Coorbital chaotic and homoclinic phenomena in the Restricted 3 Body Problem

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Outline

The restricted Planar Circular 3BP

Exponentially small breakdown

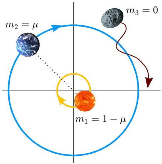
Homoclinic connections around L_3

Chaos around L_3

RestrictedPlanarCircular3BP

We consider:

- **Planar**: the motion takes place into a plane.
- **Restricted**: one body is massless, i.e. $m_3 = 0$.
- **Circular**: the two bodies with mass (primaries) move in a circular motion of the same period T.

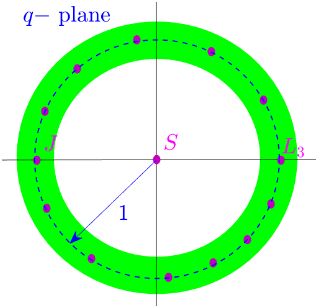


- In rotating (synodic) coordinates, the primaries are located at $(\mu, 0)$ and $(\mu - 1, 0)$ and the massless body follows a 2 degrees of freedom **autonomous** hamiltonian system.

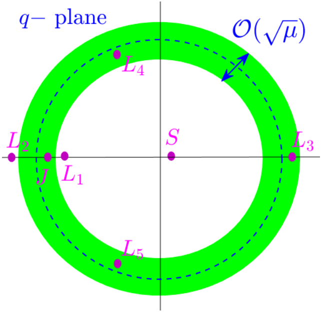
$$\frac{\|p\|^2}{2} - q^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1-\mu}{\|q-(\mu,0)\|} - \frac{\mu}{\|q-(\mu-1,0)\|}.$$

- We assume a perturbative setting, $0 < \mu \ll 1$.
- Notice that when $\mu = 0$, the third body follows a two body problem

μ as a singular parameter



$\mu = 0$. A cercle of equilibrium points



$\mu > 0$. $L_1, \cdots, L_5$ equilibrium points.

- The Lagrangian points belong to the mean motion resonance 1 : 1.

Mean Motion resonance

The mean motion resonance 1 : 1 is a region of the phase space close to the motions of the third body having the same period as the primaries. They can lead to inestabilities (diffusion) [Féjoz, Guardia, Kaloshin, Roldan, 2016]

Set up

- L_1, L_2, L_3 are center-saddle (unstable) and L_4, L_5 are stable if μ is small.
- A lot of attention has been paid to L_1, L_2 (astrodynamics interest) and L_4, L_5 have been studied for instance because there are objects orbiting around them: the trojans and greek asteroids.
- L_3 is located "at the other side" of the massive body. It has received less attention for the scientific community.

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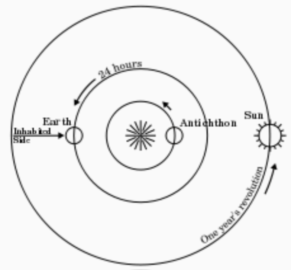

- Pythagorean philosopher Philolaus believed in the existence of a Counter-Earth or Antichthon
- An invisible planet similar to the Earth in the opposite side of a 'fire ball'

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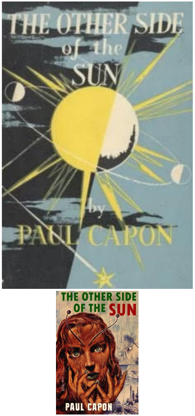
- Also L_3 is a well known location for *science fiction lovers*



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- Recently the mathematical community is starting to paid attention to L_3
  - The center-stable and center unstable manifolds, act as boundaries of stability domains, C. Simó, P. Sousa-Silva, M. Terra, 2013
  - Horseshoe shaped orbits: quasi-periodic orbits encompassing L_3, L_4 and L_5 (models co-orbital satellites): L. Niederman, A. Pousse, P. Robutel, J. Cors, J. Palacián, P. Yanguas (2019-2020). The most famous are Janus and Epimetheus and near Earth asteroids.
  - Transfer orbits from the small primary to L_3 or between primaries: Tantardini, Fantino, Ren, Pergola,G. Gómez, J. Masdemont, A. Jorba, B. Nicolás (2010-2020).
  - Existence of multiround homoclinic orbits, E. Barrabés, J.M. Mondelo, M. Ollé (2009)
- Our work relies on the study of the invariant manifolds of L_3 and of the Lyapunov orbits to it.

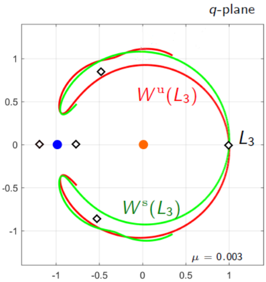
Exponentially small breakdown

- L_3 is of saddle-center type having eigenvalues with two scales when $\mu > 0$ is small:

$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i + \mathcal{O}(\mu).$$

- It has one dimensional stable and unstable manifolds, $W^{u,s}$ which either coincide or have no transversal intersection (in the figure, the projection of $W^{u,s}$ on the q-plane, the phase space is $\mathbb{R}^4$).

- **First goal:** To measure the distance between these invariant manifolds at first crossing.



Theorem

Take a section Σ as in the figure and let $(q^{u,s}, p^{u,s})$ be the intersection of $W^{u,s}(L_3)$ with Σ. When μ small enough:

$$\|q^u - q^s\| + \|p^u - p^s\| \sim K\,\mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}}.$$

Stokes constant

Known constant

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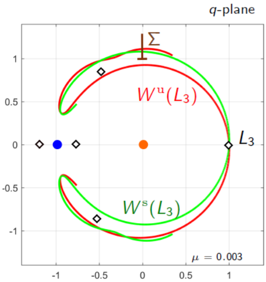

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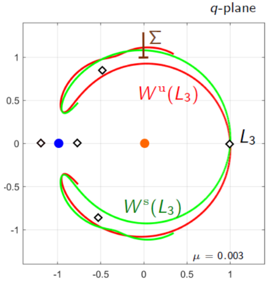

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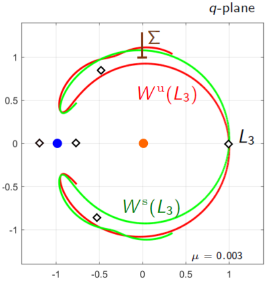

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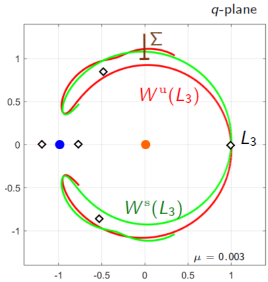

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Comments

- The motion takes place far from collision.
- The constant A has an explicit expression

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}dx \sim 0.177744$$

  it is related with a *hidden* homoclinic connection. First computed by J. Font.
- K has a different nature and it corresponds a Stokes constant, depending on the full jet of the hamiltonian by means of the so called *inner equation*.
- The hamiltonian H has no closed expression, but it can be studied by means of power series in the excentricity. However, the *inner equation* is explicit:

$$\begin{aligned}\mathcal{H}(U,W,X,Y) =& 1 + \frac{4}{9}U^{-\frac{2}{3}}W^2 - \frac{16}{27}U^{-\frac{4}{3}}W + \frac{16}{81}U^{-2} + \frac{4i}{3}U^{-\frac{2}{3}}(X-Y) \\ & - \frac{4}{9}U^{-1}W(X+Y) + \frac{8}{27}U^{-\frac{5}{3}}(X+Y) - \frac{1}{3}U^{-\frac{4}{3}}(X^2+Y^2) \\ & + \frac{10}{9}U^{-\frac{4}{3}}XY.\end{aligned}$$

- Even when the Stokes constant K is transcendental and has no explicit expression, using computer assisted proof techniques we prove that $K \neq 0$ (join work with M. Capinsky).

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- First order. We use Poincaré variables and singular scalings to transform the system $H(\lambda,\Lambda,x,y) = H_0(\lambda,\Lambda,x,y) + o(1)$ with

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Fast variables　　Slow variables

- The time parameterization of the homoclinic connection of H_0 has singularities at $\pm iA$.
- There are parameterizations of $W^{u,s}(L_3)$ in domains $\sqrt{\mu}$- close to $\pm iA$ and related with special solutions of the *inner equation* (matching complex techniques).
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Sketch of the proof

- First order. We use Poincaré variables and singular scalings to transform the system $H(\lambda, \Lambda, x, y) = H_0(\lambda, \Lambda, x, y) + o(1)$ with

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Dynamical consequences

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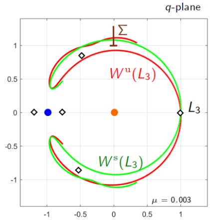
$$\mathrm{dist}(W^{s,+} \cap \Sigma, W^{u,+} \cap \Sigma) \sim K \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}}$$



- Notice that at this point we have not proven transversal intersections, but, at least, we know that there are no primary homoclinic connections.
- But we can prove the existence of
  - Two round homoclinic connection for a sequence, $(\mu_n)_n$, $\mu_n \to 0$.
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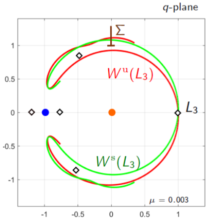

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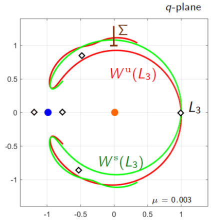

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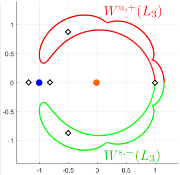
Homoclinic phenomena around L_3

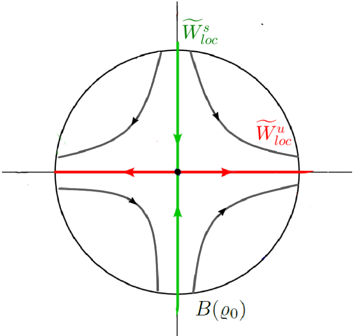
It was conjectured by E. Barrabés, M. Ollé and J.M. Mondelo (2009) that there exists a sequence of mass rations $\mu_n \to 0$ such that there exist secondary homoclinic connections.

Theorem

The RPC3BP has a 2-round homoclinc connection to L_3 between $W^{u,+}$ and $W^{s,-}$, if $K \neq 0$, for a sequence of the form

$$\mu_n = \frac{A}{n\pi}\sqrt{\frac{8}{21}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad n \gg 1$$





- Uniform normal form in a neighbourhood of the fixed point. The result is provided by a work of T. Jezequel, P. Bernard, and E. Lombardi, 2016.
- The new system is almost linear and uncoupled.
- In the picture the saddle (slow) variables.
- The fast variables travel with a velocity of $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$.

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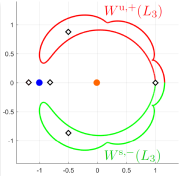

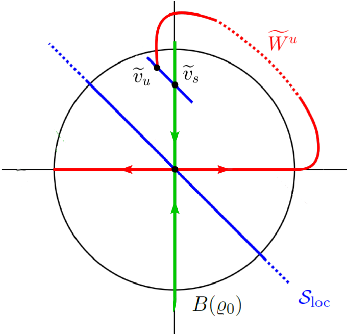

- The original system has a symmetry plane
- Choose a transversal section close enough of the equilibrium point and translate the section and the symmetry plane to the new normal form variables.
- The intersection of the stable manifold is easy to control.
- Our result asserts that the unstable manifold intersect with the transversal section
- and provides also the coordinates of its intersection.

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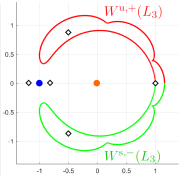

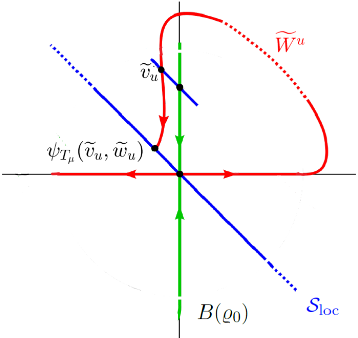

- The local system is almost uncoupled and linear, the time T_μ we need to hit the projection of the simmetry plane in the saddle plane is $T_\mu \sim \frac{1}{\mu}$.
- The fast variables pf $\psi_{T_\mu}(\widetilde{v}_u, \widetilde{w}_u)$ are approximately

$$R(\mu)\left(\cos\left(\alpha - \frac{c}{\mu}\right), \sin\left(\alpha - \frac{c}{\mu}\right)\right)$$

- They hits the symmetry axis when $\alpha - \frac{c}{\mu} = n\pi$.

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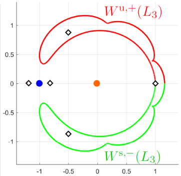

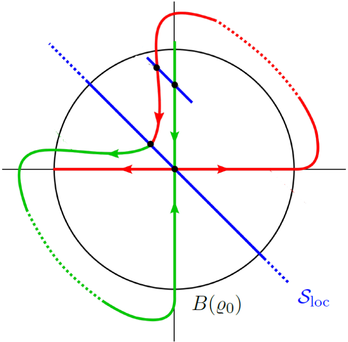

- By symmetry we are done!

Chaotic coorbital motions

The next result assures the existence of chaotic motions around L_3 and its manifolds

Theorem

There exist μ_0, E_0 *and functions* $E_{\min}, E_{\max} : (0, \mu_0) \to [0, E_0]$

$$E_{\min}(\mu) - h(L_3),\ E_{\max}(\mu) - h(L_3) \sim K^2 \mu^{2/3} e^{-\frac{2A}{\sqrt{\mu}}}$$

such that if $\mu \in (0, \mu_0)$ *and* $E \in (E_{\min}(\mu), E_{\max}(\mu)]$ *there exists a periodic Lyapunov orbit belonging to* $\{h(p, q; \mu) = E\}$*, exponentially close to* L_3*, having 2-dimensional stable and unstable manifolds that intersect transversaly.*

- We prove the existence of Lyapunov orbits in this fast-slow system.
- These orbits have two dimensional stable and unstable manifolds living in a 3 dimensional domain.
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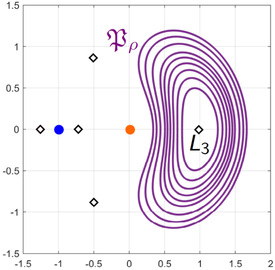

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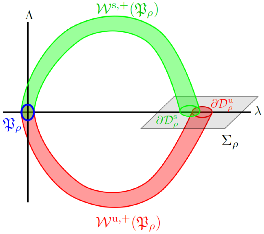

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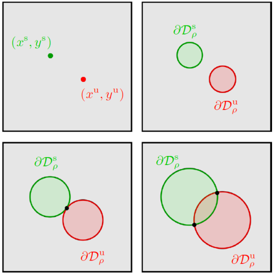

Homoclinic tangencies

Theorem

Let f_E *be the flow of our hamiltonian* h*, restricted to* $h = E$*.*
For a fixed $\mu \in (0, \mu_0)$ *and* E *close to* $E_{\min}(\mu)$*, the flow* $f_{E,\mu}$ *unfolds a quadratic homoclinic tangency between the stable and unstable manifolds of the Lyapunov periodic orbit lying in* $h = E_{\min}(\mu)$*.*
Then,

- *there exists a sequence of Newhouse intervals* H_k*,*

  $$\lim_{k\to\infty} H_k = E_{\min}(\mu),$$

  namely if $h \in H_k$*, the flow has a wild hyperbolic basic set over* H_k *(persistence of the generically unfolding homoclinic tangencies).*
- *For a residual subset of* H_k*, the homoclinic transversal points are accumulated by generic elliptic periodic orbits*

Thank you for your attention