that they go either to small or large values of the radius r or come very close to the secondary. This has been reported in [132].

But this seems only to happen up to a value $\mu \approx 0.00043$. Below that value W^s, W^u seem to be confined, even for extremely long simulations, while for larger values of μ the escape is fast or happens for moderate values of the integration time.

- Which are the objects which confine the manifolds of L_3 for sufficiently small μ?
- How to predict the critical value?

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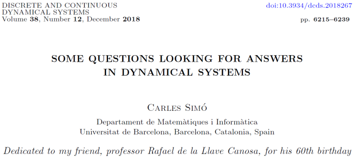

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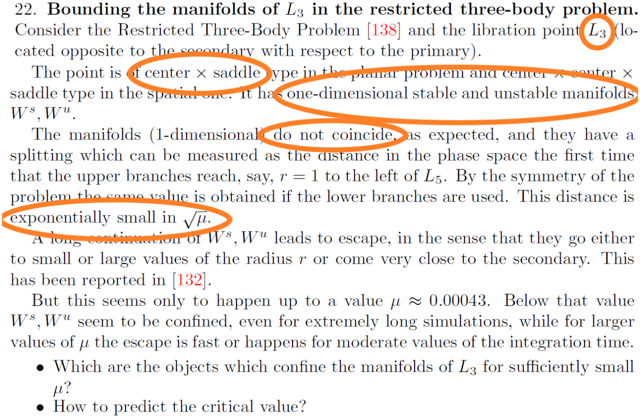

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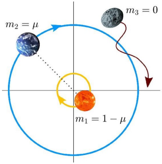
RestrictedPlanarCircular3BP

We consider:

- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.

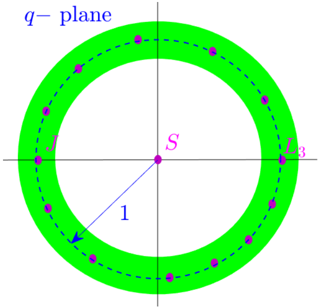


- In rotating (synodic) coordinates, the primaries are located at $(\mu, 0)$ and $(\mu - 1, 0)$ and the massless body follows a 2 degrees of freedom **autonomous** hamiltonian system.

$$\frac{\|p\|^2}{2} - q^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1-\mu}{\|q - (\mu, 0)\|} - \frac{\mu}{\|q - (\mu - 1, 0)\|}.$$

- We assume a perturbative setting, $0 < \mu \ll 1$.
- Notice that when $\mu = 0$, the third body follows a two body problem

μ as a singular parameter



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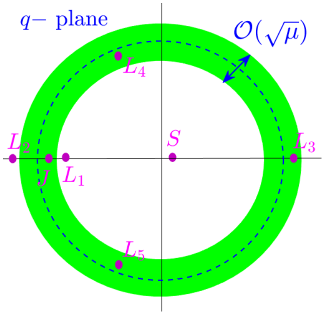


$\mu > 0$. $L_1, \cdots, L_5$ equilibrium points.

- The Lagrangian point L_3 belongs to the mean motion resonance 1 : 1.

Mean Motion resonance

The mean motion resonance 1 : 1 is a region of the phase space close to the motions of the third body having the same period as the primaries. They can lead to inestabilities (diffusion) [Féjoz, Guardia, Kaloshin, Roldan, 2016]

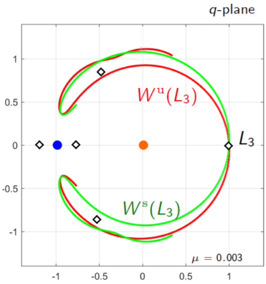
Exponentially small splitting

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no transversal intersection (In the figure is the projection of $W^{u,s}$ on the q-plane, the phase space is $\mathbb{R}^4$).

- **First goal:** To measure the distance between these invariant manifolds at first crossing.



Theorem

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

$$\|q^u - q^s\| + \|p^u - p^s\| \sim \boxed{K}\,\mu^{\frac{1}{3}}\,e^{-\frac{\boxed{A}}{\sqrt{\mu}}}.$$

Stokes constant

Known constant

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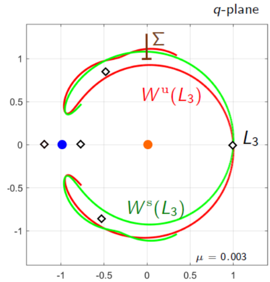

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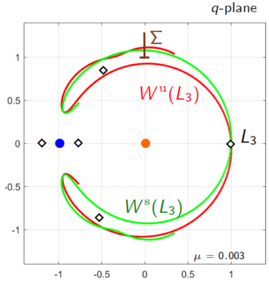

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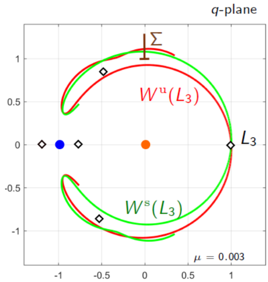

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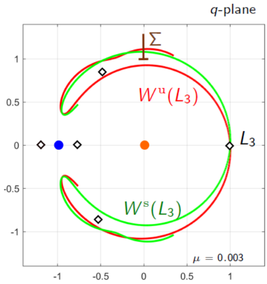

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- The motion takes place far from collision.
- The constant A has an explicit expression

  $$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}dx \sim 0.177744$$

  it is related with a *hidden* homoclinic connection. First computed by J. Font.
- K has a different nature and it corresponds a Stokes constant, depending on the full jet of the hamiltonian. Can be numerically computed by means of the so called *inner equation* that is explicit. We obtain $K \sim 1.63$. We will assume that $K \neq 0$ as a (numerical) ansatz.
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- First order. We use Poincaré variables and singular scalings to transform the system $H(\lambda, \Lambda, x, y) = H_0(\lambda, \Lambda, x, y) + o(1)$ with

$$H_0(\lambda, \Lambda, x, y; \sqrt{\mu}) = \underbrace{i\,\frac{xy}{\sqrt{\mu}}}_{\text{Fast variables}} - \underbrace{\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}}_{\text{Slow variables}}$$

- The time parameterization of the homoclinic connection of H_0 has singularities at $\pm iA$.
- There are parameterizations of $W^{u,s}(L_3)$ in domains $\sqrt{\mu}$- close to $\pm iA$ and related with special solutions of the *inner equation* (matching complex techniques).
- The *inner equation* gives a *hopefully* first order for the difference (in the fast x variable) $\Delta_0 x(u) = K e^{-\frac{A}{\sqrt{\mu}}} e^{\frac{iu}{\sqrt{\mu}}}$ for $u \in \overline{0, i(A - \sqrt{\mu})}$
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Sketch of the proof

- First order. We use Poincaré variables and singular scalings to transform the system $H(\lambda,\Lambda,x,y) = H_0(\lambda,\Lambda,x,y) + o(1)$ with

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More comments

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$$\begin{aligned}\mathcal{H}(U,W,X,Y) =& 1+\frac{4}{9}U^{-\frac{2}{3}}W^2-\frac{16}{27}U^{-\frac{4}{3}}W+\frac{16}{81}U^{-2}+\frac{4i}{3}U^{-\frac{2}{3}}(X-Y)\\ &-\frac{4}{9}U^{-1}W(X+Y)+\frac{8}{27}U^{-\frac{5}{3}}(X+Y)-\frac{1}{3}U^{-\frac{4}{3}}(X^2+Y^2)\\ &+\frac{10}{9}U^{-\frac{4}{3}}XY.\end{aligned}$$

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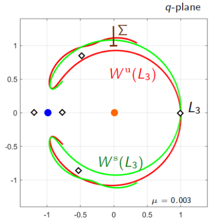

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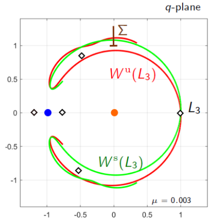

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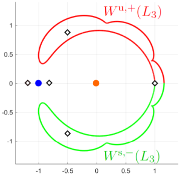
Homoclinic phenomena around L_3

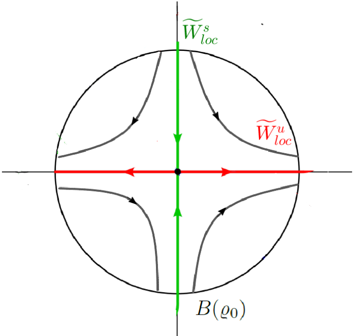
It was conjectured by E. Barrabés, M. Ollé and J.M. Mondelo (2009) that there exists a sequence of mass rations $\mu_n \to 0$ such that there exist secondary homoclinic connections.

Theorem

The RPC3BP has a 2-round homoclinc connection to L_3 between $W^{u,+}$ and $W^{s,-}$, if $K \neq 0$, for a sequence of the form

$$\mu_n = \frac{A}{n\pi}\sqrt{\frac{8}{21}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad n \gg 1$$





- Uniform normal form in a neighbourhood of the fixed point. The result is provided by a work of T. Jezequel, P. Bernard, and E. Lombardi, 2016.
- The new system is almost linear and uncoupled.
- In the picture the saddle (slow) variables.
- The fast variables travel with a velocity of $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$.

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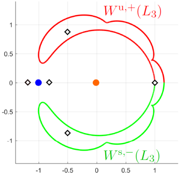

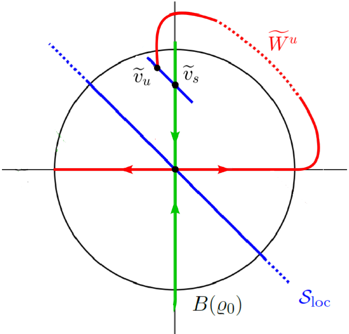

- The original system has a symmetry plane
- Choose a transversal section close enough of the equilibrium point and translate the section and the symmetry plane to the new normal form variables.
- The intersection of the stable manifold is easy to control.
- Our result asserts that the unstable manifold intersect with the transversal section
- and provides also the coordinates of its intersection.

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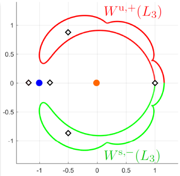

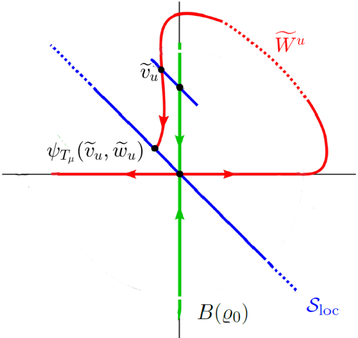

- The local system is almost uncoupled and linear, the time T_μ we need to hit the projection of the simmetry plane in the saddle plane is $T_\mu \sim \dfrac{1}{\mu}$.
- The fast variables pf $\psi_{T_\mu}(\widetilde{v}_u, \widetilde{w}_u)$ are approximately

$$R(\mu)\left(\cos\left(\alpha - \frac{c}{\mu}\right), \sin\left(\alpha - \frac{c}{\mu}\right)\right)$$

- They hits the symmetry axis when $\alpha - \dfrac{c}{\mu} = n\pi$.

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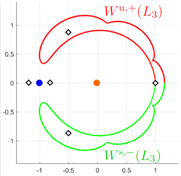
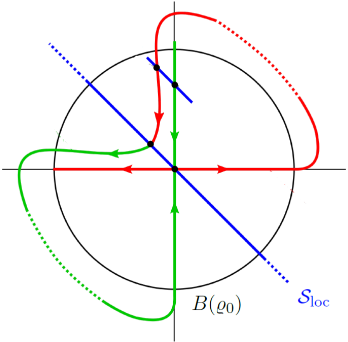




- By symmetry we are done!

Chaotic coorbital motions

The next result assures the existence of chaotic motions around L_3 and its manifolds

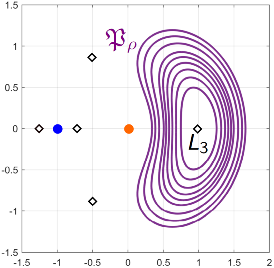
Theorem

Fix $c_1 > 1$, $c_2 > c_1$ *and assume that* $K \neq 0$. *There exists* $\mu_0 >$ *such that for* $\mu \in (0, \mu_0)$, *if the energy level* $h(p, q, \mu) = E$ *satisfies*

$$c_1 \frac{\sqrt[3]{2}}{4} K^2 \mu^{\frac{2}{3}} e^{-\frac{2A}{\sqrt{\mu}}} \leq |E - h(L_3)| \leq c_2 \frac{\sqrt[3]{2}}{4} K^2 \mu^{\frac{2}{3}} e^{-\frac{2A}{\sqrt{\mu}}},$$

there exists a periodic Lyapunov orbit belonging to $\{h(p, q; \mu) = E\}$, *exponentially close to* L_3, *having 2-dimensional stable and unstable manifolds that intersect transversally.*

- We prove the existence of Lyapunov orbits in this fast-slow system.
- These orbits have two dimensional stable and unstable manifolds living in a 3 dimensional domain.
- Following the strategy in [O. Gomide, M. Guardia, T.M. Seara, 2020] we prove the existence of transversal intersections.



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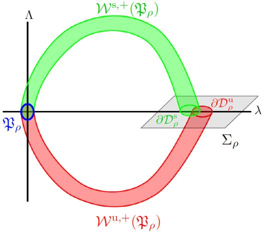

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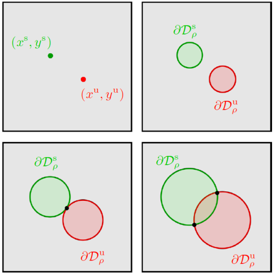
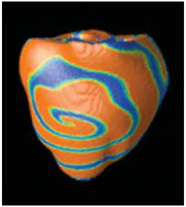


Spiral patterns

Spiral patterns are commonly observed in certain chemical, biological and physical systems

Belousov-Zhabotinskii

Social amoebas Dictyostelium discoideium



Cardiac muscle tissue

- These systems are governed by chemical or biological reaction and spatial diffusion.

$$\partial_\tau U = D\Delta U + F(U,a), \qquad D \text{ a diffusion matrix, } F \text{ the reaction nonlinearity}$$

$U = U(\tau, \vec{x}) \in \mathbb{R}^N$, $\vec{x} \in \mathbb{R}^2$ and a is a parameter (for instance some catalyst concentration).

The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
- Take $\varepsilon^2 = a - a_0 > 0$, small, $t = \varepsilon^2\tau$. Then the modulation of local oscillations with frequency ω

$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau} v_+ + c.c.] + \mathcal{O}(\varepsilon^2).$$

- and (after some scalings) the (complex) amplitude A, which can be seen as coordinates on the central manifold, satisfies the celebrated complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2,$$

  where $A(\vec{x}, t) \in \mathbb{C}$ and α, β are real parameters (dispersion parameters).

  Y. Kuramoto, *Chemical oscillations, waves and turbulence*

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The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
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Spiral waves. Definition

- We focus on infinite domains, $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$.
- The so called wave trains are solutions of the one dimensional GL in polar coordinates of the form $A(t,r) = A_*(-k_* r + \Omega t)$ with $A_*(\cdot)$ a periodic functions $A_*(\xi)$.
- Ω is the *frequency* and k_* the *wavenumber*.
- The spiral waves are bounded solutions that asymptotically tends to a wave train. Namely solutions of the form $A(t,r,\varphi) = A_s(r, n\varphi + \Omega t)$ satisfying

$$A_s(0,\psi) \text{ bounded}, \qquad \lim_{r\to\infty} \|A_s(r,\psi) - A_*(-k_* r + \theta(r) + \psi)\| = 0$$

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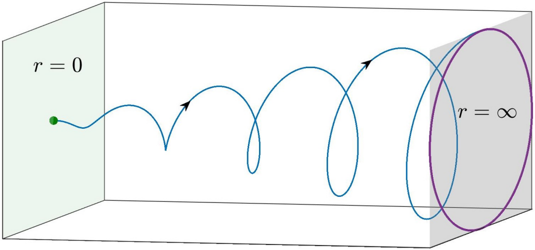

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Wave trains and spiral waves in Ginzburg-Landau equation

- The only possible wave trains are $A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}$ satisfying

$$C = \sqrt{1 - k_*^2}, \qquad \Omega = \Omega(k_*) = -\beta + k_*^2(\beta - \alpha)$$

  The last condition is the associated *dispersion relation* and the quantity $v_g := -\partial_{k_*}\Omega(k_*) = 2k_*(\alpha - \beta)$ the group velocity.

- As a consequence an spiral wave has to tend as $r \to \infty$ to

$$A_*(\Omega t + \chi(r) + n\varphi) = \sqrt{1 - k_*^2}e^{i(\Omega t + \chi(r) + n\varphi)}$$

  with $\chi(r) = -k_* r + \theta(r) \sim -k_* r$ and Ω, k_* satisfying the dispersion relation.

- We look for spirals waves n-armed of the form

$$A(t, r, \varphi) = f(r)\text{exp}\Big(i(\Omega t + \chi(r) + n\varphi)\Big),$$

  with f, χ, χ' bounded and

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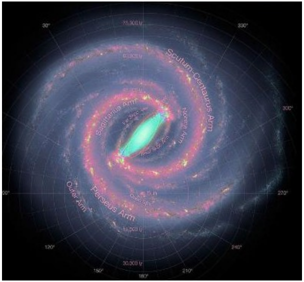
Where is the spiral shape?

- Below, the surface $\mathrm{Re}(A(t, r, \varphi)e^{-i\Omega t})$ for different values of r.

$n = 5,\ 6 \leq r \leq 20$

$n = 5,\ 20 \leq r \leq 100$



$n = 5,\ 100 \leq r \leq 500$

- The wave train $A_*(-k_* r + \Omega t + n\varphi)$ has wavelength L (distance between two spiral arms)

$$L = \frac{2\pi}{|k_*|}.$$

Since L is a constant, it is an archimedian spiral.

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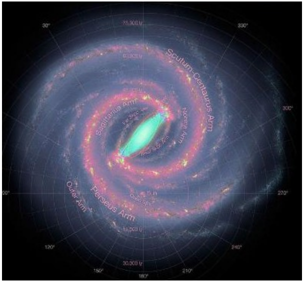

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Our result

- We introduce the *twist parameter*

$$q = \frac{\beta - \alpha}{1 + \alpha\beta}$$

Theorem

If $|q|$ is small enough, the Ginzburg-Landau equation possesses a rigidly archimedian spiral with one defect ($f(0) = 0$, $f(r;q) > 0$ for $r > 0$) and $f'(r;q) > 0$, if and only if

$$k_* = k_*(q) = \sqrt{\frac{1}{1 - \alpha q(1 - k^2(q))}}\,k(q), \qquad k(q) = \frac{2}{q} e^{-\frac{C_n}{n^2} - \gamma} e^{-\frac{\pi}{2n|q|}} (1 + \mathcal{O}(|q|)), \tag{1}$$

with γ the Euler's constant and

$$C_n = \lim_{r \to \infty} \left(\int_0^r \xi f^2(\xi;0)(1 - f^2(\xi;0))\, d\xi - n^2 \log r \right).$$

Notice that $k_*(q) = k(q)(1 + \mathcal{O}(q))$.

Remarks

- The case $q = 0$, can be reduced to the real Ginzburg Landau equation

$$\partial_t A = \Delta A + A - A|A|^2.$$

- If $q = 0$, $k_* = 0$ and there are no spiral waves.
- In our perturbative setting, these lines bend to form the spirals.

Other people dealing with spiral waves

- N. Kopell and L. N. Howard (1981). A serie of papers concerned with pattern formation in the Belousov-Zhabotinskii reaction. The existence and uniqueness of the asymtptotic wavenumber $k_* = k_*(q)$ as a function of q was proven.
- P.S. Hagan (1982), J. Greenberg (1980), M. Aguareles, M. S. Chapman, T. Witelski (2010) used asymptotic methods to compute an explicit asymptotic formula for $k(q)$. The asymptotic methods are a consistent and systematic way to conjecture true results but does not provide rigorous proofs.

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Strategy of the proof (I)

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  together with

$$\lim_{r\to\infty} v(r) = -k, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k^2}.$$

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- There are too many conditions. This indicates a selection mechanism for k.
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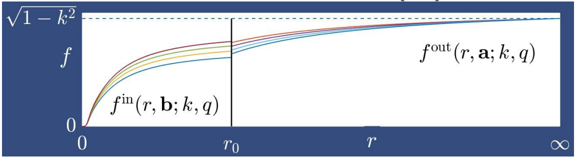
- In order to f, v being bounded at $r = 0$, we need to impose $f(0) = v(0) = 0$.
- There are too many conditions. This indicates a selection mechanism for k.
- First approach: perturbation theory with respect to $|q| \ll 1$ [M. Aguareles, I.B., Seara]. It can be seen that $\partial_q^{(n)} k(0) = 0$ for all $n \geq 0$.
- We divide the problem between $r \in [0, r_0]$ (*inner region*) $r \in [r_0, \infty)$ (*outer region*) with $r_0 = e^{\rho/q}$ with $\rho \ll 1$.
- The boundary conditions; in the *inner region* $f(0) = v(0) = 0$ and in the *outer region*,

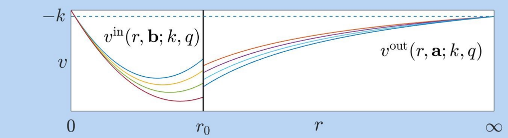
$$\lim_{r\to\infty} f(r) = \sqrt{1 - k^2}, \qquad \lim_{r\to\infty} v(r) = -k$$

  does not provide uniqueness of the solution.

Strategy of the proof (II)



- Two families of solutions depending on $(\mathbf{a}, k)$ and $(\mathbf{b}, k)$.



- Remember that the ODE is of second order; f' has also to be take into account.
- We match the two families in the common point $r = r_0$. Namely we impose that

$$f^{\rm out}(r_0, \mathbf{a}; k, q) = f^{\rm in}(r_0, \mathbf{b}; k, q)$$
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- This is a system with three unknowns $(\mathbf{a}, \mathbf{b}, k)$ and three equations (depending on q).
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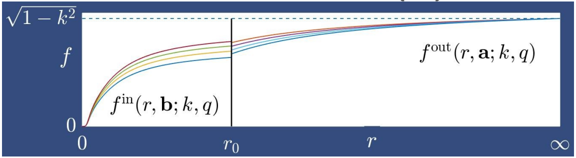

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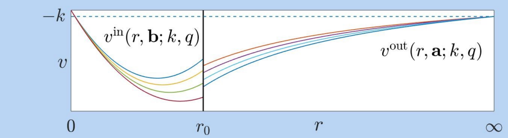

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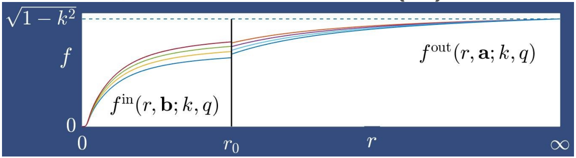

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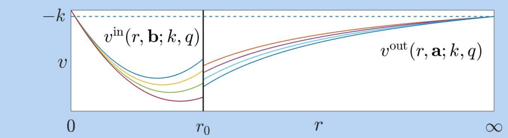

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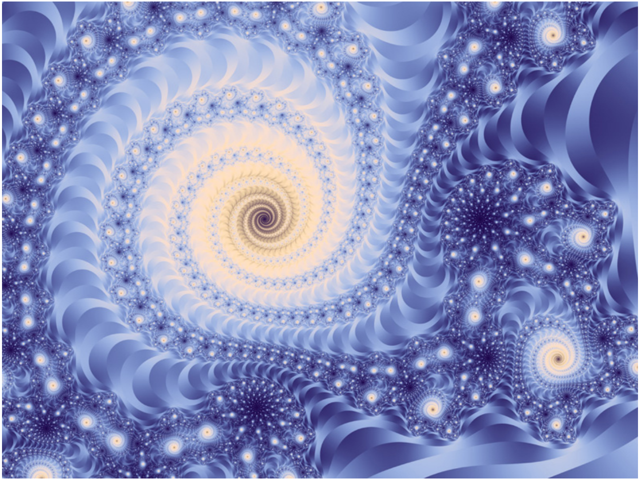

Gràcies a tothom i bona Jornada SD 2022