Chaoticity of generic analytic convex Billiards

I. Baldomá

UPC, CRM, ImTech

March 6, 2025

Transverse Homoclinic Orbits

A join work with A. Florio, M. Leguil and T.M. Seara

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We will prove that, generically, the dynamics associated with an analytic convex billiard table is chaotic and therefore generic analytic convex billiards are not integrable.

More concretely, a generic analytic convex billiard table satisfies the following property: For any m/n rational number, there is a hyperbolic periodic orbit with rotation number m/n whose stable and unstable invariant manifold intersect transversally.

Preliminary definitions and notation

Let $f : M \to M$ a diffeomorphism defined on some manifold M.

- P_0 is a **n–periodic point** if $f^n(P_0) = P_0$ and $f^m(P_0) \neq P_0$ if $m < n$. The corresponding orbit is a periodic orbit:

$$\mathcal{P} = O(P_0) = \{P_0, f(P_0), f^2(P_0), \cdots, f^{n-1}(P_0)\}$$

- P_0 is **hyperbolic** if $\operatorname{spec} Df^n(P_0) \subset \{|\lambda| \neq 1\}$
- The **unstable and stable invariant manifold** of P_0 are

$$W^{u,s}(\mathcal{P}) = \bigcup_{j=0}^{n-1} W^{u,s}(P_j), \qquad P_j = f^j(P_0)$$

with

$$W^u(P_j) = \{x \in M : \lim_{k \to -\infty} \operatorname{dist}(f^k(x), f^k(P_j)) = 0\}$$

$$W^s(P_j) = \{x \in M : \lim_{k \to \infty} \operatorname{dist}(f^k(x), f^k(P_j)) = 0\}$$

- Q is a homoclinic point associated to $O(P_0)$ if

$$Q \in W^u(P_j) \cap W^s(P_k), \qquad j,k \in \{0, \cdots, n-1\}.$$

- Q is a transverse homoclinic point if the tangent spaces of $W^u(P_j)$ and $W^s(P_k)$ generate the tangent space to M. We write

$$Q \in W^u(P_j) \pitchfork W^s(P_k)$$

- In general, to prove the existence of transverse homoclinic points for a given dynamical system is difficult.
- There are computable formulas only in the perturbative setting.

Measuring transversality (I)

- Periodic perturbations of planar systems having a saddle point P_0, with homoclinic connection ($x_0(t)$ when $\mu = 0$)

$$\dot{x} = G_0(x) + \mu G_1(x,t;\mu), \qquad G_1(x,t+1;\mu) = G_1(x,t;\mu), \qquad x \in \mathbb{R}^2$$

- The Melnikov function is

$$M(t_0) = \int_{-\infty}^{\infty} G_0(x_0(t)) \wedge G_1(x_0(t), t + t_0; 0)\,dt$$

- Simple zeros of $M(t_0)$ lead to transverse homoclinic points, for $|\mu| \ll 1$.
- More general scenarios have been considered Delshams, Gonchencko, Gutiérrez, Motonaga, Yagasaki

Measuring transversality (II)

- Planar Analytic symplectic maps. The Melnikov potential is an infinite sum depending on the generating function of the map, Delshams, Ramírez-Ros.
- Let f_ε be a family of symplectic maps with generating function $L_\varepsilon = L_0 + \varepsilon L_1 + O(\varepsilon^2)$.
- Assume that f_0 has a saddle point P_0 with eigenvalues $\{\lambda, \lambda^{-1}\}$
- Assume that P_0 has associated a homoclinic connection parameterizated by $x_0(t)$ and

$$f_0(x_0(t)) = x_0(t+h), \qquad h = \log \lambda$$

- Then the Melnikov potential is

$$M(t) = L(t), \qquad L(t) = \sum_{j \in \mathbb{Z}} L_1(x_0(t+jh), x_0(t+(j+1)h))$$

- Has been computed for some examples, McMillan map, standard like maps and elliptic billiards.

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4. For one and a half degrees of freedom rapidly forced hamiltonians the works by Baldomá, Delshams, Fiedler, Fontich, Gaivao, Gelfreich, Giralt, Guardia, Holmes, Jorba, Lombardi, Marsden, Martin, Neishtadt, Paradela, Sauzin, Sheurle, Simó, Treshev provide conditions to guarantee transverse homoclinic points, using either Melnikov function or the inner equation.

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5. Toy model for near integrable hamiltonians close to resonances and it is a crucial ingredient for the proof of Arnold's diffusion in the analytic setting.
6. The exponentially small splitting of separatrices appear in several instances of the 3-body problem.

- Analytic area preserving maps close to integrable; for instance the standard map:

$$(x, y) \mapsto f(x, y) = (x + y + \varepsilon \sin(x), y + \varepsilon \sin(x))$$

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$$\varphi \sim \frac{\pi}{\varepsilon} \Theta e^{-\pi^2/\sqrt{\varepsilon}}, \qquad \Theta \in \mathbb{C} \setminus \{0\}$$

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- The constant $\Theta = \Theta(f)$ is the Stokes constant and is not related with the Melnikov approach.
- The proof was ended by Gelfreich in 1999.
- Other works are due to

Delshams, Fontich, Gil Ramis, Martín, M-Seara, Ramírez-Ros, Sauzin, Simó

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- These methods are constructive, providing computable conditions to guarantee the existence of transverse homoclinic points.

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The question is then

- For a given class E of dynamical systems, the subset $R \subset E$ having transverse homoclinic points is residual?
- In particular, if so, when E has the Baire property, for any $f \in E$, there exists a diffeomorphism $g \in R$ as close as we want of f having transverse homoclinic points

Some previous results about genericity

- Smooth symplectic diffeomorphisms,

$\mathcal{R} = \{f : M \to M$, every hyperbolic periodic orbit has a homoclinic orbit$\}$.

Then $\mathcal{R}$ is residual in

- Takens, 1972: for $M = \mathbb{R}^2$ in C^r, $r = 1$, symplectic diffeomorphism.
- Pixton, 1982: for $M = \mathbb{S}^2$ in C^r, $r \in [1, +\infty]$, diffeomorphism.
- Oliveira, 1987: for $M = \mathbb{T}^2$, in C^r, $r \in [1, +\infty]$, diffeomorphism.
- Oliveira, 2000: for (most) genus$(M) \geq 1$.

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- Smooth convex billiards

  - Donnay, 2003. There are C^∞ curves, C^2 close to the ellipse whose billiard table have transverse homoclinics points.
  - Zhihong Xia, Pengfei Zhang, 2013. All the hyperbolic periodic points have transverse homoclinic orbits for generic C^r billiards $r \geq 1$. Also Dias Carneiro, O. Kamphorst, S. Pinto de Carvalho, 2007, for $r = 2$.
  - Bessa, del Magno, Lopes Dias, Gaivao, 2024. There exists a C^2 open and dense set of convex bodies $\mathbb{R}^d$ whose billiard maps have an hyperbolic set (positive entropy).

Genericity in the analytic setting

- Planar Analytic symplectic diffeomorphisms having an elliptic fixed point.
  - Homoclinic Points Near Elliptic Fixed points, By Zehnder, 1973. The set of analytic symplectic diffeomorphisms having the origin as an elliptic point with transverse homoclinic points in every neighborhood of the origin is residual in some analytic topology.

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  - Transversal homoclinic orbits near elliptic fixed points of area preserving diffeomorphism of the plane, by Genecand, 1993. Same result as Zehnder but using Aubrey-Mather theory. He extended the results to some analytic geodesic flows and hamiltonians.

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Those works construct small perturbations having the property we want to deal with and belonging to a desired functional space.

Billiard dynamics

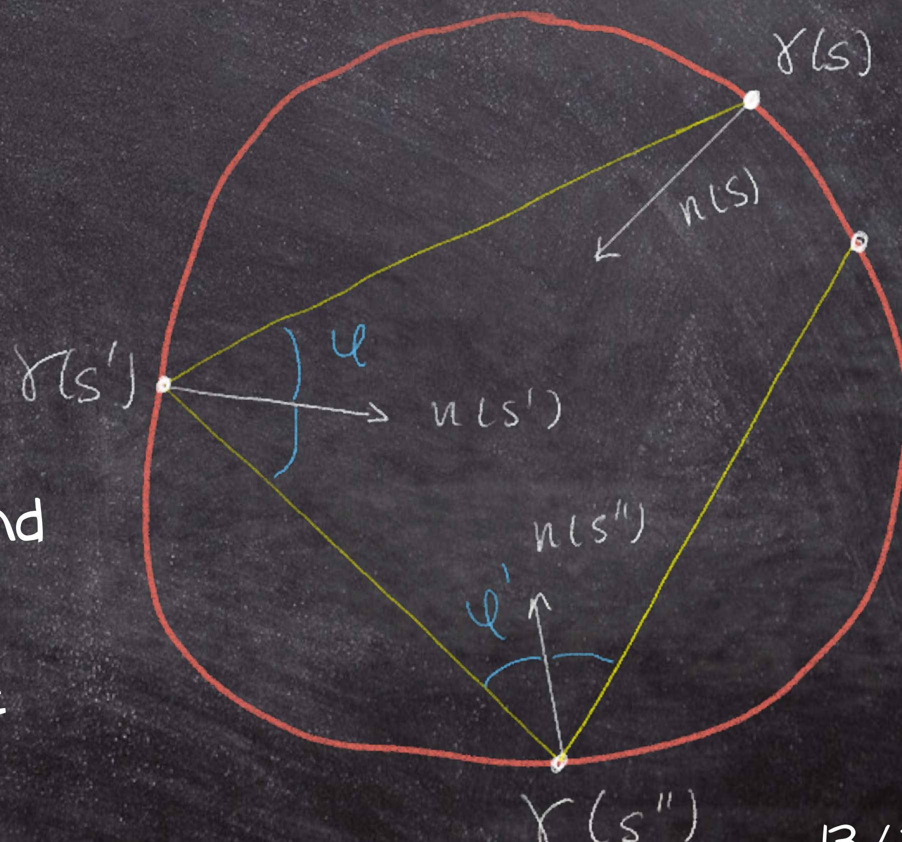
We set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

- A strictly convex billiard table $\Omega \subset \mathbb{R}^2$ is a bounded domain satisfying $\partial\Omega = \gamma(\mathbb{T})$ with $\gamma : \mathbb{T} \hookrightarrow \mathbb{R}^2$ a curve with strictly negative curvature.
- γ is an embedding and we take anticlockwise orientation

- The billiard map is

$$(s, \varphi) \mapsto f(s, \varphi) = (s', \varphi')$$

- $\gamma(s)$ is a point in $\partial\Omega$
- φ is the angle between the inward normal vector, $-n(s)$, and $\gamma(s') - \gamma(s)$
- The incidence angle is the same as the reflection



Properties of Billiard maps

Define

$$\mathbb{A} = \mathbb{T} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad \widetilde{\mathbb{A}} = \mathbb{R} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- $f : \mathbb{A} \to \mathbb{A}$, $f\left(s, \frac{\pi}{2}\right) = f\left(s, -\frac{\pi}{2}\right) = s$.
- If γ is analytic (resp. C^r), then f is analytic (resp. C^r).
- If $F : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is a lift of f, $F(s+1, \varphi) = F(x, \varphi) + 1$, we define the rotation number

$$\rho(s, \varphi) = \lim_{k \to \infty} \frac{\pi_1 F^k(s, \varphi) - s}{k}$$

- In particular for $P_0 = (s_0, \varphi_0)$ a n-periodic point, $\rho(P_0) = \frac{m}{n}$ with m the number of times that $O(P_0)$ winds around $\partial\Omega$ before closing

Topology

For a given $r > 0$, we define

$$\mathbb{T}_r = \{s \in \mathbb{C} : \operatorname{Re} s \in \mathbb{T}, |\operatorname{Im} s| < r\}$$

The functional space

$$\mathcal{C}^\omega_r(\mathbb{T}, \mathbb{R}^k) = \{\gamma : \overline{\mathbb{T}}_r \to \mathbb{R}^k, \text{ real analytic and continuous on } \overline{\mathbb{T}}_r\}$$

endowed with the norm

$$\|\gamma\|_r := \max_{s \in \overline{\mathbb{T}}_r} |\gamma(s)|$$

is a Banach space.

The space of real analytic functions on $\mathbb{T}$ satisfies

$$\mathcal{C}^\omega(\mathbb{T}, \mathbb{R}^k) = \bigcup_{r>0} \mathcal{C}^\omega_r(\mathbb{T}, \mathbb{R}^k)$$

The main result

Any strictly convex analytic Billiard Ω is characterized (not uniquely) as an element of the open set

$$B_r = \{\gamma \in C^{\omega}_r(\mathbb{T}, \mathbb{R}^k),\ \gamma : \mathbb{T} \hookrightarrow \mathbb{R}^2,\ \gamma(\mathbb{T}) \text{ strictly convex}\}.$$

We write $\Omega = \Omega(\gamma)$.

Theorem (Transverse homoclinic orbits)
Fix $r > 0$. There exists a generic set $B'_r \subset B_r$ such that for all $\gamma \in B'_r$ the following property holds:

For any rational rotation number m/n, the billiard map associated to $\Omega(\gamma)$ has at least one hyperbolic periodic orbit with rotation number m/n, having associated transverse homoclinic intersections.

The result is not perturbative

Preliminary considerations

- B_r is a Baire space because is an open set of a Banach space. Therefore, B_r' is dense.
- The result is and straightforward consequence of

Theorem (Fixing a rational rotation number)
Fix $m/n \in \mathbb{Q} \cap (0,1)$ and let $V_r^{m/n}$ be the set of $\gamma \in B_r$ such that the billiard map with table $\Omega(\gamma)$ has a transverse homoclinic orbit associated to a periodic orbit with rotation number m/n.
Then, $V_r^{m/n}$ is open and dense in B_r in the analytic topology.

- As a consequence the set

$$B_r' = \bigcap_{m/n \in \mathbb{Q}} V_r^{m/n} \subset B_r$$

is residual in the analytic topology.

- For a given $m/n \in \mathbb{Q} \cap (0,1)$, the property of being hyperbolic and transverse is open, so $V_r^{m/n}$ is open.
- To prove the density of $V_r^{m/n}$ in B_r it is enough to construct suitable analytic deformations of the billiard table having transverse intersections.

Theorem (Density using suitable analytic deformations)

Fix $m/n \in \mathbb{Q} \cap (0,1)$ and $\gamma \in B_r$. Denote by $n(s)$ the unitary outward normal vector at $\gamma(s)$.
Then for all $\varepsilon > 0$, there exists $\lambda_\varepsilon \in C_r^\omega(\mathbb{T}, \mathbb{R})$ with

$$\| \lambda_\varepsilon \|_r < \varepsilon$$

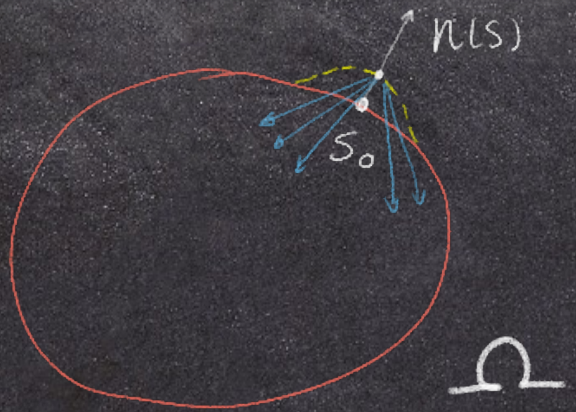
such that, letting

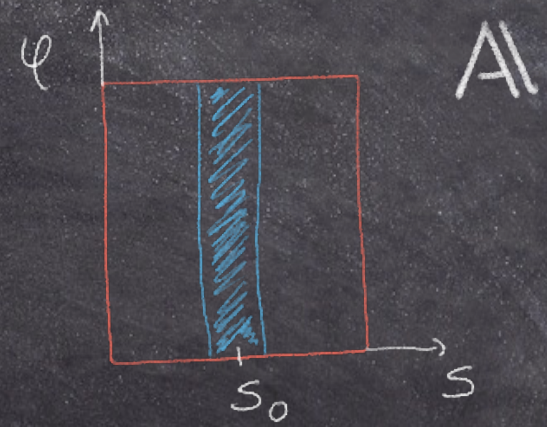
$$\gamma_\varepsilon(s) = \gamma(s) + \lambda_\varepsilon(s) n(s)$$

the billiard map of $\Omega(\gamma_\varepsilon)$ has a periodic hyperbolic orbit of rotation number m/n with transverse homoclinic points.

Comments on Billard tables

- γ will be parameterizated by arc-length, but γ_ε will not.
- Let $\lambda, \gamma, \tilde{\gamma}(s) = \gamma(s) + \lambda(s)n(s)$ and the corresponding billiards $f, \tilde{f}$. Even if λ only modify an small region of $\partial\Omega$, it affects a big region in $\mathbb{A}$:





- $\tau(s, s') = \|\gamma(s) - \gamma(s')\|$ is a generating function of f:

$$\partial_1 \tau(s, s') = -\sin\varphi, \qquad \partial_2 \tau(s, s') = \sin\varphi'$$

- The map $f = (f_1, f_2)$ satisfies the twist condition, $\partial_1 f_2 > 0$
- The Aubrey-Mather theory works for billiards.

Sketch of the proof (I)

Fix $r > 0$, $m/n \in \mathbb{Q} \cap (0,1)$ and $\gamma \in B_r$.

Sketch of the proof (1)

Fix $r > 0$, $m/n \in \mathbb{Q} \cap (0,1)$ and $\gamma \in B_r$.

- We consider $G^0 : \lambda \mapsto f + \Delta f$ the map that sends a deformation λ to the new billiard map.
- G^0 is C^1-Fréchet differentiable.

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- G^0 is $\mathcal{C}^1$-Fréchet differentiable.
- Use Aubrey-Mather theory to guarantee the existence of a hyperbolic periodic orbit $\mathcal{P}$ of rotation number m/n having a homoclinic point $Q \in W^u(\mathcal{P}) \cap W^s(\mathcal{P})$.
- Assume that Q is not transverse.

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- Construct $\mathcal{H} : \tilde{f} \mapsto (\mathcal{H}_1(\tilde{f}), \mathcal{H}_2(\tilde{f})) \in \mathbb{R}^2$ such that

$$\mathcal{H}_1(\tilde{f}) = 0 \iff Q \text{ homoclinic point}$$

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- We have that $\mathcal{H} \circ G^0(0) = (0,0)$.
- $\mathcal{H}$ is C^1–Fréchet differentiable

Sketch of the proof (II)

- Fix $\varepsilon > 0$. The goal is to prove that there exists $\lambda \in C^\omega(\mathbb{T}, \mathbb{R})$ such that

$$\|\lambda\|_r < \varepsilon, \qquad \mathcal{H} \circ G^0(\lambda) = (0, a)$$

with $a \neq 0$.

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- Compute $d(H \circ G^0)(0)\lambda = dH(f)dG^0(0)\lambda$ for functions λ C^∞ compactly supported (otherwise is impossible!)

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- Prove that there exists $\widehat{\lambda}_1, \widehat{\lambda}_2$ such that

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- Prove that there exists $\widehat{\lambda}_1, \widehat{\lambda}_2$ such that

$$d(\mathcal{H} \circ G^0)(0)\widehat{\lambda}_1 = (1, 0), \qquad d(\mathcal{H} \circ G^0)(0)\widehat{\lambda}_2 = (0, 1)$$

- By continuity, there are λ_1, λ_2 trigonometric polynomial, close to $\widehat{\lambda}_1, \widehat{\lambda}_2$ in some C^ℓ norm, such that

$$d(\mathcal{H} \circ G^0)(0)\lambda_1 = w_1, \qquad d(\mathcal{H} \circ G^0)(0)\lambda_2 = w_2$$

with $[w_1, w_2] = \mathbb{R}^2$.

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- Consider $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ the $\mathcal{C}^1$ map defined by

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- Notice that $\|\lambda_1\|_r, \|\lambda_2\|_r$ is in general (very) big. However we can take a as small as we want, so that

$$\|\lambda\|_r \lesssim |a| \cdot \|\lambda_1\|_r + |a| \cdot \|\lambda_2\|_r < \varepsilon$$

The action of the perturbation on the billiard map (I)

Fix $r > 0$, $m/n \in \mathbb{Q} \cap (0,1)$ and $\gamma \in B_r$.

- Computation of the first order in ε of the billiard map f_ε with billiard table $\Omega(\gamma_\varepsilon)$ and

$$\gamma_\varepsilon(s) = \gamma(s) + \varepsilon\lambda(s), \qquad \lambda \in C^\ell(\mathbb{T}, \mathbb{R}).$$

Denoting $(s', \varphi') = f(s, \varphi)$, $\tau = \tau(s, s') = \|\gamma(s) - \gamma(s')\|$ and $K(s)$ the curvature:

$$s'_\varepsilon = s - \varepsilon \frac{1}{\cos\varphi'}(\lambda'(s)\tau + \lambda(s)\sin\varphi - \lambda(s')\sin\varphi') + O(\varepsilon^2)$$

$$\varphi'_\varepsilon = \varphi' - \varepsilon \frac{K(s')}{\cos\varphi'}(\lambda'(s)\tau + \lambda(s)\sin\varphi - \lambda(s')\sin\varphi')$$
$$+ \varepsilon(\lambda'(s') - \lambda'(s)) + O(\varepsilon^2).$$

Uniformity away from the boundary of $\mathbb{A}$.

The action of the perturbation on the billiard map (II)

- For any $\delta > 0$, consider the set

$$\mathbb{A}_\delta = \left\{(s,\varphi) \in \mathbb{A},\ \varphi \in \left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]\right\}.$$

Let $G^\delta : \mathcal{C}^\ell(\mathbb{T},\mathbb{R}) \to \mathcal{C}^\ell(\mathbb{A}_\delta)$ the map sending λ to the billiard map restricted to $\mathbb{A}_\delta$ of billiard table with boundary

$$\gamma[\lambda](s) = \gamma(s) + \lambda(s)n(s)$$

The map G^δ is $\mathcal{C}^1$–Fréchet differentiable and

$$dG_1^\delta(0)\lambda = -\frac{1}{\cos\varphi'}(\lambda'(s)\tau + \lambda(s)\sin\varphi - \lambda(s')\sin\varphi')$$

$$dG_2^\delta(0)\lambda = -\frac{K(s')}{\cos\varphi'}(\lambda'(s)\tau + \lambda(s)\sin\varphi - \lambda(s')\sin\varphi') + \lambda'(s') - \lambda'(s).$$

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- Then, using again Aubrey-Mather theory $\mathcal{P}$ is hyperbolic and has associated homoclinic points. Let Q be one of them.
- Assume that Q is not transverse, otherwise we are done, and rename $f = G_\delta(\lambda_0)$.

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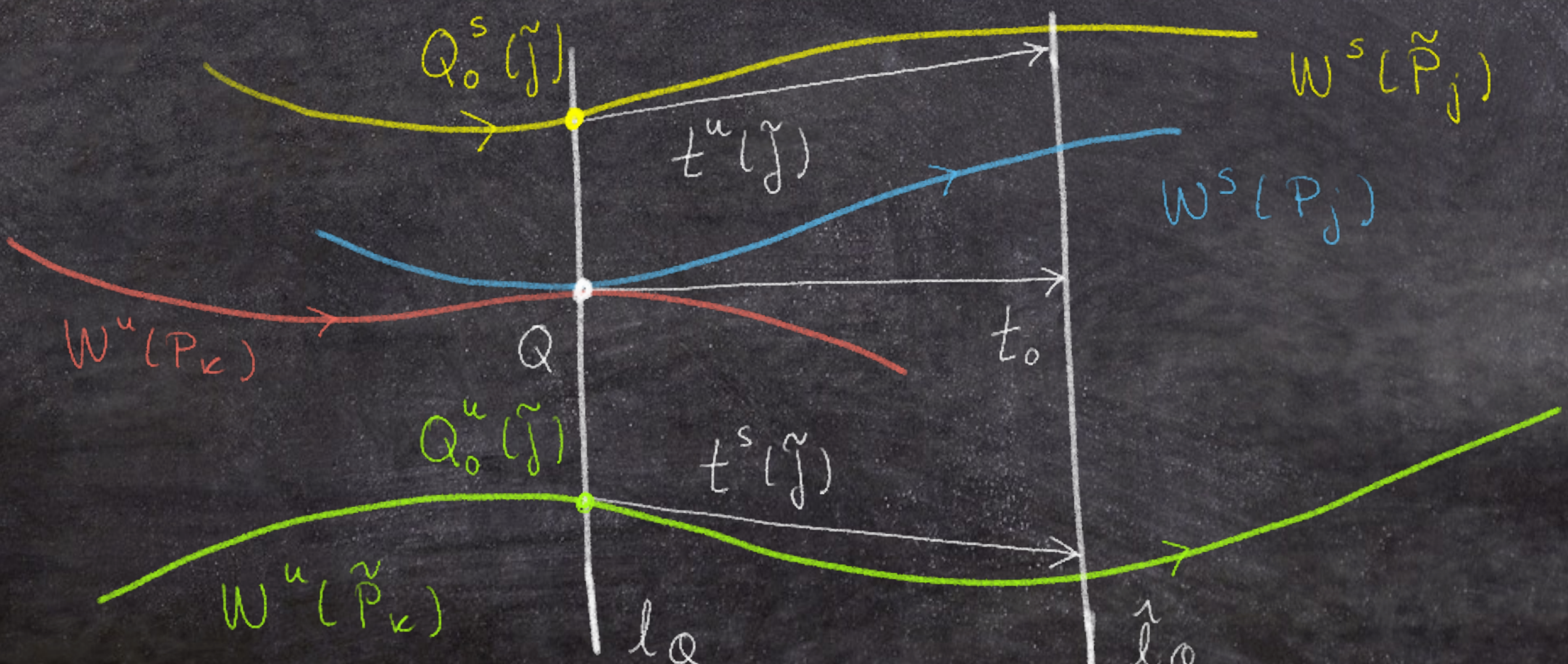
- Consider $f = G_\delta(\lambda_0)$ the (new) original billiard map,

$$E_\rho = \{\tilde{f} \in \mathcal{C}^\infty, \|f - \tilde{f}\|_{\mathcal{C}^1} \leq \rho\}$$

with ρ small enough, and the following construction:



Measuring the transversality (II)

- Let $H : E_p \to \mathbb{R}^2$ defined by

$$H(\tilde{f}) = \left(\frac{t_0}{\|t_0\|} \wedge [Q_0^u(\tilde{f}) - Q_0^s(\tilde{f})], \frac{t_0}{\|t_0\|} \wedge [t_0^u(\tilde{f}) - t_0^s(\tilde{f})] \right)$$

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- Genecand proves that $\mathcal{H}$ is C^1-Fréchet differentiable and

$$d\mathcal{H}_1(f)h = \sum_{i \in Z} t_{i+1} \wedge h(Q_i)$$

$$Q_0 = Q$$

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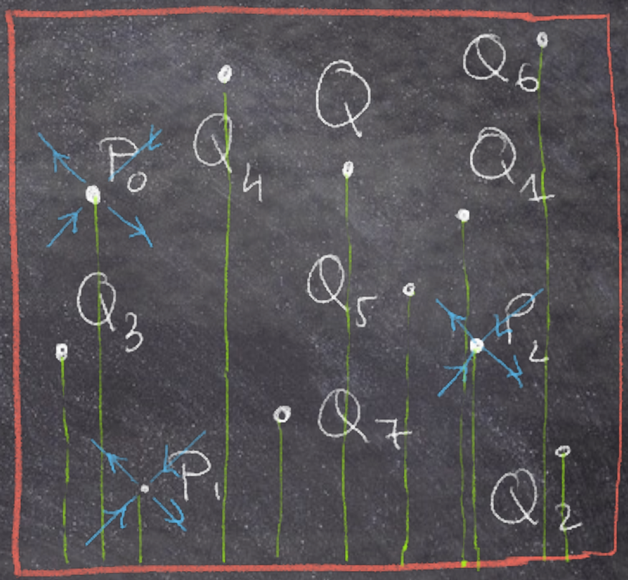
- This formula can not be computed in general

Compact supported deformations (I)

- $Q = (s_0, \varphi_0) \in W^u(P_j) \cap W^s(P_k)$
- Again using Aubrey-Mather theory, $\pi_1 : O(Q) \cup \mathcal{P} \to \mathbb{T}$ is injective.



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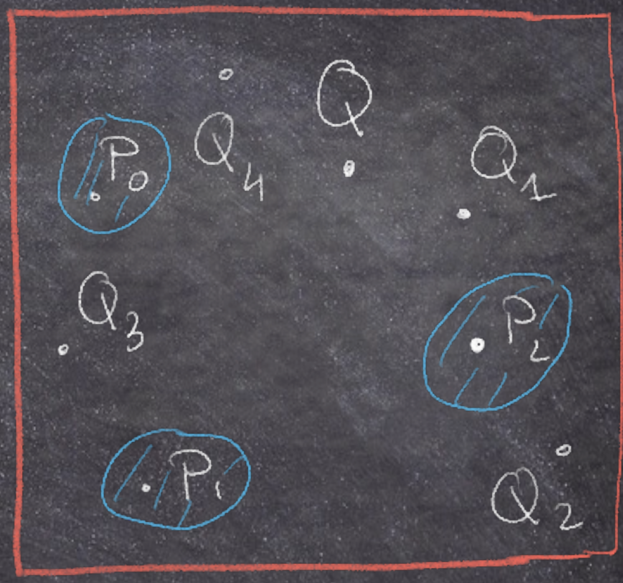

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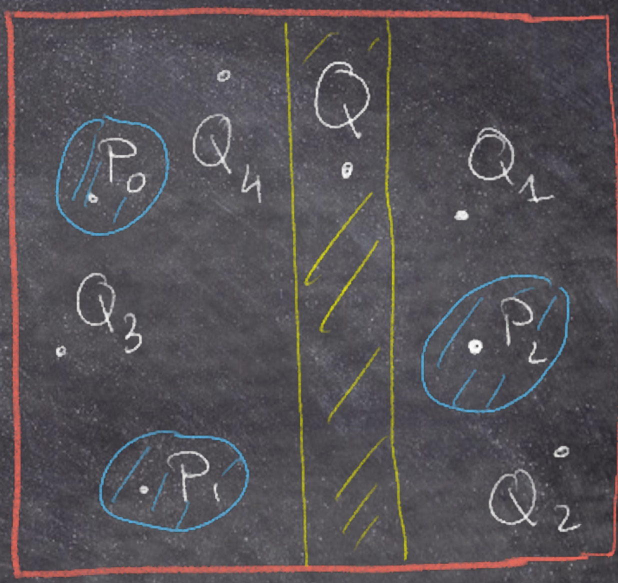

$$V = [s_0 - \eta, s + \eta] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

such that, for $i \neq 0$, $Q_i = f^i(Q) \notin V$.

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  - Let $S = [s_0 - \eta, s_0 + \eta]$ and consider λ compactly supported at S

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- Remember that, $(s', \varphi') = f(s, \varphi)$ and

$$dG^{\delta}(0)\lambda = A(s, \varphi, s', \varphi') \begin{pmatrix} \lambda(s) \\ \lambda'(s) \end{pmatrix} + B(s, s', \varphi, \varphi') \begin{pmatrix} \lambda(s') \\ \lambda'(s') \end{pmatrix}$$

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- Therefore, $h:=dG^\delta(0)\lambda$, satisfies that $h(Q_j)=0$ and $Dh(Q_j)=0$, for $j\neq 0,-1$ and

$$dH_1(f)h=\sum_{i\in Z} t_{i+1}\wedge h(Q_i)=t_1\wedge h(Q_0)+t_0\wedge h(Q_{-1})$$

$$dH_2(f)h=\sum_{i\in Z} t_{i+1}\wedge Dh(Q_i)t_i+\hat{H}$$

$$=t_1\wedge Dh(Q_0)t_0+t_0\wedge Dh(Q_{-1})t_{-1}+\hat{H}$$

is a (more or less manegable) explicit formula.

Compact supported deformations (III)

- Finally we prove that $d(\mathcal{H} \circ G_\delta)(0) : \mathcal{C}^\infty_{supp} \to \mathbb{R}^2$ is exhaustive.
- Let $Q = (s_0, \varphi_0)$. If $\lambda_1(s_0) = 0$ and $\lambda_1'(s_0) = 1$, then

$$v_1 := d(\mathcal{H} \circ G^\delta)(0)\lambda_1 = (\cos\varphi_0(\cos\varphi_0 + \cos\varphi_1), \text{something})$$

- If $\lambda_2(s_0) = \lambda_2'(s_0) = 0$ and $\lambda_2''(s_0) = 1$, then

$$v_2 := d(\mathcal{H} \circ G^\delta)(0)\lambda_2 = (0, \cos\varphi_0 \cdot [\pi_1 t_0]^2)$$

- By the twist condition, one can always assume that $\pi_1 t_0 \neq 0$ and therefore

$$\det(v_1, v_2) = \cos^2\varphi_0(\cos\varphi_0 + \cos\varphi_1) \cdot [\pi_1 t_0]^2 \neq 0.$$

Thank you!