Computation of the asymptotic wavenumber of spiral waves of the Ginzburg Landau equation

M. Aguareles[1], I. Baldomá[234], T.M Seara[234]

[1]UdG [2]UPC [3]CRM [4]IMTech

Outline

Preliminaries

The result

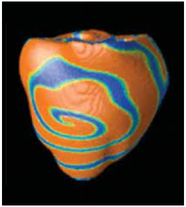
The strategy of the proof

Spiral patterns

Spiral patterns are commonly observed in certain chemical, biological and physical systems

Belousov-Zhabotinskii

Social amoebas Dictyostelium discoideium



Cardiac muscle tissue

- These systems are governed by chemical or biological reaction and spatial diffusion.

$$\partial_\tau U = D\Delta U + F(U, a), \qquad D \text{ a diffusion matrix, } F \text{ the reaction nonlinearity}$$

$U = U(\tau, \vec{x}) \in \mathbb{R}^N$, $\vec{x} \in \mathbb{R}^2$ and a is a parameter (for instance some catalyst concentration).

The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
- Take $\varepsilon^2 = a - a_0 > 0$, small, $t = \varepsilon^2\tau$. Then the modulation of local oscillations with frequency ω

$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau} v_+ + c.c.] + \mathcal{O}(\varepsilon^2).$$

- and (after some scalings) the (complex) amplitude A, satisfies the celebrated complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2,$$

where $A(\vec{x}, t) \in \mathbb{C}$ and α, β are real parameters (dispersion parameters).

Y. Kuramoto, *Chemical oscillations, waves and turbulence*

P. Hagan, *Spiral waves in Reaction-Diffusion equations*

- It appears in a wide range of different physical contexts: chemical reaction processes, as a model for pattern formation mechanisms, description of some ecological and in phase transitions in superconductivity

I.S Aranson, L. Kramer. *The world of the complex Ginzburg-Landau equation*

The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
- Take $\varepsilon^2 = a - a_0 > 0$, small, $t = \varepsilon^2\tau$. Then the modulation of local oscillations with frequency ω

$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau} v_+ + c.c.] + \mathcal{O}(\varepsilon^2).$$

- and (after some scalings) the (complex) amplitude A, satisfies the celebrated complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2,$$

where $A(\vec{x}, t) \in \mathbb{C}$ and α, β are real parameters (dispersion parameters).

Y. Kuramoto, *Chemical oscillations, waves and turbulence*

P. Hagan, *Spiral waves in Reaction-Diffusion equations*

- It appears in a wide range of different physical contexts: chemical reaction processes, as a model for pattern formation mechanisms, description of some ecological and in phase transitions in superconductivity

I.S Aranson, L. Kramer. *The world of the complex Ginzburg-Landau equation*

The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
- Take $\varepsilon^2 = a - a_0 > 0$, small, $t = \varepsilon^2\tau$. Then the modulation of local oscillations with frequency ω

$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau} v_+ + c.c.] + \mathcal{O}(\varepsilon^2).$$

- and (after some scalings) the (complex) amplitude A, satisfies the celebrated complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2,$$

  where $A(\vec{x}, t) \in \mathbb{C}$ and α, β are real parameters (dispersion parameters).

Y. Kuramoto, *Chemical oscillations, waves and turbulence*

P. Hagan, *Spiral waves in Reaction-Diffusion equations*

- It appears in a wide range of different physical contexts: chemical reaction processes, as a model for pattern formation mechanisms, description of some ecological and in phase transitions in superconductivity

I.S Aranson, L. Kramer. *The world of the complex Ginzburg-Landau equation*

The Ginzburg-Landau equation

- Assume that $\partial_\tau U = F(U, a)$ undergoes a supercritical Hopf bifurcation for (U_0, a_0) with eigenvalues $\pm i\omega$ and eigenvectors $v_\pm$.
- Take $\varepsilon^2 = a - a_0 > 0$, small, $t = \varepsilon^2 \tau$. Then the modulation of local oscillations with frequency ω
$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau} v_+ + c.c.] + \mathcal{O}(\varepsilon^2).$$
- and (after some scalings) the (complex) amplitude A, satisfies the celebrated complex Ginzburg-Landau equation
$$\frac{\partial A}{\partial t} = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2,$$
where $A(\vec{x}, t) \in \mathbb{C}$ and α, β are real parameters (dispersion parameters).

Y. Kuramoto, *Chemical oscillations, waves and turbulence*

P. Hagan, *Spiral waves in Reaction-Diffusion equations*

- It appears in a wide range of different physical contexts: chemical reaction processes, as a model for pattern formation mechanisms, description of some ecological and in phase transitions in superconductivity

I.S Aranson, L. Kramer. *The world of the complex Ginzburg-Landau equation*

Spiral waves. Definition

- We focus on infinite domains, $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$.
- The wave trains are solutions of the one dimensional GL in polar coordinates of the form $A(t,r) = A_*(-k_* r + \Omega t)$ with $A_*(\cdot)$ a periodic function.
- The spiral waves are bounded solutions that asymptotically tends to a wave train. Namely, solutions of the form $A(t,r,\varphi) = A_s(r, n\varphi + \Omega t)$ satisfying

$$A_s(0,\psi) \text{ bounded}, \qquad \lim_{r\to\infty} \|A_s(r,\psi) - A_*(-k_* r + \theta(r) + \psi)\| = 0$$

with $A_*(\cdot)$ a wave train, θ smooth and $\lim_{r\to\infty} \theta'(r) \to 0$.

- In the co-rotating frame, ($\psi = n\varphi + \Omega t$), they can be seen as an heteroclinic connection (with r as independent variable)

B. Sandstede, A. Scheel, *Spiral waves: linear and nonlinear theory*

Spiral waves. Definition

- We focus on infinite domains, $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$.
- The wave trains are solutions of the one dimensional GL in polar coordinates of the form $A(t, r) = A_*(-k_* r + \Omega t)$ with $A_*(\cdot)$ a periodic function.
- The spiral waves are bounded solutions that asymptotically tends to a wave train. Namely, solutions of the form $A(t, r, \varphi) = A_s(r, n\varphi + \Omega t)$ satisfying

$$A_s(0, \psi) \text{ bounded}, \qquad \lim_{r\to\infty} \|A_s(r, \psi) - A_*(-k_* r + \theta(r) + \psi)\| = 0$$

with $A_*(\cdot)$ a wave train, θ smooth and $\lim_{r\to\infty} \theta'(r) \to 0$.

- In the co-rotating frame, $(\psi = n\varphi + \Omega t)$, they can be seen as an heteroclinic connection (with r as independent variable)

B. Sandstede, A. Scheel, *Spiral waves: linear and nonlinear theory*

Spiral waves. Definition

- We focus on infinite domains, $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$.
- The wave trains are solutions of the one dimensional GL in polar coordinates of the form $A(t, r) = A_*(-k_* r + \Omega t)$ with $A_*(\cdot)$ a periodic function.
- The spiral waves are bounded solutions that asymptotically tends to a wave train. Namely, solutions of the form $A(t, r, \varphi) = A_s(r, n\varphi + \Omega t)$ satisfying

$$A_s(0, \psi) \text{ bounded}, \qquad \lim_{r\to\infty} \|A_s(r, \psi) - A_*(-k_* r + \theta(r) + \psi)\| = 0$$

  with $A_*(\cdot)$ a wave train, θ smooth and $\lim_{r\to\infty} \theta'(r) \to 0$.
- In the co-rotating frame, $(\psi = n\varphi + \Omega t)$, they can be seen as an heteroclinic connection (with r as independent variable)

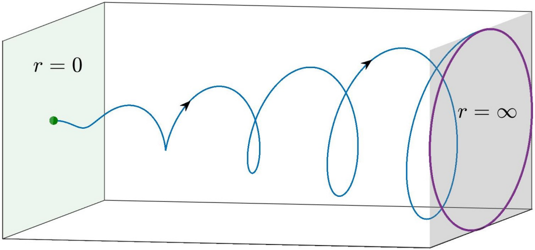
B. Sandstede, A. Scheel, *Spiral waves: linear and nonlinear theory*

Spiral waves. Definition

- We focus on infinite domains, $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$.
- The wave trains are solutions of the one dimensional GL in polar coordinates of the form $A(t, r) = A_*(-k_* r + \Omega t)$ with $A_*(\cdot)$ a periodic function.
- The spiral waves are bounded solutions that asymptotically tends to a wave train. Namely, solutions of the form $A(t, r, \varphi) = A_s(r, n\varphi + \Omega t)$ satisfying

$$A_s(0, \psi) \text{ bounded}, \qquad \lim_{r\to\infty} \|A_s(r, \psi) - A_*(-k_* r + \theta(r) + \psi)\| = 0$$

  with $A_*(\cdot)$ a wave train, θ smooth and $\lim_{r\to\infty} \theta'(r) \to 0$.
- In the co-rotating frame, ($\psi = n\varphi + \Omega t$), they can be seen as an heteroclinic connection (with r as independent variable)



B. Sandstede, A. Scheel, *Spiral waves: linear and nonlinear theory*

Wave trains and spiral waves in Ginzburg-Landau equation

- The only possible wave trains are $A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}$ with Ω and k_* satisfying

$$C = \sqrt{1-k_*^2}, \qquad \Omega = \Omega(k_*) = -\beta + k_*^2(\beta - \alpha).$$

The last condition is the associated *dispersion relation* and the quantity $v_g := -\partial_{k_*}\Omega(k_*) = 2k_*(\alpha - \beta)$ the group velocity.

- As a consequence an spiral wave has to tend as $r \to \infty$ to

$$A_*(\Omega t + \chi(r) + n\varphi) = \sqrt{1-k_*^2}e^{i(\Omega t + \chi(r) + n\varphi)}$$

with $\chi(r) = -k_* r + \theta(r) \sim -k_* r$ and Ω, k_* satisfying the dispersion relation.

- We look for spirals waves n-armed of the form

$$A(t, r, \varphi) = f(r)\exp\Big(i(\Omega t + \chi(r) + n\varphi)\Big),$$

with f, χ, χ' bounded and

$$\lim_{r\to\infty} \chi'(r) = -k_*, \qquad \lim_{r\to\infty} f(r) = \sqrt{1-k_*^2}.$$

Wave trains and spiral waves in Ginzburg-Landau equation

- The only possible wave trains are $A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}$ with Ω and k_* satisfying

$$C = \sqrt{1 - k_*^2}, \qquad \Omega = \Omega(k_*) = -\beta + k_*^2(\beta - \alpha).$$

  The last condition is the associated *dispersion relation* and the quantity $v_g := -\partial_{k_*}\Omega(k_*) = 2k_*(\alpha - \beta)$ the group velocity.

- As a consequence an spiral wave has to tend as $r \to \infty$ to

$$A_*(\Omega t + \chi(r) + n\varphi) = \sqrt{1 - k_*^2} e^{i(\Omega t + \chi(r) + n\varphi)}$$

  with $\chi(r) = -k_* r + \theta(r) \sim -k_* r$ and Ω, k_* satisfying the dispersion relation.

- We look for spirals waves n-armed of the form

$$A(t, r, \varphi) = f(r)\exp\Big(i(\Omega t + \chi(r) + n\varphi)\Big),$$

  with f, χ, χ' bounded and

$$\lim_{r\to\infty} \chi'(r) = -k_*, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k_*^2}.$$

Wave trains and spiral waves in Ginzburg-Landau equation

- The only possible wave trains are $A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}$ with Ω and k_* satisfying

$$C = \sqrt{1 - k_*^2}, \qquad \Omega = \Omega(k_*) = -\beta + k_*^2(\beta - \alpha).$$

  The last condition is the associated *dispersion relation* and the quantity $v_g := -\partial_{k_*}\Omega(k_*) = 2k_*(\alpha - \beta)$ the group velocity.

- As a consequence an spiral wave has to tend as $r \to \infty$ to

$$A_*(\Omega t + \chi(r) + n\varphi) = \sqrt{1 - k_*^2} e^{i(\Omega t + \chi(r) + n\varphi)}$$

  with $\chi(r) = -k_* r + \theta(r) \sim -k_* r$ and Ω, k_* satisfying the dispersion relation.

- We look for spirals waves n-armed of the form

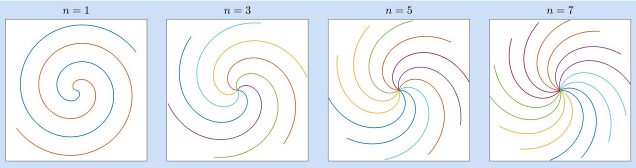
$$A(t, r, \varphi) = f(r)\exp\Big(i(\Omega t + \chi(r) + n\varphi)\Big),$$

  with f, χ, χ' bounded and

$$\lim_{r\to\infty} \chi'(r) = -k_*, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k_*^2}.$$

Where is the spiral shape?

- For any constant c, $\mathrm{Re}\Big(A_*(\Omega t - k_* r + n\varphi)e^{-i\Omega t}\Big) = c$, that is $-k_* r + n\varphi = c'$, is a archimedian spiral with wavelength (distance between two spiral arms) $2\pi n |k_*|^{-1}$

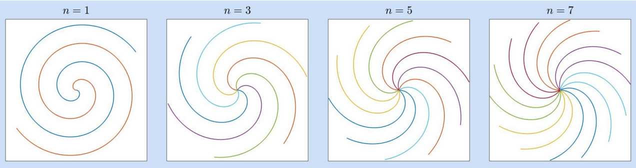


- Below, the surface $\mathrm{Re}(A(t, r, \varphi)e^{-i\Omega t})$ for different values of r.

$n = 5$, $6 \leq r \leq 20$ $n = 5$, $20 \leq r \leq 100$ $n = 5$, $100 \leq r \leq 500$

Where is the spiral shape?

- For any constant c, $\mathrm{Re}\Big(A_*(\Omega t - k_* r + n\varphi)e^{-i\Omega t}\Big) = c$, that is $-k_* r + n\varphi = c'$, is a archimedian spiral with wavelength (distance between two spiral arms) $2\pi n|k_*|^{-1}$

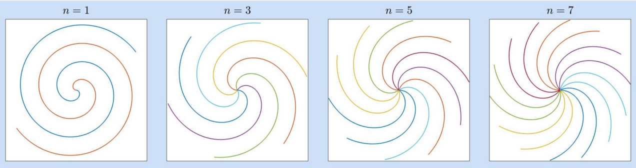


- Below, the surface $\mathrm{Re}(A(t, r, \varphi)e^{-i\Omega t})$ for different values of r.

$n = 5$, $6 \leq r \leq 20$ $n = 5$, $20 \leq r \leq 100$ $n = 5$, $100 \leq r \leq 500$

Where is the spiral shape?

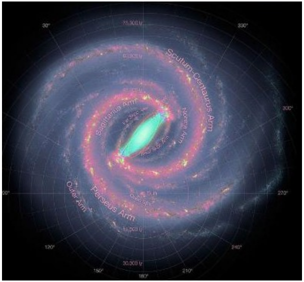
- For any constant c, $\mathrm{Re}\Big(A_*(\Omega t - k_* r + n\varphi)e^{-i\Omega t}\Big) = c$, that is $-k_* r + n\varphi = c'$, is a archimedian spiral with wavelength (distance between two spiral arms) $2\pi n|k_*|^{-1}$



- Below, the surface $\mathrm{Re}(A(t, r, \varphi)e^{-i\Omega t})$ for different values of r.

$n = 5,\ 6 \leq r \leq 20$

$n = 5,\ 20 \leq r \leq 100$



$n = 5,\ 100 \leq r \leq 500$

The result

- We introduce the *twist parameter*

$$q = \frac{\beta - \alpha}{1 + \alpha\beta}$$

Theorem

If $|q|$ is small enough, the Ginzburg-Landau equation possesses a spiral wave n-armed with one defect ($f(0; q) = 0$, $f(r; q) > 0$ for $r > 0$) and $f'(r; q) > 0$, if and only if

$$k_* = k_*(q) = \sqrt{\frac{1}{1 - \alpha q(1 - k^2(q))}}\, k(q), \qquad k(q) = \frac{2}{q} e^{-\frac{C_n}{n^2} - \gamma} e^{-\frac{\pi}{2n|q|}} (1 + \mathcal{O}(|q|)), \tag{1}$$

with γ the Euler's constant and

$$C_n = \lim_{r \to \infty} \left(\int_0^r \xi f^2(\xi; 0)(1 - f^2(\xi; 0))\, d\xi - n^2 \log r \right).$$

Notice that $k_*(q) = k(q)(1 + \mathcal{O}(q))$.

Comments

- The case $q = 0$, can be reduced to the real Ginzburg Landau equation

$$\partial_t A = \Delta A + A - A|A|^2.$$

- If $q = 0$, $k_* = 0$ and there is no spiral waves.
- In our perturbative setting, these lines bend to form the spirals.

Previous works

- N. Kopell and L. N. Howard (1981). A serie of papers concerned with pattern formation in the Belousov-Zhabotinskii reaction. The existence and uniqueness of the asymtptotic wavenumber $k_* = k_*(q)$ as a function of q was proven.
  The analytic methods used by Kopell et al, do not allow to obtain an expression for $k_*(q)$.
- P. S. Hagan (1982), J. Greenberg (1980), M. Aguareles, S. Chapman, T. Witelski (2010) used asymptotic methods to compute an explicit asymptotic formula for $k(q)$.
  The asymptotic methods are a consistent and systematic way to conjecture true results but does not provide rigorous proofs.

Comments

- The case $q = 0$, can be reduced to the real Ginzburg Landau equation

$$\partial_t A = \Delta A + A - A|A|^2.$$

- If $q = 0$, $k_* = 0$ and there is no spiral waves.
- In our perturbative setting, these lines bend to form the spirals.

Previous works

- N. Kopell and L. N. Howard (1981). A serie of papers concerned with pattern formation in the Belousov-Zhabotinskii reaction. The existence and uniqueness of the asymtptotic wavenumber $k_* = k_*(q)$ as a function of q was proven.
  The analytic methods used by Kopell et al, do not allow to obtain an expression for $k_*(q)$.
- P. S. Hagan (1982), J. Greenberg (1980), M. Aguareles, S. Chapman, T. Witelski (2010) used asymptotic methods to compute an explicit asymptotic formula for $k(q)$.
  The asymptotic methods are a consistent and systematic way to conjecture true results but does not provide rigorous proofs.

Setting

- We forget PDE because $f(r)$ and $v(r) = \chi'(r)$ has to satisfy

$$f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - v^2) = 0, \qquad v' + \frac{v}{r} + 2\frac{vf'}{f} + q(1 - f^2 - k^2) = 0.$$

together with

$$\lim_{r\to\infty} v(r) = -k, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k^2}.$$

- In order to f, v being bounded at $r = 0$, we need to impose $f(0) = v(0) = 0$.

More conditions

Notice that the equations remain by changing (v, q) to $(-v, -q)$. We set then $q > 0$.
Since we want $f'(r) > 0$, $\lim_{r\to\infty} f'(r) = 0$ and $f^2(r) < 1 - k^2$ so that

$$(f^2 vr)' = f^2 r \left(v' + \frac{v}{r} + 2\frac{vf'}{f} \right) = -f^2 rq(1 - f^2 - k^2) < 0$$

that implies $v(r) < 0$. As a consequence $k > 0$.

- There are too many conditions. This indicates a selection mechanism for k.

Setting

- We forget PDE because $f(r)$ and $v(r) = \chi'(r)$ has to satisfy

$$f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - v^2) = 0, \qquad v' + \frac{v}{r} + 2\frac{vf'}{f} + q(1 - f^2 - k^2) = 0.$$

together with

$$\lim_{r\to\infty} v(r) = -k, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k^2}.$$

- In order to f, v being bounded at $r = 0$, we need to impose $f(0) = v(0) = 0$.

More conditions

Notice that the equations remain by changing (v, q) to $(-v, -q)$. We set then $q > 0$.
Since we want $f'(r) > 0$, $\lim_{r\to\infty} f'(r) = 0$ and $f^2(r) < 1 - k^2$ so that

$$(f^2vr)' = f^2r\left(v' + \frac{v}{r} + 2\frac{vf'}{f}\right) = -f^2rq(1 - f^2 - k^2) < 0$$

that implies $v(r) < 0$. As a consequence $k > 0$.

- There are too many conditions. This indicates a selection mechanism for k.

Setting

- We forget PDE because $f(r)$ and $v(r) = \chi'(r)$ has to satisfy

$$f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - v^2) = 0, \qquad v' + \frac{v}{r} + 2\frac{vf'}{f} + q(1 - f^2 - k^2) = 0.$$

together with

$$\lim_{r\to\infty} v(r) = -k, \qquad \lim_{r\to\infty} f(r) = \sqrt{1 - k^2}.$$

- In order to f, v being bounded at $r = 0$, we need to impose $f(0) = v(0) = 0$.

More conditions

Notice that the equations remain by changing (v, q) to $(-v, -q)$. We set then $q > 0$. Since we want $f'(r) > 0$, $\lim_{r\to\infty} f'(r) = 0$ and $f^2(r) < 1 - k^2$ so that

$$(f^2vr)' = f^2r\left(v' + \frac{v}{r} + 2\frac{vf'}{f}\right) = -f^2rq(1 - f^2 - k^2) < 0$$

that implies $v(r) < 0$. As a consequence $k > 0$.

- There are too many conditions. This indicates a selection mechanism for k.

Counting dimensions

- We want to connect $(f, f', v, r) = (0,0,0,0)$ to $(f, f', v, r) = (\sqrt{1-k^2}, 0, -k, \infty)$.

Dynamics around $r \sim 0$

- $w = f^2 vr$ and $r = e^s$.
- Dominant dynamics

$$\tilde{f}'' = n^2 \tilde{f}, \ \tilde{w}' = -\tilde{f}^2 q(1-k^2).$$

- $(f, f', v) = (0,0,0)$ has 1 unstable direction.

Dynamics around $r \sim \infty$

- Dominant dynamics

$$f'' = -f(1 - f^2 - v^2),$$

$$v' = -2\frac{vf'}{f} - q(1 - f^2 - k^2)$$

- $(f, f', v) = (\sqrt{1-k^2}, 0, -k)$ has 1 stable direction.

In the extended phase space, $\mathbb{R}^5$ $(r' = 1,\ k' = 0)$

- $W^{s,u}$ have dimension 3.
- Generically they intersect in a curve (a solution).
- We need to select k.

Counting dimensions

- We want to connect $(f, f', v, r) = (0,0,0,0)$ to $(f, f', v, r) = (\sqrt{1-k^2}, 0, -k, \infty)$.

Dynamics around $r \sim 0$

- $w = f^2 vr$ and $r = e^s$.
- Dominant dynamics

$$\tilde{f}'' = n^2 \tilde{f}, \ \tilde{w}' = -\tilde{f}^2 q(1-k^2).$$

- $(f, f', v) = (0,0,0)$ has 1 unstable direction.

Dynamics around $r \sim \infty$

- Dominant dynamics

$$f'' = -f(1-f^2-v^2),$$

$$v' = -2\frac{vf'}{f} - q(1-f^2-k^2)$$

- $(f, f', v) = (\sqrt{1-k^2}, 0, -k)$ has 1 stable direction.

In the extended phase space, $\mathbb{R}^5$ $(r' = 1, \ k' = 0)$

- $W^{s,u}$ have dimension 3.
- Generically they intersect in a curve (a solution).
- We need to select k.

Counting dimensions

- We want to connect $(f, f', v, r) = (0, 0, 0, 0)$ to $(f, f', v, r) = (\sqrt{1-k^2}, 0, -k, \infty)$.

Dynamics around $r \sim 0$

- $w = f^2 vr$ and $r = e^s$.
- Dominant dynamics

$$\tilde{f}'' = n^2 \tilde{f}, \ \tilde{w}' = -\tilde{f}^2 q(1-k^2).$$

- $(f, f', v) = (0, 0, 0)$ has 1 unstable direction.

Dynamics around $r \sim \infty$

- Dominant dynamics

$$f'' = -f(1 - f^2 - v^2),$$
$$v' = -2\frac{vf'}{f} - q(1 - f^2 - k^2)$$

- $(f, f', v) = (\sqrt{1-k^2}, 0, -k)$ has 1 stable direction.

In the extended phase space, $\mathbb{R}^5$ $(r' = 1, \ k' = 0)$

- $W^{s,u}$ have dimension 3.
- Generically they intersect in a curve (a solution).
- We need to select k.

Counting dimensions

- We want to connect $(f, f', v, r) = (0, 0, 0, 0)$ to $(f, f', v, r) = (\sqrt{1-k^2}, 0, -k, \infty)$.

Dynamics around $r \sim 0$

- $w = f^2 vr$ and $r = e^s$.
- Dominant dynamics

$$\tilde{f}'' = n^2 \tilde{f}, \;\; \tilde{w}' = -\tilde{f}^2 q(1-k^2).$$

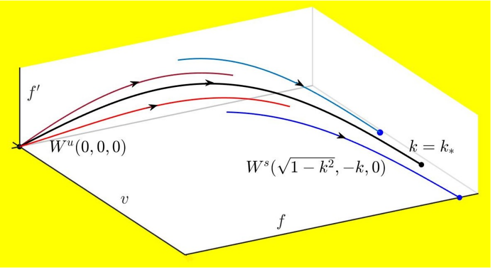
- $(f, f', v) = (0, 0, 0)$ has 1 unstable direction.

Dynamics around $r \sim \infty$

- Dominant dynamics

$$f'' = -f(1 - f^2 - v^2),$$
$$v' = -2\frac{vf'}{f} - q(1 - f^2 - k^2)$$

- $(f, f', v) = (\sqrt{1-k^2}, 0, -k)$ has 1 stable direction.



In the extended phase space, $\mathbb{R}^5$ $(r' = 1, \; k' = 0)$

- $W^{s,u}$ have dimension 3.
- Generically they intersect in a curve (a solution).
- We need to select k.

Counting dimensions

- We want to connect $(f, f', v, r) = (0,0,0,0)$ to $(f, f', v, r) = (\sqrt{1-k^2}, 0, -k, \infty)$.

Dynamics around $r \sim 0$

- $w = f^2 vr$ and $r = e^s$.
- Dominant dynamics

$$\tilde{f}'' = n^2 \tilde{f}, \ \tilde{w}' = -\tilde{f}^2 q(1-k^2).$$

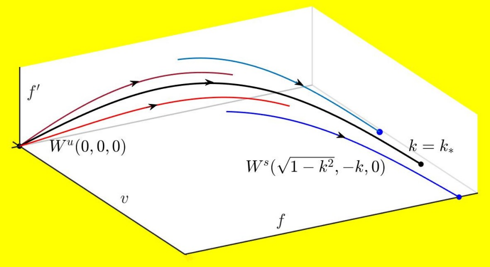
- $(f, f', v) = (0,0,0)$ has 1 unstable direction.

Dynamics around $r \sim \infty$

- Dominant dynamics

$$f'' = -f(1-f^2-v^2),$$
$$v' = -2\frac{vf'}{f} - q(1-f^2-k^2)$$

- $(f, f', v) = (\sqrt{1-k^2}, 0, -k)$ has 1 stable direction.



In the extended phase space, $\mathbb{R}^5$ $(r' = 1, \ k' = 0)$

- $W^{s,u}$ have dimension 3.
- Generically they intersect in a curve (a solution).
- We need to select k.

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r \to \infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r \to \infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r \to \infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Beyond all order phenomenon

First approach: perturbation theory with respect to $|q| \ll 1$.

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- For $q = 0$, we have that there exists f_0 such that $f_0(0) = 0$ and $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$ and computing the ODE for f_m, v_m, it turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = -f_0 k_m + o(r^{-1})$$

- Since $f_m', v_m' \to 0$ as $r \to \infty$, taking $r \to \infty$ we have that $k_m = 0$.
- The previous analysis provides that $k = k(q)$ satisfies $k(q) = \mathcal{O}(q^m)$ for any $m \geq 0$.

M. Aguareles, I.B., T.M.Seara, *On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems*, (2017).

M. Aguareles, I.B., *Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations*, (2011).

Idea of the proof (I)

- We are not able to deal directly with the existence of solutions for $r \geq 0$.
- We are forced to divide the problem between $r \in [0, r_0]$ (*inner region*) and $r \in [r_0, \infty)$ (*outer region*) with $r_0 = e^{\rho/q}$, $0 < \rho \ll 1$.
- The boundary conditions; in the *inner region* $f(0) = v(0) = 0$ and in the *outer region*

$$\lim_{r\to\infty} f(r) = \sqrt{1-k^2}, \qquad \lim_{r\to\infty} v(r) = -k$$

- These boundary conditions, does not provide uniqueness of the solution.
- Two families of solutions depending on $(\mathbf{a}, k)$ and $(\mathbf{b}, k)$.
- Remember that the ODE is of second order; f' has also to be take into account.

Idea of the proof (I)

- We are not able to deal directly with the existence of solutions for $r \geq 0$.
- We are forced to divide the problem between $r \in [0, r_0]$ (*inner region*) and $r \in [r_0, \infty)$ (*outer region*) with $r_0 = e^{\rho/q}$, $0 < \rho \ll 1$.
- The boundary conditions; in the *inner region* $f(0) = v(0) = 0$ and in the *outer region*

$$\lim_{r\to\infty} f(r) = \sqrt{1-k^2}, \qquad \lim_{r\to\infty} v(r) = -k$$

- These boundary conditions, does not provide uniqueness of the solution.

- Two families of solutions depending on $(\mathbf{a}, k)$ and $(\mathbf{b}, k)$.

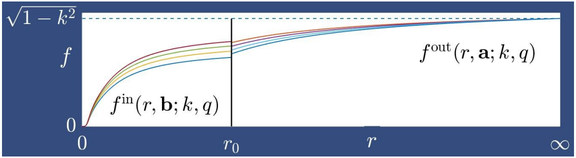
- Remember that the ODE is of second order; f' has also to be take into account.

Idea of the proof (I)

- We are not able to deal directly with the existence of solutions for $r \geq 0$.
- We are forced to divide the problem between $r \in [0, r_0]$ (*inner region*) and $r \in [r_0, \infty)$ (*outer region*) with $r_0 = e^{\rho/q}$, $0 < \rho \ll 1$.
- The boundary conditions; in the *inner region* $f(0) = v(0) = 0$ and in the *outer region*

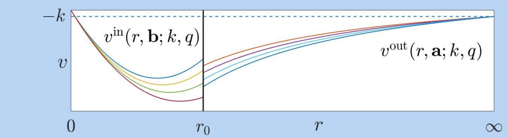
$$\lim_{r\to\infty} f(r) = \sqrt{1-k^2}, \qquad \lim_{r\to\infty} v(r) = -k$$

- These boundary conditions, does not provide uniqueness of the solution.



- Two families of solutions depending on $(\mathbf{a}, k)$ and $(\mathbf{b}, k)$.

- Remember that the ODE is of second order; f' has also to be take into account.



Idea of the proof (II)

- We match the two families in the common point $r = r_0$. Namely we impose that

$$f^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = f^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$$
$$\partial_r f^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = \partial_r f^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$$
$$v^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = v^{\mathrm{in}}(r_0, \mathbf{b}; k, q).$$

- This is a system with three unknowns $(\mathbf{a}, \mathbf{b}, k)$ and three equations (depending on q).

We start by matching $v^{\mathrm{out}}, v^{\mathrm{in}}$

- We manage to prove that for $r \sim r_0$

$$v^{\mathrm{out}}(r, \mathbf{a}; k, q) = -k\frac{K'_{inq}(kqr)}{K_{inq}(kqr)} + \cdots$$
$$= -\frac{n}{r}\tan\left(nq\log r + nq\log(kq) + \frac{\pi}{2} + nq\gamma + \cdots\right) + \cdots$$
$$v^{\mathrm{in}}(r, \mathbf{b}; k, q) = -q\frac{n^2}{r}\log r + \frac{qC_n}{r} + \cdots$$

with K_{inq} the Bessel function of second kind.

- Then we see that $v^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = v^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$ if and only if

$$k(q) = \mu\frac{1}{q}e^{-\frac{\pi}{2nq}}, \qquad \mu = 2\exp\left(-\gamma + \frac{C_n}{n^2} + \mathcal{V}(\mathbf{a}, \mathbf{b}, \mu; q)\right)$$

Idea of the proof (II)

- We match the two families in the common point $r = r_0$. Namely we impose that

$$f^{\rm out}(r_0, \mathbf{a}; k, q) = f^{\rm in}(r_0, \mathbf{b}; k, q)$$
$$\partial_r f^{\rm out}(r_0, \mathbf{a}; k, q) = \partial_r f^{\rm in}(r_0, \mathbf{b}; k, q)$$
$$v^{\rm out}(r_0, \mathbf{a}; k, q) = v^{\rm in}(r_0, \mathbf{b}; k, q).$$

- This is a system with three unknowns $(\mathbf{a}, \mathbf{b}, k)$ and three equations (depending on q).

We start by matching $v^{\rm out}, v^{\rm in}$

- We manage to prove that for $r \sim r_0$

$$v^{\rm out}(r, \mathbf{a}; k, q) = -k\frac{K'_{inq}(kqr)}{K_{inq}(kqr)} + \cdots$$
$$= -\frac{n}{r}\tan\left(nq\log r + nq\log(kq) + \frac{\pi}{2} + nq\gamma + \cdots\right) + \cdots$$
$$v^{\rm in}(r, \mathbf{b}; k, q) = -q\frac{n^2}{r}\log r + \frac{qC_n}{r} + \cdots$$

with K_{inq} the Bessel function of second kind.

- Then we see that $v^{\rm out}(r_0, \mathbf{a}; k, q) = v^{\rm in}(r_0, \mathbf{b}; k, q)$ if and only if

$$k(q) = \mu\frac{1}{q}e^{-\frac{\pi}{2nq}}, \qquad \mu = 2\exp\left(-\gamma + \frac{C_n}{n^2} + \mathcal{V}(\mathbf{a}, \mathbf{b}, \mu; q)\right)$$

Idea of the proof (II)

- We match the two families in the common point $r = r_0$. Namely we impose that

$$f^{\text{out}}(r_0, \mathbf{a}; k, q) = f^{\text{in}}(r_0, \mathbf{b}; k, q)$$
$$\partial_r f^{\text{out}}(r_0, \mathbf{a}; k, q) = \partial_r f^{\text{in}}(r_0, \mathbf{b}; k, q)$$
$$v^{\text{out}}(r_0, \mathbf{a}; k, q) = v^{\text{in}}(r_0, \mathbf{b}; k, q).$$

- This is a system with three unknowns $(\mathbf{a}, \mathbf{b}, k)$ and three equations (depending on q).

We start by matching $v^{\text{out}}, v^{\text{in}}$

- We manage to prove that for $r \sim r_0$

$$v^{\text{out}}(r, \mathbf{a}; k, q) = -k \frac{K'_{inq}(kqr)}{K_{inq}(kqr)} + \cdots$$
$$= -\frac{n}{r} \tan\left(nq \log r + nq \log(kq) + \frac{\pi}{2} + nq\gamma + \cdots\right) + \cdots$$
$$v^{\text{in}}(r, \mathbf{b}; k, q) = -q\frac{n^2}{r} \log r + \frac{qC_n}{r} + \cdots$$

with K_{inq} the Bessel function of second kind.

- Then we see that $v^{\text{out}}(r_0, \mathbf{a}; k, q) = v^{\text{in}}(r_0, \mathbf{b}; k, q)$ if and only if

$$k(q) = \mu \frac{1}{q} e^{-\frac{\pi}{2nq}}, \qquad \mu = 2\exp\left(-\gamma + \frac{C_n}{n^2} + \mathcal{V}(\mathbf{a}, \mathbf{b}, \mu; q)\right)$$

Idea of the proof (II)

- We match the two families in the common point $r = r_0$. Namely we impose that

$$f^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = f^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$$
$$\partial_r f^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = \partial_r f^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$$
$$v^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = v^{\mathrm{in}}(r_0, \mathbf{b}; k, q).$$

- This is a system with three unknowns $(\mathbf{a}, \mathbf{b}, k)$ and three equations (depending on q).

We start by matching $v^{\mathrm{out}}, v^{\mathrm{in}}$

- We manage to prove that for $r \sim r_0$

$$\begin{aligned} v^{\mathrm{out}}(r, \mathbf{a}; k, q) &= -k \frac{K'_{inq}(kqr)}{K_{inq}(kqr)} + \cdots \\ &= -\frac{n}{r} \tan\left(nq \log r + nq \log(kq) + \frac{\pi}{2} + nq\gamma + \cdots \right) + \cdots \\ v^{\mathrm{in}}(r, \mathbf{b}; k, q) &= -q \frac{n^2}{r} \log r + \frac{qC_n}{r} + \cdots \end{aligned}$$

  with K_{inq} the Bessel function of second kind.

- Then we see that $v^{\mathrm{out}}(r_0, \mathbf{a}; k, q) = v^{\mathrm{in}}(r_0, \mathbf{b}; k, q)$ if and only if

$$k(q) = \mu \frac{1}{q} e^{-\frac{\pi}{2nq}}, \qquad \mu = 2\exp\left(-\gamma + \frac{C_n}{n^2} + \mathcal{V}(\mathbf{a}, \mathbf{b}, \mu; q) \right)$$

Idea of the proof (III)

We match now $f^{\mathrm{out}}, f^{\mathrm{in}}$ and their derivatives.

- We prove that

$$f^{\mathrm{out}}(r,\mathbf{a};\mu,q) = K_0(r\sqrt{2})\mathbf{a} + \sqrt{1 - \frac{n^2}{r^2} - (v^{\mathrm{out}}(r,\mathbf{a};\mu,q)^2} + \cdots .$$

$$f^{\mathrm{in}}(r,\mathbf{b};\mu,q) = I_n(r\sqrt{2})\mathbf{b} + 1 - \frac{n^2}{2r^2} + \cdots$$

- Then, matching the solutions at $r = r_0$ we have that

$$K_0(r_0\sqrt{2})\mathbf{a} - I_n(r_0\sqrt{2})\mathbf{b} = \mathcal{F}(\mathbf{a},\mathbf{b},\mu;q)$$

$$K_0'(r_0\sqrt{2})\mathbf{a} - I_n'(r_0\sqrt{2})\mathbf{b} = \mathcal{G}(\mathbf{a},\mathbf{b},\mu;q)$$

- As a consequence we can write

$$(\mathbf{a},\mathbf{b},\mu) = \mathcal{H}(\mathbf{a},\mathbf{b},\mu;q).$$

- A thorough control of the error terms, allow us to prove the existence of a fixed point solution by the Brouwer's theorem with

$$\mu = 2\exp\left(-\gamma + \frac{C_n}{n^2}\right)(1 + \mathcal{O}(q))$$

.

Idea of the proof (III)

We match now $f^{\mathrm{out}}, f^{\mathrm{in}}$ and their derivatives.

- We prove that

$$f^{\mathrm{out}}(r, \mathbf{a}; \mu, q) = K_0(r\sqrt{2})\mathbf{a} + \sqrt{1 - \frac{n^2}{r^2} - (v^{\mathrm{out}}(r, \mathbf{a}; \mu, q)^2} + \cdots .$$

$$f^{\mathrm{in}}(r, \mathbf{b}; \mu, q) = I_n(r\sqrt{2})\mathbf{b} + 1 - \frac{n^2}{2r^2} + \cdots$$

- Then, matching the solutions at $r = r_0$ we have that

$$K_0(r_0\sqrt{2})\mathbf{a} - I_n(r_0\sqrt{2})\mathbf{b} = \mathcal{F}(\mathbf{a}, \mathbf{b}, \mu; q)$$
$$K_0'(r_0\sqrt{2})\mathbf{a} - I_n'(r_0\sqrt{2})\mathbf{b} = \mathcal{G}(\mathbf{a}, \mathbf{b}, \mu; q)$$

- As a consequence we can write

$$(\mathbf{a}, \mathbf{b}, \mu) = \mathcal{H}(\mathbf{a}, \mathbf{b}, \mu; q).$$

- A thorough control of the error terms, allow us to prove the existence of a fixed point solution by the Brouwer's theorem with

$$\mu = 2\exp\left(-\gamma + \frac{C_n}{n^2}\right)(1 + \mathcal{O}(q))$$

.

Idea of the proof (III)

We match now $f^{\mathrm{out}}, f^{\mathrm{in}}$ and their derivatives.

- We prove that

$$f^{\mathrm{out}}(r, \mathbf{a}; \mu, q) = K_0(r\sqrt{2})\mathbf{a} + \sqrt{1 - \frac{n^2}{r^2} - (v^{\mathrm{out}}(r, \mathbf{a}; \mu, q)^2} + \cdots .$$

$$f^{\mathrm{in}}(r, \mathbf{b}; \mu, q) = I_n(r\sqrt{2})\mathbf{b} + 1 - \frac{n^2}{2r^2} + \cdots$$

- Then, matching the solutions at $r = r_0$ we have that

$$K_0(r_0\sqrt{2})\mathbf{a} - I_n(r_0\sqrt{2})\mathbf{b} = \mathcal{F}(\mathbf{a}, \mathbf{b}, \mu; q)$$
$$K_0'(r_0\sqrt{2})\mathbf{a} - I_n'(r_0\sqrt{2})\mathbf{b} = \mathcal{G}(\mathbf{a}, \mathbf{b}, \mu; q)$$

- As a consequence we can write

$$(\mathbf{a}, \mathbf{b}, \mu) = \mathcal{H}(\mathbf{a}, \mathbf{b}, \mu; q).$$

- A thorough control of the error terms, allow us to prove the existence of a fixed point solution by the Brouwer's theorem with

$$\mu = 2\exp\left(-\gamma + \frac{C_n}{n^2}\right)(1 + \mathcal{O}(q))$$

.

Idea of the proof (III)

We match now $f^{\mathrm{out}}, f^{\mathrm{in}}$ and their derivatives.

- We prove that

$$f^{\mathrm{out}}(r, \mathbf{a}; \mu, q) = K_0(r\sqrt{2})\mathbf{a} + \sqrt{1 - \frac{n^2}{r^2} - (v^{\mathrm{out}}(r, \mathbf{a}; \mu, q)^2} + \cdots .$$

$$f^{\mathrm{in}}(r, \mathbf{b}; \mu, q) = I_n(r\sqrt{2})\mathbf{b} + 1 - \frac{n^2}{2r^2} + \cdots$$

- Then, matching the solutions at $r = r_0$ we have that

$$K_0(r_0\sqrt{2})\mathbf{a} - I_n(r_0\sqrt{2})\mathbf{b} = \mathcal{F}(\mathbf{a}, \mathbf{b}, \mu; q)$$
$$K_0'(r_0\sqrt{2})\mathbf{a} - I_n'(r_0\sqrt{2})\mathbf{b} = \mathcal{G}(\mathbf{a}, \mathbf{b}, \mu; q)$$

- As a consequence we can write

$$(\mathbf{a}, \mathbf{b}, \mu) = \mathcal{H}(\mathbf{a}, \mathbf{b}, \mu; q).$$

- A thorough control of the error terms, allow us to prove the existence of a fixed point solution by the Brouwer's theorem with

$$\mu = 2\exp\left(-\gamma + \frac{C_n}{n^2}\right)(1 + \mathcal{O}(q))$$

.

Thanks for your attention