SOME INSTANCES WHERE WE CAN ENCOUNTER A BEYOND ALL ORDER PHENOMENON

I. Baldomá[1][2][3]

[1]Universitat Politècnica de Catalunya (UPC)
[2]Centre de Recerca Matemàtica (CRM)
[3]Institute of Mathematics of UPC-BarcelonaTech (IMTech)

CRM International Conference on Dynamics in Systems and Synthetic Biology, 2021

OUTLINE

1. BEYOND ALL ORDER PHENOMENON
2. THE INVARIANT MANIFOLDS OF L_3
3. THE UNFOLDINGS OF THE HOPF-ZERO SINGULARITY
4. ASYMPTOTIC WAVENUMBER OF SPIRAL WAVES

BEYOND ALL ORDERS PHENOMENON

BEYOND ALL ORDERS PHENOMENON

In a family $\dot{x} = X(x, \varepsilon)$ ($\varepsilon \sim 0$) if a phenomenon can be described by a flat function $\psi(\varepsilon)$ we say that it is a *beyond all orders phenomenon (BOP)*. Namely $\psi(\varepsilon) = \mathcal{O}(|\varepsilon|^m)$ for all $m \geq 0$.

BEYOND ALL ORDERS PHENOMENON

BEYOND ALL ORDERS PHENOMENON

In a family $\dot{x} = X(x, \varepsilon)$ ($\varepsilon \sim 0$) if a phenomenon can be described by a flat function $\psi(\varepsilon)$ we say that it is a *beyond all orders phenomenon (BOP)*. Namely $\psi(\varepsilon) = \mathcal{O}(|\varepsilon|^m)$ for all $m \geq 0$.

A popular setting for BOP are **singularly perturbed** systems with two different scales:

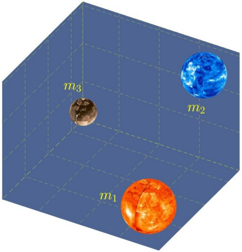
$$\frac{dx}{dt} = f(x, y, \varepsilon),\ \frac{dy}{dt} = \varepsilon g(x, y, \varepsilon), \quad \text{equivalent to } \tau = \varepsilon t \quad \varepsilon \frac{dx}{d\tau} = f(x, y, \varepsilon),\ \frac{dy}{d\tau} = g(x, y, \varepsilon),$$

- See that as $\varepsilon = 0$ we get

$$\dot{x} = f(x, y, 0),\ \dot{y} = 0, \quad \text{not equivalent to} \quad 0 = f(x, y, 0),\ y' = g(x, y, 0).$$

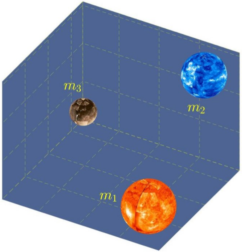
- Fenichel's geometric singular perturbation theory is a really useful tool (see *Geometric singular perturbation theory in biological practice* (2010) by Geertje Hek).

THE INVARIANT MANIFOLDS OF L_3



- We consider a configuration of the 3-body problem (RPC3BP) having a saddle-center equilibrium point called L_3 with a 1-dimensional stable and unstable manifold.
- The distance between these manifolds is exponentially small with respect to some mass parameter.
- Authors dealing with L_3 J. Font (1984), C. Simó, P. Sousa-Silva and M. Terra (2013), L. Niederman, A. Pousse and P. Robutel (2020) and E. Barrabés, J. M. Mondelo and M. Ollé (2013).

THE INVARIANT MANIFOLDS OF L_3



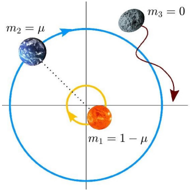
- We consider a configuration of the 3-body problem (RPC3BP) having a saddle-center equilibrium point called L_3 with a 1-dimensional stable and unstable manifold.
- The distance between these manifolds is exponentially small with respect to some mass parameter.
- Authors dealing with L_3 J. Font (1984), C. Simó, P. Sousa-Silva and M. Terra (2013), L. Niederman, A. Pousse and P. Robutel (2020) and E. Barrabés, J. M. Mondelo and M. Ollé (2013).

This is a joint work with

RESTRICTEDPLANARCIRCULAR3BP

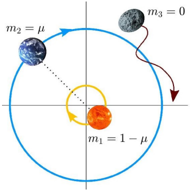
We consider:

- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.
- Changing unities: $m_1 = 1 - \mu$, $m_2 = \mu$. We assume $\mu \ll 1$.



RESTRICTEDPLANARCIRCULAR3BP

We consider:

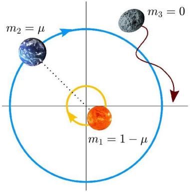


- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.
- Changing unities: $m_1 = 1 - \mu$, $m_2 = \mu$. We assume $\mu \ll 1$.
- In rotating (synodic) coordinates, the primaries are located at $(\mu, 0)$ and $(\mu - 1, 0)$ and the massless body follows a 2 degrees of freedom **autonomous** hamiltonian system.

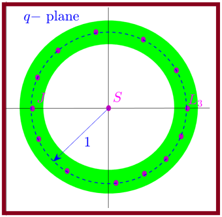
RESTRICTEDPLANARCIRCULAR3BP

We consider:

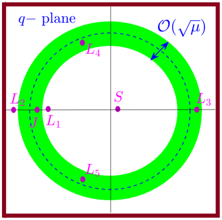
- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.
- Changing unities: $m_1 = 1 - \mu$, $m_2 = \mu$. We assume $\mu \ll 1$.



- In rotating (synodic) coordinates, the primaries are located at $(\mu, 0)$ and $(\mu - 1, 0)$ and the massless body follows a 2 degrees of freedom **autonomous** hamiltonian system.



$q \in \mathbb{R}^2$ position
$p \in \mathbb{R}^2$ momenta



$\mu = 0$. A cercle of equilibrium points

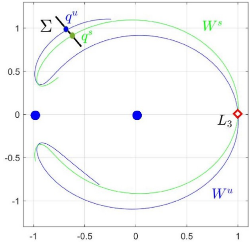
$\mu > 0$. $L_1, \cdots, L_5$ equilibrium points.

THE LAGRANGIAN POINT L_3

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no intersection (In the figure is the projection of $W^{u,s}$ on the q-plane).



THEOREM

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

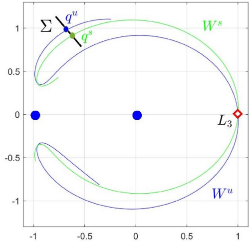
$$\|q^u-q^s\|+\|p^u-p^s\| \sim K\mu^{\frac{1}{3}}e^{-\frac{A}{\sqrt{\mu}}}.$$

THE LAGRANGIAN POINT L_3

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no intersection (In the figure is the projection of $W^{u,s}$ on the q-plane).



THEOREM

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

$$\|q^u - q^s\| + \|p^u - p^s\| \sim K\,\mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}}.$$

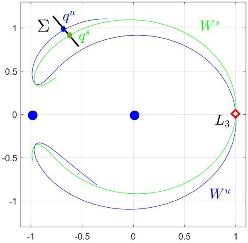
Stokes constant

THE LAGRANGIAN POINT L_3

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i+\mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no intersection (In the figure is the projection of $W^{u,s}$ on the q-plane).



THEOREM

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

$$\|q^u - q^s\| + \|p^u - p^s\| \sim K\,\mu^{\frac{1}{3}}\,e^{-\frac{A}{\sqrt{\mu}}}.$$

Stokes constant

Known constant

DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y) \sim i\frac{xy}{\sqrt{\mu}} - \frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}$$

DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda, \Lambda, x, y) \sim i\frac{xy}{\sqrt{\mu}} - \frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}$$

Fast variables

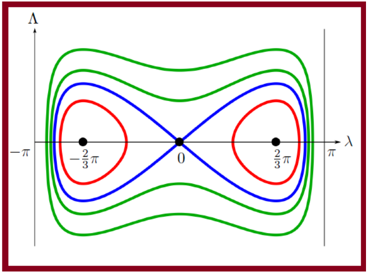
DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y) \sim i\frac{xy}{\sqrt{\mu}} - \frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}$$

Fast variables

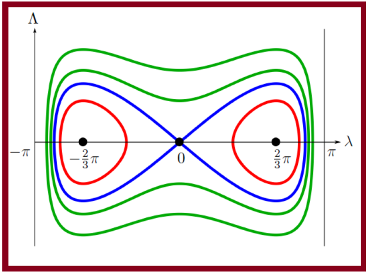
Slow variables



DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y)\sim \underbrace{i\frac{xy}{\sqrt{\mu}}}_{\text{Fast variables}}-\underbrace{\frac{3}{2}\Lambda^2+1-\cos\lambda-\frac{1}{\sqrt{2+2\cos\lambda}}}_{\text{Slow variables}}$$



- The homoclinic connection is the approximation of the invariant manifolds.

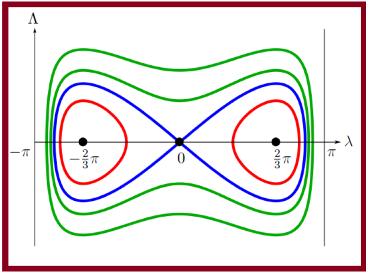
DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

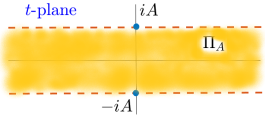
$$H(\lambda, \Lambda, x, y) \sim \boxed{i\frac{xy}{\sqrt{\mu}}} - \boxed{\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}}$$

Fast variables

Slow variables



- The homoclinic connection is the approximation of the invariant manifolds.
- The invariant manifolds can be analytically extended to Π_A.
- The difference between them is a solution of a linear homogeneous system satisfying



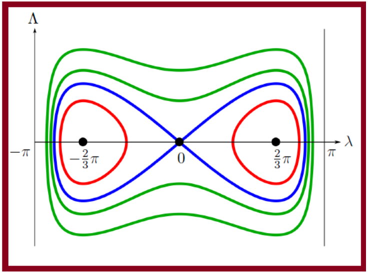
$$\dot{\Delta x} \sim \frac{i}{\sqrt{\mu}}\Delta x, \qquad \Delta x(t) \sim e^{i\frac{t}{\sqrt{\mu}}} C.$$

- Then $\Delta x(-iA) \sim e^{\frac{A}{\sqrt{\mu}}} C$ implies $C \sim e^{-\frac{A}{\sqrt{\mu}}}$.

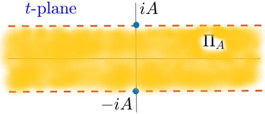
DIFFERENT SCALES

- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y) \sim \underbrace{i\frac{xy}{\sqrt{\mu}}}_{\text{Fast variables}} - \underbrace{\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}}_{\text{Slow variables}}$$



- The homoclinic connection is the approximation of the invariant manifolds.
- The invariant manifolds can be analytically extended to Π_A.
- The difference between them is a solution of a linear homogeneous system satisfying



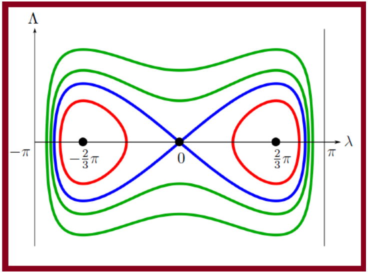
$$\dot{\Delta x} \sim \frac{i}{\sqrt{\mu}}\Delta x, \qquad \Delta x(t) \sim e^{i\frac{t}{\sqrt{\mu}}}C.$$

- Then $\Delta x(-iA) \sim e^{\frac{A}{\sqrt{\mu}}}C$ implies $C \sim e^{-\frac{A}{\sqrt{\mu}}}$.

DIFFERENT SCALES

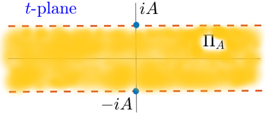
- Using Poincaré variables and singular scalings to write the system as

$$H(\lambda,\Lambda,x,y) \sim \underbrace{i\frac{xy}{\sqrt{\mu}}}_{\text{Fast variables}} - \underbrace{\frac{3}{2}\Lambda^2 + 1 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}}_{\text{Slow variables}}$$



- The homoclinic connection is the approximation of the invariant manifolds.
- The invariant manifolds can be analytically extended to Π_A.
- The difference between them is a solution of a linear homogeneous system satisfying

$$\dot{\Delta x} \sim \frac{i}{\sqrt{\mu}}\Delta x, \qquad \Delta x(t) \sim e^{i\frac{t}{\sqrt{\mu}}} C.$$



$\pm iA$ are the singularities of the homoclinic connection.

- Then $\Delta x(-iA) \sim e^{\frac{A}{\sqrt{\mu}}} C$ implies $C \sim e^{-\frac{A}{\sqrt{\mu}}}$.

HOPF-ZERO SINGULARITIES TRULY UNFOLD CHAOS

- We give sufficient and computable conditions to guarantee the occurrence of chaos in generic analytic unfoldings of some Hopf-Zero singularities.
- Authors dealing with these unfoldings: Takens, Guckenheimer, Kutnesov, Broer, Vegter, Dumortier, Simó.

HOPF-ZERO SINGULARITIES TRULY UNFOLD CHAOS

- We give sufficient and computable conditions to guarantee the occurrence of chaos in generic analytic unfoldings of some Hopf-Zero singularities.
- Authors dealing with these unfoldings: Takens, Guckenheimer, Kutnesov, Broer, Vegter, Dumortier, Simó.

This is a joint work

SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0.
$X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

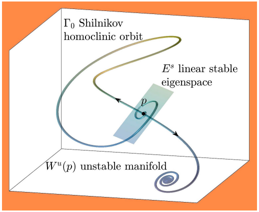
SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0.
$X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

We look for $(\mu_*, \nu_*) \sim \mathbf{0}$ such that X_{μ_*,ν_*} undergoes a Šilnikov orbit:

- X_{μ_*,ν_*} has a saddle-focus equilibrium point with eigenvalues $\lambda, -\rho \pm i\omega$ with $\lambda, \rho > 0$.
- X_{μ_*,ν_*} has an homoclinic orbit $\Gamma_0 \subset W^u(p)$.
- If $\lambda - \rho > 0$ and $(\mu, \nu) \sim (\mu_*, \nu_*)$, $X_{\mu,\nu}$ is chaotic.



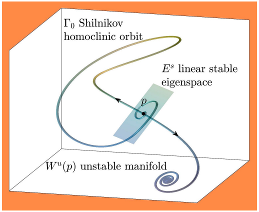
SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0. $X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

We look for $(\mu_*, \nu_*) \sim \mathbf{0}$ such that X_{μ_*,ν_*} undergoes a Šilnikov orbit:

- X_{μ_*,ν_*} has a saddle-focus equilibrium point with eigenvalues $\lambda, -\rho \pm i\omega$ with $\lambda, \rho > 0$.
- X_{μ_*,ν_*} has an homoclinic orbit $\Gamma_0 \subset W^u(p)$.
- If $\lambda - \rho > 0$ and $(\mu, \nu) \sim (\mu_*, \nu_*)$, $X_{\mu,\nu}$ is chaotic.



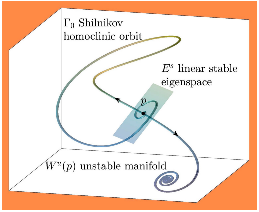
SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0.
$X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

We look for $(\mu_*, \nu_*) \sim \mathbf{0}$ such that X_{μ_*,ν_*} undergoes a Šilnikov orbit:

- X_{μ_*,ν_*} has a saddle-focus equilibrium point with eigenvalues $\lambda, -\rho \pm i\omega$ with $\lambda, \rho > 0$.
- X_{μ_*,ν_*} has an homoclinic orbit $\Gamma_0 \subset W^u(p)$.
- If $\lambda - \rho > 0$ and $(\mu, \nu) \sim (\mu_*, \nu_*)$, $X_{\mu,\nu}$ is chaotic.



- $X_{\mu,\nu} = X^k_{\mu,\nu} + \mathcal{O}(\|x, \mu, \nu\|^{k+1})$ with $X^k_{\mu,\nu}$, the truncation of the normal form up to order k. For $k = 2$

$$\dot{z} = -\mu + z^2 + br^2, \quad \dot{r} = r(\nu - az + z^2), \quad \dot{\theta} = \alpha.$$

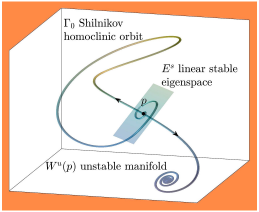
SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0.
$X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

We look for $(\mu_*, \nu_*) \sim \mathbf{0}$ such that X_{μ_*,ν_*} undergoes a Šilnikov orbit:

- X_{μ_*,ν_*} has a saddle-focus equilibrium point with eigenvalues $\lambda, -\rho \pm i\omega$ with $\lambda, \rho > 0$.
- X_{μ_*,ν_*} has an homoclinic orbit $\Gamma_0 \subset W^u(p)$.
- If $\lambda - \rho > 0$ and $(\mu, \nu) \sim (\mu_*, \nu_*)$, $X_{\mu,\nu}$ is chaotic.



- $X_{\mu,\nu} = X^k_{\mu,\nu} + \mathcal{O}(\|x, \mu, \nu\|^{k+1})$ with $X^k_{\mu,\nu}$, the truncation of the normal form up to order k. For $k = 2$
$$\dot{z} = -\mu + z^2 + br^2, \quad \dot{r} = r(\nu - az + z^2), \quad \dot{\theta} = \alpha.$$
- In this case, for $\mu > 0$, the equilibrium points are $S^2_\pm = (0, 0, \pm\sqrt{\mu})$ with corresponding two scales eigenvalues $\sim \pm 2\sqrt{\mu}$, $\nu \mp a\sqrt{\mu} \pm i\alpha$.

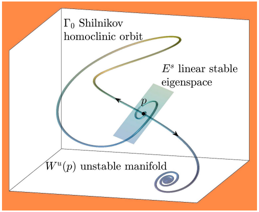
SETTING

UNFOLDINGS

Families $X_{\mu,\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $X_{0,0}(\mathbf{0}) = \mathbf{0}$ and $DX_{0,0}(\mathbf{0})$ has eigenvalues $\pm i\alpha$, 0.
$X_{0,0}$ is called Hopf-zero singularity and $X_{\mu,\nu}$ is called unfolding.

We look for $(\mu_*, \nu_*) \sim \mathbf{0}$ such that X_{μ_*,ν_*} undergoes a Šilnikov orbit:

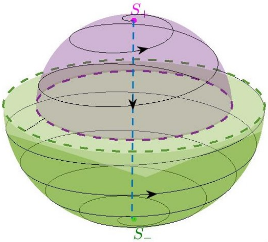
- X_{μ_*,ν_*} has a saddle-focus equilibrium point with eigenvalues $\lambda, -\rho \pm i\omega$ with $\lambda, \rho > 0$.
- X_{μ_*,ν_*} has an homoclinic orbit $\Gamma_0 \subset W^u(p)$.
- If $\lambda - \rho > 0$ and $(\mu, \nu) \sim (\mu_*, \nu_*)$, $X_{\mu,\nu}$ is chaotic.



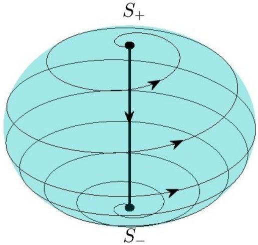
- $X_{\mu,\nu} = X^k_{\mu,\nu} + \mathcal{O}(\|x, \mu, \nu\|^{k+1})$ with $X^k_{\mu,\nu}$, the truncation of the normal form up to order k. For $k = 2$
$$\dot{z} = -\mu + z^2 + br^2, \quad \dot{r} = r(\nu - az + z^2), \quad \dot{\theta} = \alpha.$$
- In this case, for $\mu > 0$, the equilibrium points are $S^2_\pm = (0, 0, \pm\sqrt{\mu})$ with corresponding two scales eigenvalues $\sim \pm 2\sqrt{\mu}$, $\nu \mp a\sqrt{\mu} \pm i\alpha$.
- We want $S^2_\pm$ to be saddle-focus equilibrium points, so we assume the open conditions $\mu > 0$, $0 < a < 2$, $b > 0$, $|\nu| < a\sqrt{\mu}$.

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

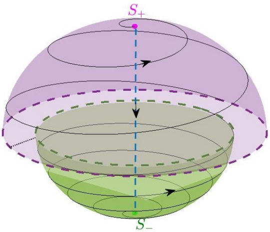
The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$



$\nu < \nu_k(\mu)$



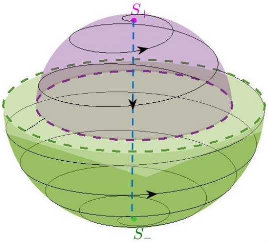
$\nu = \nu_k(\mu)$



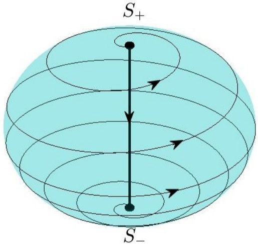
$\nu > \nu_k(\mu)$

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

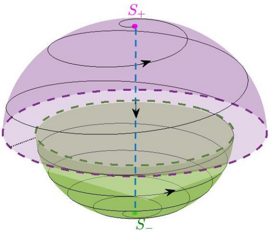
The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$



$\nu < \nu_k(\mu)$



$\nu = \nu_k(\mu)$

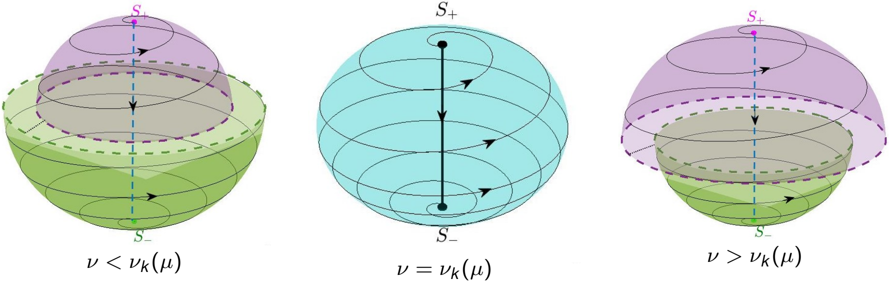


$\nu > \nu_k(\mu)$

- One dimensional heteroclinic connection, $k \geq 2$

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

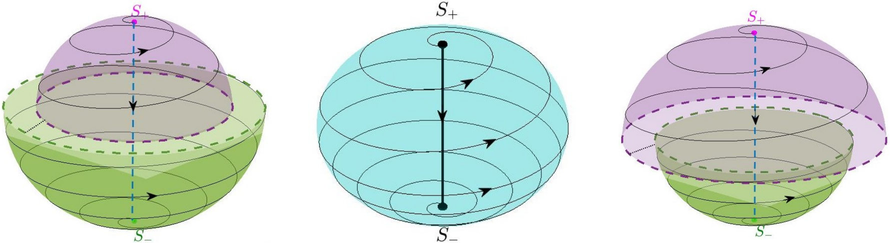
The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$



- One dimensional heteroclinic connection, $k \geq 2$.
- The two invariant manifolds either coincide or do not intersect.

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$

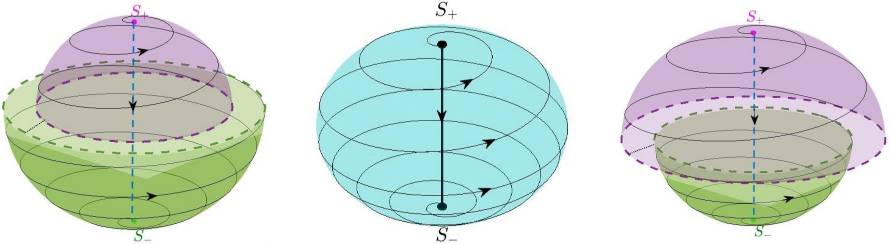


$\nu < \nu_k(\mu)$ $\nu = \nu_k(\mu)$ $\nu > \nu_k(\mu)$

- One dimensional heteroclinic connection, $k \geq 2$.
- The two invariant manifolds either coincide or do not intersect.
- $X^k_{\mu,\nu}$ has no Šilnikov orbit.

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$

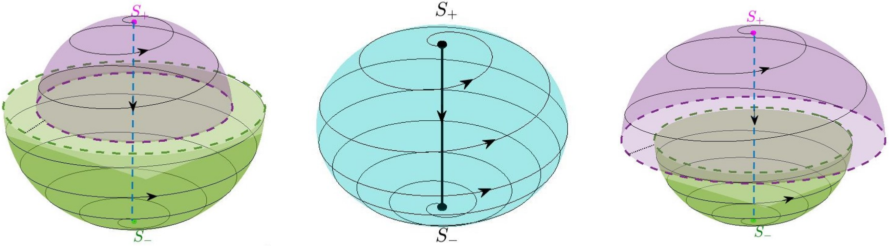


$\nu < \nu_k(\mu)$ $\nu = \nu_k(\mu)$ $\nu > \nu_k(\mu)$

- One dimensional heteroclinic connection, $k \geq 2$.
- The two invariant manifolds either coincide or do not intersect.
- $X^k_{\mu,\nu}$ has no Šilnikov orbit.
- The one dimensional heteroclinic connection has to be destroyed when the full $X_{\mu,\nu}$ is considered.

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$

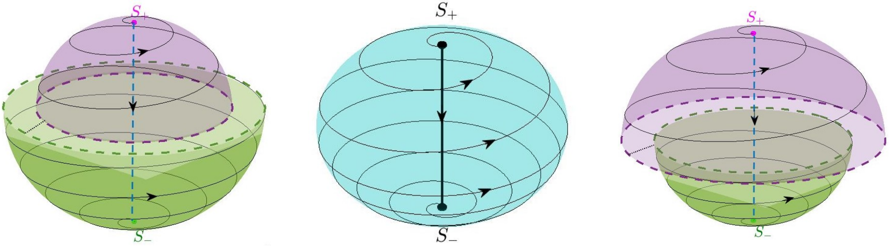


$\nu < \nu_k(\mu)$ $\nu = \nu_k(\mu)$ $\nu > \nu_k(\mu)$

- One dimensional heteroclinic connection, $k \geq 2$.
- The two invariant manifolds either coincide or do not intersect.
- $X^k_{\mu,\nu}$ has no Šilnikov orbit.
- The one dimensional heteroclinic connection has to be destroyed when the full $X_{\mu,\nu}$ is considered.

NORMAL FORM AND BEYOND ALL ORDER PHENOMENON

The normal form $X^k_{\mu,\nu}$ is $\dot{z} = Z^k(r^2, z)$, $\dot{r} = rR^k(r^2, z)$, $\dot{\theta} = \alpha$

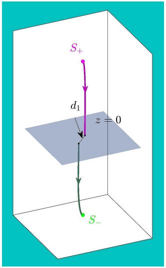


$\nu < \nu_k(\mu)$ $\nu = \nu_k(\mu)$ $\nu > \nu_k(\mu)$

- One dimensional heteroclinic connection, $k \geq 2$.
- The two invariant manifolds either coincide or do not intersect.
- $X^k_{\mu,\nu}$ has no Šilnikov orbit.
- The one dimensional heteroclinic connection has to be destroyed when the full $X_{\mu,\nu}$ is considered.

Since $S_\pm, \nu = \mathcal{O}(\sqrt{\mu})$, then $X_{\mu,\nu} - X^k_{\mu,\nu} = \mathcal{O}((\sqrt{\mu})^{k+1})$. The breakdown of the one dimensional heteroclinic connection has to be $\mathcal{O}((\sqrt{\mu})^k)$ for any k.

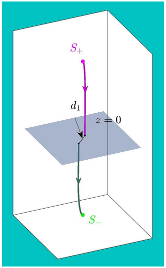
QUANTITATIVE RESULTS



$$d_1(\mu,\nu) \sim K\mu^{(-1+a)/2}e^{-\frac{\alpha\pi}{2\sqrt{\mu}}}.$$

With K a Stokes constant which satisfies, generically, $K \neq 0$.

QUANTITATIVE RESULTS



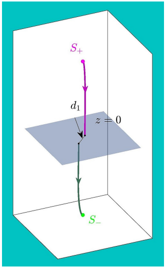
$$d_1(\mu,\nu) \sim K\mu^{(-1+a)/2} e^{-\frac{\alpha\pi}{2\sqrt{\mu}}}.$$

With K a Stokes constant which satisfies, generically, $K \neq 0$.

THEOREM (ŠILNIKOV HOMOCLINIC ORBITS)

For $0 < a < 2$ and $K \neq 0$, there is a curve $\gamma = \{\nu = \nu(\mu)\}$ such that $X_{\mu,\nu(\mu)}$ has a Šilnikov homoclinic orbit.

QUANTITATIVE RESULTS



$$d_1(\mu,\nu) \sim K\mu^{(-1+a)/2} e^{-\frac{\alpha\pi}{2\sqrt{\mu}}}.$$

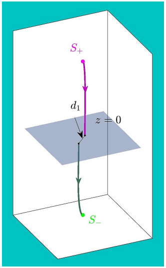
With K a Stokes constant which satisfies, generically, $K \neq 0$.

THEOREM (ŠILNIKOV HOMOCLINIC ORBITS)

For $0 < a < 2$ and $K \neq 0$, there is a curve $\gamma = \{\nu = \nu(\mu)\}$ such that $X_{\mu,\nu(\mu)}$ has a Šilnikov homoclinic orbit.

- To prove the existence of Šilnikov orbits from the formula for $d_1(\mu,\nu)$ we use classical arguments mainly Bolzano and inclination lemma.

QUANTITATIVE RESULTS



$$d_1(\mu,\nu) \sim K\mu^{(-1+a)/2} e^{-\frac{\alpha\pi}{2\sqrt{\mu}}}.$$

With K a Stokes constant which satisfies, generically, $K \neq 0$.

THEOREM (ŠILNIKOV HOMOCLINIC ORBITS)

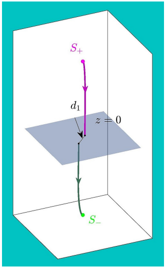
For $0 < a < 2$ and $K \neq 0$, there is a curve $\gamma = \{\nu = \nu(\mu)\}$ such that $X_{\mu,\nu(\mu)}$ has a Šilnikov homoclinic orbit.

- To prove the existence of Šilnikov orbits from the formula for $d_1(\mu,\nu)$ we use classical arguments mainly Bolzano and inclination lemma.
- We can also prove that the distance between the two invariant manifolds is

$$d_2(\theta,\mu,\nu) \sim \overline{d}_2(\mu,\nu) + \mu^{(-2-2a^{-1})/2} e^{-\frac{\pi\alpha}{2a\sqrt{\mu}}} \left[C_1 \cos(\theta - c\log\mu) + C_2 \sin(\theta - c\log\mu)\right]$$

where $\overline{d}_2(\mu,\nu) \sim c_1\mu + c_2\nu$, $c_1, c_2 \neq 0$.

QUANTITATIVE RESULTS



$$d_1(\mu,\nu) \sim K\mu^{(-1+a)/2} e^{-\frac{\alpha\pi}{2\sqrt{\mu}}}.$$

With K a Stokes constant which satisfies, generically, $K \neq 0$.

THEOREM (ŠILNIKOV HOMOCLINIC ORBITS)

For $0 < a < 2$ and $K \neq 0$, there is a curve $\gamma = \{\nu = \nu(\mu)\}$ such that $X_{\mu,\nu(\mu)}$ has a Šilnikov homoclinic orbit.

- To prove the existence of Šilnikov orbits from the formula for $d_1(\mu,\nu)$ we use classical arguments mainly Bolzano and inclination lemma.
- We can also prove that the distance between the two invariant manifolds is

$$d_2(\theta,\mu,\nu) \sim \overline{d}_2(\mu,\nu) + \mu^{(-2-2a^{-1})/2} e^{-\frac{\pi\alpha}{2a\sqrt{\mu}}} [C_1 \cos(\theta - c\log\mu) + C_2 \sin(\theta - c\log\mu)]$$

where $\overline{d}_2(\mu,\nu) \sim c_1\mu + c_2\nu$, $c_1, c_2 \neq 0$.

- Using this formula we can deal also with the volume preserving case ($\nu = 0$) and to obtain a better knowledge of the curve $\gamma = \{\nu = \nu(\mu)\}$.

ASYMPTOTIC WAVENUMBER OF SPIRAL WAVES

- We consider a class of reaction-diffusion systems
- We prove that these systems have rotating spiral waves only if some quantity (*the asymptotic wavenumber*) is exponentially small with respect to some parameter (*the twist parameter*)
- These systems has been studied by many authors, Koppel, Hagan, Greenberg, Coen, Neu, Rosales, Howards, Fife, Chapman, Paullet, Ermentrout, Troy, etc. Different techniques have be used (Fenichel's theory, asymptotic methods, numerical methods).

ASYMPTOTIC WAVENUMBER OF SPIRAL WAVES

- We consider a class of reaction-diffusion systems
- We prove that these systems have rotating spiral waves only if some quantity (*the asymptotic wavenumber*) is exponentially small with respect to some parameter (*the twist parameter*)
- These systems has been studied by many authors, Koppel, Hagan, Greenberg, Coen, Neu, Rosales, Howards, Fife, Chapman, Paullet, Ermentrout, Troy, etc. Different techniques have be used (Fenichel's theory, asymptotic methods, numerical methods).

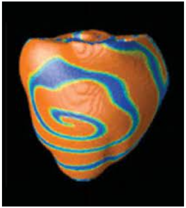
A joint work with

SPIRAL PATTERNS

Spiral patterns are commonly observed in certain chemical, biological and physical systems

Belousov-Zhabotinskii reaction

Social amoebas Dictyostelium discoideium



Cardiac muscle tissue

- These systems are governed by chemical or biological reaction and spatial diffusion.

$$\partial_\tau U = D\Delta U + F(U,a), \qquad D \text{ a diffusion matrix, } F \text{ the reaction nonlinearity}$$

$U = U(\tau, x, y) \in \mathbb{R}^2$ and a is a parameter (for instance some catalyst concentration).

SPIRAL WAVES

We focus on the Ginzburg-Landau systems

$$\begin{aligned} u_t &= \Delta u + \lambda\left(\sqrt{u^2+v^2}\right)u - \omega\left(\sqrt{u^2+v^2}\right)w, \\ w_t &= \Delta w + \omega\left(\sqrt{u^2+v^2}\right)u + \lambda\left(\sqrt{u^2+v^2}\right)w, \end{aligned}$$

with $\lambda(z) = 1 - z^2$, $\omega(z) = \omega_0 + qz^2$ and q the *small* twist parameter.

SPIRAL WAVES

We focus on the Ginzburg-Landau systems

$$\begin{aligned} u_t &= \Delta u + \lambda\big(\sqrt{u^2+v^2}\big)u - \omega\big(\sqrt{u^2+v^2}\big)w, \\ w_t &= \Delta w + \omega\big(\sqrt{u^2+v^2}\big)u + \lambda\big(\sqrt{u^2+v^2}\big)w, \end{aligned}$$

with $\lambda(z) = 1 - z^2$, $\omega(z) = \omega_0 + qz^2$ and q the *small* twist parameter.

- The first order of a reaction-diffusion equation near a Hopf bifurcation.

SPIRAL WAVES

We focus on the Ginzburg-Landau systems

$$\begin{aligned} u_t &= \Delta u + \lambda\left(\sqrt{u^2+v^2}\right)u - \omega\left(\sqrt{u^2+v^2}\right)w, \\ w_t &= \Delta w + \omega\left(\sqrt{u^2+v^2}\right)u + \lambda\left(\sqrt{u^2+v^2}\right)w, \end{aligned}$$

with $\lambda(z) = 1 - z^2$, $\omega(z) = \omega_0 + qz^2$ and q the *small* twist parameter.

- The first order of a reaction-diffusion equation near a Hopf bifurcation.

ROTATING SPIRAL WAVES

Are $\mathcal{C}^2$ solutions of the form $U(t, r, \theta) = (u(t, r, \theta), w(t, r, \theta)) = f(r)\exp\left(i\left[\Omega t + n\theta - \chi(r)\right]\right)$.

SPIRAL WAVES

We focus on the Ginzburg-Landau systems

$$\begin{aligned} u_t &= \Delta u + \lambda(\sqrt{u^2+v^2})u - \omega(\sqrt{u^2+v^2})w, \\ w_t &= \Delta w + \omega(\sqrt{u^2+v^2})u + \lambda(\sqrt{u^2+v^2})w, \end{aligned}$$

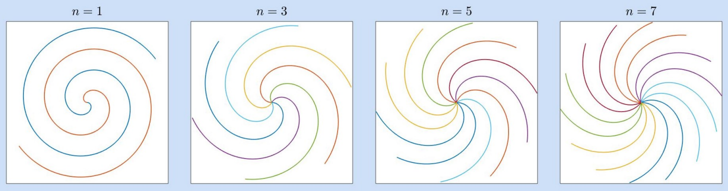
with $\lambda(z) = 1 - z^2$, $\omega(z) = \omega_0 + qz^2$ and q the *small* twist parameter.

- The first order of a reaction-diffusion equation near a Hopf bifurcation.

ROTATING SPIRAL WAVES

Are $\mathcal{C}^2$ solutions of the form $U(t,r,\theta) = (u(t,r,\theta), w(t,r,\theta)) = f(r)\exp\left(i\left[\Omega t + n\theta - \chi(r)\right]\right)$.

- In the rotating framework, $(\tilde{u},\tilde{w})(r,\theta) = e^{-it\Omega}U(t,r,\theta)$ we encounter the spiral patterns by setting $\tilde{u} = ctt$ or $\tilde{w} = ctt$.



FROM PDE TO ODE. BOUNDARY CONDITIONS

- We define the *asymptotic wavenumber* k as $q(1-k^2)=\Omega-\omega_0$.
- We forget PDE because $f(r)$ and $v(r)=\chi'(r)$ has to satisfy

$$\begin{aligned} f''+\frac{f'}{r}-f\frac{n^2}{r^2}+f(1-f^2-v^2)=0, \\ v'+\frac{v}{r}+2\frac{vf'}{f}+q(1-k^2-f^2)=0. \end{aligned}$$

BOUNDARY CONDITIONS

To guarantee that the solutions $f(r)e^{i(n\theta-\chi(r))}$ are $\mathcal{C}^2$ and archimedian spirals:

$$f(0)=v(0)=0, \qquad \exists \lim_{r\to\infty} f(r), \lim_{r\to\infty} v(r)$$

$f(r)>0$, $r>0$, and $v(r)$ has constant sign.

- These are too many restrictions to a third order system of ODE. This suggests that there exists a selection mechanism for k.

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots , \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r\to\infty} f_0(r) = 1$.

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots , \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r \to \infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$, compute the ODE for f_m, v_m. It turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = (c_m(r) - f_0 k_m)$$

with $c_m(r) = o(r^{-1})$ as $r \to \infty$ a known function (depending $f_0, \cdots , f_{m-1}, v_0, \cdots , v_{m-1}$)

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$, compute the ODE for f_m, v_m. It turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = (c_m(r) - f_0 k_m)$$

with $c_m(r) = o(r^{-1})$ as $r \to \infty$ a known function (depending $f_0, \cdots, f_{m-1}, v_0, \cdots, v_{m-1}$)

- We have that

$$v_m(r) = \frac{1}{r f_0^2(r)} \int_0^r \xi f_0(\xi)(c_m(\xi) - f_0(\xi) k_m)\, d\xi.$$

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$, compute the ODE for f_m, v_m. It turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = (c_m(r) - f_0 k_m)$$

with $c_m(r) = o(r^{-1})$ as $r \to \infty$ a known function (depending $f_0, \cdots, f_{m-1}, v_0, \cdots, v_{m-1}$)

- We have that

$$v_m(r) = \frac{1}{r f_0^2(r)} \int_0^r \xi f_0(\xi)(c_m(\xi) - f_0(\xi) k_m)\, d\xi.$$

- Since v_m has to be bounded as $r \to \infty$, $k_m = 0$.

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$, compute the ODE for f_m, v_m. It turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = (c_m(r) - f_0 k_m)$$

with $c_m(r) = o(r^{-1})$ as $r \to \infty$ a known function (depending $f_0, \cdots, f_{m-1}, v_0, \cdots, v_{m-1}$)

- We have that

$$v_m(r) = \frac{1}{r f_0^2(r)} \int_0^r \xi f_0(\xi)(c_m(\xi) - f_0(\xi) k_m)\, d\xi.$$

- Since v_m has to be bounded as $r \to \infty$, $k_m = 0$.

BEYOND ALL ORDER PHENOMENON

- By symmetry write $k(q) = k_0 + q^2 k_1 + q^4 k_2 + \cdots$

$$f(r) = f_0(r) + q^2 f_1(r) + q^4 f_2(r) \cdots, \qquad v(r) = q(v_0(r) + q^2 v_1(r) + q^4 v_2(r) + \cdots).$$

- Only imposing that $f_0(0) = 0$ and it is bounded, $\lim_{r\to\infty} f_0(r) = 1$.
- Equating orders $\mathcal{O}(q^m)$, compute the ODE for f_m, v_m. It turns out that, for $m \geq 1$

$$v_m' + \frac{v_m}{r} + 2\frac{v_m f_0'}{f_0} = (c_m(r) - f_0 k_m)$$

with $c_m(r) = o(r^{-1})$ as $r \to \infty$ a known function (depending $f_0, \cdots, f_{m-1}, v_0, \cdots, v_{m-1}$)

- We have that

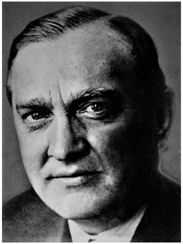
$$v_m(r) = \frac{1}{r f_0^2(r)} \int_0^r \xi f_0(\xi)(c_m(\xi) - f_0(\xi) k_m)\, d\xi.$$

- Since v_m has to be bounded as $r \to \infty$, $k_m = 0$.

THEOREM

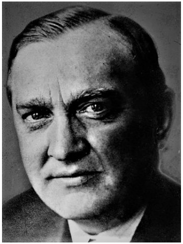
We rigorously prove that $k(q) \sim \frac{A}{q} e^{-\frac{\pi}{2nq}}$ with A a constant that only depends on f_0.

THANKS!



For its Fixed Point Theorem in Banach spaces which is the core of our proofs.

THANKS!



For its Fixed Point Theorem in Banach spaces which is the core of our proofs.

and to you...