



BREAKDOWN OF HOMOCLINIC ORBITS TO L_3 : NONVANISHING OF THE STOKES CONSTANT

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ABSTRACT. The Restricted Planar Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform circular orbits coplanar with that of the massless body. In rotating coordinates, it can be modelled by a two degrees of freedom Hamiltonian system, which has five critical points called the Lagrange points. Among them, the point L_3 is a saddle-center which is collinear with the primaries and beyond the largest of the two. The papers [3, 4] provide an asymptotic formula for the distance between the one dimensional stable and unstable manifolds of L_3 in a transverse section for small values of the mass ratio $0 < \mu \ll 1$. This distance is exponentially small with respect to μ and its first order depends on what is usually called a Stokes constant. The non-vanishing of this constant implies that the distance between the invariant manifolds at the section is not zero. In this paper, we prove that the Stokes constant is non-zero. The proof is computer assisted.

1. Introduction and main result. The Restricted Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform a circular motion. If one also assumes that the massless body moves on the same plane as the primaries one has the Restricted Planar Circular 3-Body Problem (RPC3BP).

Let us name the two primaries S (star) and P (planet) and normalize their masses so that $m_S = 1 - \mu$ and $m_P = \mu$, with $\mu \in (0, \frac{1}{2}]$. In a rotating coordinate system, the positions of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$. Then,

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the position and momenta of the third body, $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the autonomous Hamiltonian

$$H(q, p; \mu) = \frac{\|p\|^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1-\mu)}{\|q - (\mu, 0)\|} - \frac{\mu}{\|q - (\mu-1, 0)\|}. \quad (1)$$

For $\mu > 0$, it is a well known fact that (1) has five critical points, usually called Lagrange points (see Figure 1). The three collinear Lagrange points, L_1 , L_2 and L_3 , are of center-saddle type whereas, for small μ , the triangular ones, L_4 and L_5 , are of center-center type (see, for instance, [17]).

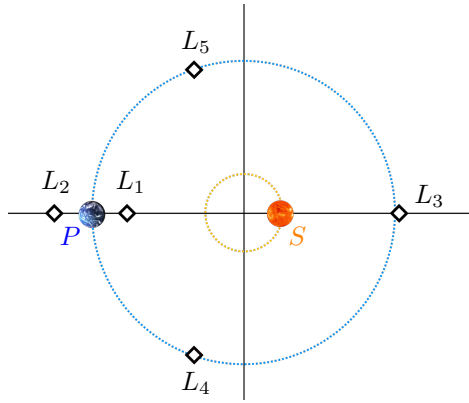


FIGURE 1. Projection onto the q -plane of the Lagrange equilibrium points for the RPC3BP on rotating coordinates.

The invariant manifolds of the (unstable) Lagrange points are of fundamental importance for understanding the dynamics of the RPC3BP. In particular, those of the point L_3 (more precisely its center-stable and center-unstable invariant manifolds) act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [8, 16]). They also allow to create transfer orbits from the small primary to L_3 in the RPC3BP (see [9, 18]) or between primaries in the Bicircular 4-Body Problem (see [10, 11]).

In understanding how the invariant manifolds of L_3 structure the global dynamics, it is fundamental to know whether they coincide or not. The purpose of the papers [3, 4] and the present one is to prove that these invariant manifolds do not coincide the first time they hit a suitable transverse section. This is the content of the Theorems 1.1 and 1.2 below. This fact has several dynamical implications, which are proven in the paper [5] and are explained in Remarks 1.3 and 1.4 below.

The manifolds $W^u(L_3)$ and $W^s(L_3)$ lie in the so called 1 : 1 mean motion resonance and have two branches each. One pair, which we denote by $W^{u,+}(L_3)$ and $W^{s,+}(L_3)$, circumvents L_5 whereas the other circumvents L_4 and it is denoted as $W^{u,-}(L_3)$ and $W^{s,-}(L_3)$, see Figure 2. These branches are symmetric with respect to

$$\Psi(q, p) = (q_1, -q_2, -p_1, p_2).$$

Thus, to compute the distance between the manifolds, one can restrict the study to the first ones, $W^{u,+}(L_3)$ and $W^{s,+}(L_3)$. We measure this distance in symplectic

polar coordinates, defined as

$$q = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad p = R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} - \frac{G}{r} \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix},$$

where r is the distance of the third body from the origin, θ its argument, R is the radial linear momentum and G is the angular momentum.

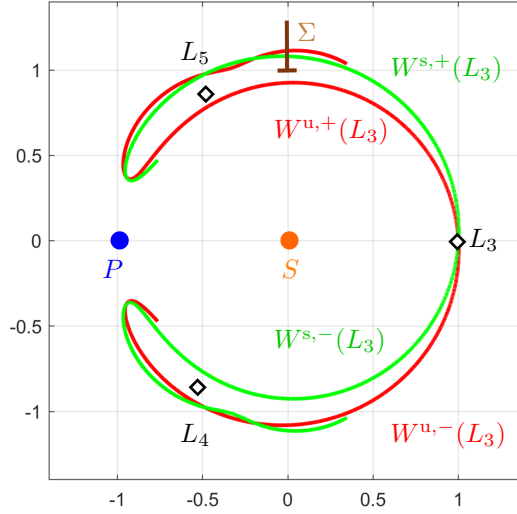


FIGURE 2. Projection onto the q -plane of the unstable (red) and stable (green) manifolds of L_3 , for $\mu = 0.0028$.

We consider as well the 3-dimensional section

$$\Sigma = \left\{ (r, \theta, R, G) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 : r > 1, \theta = \frac{\pi}{2} \right\}$$

and denote by $(r_*^u, \frac{\pi}{2}, R_*^u, G_*^u)$ and $(r_*^s, \frac{\pi}{2}, R_*^s, G_*^s)$ the first crossing of the invariant manifolds with this section (see Figure 2). The next theorem, proven in [3, 4], measures the distance between these points for $0 < \mu \ll 1$.

Theorem 1.1. *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$\|(r_*^u, R_*^u, G_*^u) - (r_*^s, R_*^s, G_*^s)\| = \sqrt[3]{4} \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|}\right) \right],$$

where

- The constant $A > 0$ is given by the real-valued integral

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx \approx 0.177744. \quad (2)$$

- The constant $\Theta \in \mathbb{C}$ is the Stokes constant associated to the inner equation analyzed in [3, Theorem 2.7] (see also Theorem 2.2).

Theorem 1.1 provides a first order for the distance between the invariant manifolds of L_3 , at the first crossing with Σ , provided the Stokes constant Θ is not zero. The main result of the present paper is the following.

Theorem 1.2. *The constant $\Theta \in \mathbb{C}$ introduced in Theorem 1.1 satisfies*

$$\Theta \neq 0.$$

The proof of this theorem relies on recent techniques developed by some of the authors in [2]. Note however, that the models considered in that paper are rather simple unfoldings of the Hopf-zero singularity whereas here we deal with a Celestial Mechanics model, which makes the analysis considerably more delicate.

Remark 1.3. Theorem 1.1 and Theorem 1.2 imply that the unstable and stable manifolds of L_3 do not meet at their first encounter with the section Σ . This is expected as, in generic systems, homoclinic connections are likely to breakdown when subjected to small perturbations. However, this does not prevent the existence of homoclinic connections. Indeed, in [5], by relying on the results in the present paper and [3, 4], some of the authors of this paper construct homoclinic orbits to L_3 for a sequence of mass ratios μ tending to zero. These homoclinic orbits are two round, that is cross the section Σ twice.

Remark 1.4. A fundamental question in dynamical systems is to prove that a system exhibits chaotic dynamics. One of the classical ways to prove this it is to show the existence of transverse intersections between the stable and unstable manifolds of a particular invariant object (see [7]). In the case of L_3 , its stable and unstable manifolds are 1-dimensional and, therefore, they have too small dimension to allow for transverse intersections. However, in [5], we are able to prove that the splitting of the invariant manifolds of L_3 imply that there exist transverse intersections between the stable and unstable manifolds of certain periodic Lyapunov orbits close to L_3 . This proves the existence of chaotic dynamics close to L_3 and its invariant manifolds. Moreover, we are also able to prove that some of these Lyapunov periodic orbits possess quadratic homoclinic tangencies, which imply the existence of Newhouse domains in the RPC3BP.

The paper is organized as follows. In Section 2 we summarize the main steps to prove Theorem 1.1, performed in the previous works [3, 4]; we present the inner equation, which is independent on the small parameter μ , and, finally, we describe the relation between the Stokes constant Θ and suitable solutions, $Z^{u,s}$ of the inner equation (see Theorem 2.2) below. The (short) Section 3 is devoted to explain the strategy to prove Theorem 1.2 which consists in two main steps: characterization of the complex domains where $Z^{u,s}$ are defined (Theorem 3.1) and analysis of its difference (Theorem 3.2). Then, in Section 4, relying on the approach developed in [2], we prove Theorem 3.1. In Section 5 we give the proof of Theorem 3.2, which in part is computer assisted¹.

2. The invariant manifolds of L_3 and the inner equation. Theorem 1.1 falls into what is usually called exponentially small splitting of separatrices. That is, on the perturbative analysis of the distance between the stable and unstable manifold of an invariant object when it is exponentially small. Its proof follows the original approach proposed by Lazutkin in his seminal work on the Standard Map [14]. Note that the papers [3, 4] are the first ones where such approach has been implemented in a Celestial Mechanics model. We refer to [3] for a detailed list of references on the exponentially small splitting of separatrices phenomenon.

¹The code for the computer assisted part of the proof is available on the personal web page of M. J. Capiński.

Let us summarize the main steps of this proof and explain where the Stokes constant arises.

Note first that, in the limit problem H in (1) with $\mu = 0$, the five Lagrange points and the associated invariant manifolds “collapse” into the circle of (degenerate) critical points $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$. Therefore, to analyze the invariant manifolds, it is convenient to perform a singular change of coordinates to obtain a “new first order” Hamiltonian which has a saddle-center equilibrium point with stable and unstable manifolds that coincide along a separatrix. This change of coordinates boils down to a suitable (singular with respect to μ) scaling of the classical Poincaré planar elements (see [15]). These coordinates are explained in full detail in [3].

In these coordinates, the proof of Theorem 1.1 relies on the following steps.

- A. We perform the aforementioned change of coordinates which captures the slow-fast dynamics of the system. The new Hamiltonian becomes a (fast) oscillator weakly coupled to a 1-degree of freedom Hamiltonian with a saddle point and a separatrix associated to it.
- B. We analyze the analytical continuation of a time-parametrization of the separatrix. In particular, we obtain its maximal strip of analyticity, which is given by $|\operatorname{Im} t| < A$ where A is the constant introduced in (2) and t the time of the parametrization. We also describe the character and location of the complex singularities at the boundaries of this region.
- C. We derive the inner equation, which gives the first order of the original system close to the singularities of the separatrix described in Step B. This equation is independent of the perturbative parameter μ .
- D. We study two special solutions of the inner equation which are approximations of the perturbed invariant manifolds near the singularities. Moreover, we provide an asymptotic formula for the difference between these two solutions of the inner equation. This difference is given in terms of the Stokes constant Θ introduced in Theorem 1.1.
- E. We prove the existence of the analytic continuation of suitable parametrizations of $W^{u,+}(L_3)$ and $W^{s,+}(L_3)$ in appropriate complex domains (and as graphs). These domains contain a segment of the real line and intersect a neighborhood sufficiently close to the singularities of the separatrix.
- F. By using complex matching techniques, we compare the solutions of the inner equation with the graph parametrizations of the perturbed invariant manifolds.
- G. Finally, we prove that the dominant term of the difference between manifolds is given by the term obtained from the difference of the solutions of the inner equation.

Steps A, B, C and D are performed in [3] whereas the Steps E, F and G are performed in [4]. In particular, [3] showed that the constant $\Theta \in \mathbb{C}$ exists but no proof of its non-vanishing is provided.

To prove that the Stokes constant Θ is not zero, we have to perform a deeper analysis of the two special solutions of the inner equation mentioned in Step D. To this end, we first introduce the so-called inner Hamiltonian, the computation of which (Step C) is explained in full detail in [3], given by

$$\mathcal{H}(U, W, X, Y) = W + XY + \mathcal{K}(U, W, X, Y), \quad (3)$$

with

$$\mathcal{K}(U, W, X, Y) = -\frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}} \left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1 \right) \quad (4)$$

and

$$\begin{aligned} \mathcal{J}(U, W, X, Y) = & \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U} \left(W - \frac{2}{3U^{\frac{2}{3}}} \right) \\ & - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2+Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}, \end{aligned} \quad (5)$$

and the symplectic form

$$\Omega = dU \wedge dW + idX \wedge dY.$$

Let us now summarize the main features of the analysis of the inner equation performed previously in [3] (see Theorem 2.2 below). Following the approach presented in [6] (see also [1]), we look for two suitable solutions of the Hamiltonian system associated to \mathcal{H} , analyzing their orbits as graphs over U . That is, we do not analyze the trajectories of \mathcal{H} directly. To this end, we introduce $Z = (W, X, Y)$ and the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

With this notation, the equation associated to the Hamiltonian \mathcal{H} can be written as

$$\begin{cases} \dot{U} = 1 + g(U, Z), \\ \dot{Z} = \mathcal{A}Z + f(U, Z), \end{cases} \quad (6)$$

where $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g = \partial_W \mathcal{K}$. Therefore, to look for solutions of this equation parametrized as graphs with respect to U , we search functions

$$Z^\diamond(U) = (W^\diamond(U), X^\diamond(U), Y^\diamond(U))^T, \quad \text{for } \diamond = \text{u, s},$$

satisfying the invariance condition given by (6), that is

$$\partial_U Z^\diamond = \mathcal{A}Z^\diamond + \mathcal{R}[Z^\diamond], \quad \text{for } \diamond = \text{u, s}, \quad (7)$$

where

$$\mathcal{R}[\varphi](U) = \frac{f(U, \varphi) - g(U, \varphi)\mathcal{A}\varphi}{1 + g(U, \varphi)}. \quad (8)$$

The special solutions Z^\diamond we are interested in, satisfy the asymptotic conditions

$$\lim_{\text{Re } U \rightarrow -\infty} Z^{\text{u}}(U) = 0, \quad \lim_{\text{Re } U \rightarrow +\infty} Z^{\text{s}}(U) = 0. \quad (9)$$

In fact, for a fixed $\gamma \in (0, \frac{\pi}{2})$, we look for functions Z^{u} and Z^{s} satisfying (7), (9) and defined in the domains

$$\mathcal{D}_\kappa^{\text{u}} = \left\{ U \in \mathbb{C} : |\text{Im } U| \geq \tan \gamma \text{Re } U + \frac{\kappa}{\cos \gamma}, \text{Re } U \leq 0 \right\}, \quad \mathcal{D}_\kappa^{\text{s}} = -\mathcal{D}_\kappa^{\text{u}}, \quad (10)$$

respectively, for some $\kappa > 0$ big enough (see Figure 3).

As a consequence, the difference $\Delta Z = Z^{\text{u}} - Z^{\text{s}}$ can be analyzed in the overlapping domain

$$\mathcal{E}_\kappa = \{U \in \mathbb{C} : \text{Im } U \leq -\rho(\kappa, \gamma), \text{Re } U = 0\},$$

where

$$\rho(\kappa, \gamma) = \frac{\kappa}{\cos \gamma}. \quad (11)$$

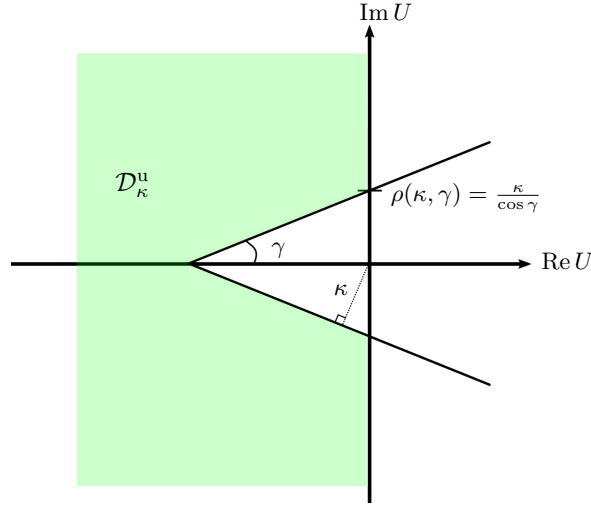


FIGURE 3. The inner domain, \mathcal{D}_κ^u , for the unstable case (see (10)).

Remark 2.1. The domains used in [3] are bigger than the ones consider in the present work and they have an open overlapping domain. We have chosen these smaller domains, \mathcal{D}_κ^u and \mathcal{D}_κ^s . The resason is that, on the one hand, we will see that it is enough to analyze the difference $Z^u - Z^s$ on $\mathcal{E}_\kappa = \mathcal{D}_\kappa^u \cap \mathcal{D}_\kappa^s$ (contained in the imaginary axis) and, on the other, to restrict the analysis to smaller domains, makes the explicit computation of all the constants that appear in our analysis easier.

The following result is proven in [3].

Theorem 2.2. *There exist $\kappa_0, b_1, b_2 > 0$ such that for any $\kappa \geq \kappa_0$, equation (7) has analytic solutions $Z^\diamond(U) = (W^\diamond(U), X^\diamond(U), Y^\diamond(U))^T$, for $U \in \mathcal{D}_\kappa^\diamond$, $\diamond = u, s$, satisfying*

$$|U^{\frac{8}{3}}W^\diamond(U)| \leq b_1, \quad |U^{\frac{4}{3}}X^\diamond(U)| \leq b_2, \quad |U^{\frac{4}{3}}Y^\diamond(U)| \leq b_2.$$

In addition, there exist $\Theta \in \mathbb{C}$ and $b_3 > 0$ independent of κ , and a function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that

$$\Delta Z(U) = Z^u(U) - Z^s(U) = \Theta e^{-iU} \left((0, 0, 1)^T + \chi(U) \right) \quad (12)$$

and, for $U \in \mathcal{E}_\kappa$,

$$|U^{\frac{7}{3}}\chi_1(U)| \leq b_3, \quad |U^2\chi_2(U)| \leq b_3, \quad |U\chi_3(U)| \leq b_3.$$

Moreover, $\Theta \neq 0$ if and only if $\Delta Z \neq 0$.

Remark 2.3. In the paper [3], the authors compute numerically the constant Θ which is approximately $\Theta \approx 1.63$. To compute it, it suffices to take into account that $\Theta = \lim_{\text{Im } U \rightarrow -\infty} \Delta Y(U)e^{iU}$. Then, we consider

$$\Theta_\rho = |\Delta Y(-i\rho)| e^\rho,$$

which, for ρ big enough, satisfies $\Theta_\rho \approx |\Theta|$.

Notice that, as we already claimed in Remark 2.1, we only need to analyze the difference on \mathcal{E}_κ , contained in the imaginary axis.

Theorem 2.2 is proven in [3] in two steps. First, for $\diamond = u, s$, one proves the existence of the functions Z^\diamond in domains $\mathcal{D}_\kappa^\diamond$ with κ large enough. This is achieved through a fixed point argument of Perron type. Once the existence of the two functions in a common domain is proved, one looks for an equation for its difference that is used to derive the asymptotic formula (12).

Next section specify the concrete steps for proving Theorem 1.2.

3. Strategy to prove the main result. To prove Theorem 1.2, we follow the strategy developed in [2] to prove that the Stokes constant associated to unfoldings of the Hopf-zero singularity does not vanish. However, note that in [2] the strategy is tested on rather “simple” unfoldings of the singularity. On the contrary, in the present paper we deal with a given model of Celestial Mechanics, which requires more accurate estimates.

The first step to prove Theorem 1.2 is to provide a more quantitative version of Theorem 2.2. It provides a larger domain of definition of the functions Z^\diamond given by Theorem 2.2 and relies on carrying out a more detailed analysis of the two solutions Z^\diamond for $\diamond = u, s$.

Theorem 3.1. *The functions $Z^\diamond(U) = (W^\diamond(U), X^\diamond(U), Y^\diamond(U))^T$, $\diamond = u, s$, introduced in Theorem 2.2 are defined in $\mathcal{D}_{\kappa^*}^\diamond$ with (see (11))*

$$\kappa^* = 6.24, \quad \gamma = \frac{1}{2} \quad \text{and} \quad \rho^* = \rho(\kappa^*, \gamma) = \frac{\kappa^*}{\cos \gamma} < \rho_0 = 7.12. \quad (13)$$

The proof of this theorem is deferred to Section 4. It follows the approach developed in [2] for the inner equation associated to the Hopf-zero singularity.

The second step we perform to prove Theorem 1.2 is to analyze the difference ΔZ (see (12)). This is provided by the next theorem, whose proof is computer assisted and is deferred to Section 5.

Theorem 3.2. *The function $\Delta Z(U) = Z^u(U) - Z^s(U)$ introduced in (12) satisfies*

$$\Delta Z(-i\rho) \neq 0, \quad (14)$$

for a $\rho > \rho_0$, where ρ_0 is the constant introduced in (13).

By Theorem 2.2, $\Delta Z(-i\rho) \neq 0$ implies that

$$\Theta \neq 0$$

and this completes the proof of Theorem 1.2.

4. The domain of the solutions of the inner equation. The proof of Theorem 3.1 relies on a fixed point argument, and follows the same lines as the proof of the first part of Theorem 2.2 in [3]. The main difference between the two proofs is that now we need explicit estimates for the fixed point argument. Moreover, they have to be rather accurate so that we obtain the κ^* given by Theorem 3.1. Note that a larger κ^* would lead to a larger ρ^* and ρ_0 (see (13)), which would make harder to prove Theorem 3.2, since the difference ΔZ is exponentially small with respect to ρ_0 .

We denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3)^T$, unless stated otherwise. Moreover, we deal only with the analysis for Z^u . The analysis for Z^s is analogous and leads to exactly the same estimates.

4.1. The fixed point equation and the functional setting. The invariance equation (7) can be written as $\mathcal{L}Z^u = \mathcal{R}[Z^u]$ where \mathcal{L} is the linear operator

$$\mathcal{L}\varphi = (\partial_U - \mathcal{A})\varphi.$$

To construct a fixed point equation from (7), we consider the following left inverse operator of \mathcal{L} ,

$$\begin{aligned} \mathcal{G}[\varphi](U) &= \left(\int_{-\infty}^0 \varphi_1(s+U)ds, \int_{-\infty}^0 e^{-is} \varphi_2(s+U)ds, \int_{-\infty}^0 e^{is} \varphi_3(s+U)ds \right)^T, \end{aligned} \quad (15)$$

(see Lemma 4.4 below). We look for a fixed point of the operator

$$\mathcal{F} = \mathcal{G} \circ \mathcal{R}, \quad (16)$$

in a suitable Banach space.

Given $\nu \in \mathbb{R}$ and $\kappa > 0$, we define the norm

$$\|\varphi\|_\nu = \sup_{U \in \mathcal{D}_\kappa^u} |U^\nu \varphi(U)|,$$

where the domain \mathcal{D}_κ^u is given in (10), and we introduce the Banach space

$$\mathcal{X}_\nu^u = \{\varphi : \mathcal{D}_\kappa^u \rightarrow \mathbb{C} : \varphi \text{ analytic, } \|\varphi\|_\nu < +\infty\}. \quad (17)$$

A solution of $Z^u = \mathcal{F}[Z^u]$ belonging to $\mathcal{X}_\eta^u \times \mathcal{X}_\nu^u \times \mathcal{X}_\nu^u$ with $\eta, \nu > 0$ satisfies equation (7) and the asymptotic condition (9). Then, to prove Theorem 3.1, we look for a fixed point of the operator \mathcal{F} in the Banach space

$$\mathcal{X}_\times^u = \mathcal{X}_{\frac{8}{3}}^u \times \mathcal{X}_{\frac{4}{3}}^u \times \mathcal{X}_{\frac{4}{3}}^u, \quad (18)$$

endowed with the norm

$$\|\varphi\|_\times = \max \left\{ \|\varphi_1\|_{\frac{8}{3}}, \|\varphi_2\|_{\frac{4}{3}}, \|\varphi_3\|_{\frac{4}{3}} \right\}. \quad (19)$$

Theorem 3.1 is a direct consequence of the following proposition.

Proposition 4.1. *For any $\kappa \geq \kappa^*$ where κ^* is the constant introduced in (13), the fixed point equation $Z^u = \mathcal{F}[Z^u]$ has a solution $Z^u \in \mathcal{X}_\times^u$.*

We devote the rest of this section to prove Proposition 4.1. Note that our goal is to compute explicit estimates throughout the proof to obtain a “not too large” κ^* .

Let us explain the main steps of the proof, which are carried out in the forthcoming sections.

- In Section 4.2, we provide properties of the Banach spaces introduced in (17) and give explicit bounds for the norm of the linear operator \mathcal{G} in (15).
- In Section 4.3, we give estimates for $\mathcal{F}[0]$.
- In Section 4.4, we provide estimates for the derivatives of \mathcal{R} (see (8)) in a suitable domain.
- In Section 4.5, we provide explicit expressions for the Lipschitz constant of the operator \mathcal{F} in a suitable region of the Banach space (18). Finally, we prove that the Lipschitz constant is smaller than 1, and therefore \mathcal{F} is contractive and has a unique fixed point.

To have accurate estimates in these steps, we rely on explicit formulae for the non-linear operator \mathcal{R} in (8) and its first and second derivatives. They are given in Appendix A.

Note that in [3], we were not able to prove that \mathcal{F} is contractive. Instead, we considered a slightly modified operator which had the same fixed points as \mathcal{F} and a smaller Lipschitz constant. Since the estimates in the present paper are more accurate, we are able to prove that the original operator \mathcal{F} is contractive.

4.2. Banach space properties and the integral operator. Throughout this and the forthcoming sections we use without mentioning the following remark and lemma.

Remark 4.2. Let $U \in \mathcal{D}_\kappa^u$, then $|U| \geq \kappa$.

Next lemma, proven in [1], gives some properties of the Banach spaces \mathcal{X}_η^u introduced in (17).

Lemma 4.3. Let $\kappa > 0$ and $\nu, \eta \in \mathbb{R}$. The following statements hold:

1. If $\nu > \eta$, then $\mathcal{X}_\nu^u \subset \mathcal{X}_\eta^u$ and $\|\varphi\|_\eta \leq \kappa^{\eta-\nu} \|\varphi\|_\nu$.
2. If $\varphi \in \mathcal{X}_\nu^u$ and $\zeta \in \mathcal{X}_\eta^u$, then the product $\varphi\zeta \in \mathcal{X}_{\nu+\eta}^u$ and $\|\varphi\zeta\|_{\nu+\eta} \leq \|\varphi\|_\nu \|\zeta\|_\eta$.

Now we provide estimates for the integral operator introduced in (15).

Lemma 4.4. Consider the linear operator $\mathcal{G}[\varphi] = (\mathcal{G}_1[\varphi_1], \mathcal{G}_2[\varphi_2], \mathcal{G}_3[\varphi_3])^T$ as defined in (15) and fix $\eta > 1$, $\nu > 0$ and $\kappa \geq 1$. Then, $\mathcal{G} : \mathcal{X}_\eta^u \times \mathcal{X}_\nu^u \times \mathcal{X}_\nu^u \rightarrow \mathcal{X}_{\eta-1}^u \times \mathcal{X}_\nu^u \times \mathcal{X}_\nu^u$ is a continuous linear operator and is a left-inverse of \mathcal{L} .

Moreover,

1. For $j = 1, 2, 3$ and $\eta > 1$, the linear operator $\mathcal{G}_j : \mathcal{X}_\eta^u \rightarrow \mathcal{X}_{\eta-1}^u$ is continuous and satisfies that, for $\varphi \in \mathcal{X}_\eta^u$,

$$\|\mathcal{G}_j[\varphi]\|_{\eta-1} \leq G_\eta \|\varphi\|_\eta, \quad G_\eta = \frac{\sqrt{\pi} \Gamma\left(\frac{\eta-1}{2}\right)}{2\Gamma\left(\frac{\eta}{2}\right)}.$$

2. For $j = 2, 3$ and $\nu > 0$, the linear operator $\mathcal{G}_j : \mathcal{X}_\nu^u \rightarrow \mathcal{X}_\nu^u$ is continuous and, for all $0 < \sigma \leq \gamma$ (see (10)) and $\varphi \in \mathcal{X}_\nu^u$, satisfies that

$$\|\mathcal{G}_j[\varphi]\|_\nu \leq \frac{\|\varphi\|_\nu}{\sin \sigma (\cos \sigma)^\nu}.$$

The proof of this lemma follows the same lines of the proof of [2, Lemma 2.1]. We first state the following lemma.

Lemma 4.5. Let $\eta > 1$. Then,

$$\int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{\eta}{2}}} \leq \frac{\sqrt{\pi} \Gamma\left(\frac{\eta-1}{2}\right)}{2\Gamma\left(\frac{\eta}{2}\right)}.$$

Proof. Considering the change of coordinates $s = \tan x$,

$$\int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{\eta}{2}}} \leq \int_{-\frac{\pi}{2}}^0 \frac{dx}{(1+\tan(x)^2)^{\frac{\eta}{2}} \cos^2(x)} = \int_0^{\frac{\pi}{2}} \cos^{\eta-2}(x) dx.$$

To complete the proof of the lemma, it only remains to recall that

$$\int_0^{\frac{\pi}{2}} \cos^{\eta-2}(x) dx = \frac{1}{2} B\left(\frac{\eta-1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{\eta-1}{2}\right)}{2\Gamma\left(\frac{\eta}{2}\right)},$$

where B is the classical beta function. \square

Proof of Lemma 4.4. To prove the first item of the lemma, we consider $\eta > 1$ and $\varphi \in \mathcal{X}_\eta^u$. Then, for $j = 1, 2, 3$ and $U \in \mathcal{D}_\kappa^u$, one has that

$$|\mathcal{G}_j[\varphi](U)| \leq \|\varphi\|_\eta^u \int_{-\infty}^0 \frac{ds}{|s+U|^\eta}.$$

Moreover, for $s \in (-\infty, 0]$, one has that $|s+U|^2 \geq s^2 + |U|^2$. Then, by Lemma 4.5,

$$|\mathcal{G}_j[\varphi](U)| \leq \int_{-\infty}^0 \frac{\|\varphi\|_\eta^u}{(s^2 + |U|^2)^{\frac{\eta}{2}}} ds = \frac{\|\varphi\|_\eta^u}{|U|^{\eta-1}} \int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{\eta}{2}}} \leq \frac{\|\varphi\|_\eta^u}{|U|^{\eta-1}} \frac{\sqrt{\pi} \Gamma(\frac{\eta-1}{2})}{2\Gamma(\frac{\eta}{2})}.$$

Next, we prove the second item. By the geometry of \mathcal{D}_κ^u and using the Cauchy's theorem, we can change the path of integration in the integral (15) to $te^{i\sigma}$, $t \in (-\infty, 0]$, with $0 \leq \sigma \leq \gamma$. Then, for $\nu > 0$ and $\varphi \in \mathcal{X}_\nu^u$,

$$\mathcal{G}_2[\varphi](U) = \int_{-\infty}^0 e^{-it \cos \sigma} e^{t \sin \sigma} \varphi(U + te^{i\sigma}) e^{i\sigma} dt.$$

Notice that $U + te^{i\sigma} \in \mathcal{D}_\kappa^u$. Then, the function $f(t) = |U + te^{i\sigma}|$ has a minimum at $t_* = -|U| \cos(\arg U - \sigma)$ and $f(t_*) = |U \sin(\arg U - \sigma)|$. Notice that, since $\arg U \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and t_* is negative only if $\arg U \in [\frac{\pi}{2}, \frac{\pi}{2} + \sigma]$, one has that

$$|U + te^{i\sigma}| \geq |U| \sin\left(\frac{\pi}{2} - \sigma\right) = |U| \cos \sigma.$$

Therefore,

$$|\mathcal{G}_2[\varphi](U)| \leq \frac{\|\varphi\|_\nu^u}{|U| (\cos \sigma)^\nu} \int_{-\infty}^0 e^{t \sin \sigma} dt = \frac{\|\varphi\|_\nu^u}{|U| \sin \sigma (\cos \sigma)^\nu}.$$

The norm for $\mathcal{G}_3[\varphi]$ can be treated in the same way changing the integration path to $te^{-i\sigma}$. \square

4.3. Estimates for $\mathcal{F}(0)$. The next proposition gives estimates for $\mathcal{F}(0)$.

Proposition 4.6. *Fix $\kappa \geq 1$. Then, the operator \mathcal{F} in (16) satisfies*

$$\|\mathcal{F}_1[0]\|_{\frac{8}{3}} \leq \alpha_0(\kappa), \quad \|\mathcal{F}_j[0]\|_{\frac{4}{3}} \leq \beta_0(\kappa), \quad j = 2, 3$$

and

$$\|\mathcal{F}[0]\|_\infty = \max\{\alpha_0(\kappa), \beta_0(\kappa)\},$$

where $\alpha_0(\kappa)$ and $\beta_0(\kappa)$ are decreasing functions satisfying that

$$\begin{aligned} \alpha_0(\kappa) &= \frac{8}{243} + \frac{32\sqrt{\pi} \Gamma(\frac{7}{3})}{729 \Gamma(\frac{17}{6})} \frac{1}{\kappa^2} \left(\frac{16}{81(2 - \zeta_0(\kappa))} + \frac{\zeta_3(\kappa)}{8 - \zeta_2(\kappa)} + \frac{16}{81\kappa^2} \frac{\zeta_3(\kappa)}{16 - \zeta_4(\kappa)} \right), \\ \beta_0(\kappa) &= \frac{2}{9} + \frac{14\sqrt{\pi} \Gamma(\frac{2}{3})}{81 \Gamma(\frac{7}{6})} + \left(\frac{\sqrt{\pi} \Gamma(\frac{7}{6})}{9 \Gamma(\frac{5}{3})} \frac{1}{\kappa} + \frac{2\sqrt{\pi} \Gamma(\frac{5}{3})}{81 \Gamma(\frac{13}{6})} \frac{1}{\kappa^2} \right) \frac{\zeta_1(\kappa)}{2 - \zeta_0(\kappa)}, \end{aligned}$$

with

$$\begin{aligned} \zeta_0(\kappa) &= \frac{16}{81\kappa^2} \left(1 + 3\sqrt{1 + \frac{16}{81\kappa^2}} \right), & \zeta_1(\kappa) &= \frac{16}{81} \left(1 + 3\sqrt{1 + \frac{16}{81\kappa^2}} \right), \\ \zeta_2(\kappa) &= \frac{16}{81\kappa^2} \left(6\sqrt{1 + \frac{16}{81\kappa^2}} + 8 + \frac{128}{81\kappa^2} \right), \\ \zeta_3(\kappa) &= \frac{16}{81} \left(\frac{3}{2\sqrt{1 - \frac{16}{81\kappa^2}}} + 6\sqrt{1 + \frac{16}{81\kappa^2}} + 6 + \frac{128}{81\kappa^2} \right), \end{aligned}$$

$$\zeta_4(\kappa) = \frac{32}{81\kappa^2} \left(10 + \frac{256}{81\kappa^2} + \frac{2^9}{3^7\kappa^4} + \sqrt{1 + \frac{16}{81\kappa^2}} \left(12 + \frac{368}{81\kappa^2} + \frac{2^9 7}{3^8\kappa^4} \right) \right).$$

We devote the rest of the section to prove Proposition 4.6. We first provide estimates for $\mathcal{R}[0]$.

Lemma 4.7. *Let the operator \mathcal{R} be as defined in (8). Then,*

$$\begin{aligned} \mathcal{R}_1[0](U) &= \frac{2^6}{3^6 U^{\frac{11}{3}}} (1 + R_1^0(U)), \\ \mathcal{R}_2[0](U) &= \left(-\frac{2}{9U^{\frac{4}{3}}} - \frac{4i}{81U^{\frac{7}{3}}} \right) (1 + R_{2,3}^0(U)), \\ \mathcal{R}_3[0](U) &= \left(-\frac{2}{9U^{\frac{4}{3}}} + \frac{4i}{81U^{\frac{7}{3}}} \right) (1 + R_{2,3}^0(U)), \end{aligned}$$

where

$$\begin{aligned} R_1^0(U) &= \frac{3}{4} \left(\frac{2}{3\sqrt{1 + \mathcal{J}_0}(1 + \sqrt{1 + \mathcal{J}_0})} + \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}} \right) \left(\frac{2(1 + \mathcal{J}_0)^{\frac{3}{2}}}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right) - 1, \\ R_{2,3}^0(U) &= \frac{2}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} - 1, \end{aligned}$$

with $\mathcal{J}_0 = \mathcal{J}_0(U) = \frac{16}{81U^2}$.

Proof. By the expression of \mathcal{R} in (8), one sees that

$$\mathcal{R}[0](U) = \left(\frac{-\partial_U \mathcal{K}(U, 0)}{1 + \partial_W \mathcal{K}(U, 0)}, \frac{i\partial_Y \mathcal{K}(U, 0)}{1 + \partial_W \mathcal{K}(U, 0)}, \frac{-i\partial_X \mathcal{K}(U, 0)}{1 + \partial_W \mathcal{K}(U, 0)} \right). \quad (20)$$

Notice that, by the definition of \mathcal{J} in (5), one has that $\mathcal{J}_0(U) := \mathcal{J}(U, 0) = \frac{16}{81U^2}$. Moreover, using the formulae in Appendix A for function \mathcal{J} , one sees that

$$\begin{aligned} \partial_U \mathcal{J}(U, 0) &= -\frac{32}{81U^3}, & \partial_W \mathcal{J}(U, 0) &= -\frac{16}{27U^{\frac{4}{3}}}, \\ \partial_X \mathcal{J}(U, 0) &= -\frac{4i}{3U^{\frac{2}{3}}} - \frac{8}{27U^{\frac{5}{3}}}, & \partial_Y \mathcal{J}(U, 0) &= \frac{4i}{3U^{\frac{2}{3}}} - \frac{8}{27U^{\frac{5}{3}}}. \end{aligned}$$

In the same way, using the formulae in Appendix A, one obtains that

$$\begin{aligned} \partial_U \mathcal{K}(U, 0) &= -\frac{2^5}{3^6 U^{\frac{11}{3}}} \frac{1}{\sqrt{1 + \mathcal{J}_0}(1 + \sqrt{1 + \mathcal{J}_0})} - \frac{2^4}{3^5 U^{\frac{11}{3}}} \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}}, \\ &= -\frac{2^4}{3^5 U^{\frac{11}{3}}} \left(\frac{2}{3\sqrt{1 + \mathcal{J}_0}(1 + \sqrt{1 + \mathcal{J}_0})} + \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}} \right), \\ \partial_W \mathcal{K}(U, 0) &= -\frac{8}{81U^2} \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}} = -\frac{\mathcal{J}_0}{2(1 + \mathcal{J}_0)^{\frac{3}{2}}}, \\ \partial_X \mathcal{K}(U, 0) &= \left(-\frac{2i}{9U^{\frac{4}{3}}} - \frac{4}{81U^{\frac{7}{3}}} \right) \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}}, \\ \partial_Y \mathcal{K}(U, 0) &= \left(\frac{2i}{9U^{\frac{4}{3}}} - \frac{4}{81U^{\frac{7}{3}}} \right) \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}}. \end{aligned}$$

Finally, applying these expressions to (20),

$$\mathcal{R}_1[0](U) = \frac{2^4}{3^5 U^{\frac{11}{3}}} \left(\frac{2}{3\sqrt{1+\mathcal{J}_0}(1+\sqrt{1+\mathcal{J}_0})} + \frac{1}{(1+\mathcal{J}_0)^{\frac{3}{2}}} \right) \left(\frac{2(1+\mathcal{J}_0)^{\frac{3}{2}}}{2(1+\mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right),$$

$$\mathcal{R}_2[0](U) = \left(-\frac{2}{9U^{\frac{4}{3}}} - \frac{4i}{81U^{\frac{7}{3}}} \right) \frac{2}{2(1+\mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0},$$

$$\mathcal{R}_3[0](U) = \left(-\frac{2}{9U^{\frac{4}{3}}} + \frac{4i}{81U^{\frac{7}{3}}} \right) \frac{2}{2(1+\mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0}.$$

□

Notice that, for $|U| \rightarrow \infty$,

$$\begin{aligned} R_1^0(U) &= -\frac{13}{16}\mathcal{J}_0 + \frac{13}{32}\mathcal{J}_0^2 + \mathcal{O}(\mathcal{J}_0^3) = -\frac{13}{81U^2} + \frac{104}{6561U^4} + \mathcal{O}\left(\frac{1}{U^6}\right), \\ R_{2,3}^0(U) &= -\mathcal{J}_0 + \frac{5}{8}\mathcal{J}_0^2 + \mathcal{O}(\mathcal{J}_0^3) = -\frac{16}{81U^2} + \frac{160}{6561U^4} + \mathcal{O}\left(\frac{1}{U^6}\right). \end{aligned}$$

Next lemma provides explicit estimates for these functions in terms of the parameter κ .

Lemma 4.8. *For $\kappa \geq 1$ and $U \in \mathcal{D}_\kappa^u$ (see (10)) one has that*

$$|R_1^0(U)| \leq \frac{C_1^0(\kappa)}{|U|^2}, \quad |R_{2,3}^0(U)| \leq \frac{C_{2,3}^0(\kappa)}{|U|^2},$$

where $C_1^0(\kappa)$ and $C_{2,3}^0(\kappa)$ are decreasing functions satisfying that

$$C_{2,3}^0(\kappa) = \frac{\zeta_1(\kappa)}{2 - \zeta_0(\kappa)}, \quad C_1^0(\kappa) = \frac{\zeta_3(\kappa)}{8 - \zeta_2(\kappa)} + \frac{16}{81(2 - \zeta_0(\kappa))} + \frac{16}{81\kappa^2} \frac{\zeta_3(\kappa)}{16 - \zeta_4(\kappa)},$$

with

$$\begin{aligned} \zeta_0(\kappa) &= \frac{16}{81\kappa^2} \left(3\sqrt{1 + \frac{16}{81\kappa^2}} + 1 \right), & \zeta_1(\kappa) &= \frac{16}{81} \left(3\sqrt{1 + \frac{16}{81\kappa^2}} + 1 \right), \\ \zeta_2(\kappa) &= \frac{16}{81\kappa^2} \left(6\sqrt{1 + \frac{16}{81\kappa^2}} + 8 + \frac{128}{81\kappa^2} \right), \\ \zeta_3(\kappa) &= \frac{16}{81} \left(\frac{3}{2\sqrt{1 - \frac{16}{81\kappa^2}}} + 6\sqrt{1 + \frac{16}{81\kappa^2}} + 6 + \frac{128}{81\kappa^2} \right), \\ \zeta_4(\kappa) &= \frac{32}{81\kappa^2} \left(10 + \frac{256}{81\kappa^2} + \frac{2^9}{3^7\kappa^4} + \sqrt{1 + \frac{16}{81\kappa^2}} \left(12 + \frac{368}{81\kappa^2} + \frac{2^9 7}{3^8\kappa^4} \right) \right). \end{aligned}$$

Proof. First, we compute $C_{2,3}^0(\kappa)$. By Lemma 4.7,

$$R_{2,3}^0(U) = \frac{2 + \mathcal{J}_0 - 2(1 + \mathcal{J}_0)^{\frac{3}{2}}}{2 + \left\{ 2(1 + \mathcal{J}_0)^{\frac{3}{2}} - 2 - \mathcal{J}_0 \right\}} = \frac{r_1(\mathcal{J}_0)}{2 + r_0(\mathcal{J}_0)}.$$

Then, by the mean value theorem,

$$\begin{aligned} |r_0(\mathcal{J}_0)| &\leq |\mathcal{J}_0| \sup_{\sigma \in [0,1]} \left| 3\sqrt{1 + \sigma\mathcal{J}_0} - 1 \right| \leq |\mathcal{J}_0| \left(3\sqrt{1 + |\mathcal{J}_0|} + 1 \right), \\ |r_1(\mathcal{J}_0)| &\leq |\mathcal{J}_0| \sup_{\sigma \in [0,1]} \left| 1 - 3\sqrt{1 + \sigma\mathcal{J}_0} \right| \leq |\mathcal{J}_0| \left(1 + 3\sqrt{1 + |\mathcal{J}_0|} \right). \end{aligned}$$

Since, $U \in \mathcal{D}_\kappa^u$, one has that $|U| \geq \kappa$. Then, $|\mathcal{J}_0(U)| \leq \frac{16}{81\kappa^2}$ and, as a result,

$$\begin{aligned} |r_0(\mathcal{J}_0)| &\leq \frac{16}{81\kappa^2} \left(3 \left(1 + \frac{16}{81\kappa^2} \right)^{\frac{1}{2}} + 1 \right) = \zeta_0(\kappa), \\ |r_1(\mathcal{J}_0)U^2| &\leq \frac{16}{81} \left(3 \left(1 + \frac{16}{81\kappa^2} \right)^{\frac{1}{2}} + 1 \right) = \zeta_1(\kappa). \end{aligned}$$

Notice that $\zeta_0(\kappa)$ and $\zeta_1(\kappa)$ are decreasing functions for positive κ and $\zeta_0(1) < 2$. Therefore, applying the triangular inequality, we denote

$$C_{2,3}^0(\kappa) = \frac{\zeta_1(\kappa)}{2 - \zeta_0(\kappa)},$$

which, by construction, is a decreasing function as well.

Analogously, we compute $C_1^0(\kappa)$. By Lemma 4.7,

$$\begin{aligned} R_1^0(U) &= \frac{3}{4} \left(\frac{2}{3\sqrt{1 + \mathcal{J}_0}(1 + \sqrt{1 + \mathcal{J}_0})} + \frac{1}{(1 + \mathcal{J}_0)^{\frac{3}{2}}} \right) \left(\frac{2(1 + \mathcal{J}_0)^{\frac{3}{2}}}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right) - 1 \\ &= \frac{3}{4} \left(\frac{2(1 + \mathcal{J}_0) + 3(1 + \sqrt{1 + \mathcal{J}_0})}{3(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0})} \right) \left(\frac{2(1 + \mathcal{J}_0)^{\frac{3}{2}}}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right) - 1 \\ &= \frac{3}{4} \left(\frac{4}{3} + \frac{2(1 + \mathcal{J}_0) + 3(1 + \sqrt{1 + \mathcal{J}_0}) - 4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0})}{3(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0})} \right) \\ &\quad \left(1 + \frac{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - 2(1 + \mathcal{J}_0)^{\frac{3}{2}} + \mathcal{J}_0}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right) - 1 \end{aligned}$$

and

$$\begin{aligned} R_1^0(U) &= \left(1 + \frac{1 + 3\sqrt{1 + \mathcal{J}_0} - 4(1 + \mathcal{J}_0)^{\frac{3}{2}} - 6\mathcal{J}_0 - 4\mathcal{J}_0^2}{4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0})} \right) \left(1 + \frac{\mathcal{J}_0}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \right) - 1 \\ &= \frac{1 + 3\sqrt{1 + \mathcal{J}_0} - 4(1 + \mathcal{J}_0)^{\frac{3}{2}} - 6\mathcal{J}_0 - 4\mathcal{J}_0^2}{4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0})} + \frac{\mathcal{J}_0}{2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0} \\ &\quad + \mathcal{J}_0 \frac{1 + 3\sqrt{1 + \mathcal{J}_0} - 4(1 + \mathcal{J}_0)^{\frac{3}{2}} - 6\mathcal{J}_0 - 4\mathcal{J}_0^2}{4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0}) \left(2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0 \right)} \\ &= \frac{r_3(\mathcal{J}_0)}{8 + r_2(\mathcal{J}_0)} + \frac{\mathcal{J}_0}{2 + r_0(\mathcal{J}_0)} + \mathcal{J}_0 \frac{r_3(\mathcal{J}_0)}{16 + r_4(\mathcal{J}_0)}. \end{aligned}$$

Then, the function

$$r_2(\mathcal{J}_0) = 4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0}) - 8 = 4(1 + \mathcal{J}_0)^{\frac{3}{2}} - 4 + 8\mathcal{J}_0 + 4\mathcal{J}_0^2,$$

by the mean value theorem and taking into account that $|\mathcal{J}_0(U)| \leq \frac{16}{81\kappa^2}$, satisfies

$$\begin{aligned} |r_2(\mathcal{J}_0)| &\leq |\mathcal{J}_0| \sup_{\sigma \in [0,1]} \left| 6\sqrt{1 + \sigma\mathcal{J}_0} + 8 + 8\sigma\mathcal{J}_0 \right| \leq |\mathcal{J}_0| \left(6\sqrt{1 + |\mathcal{J}_0|} + 8 + 8|\mathcal{J}_0| \right), \\ &\leq \frac{16}{81\kappa^2} \left(6\sqrt{1 + \frac{16}{81\kappa^2}} + 8 + \frac{128}{81\kappa^2} \right) = \zeta_2(\kappa). \end{aligned}$$

Notice that, $|\mathcal{J}_0(U)| \leq \frac{16}{81\kappa^2} < 1$ for $\kappa \geq 1$. Then, the functions

$$\begin{aligned} r_3(\mathcal{J}_0) &= 1 + 3\sqrt{1 + \mathcal{J}_0} - 4(1 + \mathcal{J}_0)^{\frac{3}{2}} - 6\mathcal{J}_0 - 4\mathcal{J}_0^2, \\ r_4(\mathcal{J}_0) &= 4(1 + \mathcal{J}_0)^{\frac{3}{2}}(1 + \sqrt{1 + \mathcal{J}_0}) \left(2(1 + \mathcal{J}_0)^{\frac{3}{2}} - \mathcal{J}_0 \right) - 16 \\ &= 4(-2 + 5\mathcal{J}_0 + 4\mathcal{J}_0^2 + \mathcal{J}_0^3) + 4(1 + \mathcal{J}_0)^{\frac{3}{2}}(2 + 3\mathcal{J}_0 + 2\mathcal{J}_0^2), \end{aligned}$$

satisfy

$$\begin{aligned} |r_3(\mathcal{J}_0)| &\leq |\mathcal{J}_0| \sup_{\sigma \in [0,1]} \left| \frac{3}{2\sqrt{1 + \sigma\mathcal{J}_0}} - 6\sqrt{1 + \sigma\mathcal{J}_0} - 6 - 8\sigma\mathcal{J}_0 \right| \\ &\leq |\mathcal{J}_0| \left(\frac{3}{2\sqrt{1 - |\mathcal{J}_0|}} + 6\sqrt{1 + |\mathcal{J}_0|} + 6 + 8|\mathcal{J}_0| \right), \\ |r_3(\mathcal{J}_0)U^2| &\leq \frac{16}{81} \left(\frac{3}{2\sqrt{1 - \frac{16}{81\kappa^2}}} + 6\sqrt{1 + \frac{16}{81\kappa^2}} + 6 + \frac{128}{81\kappa^2} \right) = \zeta_3(\kappa) \end{aligned}$$

and

$$\begin{aligned} |r_4(\mathcal{J}_0)| &\leq 4|\mathcal{J}_0| \sup_{\sigma \in [0,1]} \left| 5 + 8\sigma\mathcal{J}_0 + 3\sigma^2\mathcal{J}_0^2 + \frac{\sqrt{1 + \sigma\mathcal{J}_0}}{2}(12 + 23\sigma\mathcal{J}_0 + 14\sigma^2\mathcal{J}_0^2) \right| \\ &\leq 2|\mathcal{J}_0| \left(10 + 16|\mathcal{J}_0| + 6|\mathcal{J}_0|^2 + \sqrt{1 + |\mathcal{J}_0|} (12 + 23|\mathcal{J}_0| + 14|\mathcal{J}_0|^2) \right) \\ &\leq \frac{32}{81\kappa^2} \left(10 + \frac{256}{81\kappa^2} + \frac{2^9}{3^7\kappa^4} + \sqrt{1 + \frac{16}{81\kappa^2}} \left(12 + \frac{368}{81\kappa^2} + \frac{2^9}{3^8\kappa^4} \right) \right) \\ &= \zeta_4(\kappa). \end{aligned}$$

Notice that $\zeta_0(\kappa)$, $\zeta_2(\kappa)$ and $\zeta_4(\kappa)$ are decreasing functions for positive κ and one can easily checked that $\zeta_0(1) < 2$, $\zeta_2(1) < 8$ and $\zeta_4(1) < 16$. Then, we denote

$$C_1^0(\kappa) = \frac{\zeta_3(\kappa)}{8 - \zeta_2(\kappa)} + \frac{16}{81} \frac{1}{2 - \zeta_0(\kappa)} + \frac{16}{81\kappa^2} \frac{\zeta_3(\kappa)}{16 - \zeta_4(\kappa)},$$

which, by construction, is a decreasing function for $\kappa \geq 1$. \square

Proof of Proposition 4.6. Let us recall that $\mathcal{F} = \mathcal{G} \circ \mathcal{R}$. Then, by Lemmas 4.5, 4.7 and 4.8, and proceeding as in the proof of Lemma 4.4,

$$\begin{aligned} |\mathcal{F}_1[0](U)| &= \left| \int_{-\infty}^0 \mathcal{R}_1[0](s + U) ds \right| \\ &\leq \frac{2^6}{3^6} \left| \int_{-\infty}^0 \frac{ds}{(s + U)^{\frac{11}{3}}} \right| + \frac{2^6}{3^6} \left| \int_{-\infty}^0 \frac{R_1(s + U)}{(s + U)^{\frac{11}{3}}} ds \right| \\ &\leq \frac{2^6}{3^6} \left| \frac{3}{8U^{\frac{8}{3}}} \right| + \frac{2^6}{3^6} C_1^0(\kappa) \int_{-\infty}^0 \frac{ds}{|s + U|^{\frac{17}{3}}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2^3}{3^5 |U|^{\frac{8}{3}}} + \frac{2^6}{3^6} C_1^0(\kappa) \int_{-\infty}^0 \frac{ds}{(s^2 + |U|^2)^{\frac{17}{6}}} \\
&\leq \frac{8}{3^5 |U|^{\frac{8}{3}}} + \frac{2^6 C_1^0(\kappa)}{3^6 |U|^{\frac{14}{3}}} \int_{-\infty}^0 \frac{ds}{(1 + s^2)^{\frac{17}{6}}} \\
&\leq \frac{8}{3^5 |U|^{\frac{8}{3}}} + \frac{2^6 C_1^0(\kappa) \sqrt{\pi} \Gamma(\frac{7}{3})}{3^6 |U|^{\frac{14}{3}} 2\Gamma(\frac{17}{6})}.
\end{aligned}$$

Then

$$\|\mathcal{F}_1[0]\|_{\frac{8}{3}} \leq \frac{8}{243} + \frac{32\sqrt{\pi} \Gamma(\frac{7}{3}) C_1^0(\kappa)}{729 \Gamma(\frac{17}{6}) \kappa^2}.$$

Following the same ideas, one has that

$$\begin{aligned}
|\mathcal{F}_2[0](U)| &= \left| \int_{-\infty}^0 e^{-is} \mathcal{R}_2[0](s+U) ds \right| \leq \frac{2}{9} \left| \int_{-\infty}^0 \frac{e^{-is}}{(s+U)^{\frac{4}{3}}} ds \right| \\
&\quad + \frac{2}{9} \left| \int_{-\infty}^0 \frac{e^{-is} R_{2,3}(s+U)}{(s+U)^{\frac{4}{3}}} ds \right| + \frac{4}{81} \left| \int_{-\infty}^0 \frac{e^{-is} (1 + R_{2,3}(s+U))}{(s+U)^{\frac{7}{3}}} ds \right|.
\end{aligned}$$

Notice that the first integral in the inequality satisfies that

$$\begin{aligned}
\int_{-\infty}^0 \frac{e^{-is}}{(s+U)^{\frac{4}{3}}} ds &= i \left[\frac{e^{-is}}{(s+U)^{\frac{4}{3}}} \right]_{s=-\infty}^{s=0} + \frac{4i}{3} \int_{-\infty}^0 \frac{e^{-is}}{(s+U)^{\frac{7}{3}}} ds \\
&= \frac{i}{U^{\frac{4}{3}}} + \frac{4i}{3} \int_{-\infty}^0 \frac{e^{-is}}{(s+U)^{\frac{7}{3}}} ds.
\end{aligned}$$

Then, proceeding as in the proof of Lemma 4.4,

$$\begin{aligned}
|\mathcal{F}_2[0](U)| &\leq \frac{2}{9 |U|^{\frac{4}{3}}} + \frac{8}{27} \int_{-\infty}^0 \frac{ds}{|s+U|^{\frac{7}{3}}} + \frac{2C_{2,3}^0(\kappa)}{9} \int_{-\infty}^0 \frac{ds}{|s+U|^{\frac{10}{3}}} \\
&\quad + \frac{4}{81} \int_{-\infty}^0 \frac{ds}{|s+U|^{\frac{7}{3}}} + \frac{4C_{2,3}^0(\kappa)}{81} \int_{-\infty}^0 \frac{ds}{|s+U|^{\frac{13}{3}}} \\
&\leq \frac{2}{9 |U|^{\frac{4}{3}}} + \frac{28}{81 |U|^{\frac{4}{3}}} \int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{7}{6}}} + \frac{2C_{2,3}^0(\kappa)}{9 |U|^{\frac{7}{3}}} \int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{5}{3}}} \\
&\quad + \frac{4C_{2,3}^0(\kappa)}{81 |U|^{\frac{10}{3}}} \int_{-\infty}^0 \frac{ds}{(1+s^2)^{\frac{13}{6}}} \\
&= \left(\frac{2}{9} + \frac{14\sqrt{\pi} \Gamma(\frac{2}{3})}{81\Gamma(\frac{7}{6})} \right) \frac{1}{|U|^{\frac{4}{3}}} + \frac{\sqrt{\pi} \Gamma(\frac{7}{6}) C_{2,3}^0(\kappa)}{9\Gamma(\frac{5}{3})} \frac{1}{|U|^{\frac{7}{3}}} \\
&\quad + \frac{2\sqrt{\pi} \Gamma(\frac{5}{3}) C_{2,3}^0(\kappa)}{81\Gamma(\frac{13}{6})} \frac{1}{|U|^{\frac{10}{3}}}.
\end{aligned}$$

Then

$$\|\mathcal{F}_2[0]\|_{\frac{4}{3}} \leq \frac{2}{9} + \frac{14\sqrt{\pi} \Gamma(\frac{2}{3})}{81 \Gamma(\frac{7}{6})} + \frac{\sqrt{\pi} \Gamma(\frac{7}{6}) C_{2,3}^0(\kappa)}{9 \Gamma(\frac{5}{3})} \frac{1}{\kappa} + \frac{2\sqrt{\pi} \Gamma(\frac{5}{3}) C_{2,3}^0(\kappa)}{81 \Gamma(\frac{13}{6})} \frac{1}{\kappa^2}.$$

An analogous result holds for $\mathcal{F}_3[0]$. \square

4.4. **Estimates for the derivatives of \mathcal{R} .** Let $\varrho_1, \varrho_2 > 1$ and denote

$$\alpha(\kappa, \varrho_1) = \alpha_0(\kappa)\varrho_1, \quad \beta(\kappa, \varrho_2) = \beta_0(\kappa)\varrho_2,$$

where functions α_0 and β_0 as given in Proposition 4.6. Notice that, for $\kappa \geq 1$, the functions $\alpha(\kappa, \varrho_1)$ and $\beta(\kappa, \varrho_2)$ are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 , respectively.

Then, we consider the closed set defined by

$$\mathbf{R}_\kappa(\varrho_1, \varrho_2) = \{(W^u, X^u, Y^u) \in \mathcal{X}_\times^u : \|W^u\|_{\frac{3}{2}} \leq \alpha(\kappa, \varrho_1), \|X^u\|_{\frac{4}{3}}, \|Y^u\|_{\frac{4}{3}} \leq \beta(\kappa, \varrho_2)\},$$

where \mathcal{X}_\times^u and $\|\cdot\|_\times$ were defined in (18) and (19), respectively. The next two lemmas give estimates for the functions \mathcal{J} and \mathcal{K} , introduced in (5) and (4) respectively, and their derivatives for functions in the closed domain $\mathbf{R}_\kappa(\varrho_1, \varrho_2)$.

In the following, we omit the dependence of certain functions in ϱ_1, ϱ_2 and κ to simplify notation.

Lemma 4.9. *Let $\kappa \geq 1$ and $\varrho_1, \varrho_2 > 1$ and define the functions*

$$\begin{aligned} \xi_0 &= \frac{16 + 216\beta}{81} + \frac{16\beta}{27\kappa} + \frac{16\alpha + 48\beta^2}{27\kappa^2} + \frac{8\alpha\beta}{9\kappa^3} + \frac{4\alpha^2}{9\kappa^4}, \\ \xi_1 &= \frac{32 + 144\beta}{81} + \frac{80\beta}{81\kappa} + \frac{64\alpha + 192\beta^2}{81\kappa^2} + \frac{8\alpha\beta}{9\kappa^3} + \frac{8\alpha^2}{27\kappa^4}, \\ \xi_2 &= \frac{16}{27} + \frac{8\beta}{9\kappa} + \frac{8\alpha}{9\kappa^2}, \\ \xi_3 &= \frac{4}{3} + \frac{8}{27\kappa} + \frac{16\beta}{9\kappa^2} + \frac{4\alpha}{9\kappa^3}, \\ \xi_4 &= \frac{64}{81} + \frac{8\beta}{9\kappa} + \frac{16\alpha}{27\kappa^2}, \\ \xi_5 &= \frac{8}{9} + \frac{40}{81\kappa} + \frac{64\beta}{27\kappa^2} + \frac{4\alpha}{9\kappa^3}, \end{aligned}$$

which are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 .

Then, for $Z^u \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$, the following estimate for $\mathcal{J}(U, Z^u)$ in (5) holds,

$$\|\mathcal{J}(\cdot, Z^u)\|_2 \leq \xi_0.$$

Moreover, its first and second derivatives satisfy

$$\|\partial_U \mathcal{J}(\cdot, Z^u)\|_3 \leq \xi_1, \quad \|\partial_W \mathcal{J}(\cdot, Z^u)\|_{\frac{4}{3}} \leq \xi_2, \quad \|\partial_X \mathcal{J}(\cdot, Z^u)\|_{\frac{2}{3}}, \|\partial_Y \mathcal{J}(\cdot, Z^u)\|_{\frac{2}{3}} \leq \xi_3,$$

and

$$\begin{aligned} \|\partial_{UW} \mathcal{J}(\cdot, Z^u)\|_{\frac{7}{3}} &\leq \xi_4, & \|\partial_{UX} \mathcal{J}(\cdot, Z^u)\|_{\frac{5}{3}}, \|\partial_{UY} \mathcal{J}(\cdot, Z^u)\|_{\frac{5}{3}} &\leq \xi_5, \\ \|\partial_W^2 \mathcal{J}(\cdot, Z^u)\|_{\frac{2}{3}} &= \frac{8}{9}, & \|\partial_{WX} \mathcal{J}(\cdot, Z^u)\|_1, \|\partial_{WY} \mathcal{J}(\cdot, Z^u)\|_1 &= \frac{4}{9}, \\ \|\partial_{XY} \mathcal{J}(\cdot, Z^u)\|_{\frac{4}{3}} &= \frac{10}{9}, & \|\partial_X^2 \mathcal{J}(\cdot, Z^u)\|_{\frac{4}{3}}, \|\partial_Y^2 \mathcal{J}(\cdot, Z^u)\|_{\frac{4}{3}} &= \frac{2}{3}. \end{aligned}$$

Proof. The estimate for \mathcal{J} is a consequence of the following,

$$\begin{aligned} |\mathcal{J}(U, Z^u)| &\leq \frac{4\alpha^2}{9|U|^6} + \frac{16\alpha}{27|U|^4} + \frac{16}{81|U|^2} + \frac{8\alpha\beta}{9|U|^5} + \frac{16\beta}{27|U|^3} + \frac{8\beta}{3|U|^2} + \frac{2\beta^2}{3|U|^4} \\ &\quad + \frac{10\beta^2}{9|U|^4} \\ &= \frac{1}{|U|^2} \left(\frac{16 + 216\beta}{81} + \frac{16\beta}{27\kappa} + \frac{16\alpha + 48\beta^2}{27\kappa^2} + \frac{8\alpha\beta}{9\kappa^3} + \frac{4\alpha^2}{9\kappa^4} \right) := \frac{\xi_0}{|U|^2}. \end{aligned}$$

The first derivatives can be bounded by

$$\begin{aligned}
|\partial_U \mathcal{J}(U, Z^u)| &\leq \frac{8\alpha^2}{27|U|^7} + \frac{64\alpha}{81|U|^5} + \frac{32}{81|U|^3} + \frac{8\alpha\beta}{9|U|^6} + \frac{80\beta}{81|U|^4} + \frac{16\beta}{9|U|^3} \\
&\quad + \frac{8\beta^2}{9|U|^5} + \frac{40\beta^2}{27|U|^5} \\
&= \frac{1}{|U|^3} \left(\frac{32 + 144\beta}{81} + \frac{80\beta}{81\kappa} + \frac{64\alpha + 192\beta^2}{81\kappa^2} + \frac{8\alpha\beta}{9\kappa^3} + \frac{8\alpha^2}{27\kappa^4} \right) := \frac{\xi_1}{|U|^3}, \\
|\partial_W \mathcal{J}(U, Z^u)| &\leq \frac{8\alpha}{9|U|^{\frac{10}{3}}} + \frac{16}{27|U|^{\frac{4}{3}}} + \frac{8\beta}{9|U|^{\frac{7}{3}}} = \frac{1}{|U|^{\frac{4}{3}}} \left(\frac{16}{27} + \frac{8\beta}{9\kappa} + \frac{8\alpha}{9\kappa^2} \right) := \frac{\xi_2}{|U|^{\frac{4}{3}}}, \\
|\partial_X \mathcal{J}(U, Z^u)| &\leq \frac{4\alpha}{9|U|^{\frac{11}{3}}} + \frac{8}{27|U|^{\frac{5}{3}}} + \frac{4}{3|U|^{\frac{2}{3}}} + \frac{2\beta}{3|U|^{\frac{8}{3}}} + \frac{10\beta}{9|U|^{\frac{8}{3}}} \\
&= \frac{1}{|U|^{\frac{2}{3}}} \left(\frac{4}{3} + \frac{8}{27\kappa} + \frac{16\beta}{9\kappa^2} + \frac{4\alpha}{9\kappa^3} \right) := \frac{\xi_3}{|U|^{\frac{2}{3}}}, \\
|\partial_Y \mathcal{J}(U, Z^u)| &\leq \frac{\xi_3}{|U|^{\frac{2}{3}}},
\end{aligned}$$

and the second derivatives by

$$\begin{aligned}
|\partial_{UW} \mathcal{J}(U, Z^u)| &\leq \frac{16\alpha}{27|U|^{\frac{13}{3}}} + \frac{64}{81|U|^{\frac{7}{3}}} + \frac{8\beta}{9|U|^{\frac{10}{3}}} = \frac{1}{|U|^{\frac{7}{3}}} \left(\frac{64}{81} + \frac{8\beta}{9\kappa} + \frac{16\alpha}{27\kappa^2} \right) \\
&:= \frac{\xi_4}{|U|^{\frac{7}{3}}}, \\
|\partial_{UX} \mathcal{J}(U, Z^u)| &\leq \frac{4\alpha}{9|U|^{\frac{14}{3}}} + \frac{40}{81|U|^{\frac{8}{3}}} + \frac{8}{9|U|^{\frac{5}{3}}} + \frac{8\beta}{9|U|^{\frac{11}{3}}} + \frac{40\beta}{27|U|^{\frac{11}{3}}} \\
&= \frac{1}{|U|^{\frac{5}{3}}} \left(\frac{8}{9} + \frac{40}{81\kappa} + \frac{64\beta}{27\kappa^2} + \frac{4\alpha}{9\kappa^3} \right) := \frac{\xi_5}{|U|^{\frac{5}{3}}}, \\
|\partial_{UY} \mathcal{J}(U, Z^u)| &\leq \frac{\xi_5}{|U|^{\frac{5}{3}}}, \\
|\partial_W^2 \mathcal{J}(U, Z^u)| &= \frac{8}{9|U|^{\frac{2}{3}}}, \quad |\partial_{WX} \mathcal{J}(U, Z^u)| = \frac{4}{9|U|}, \quad |\partial_{WY} \mathcal{J}(U, Z^u)| = \frac{4}{9|U|}, \\
|\partial_X^2 \mathcal{J}(U, Z^u)| &= \frac{2}{3|U|^{\frac{4}{3}}}, \quad |\partial_{XY} \mathcal{J}(U, Z^u)| = \frac{10}{9|U|^{\frac{4}{3}}}, \quad |\partial_Y^2 \mathcal{J}(U, Z^u)| = \frac{2}{3|U|^{\frac{4}{3}}}.
\end{aligned}$$

Finally notice that, $\kappa \geq 1$ and $\varrho_1, \varrho_2 > 1$, by Proposition 4.6, $\alpha(\kappa, \varrho_1) = \varrho_1 \alpha_0(\kappa)$ and $\beta(\kappa, \varrho_2) = \varrho_2 \beta_0(\kappa)$ are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 . Therefore, the auxiliary functions ξ_0, \dots, ξ_5 are as well. \square

Lemma 4.10. *Let $\varrho_1 \in (1, 60)$, $\varrho_2 \in (1, 3)$ and $\kappa \geq 3$. Then, for $Z^u \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$, the derivatives of the Hamiltonian \mathcal{K} in (4) satisfy*

$$\|\partial_U \mathcal{K}(\cdot, Z^u)\|_{\frac{1}{3}} \leq \eta_1, \quad \|\partial_W \mathcal{K}(\cdot, Z^u)\|_2 \leq \eta_2, \quad \|\partial_X \mathcal{K}(\cdot, Z^u)\|_{\frac{4}{3}}, \|\partial_Y \mathcal{K}(\cdot, Z^u)\|_{\frac{4}{3}} \leq \eta_3$$

and

$$\begin{aligned}
\|\partial_{UW} \mathcal{K}(\cdot, Z^u)\|_3 &\leq \eta_4, & \|\partial_{UX} \mathcal{K}(\cdot, Z^u)\|_{\frac{7}{3}}, \|\partial_{UY} \mathcal{K}(\cdot, Z^u)\|_{\frac{7}{3}} &\leq \eta_5, \\
\|\partial_W^2 \mathcal{K}(\cdot, Z^u)\|_{-\frac{2}{3}} &\leq \eta_6, & \|\partial_{WX} \mathcal{K}(\cdot, Z^u)\|_{\frac{5}{3}}, \|\partial_{WY} \mathcal{K}(\cdot, Z^u)\|_{\frac{5}{3}} &\leq \eta_7,
\end{aligned}$$

$$\|\partial_{XY}\mathcal{K}(\cdot, Z^u)\|_2 \leq \eta_8, \quad \|\partial_X^2\mathcal{K}(\cdot, Z^u)\|_2, \|\partial_Y^2\mathcal{K}(\cdot, Z^u)\|_2 \leq \eta_9,$$

where

$$\begin{aligned} \eta_0 &= \sqrt{1 - \frac{\xi_0}{\kappa^2}}, & \eta_1 &= \frac{4\xi_0}{9\eta_0(4 - \xi_0\eta_0^{-1}\kappa^{-2})} + \frac{\xi_1}{6\eta_0^3} + \frac{\alpha^2}{2\kappa^2}, \\ \eta_2 &= \frac{3\alpha}{2} + \frac{\xi_2}{6\eta_0^3}, & \eta_3 &= \frac{\xi_3}{6\eta_0^3}, \\ \eta_4 &= \alpha + \frac{\xi_2}{9\eta_0^3} + \frac{\xi_4}{6\eta_0^3} + \frac{\xi_1\xi_2}{4\eta_0^5\kappa^2}, & \eta_5 &= \frac{\xi_3}{9\eta_0^3} + \frac{\xi_5}{6\eta_0^3} + \frac{\xi_1\xi_3}{4\eta_0^5\kappa^2}, \\ \eta_6 &= \frac{3}{2} + \frac{4}{27\eta_0^3\kappa^2} + \frac{\xi_2^2}{4\eta_0^5\kappa^4}, & \eta_7 &= \frac{2}{27\eta_0^3} + \frac{\xi_2\xi_3}{4\eta_0^5\kappa}, \\ \eta_8 &= \frac{5}{27\eta_0^3} + \frac{\xi_3^2}{4\eta_0^5}, & \eta_9 &= \frac{1}{9\eta_0^3} + \frac{\xi_3^2}{4\eta_0^5}. \end{aligned}$$

Moreover, for $\kappa \geq 3$, $\varrho_1 \in (1, 60)$ and $\varrho_2 \in (1, 3)$, the functions η_1, \dots, η_9 are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 .

Proof. Let us consider first lower bounds for $1 + \mathcal{J}(U, Z^u)$. By Lemma 4.9, one has that

$$|\mathcal{J}(U, Z^u)| \leq \frac{\xi_0}{\kappa^2}.$$

Notice that, by Proposition 4.6, $\alpha_0(\kappa)$ and $\beta_0(\kappa)$ are decreasing for positive values of κ . Therefore, $\alpha(\kappa, \varrho_1)$ and $\beta(\kappa, \varrho_2)$ are decreasing for $\kappa > 0$ and increasing for $\varrho_1, \varrho_2 > 0$. Then, by the definition of ξ_0 in Lemma 4.9,

$$|\mathcal{J}(U, Z^u)| \leq \frac{\xi_0}{\kappa^2} \Bigg|_{\kappa=3, \varrho_1=60, \varrho_2=3} < 1. \quad (21)$$

Therefore, by the triangular inequality,

$$|\sqrt{1 + \mathcal{J}(U, Z^u)}| \geq \sqrt{1 - |\mathcal{J}(U, Z^u)|} \geq \sqrt{1 - \frac{\xi_0}{\kappa^2}} := \eta_0. \quad (22)$$

Next, we consider lower bounds for the denominator $1 + \sqrt{1 + \mathcal{J}(U, Z^u)}$. Notice that, by the mean value theorem and taking into account (21),

$$\begin{aligned} \left| \sqrt{1 + \mathcal{J}(U, Z^u)} - 1 \right| &\leq |\mathcal{J}(U, Z^u)| \sup_{\sigma \in [0, 1]} \left| \frac{1}{2\sqrt{1 + \sigma\mathcal{J}(U, Z^u)}} \right| \\ &\leq \frac{\xi_0}{2\kappa^2} \frac{1}{\sqrt{1 - |\mathcal{J}(U, Z^u)|}} \leq \frac{\xi_0}{2\eta_0\kappa^2}. \end{aligned}$$

In addition, one can see that,

$$\left| \sqrt{1 + \mathcal{J}(U, Z^u)} - 1 \right| \leq \frac{\xi_0}{2\eta_0\kappa^2} \Bigg|_{\kappa=3, \varrho_1=60, \varrho_2=3} < 2.$$

Then, by the triangular inequality,

$$\left| 1 + \sqrt{1 + \mathcal{J}(U, Z^u)} \right| \geq 2 - \frac{\xi_0}{2\eta_0\kappa^2}. \quad (23)$$

We estimate now the first derivatives of \mathcal{K} . By the formulae in Appendix A, Lemma 4.9, (22) and (23), one has that

$$\begin{aligned} |\partial_U \mathcal{K}(U, Z^u)| &\leq \frac{\alpha^2}{2|U|^{\frac{17}{3}}} + \frac{2\xi_0}{9\eta_0(2 - \xi_0 2^{-1}\eta_0^{-1}\kappa^{-2})|U|^{\frac{11}{3}}} + \frac{\xi_1}{6\eta_0^3|U|^{\frac{11}{3}}} \\ &\leq \frac{1}{|U|^{\frac{11}{3}}} \left(\frac{4\xi_0}{9\eta_0(4 - \xi_0\eta_0^{-1}\kappa^{-2})} + \frac{\xi_1}{6\eta_0^3} + \frac{\alpha^2}{2\kappa^2} \right) := \frac{\eta_1}{|U|^{\frac{11}{3}}} \end{aligned}$$

and

$$\begin{aligned} |\partial_W \mathcal{K}(U, Z^u)| &= \frac{3\alpha}{2|U|^2} + \frac{\xi_2}{6\eta_0^3|U|^2} := \frac{\eta_2}{|U|^2}, \\ |\partial_X \mathcal{K}(U, Z^u)| &= \frac{\xi_3}{|U|^{\frac{4}{3}}6\eta_0^3} := \frac{\eta_3}{|U|^{\frac{4}{3}}}, \\ |\partial_Y \mathcal{K}(U, Z^u)| &\leq \frac{\eta_3}{|U|^{\frac{4}{3}}}. \end{aligned}$$

Finally, we consider estimates for the second derivatives of \mathcal{K} ,

$$\begin{aligned} |\partial_{UW} \mathcal{K}(U, Z^u)| &\leq \frac{\alpha}{|U|^3} + \frac{\xi_2}{9\eta_0^3|U|^3} + \frac{\xi_4}{6\eta_0^3|U|^3} + \frac{\xi_1\xi_2}{4\eta_0^5|U|^5} \\ &\leq \frac{1}{|U|^3} \left(\alpha + \frac{\xi_2}{9\eta_0^3} + \frac{\xi_4}{6\eta_0^3} + \frac{\xi_1\xi_2}{4\eta_0^5\kappa^2} \right) := \frac{\eta_4}{|U|^3}, \\ |\partial_{UX} \mathcal{K}(U, Z^u)| &\leq \frac{\xi_3}{9\eta_0^3|U|^{\frac{7}{3}}} + \frac{\xi_5}{6\eta_0^3|U|^{\frac{7}{3}}} + \frac{\xi_1\xi_3}{4\eta_0^5|U|^{\frac{13}{3}}} \\ &\leq \frac{1}{|U|^{\frac{7}{3}}} \left(\frac{\xi_3}{9\eta_0^3} + \frac{\xi_5}{6\eta_0^3} + \frac{\xi_1\xi_3}{4\eta_0^5\kappa^2} \right) := \frac{\eta_5}{|U|^{\frac{7}{3}}}, \\ |\partial_{UY} \mathcal{K}(U, Z^u)| &\leq \frac{\eta_5}{|U|^{\frac{7}{3}}} \end{aligned}$$

and

$$\begin{aligned} |\partial_W^2 \mathcal{K}(U, Z^u)| &\leq \frac{3|U|^{\frac{2}{3}}}{2} + \frac{4}{27\eta_0^3|U|^{\frac{4}{3}}} + \frac{\xi_2^2}{4\eta_0^5|U|^{\frac{10}{3}}} \\ &\leq |U|^{\frac{2}{3}} \left(\frac{3}{2} + \frac{4}{27\eta_0^3\kappa^2} + \frac{\xi_2^2}{4\eta_0^5\kappa^4} \right) := |U|^{\frac{2}{3}} \eta_6, \\ |\partial_{WX} \mathcal{K}(U, Z^u)| &\leq \frac{2}{27\eta_0^3|U|^{\frac{5}{3}}} + \frac{\xi_2\xi_3}{4\eta_0^5|U|^{\frac{8}{3}}} \leq \frac{1}{|U|^{\frac{5}{3}}} \left(\frac{2}{27\eta_0^3} + \frac{\xi_2\xi_3}{4\eta_0^5\kappa} \right) := \frac{\eta_7}{|U|^{\frac{5}{3}}}, \\ |\partial_{WY} \mathcal{K}(U, Z^u)| &\leq \frac{\eta_7}{|U|^{\frac{5}{3}}} \end{aligned}$$

and

$$\begin{aligned} |\partial_{XY} \mathcal{K}(U, Z^u)| &\leq \frac{5}{27\eta_0^3|U|^2} + \frac{\xi_3^2}{4\eta_0^5|U|^2} := \frac{\eta_8}{|U|^2}, \\ |\partial_X^2 \mathcal{K}(U, Z^u)| &\leq \frac{1}{9\eta_0^3|U|^2} + \frac{\xi_3^2}{4\eta_0^5|U|^2} := \frac{\eta_9}{|U|^2}, \\ |\partial_Y^2 \mathcal{K}(U, Z^u)| &\leq \frac{\eta_9}{|U|^2}. \end{aligned}$$

Since ξ_0, \dots, ξ_5 are decreasing functions for κ and increasing for ϱ , by construction one can see that the functions η_1, \dots, η_9 are increasing for $\varrho_1 \in (1, 60)$ and $\varrho_2 \in (1, 3)$ and decreasing for $\kappa \geq 3$. \square

The two previous lemmas provide the necessary bounds to estimate the derivatives of the function \mathcal{R} .

Lemma 4.11. *Assume $\varrho_1 \in (1, 60)$, $\varrho_2 \in (1, 3)$ and $\kappa \geq 3$. Then, for $Z^u \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$, one has that*

$$\begin{aligned} \|\partial_W \mathcal{R}_1[Z^u]\|_3 &\leq \nu_1, & \|\partial_X \mathcal{R}_1[Z^u]\|_{\frac{7}{3}} &\leq \nu_2, & \|\partial_Y \mathcal{R}_1[Z^u]\|_{\frac{7}{3}} &\leq \nu_2, \\ \|\partial_W \mathcal{R}_2[Z^u]\|_{\frac{2}{3}} &\leq \nu_3, & \|\partial_X \mathcal{R}_2[Z^u]\|_2 &\leq \nu_4, & \|\partial_Y \mathcal{R}_2[Z^u]\|_2 &\leq \nu_5, \\ \|\partial_W \mathcal{R}_3[Z^u]\|_{\frac{2}{3}} &\leq \nu_3, & \|\partial_X \mathcal{R}_3[Z^u]\|_2 &\leq \nu_5, & \|\partial_Y \mathcal{R}_3[Z^u]\|_2 &\leq \nu_4, \end{aligned}$$

where

$$\begin{aligned} \nu_0 &= \left(1 - \frac{\eta_2}{\kappa^2}\right)^2, \\ \nu_1 &= \frac{\eta_4}{\nu_0} + \frac{\eta_2 \eta_4}{\nu_0 \kappa^2} + \frac{\eta_1 \eta_6}{\nu_0}, \\ \nu_2 &= \frac{\eta_5}{\nu_0} + \frac{\eta_2 \eta_5}{\nu_0 \kappa^2} + \frac{\eta_1 \eta_7}{\nu_0 \kappa^3}, \\ \nu_3 &= \frac{\eta_7}{\nu_0 \kappa} + \frac{\eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\beta \eta_6}{\nu_0} + \frac{2\beta \eta_2 \eta_6}{\nu_0 \kappa^2} + \frac{\eta_3 \eta_6}{\nu_0}, \\ \nu_4 &= \frac{\eta_8}{\nu_0} + \frac{\eta_2 \eta_8}{\nu_0 \kappa^2} + \frac{\eta_2}{\nu_0} + \frac{\eta_2^2}{\nu_0 \kappa^2} + \frac{\beta \eta_7}{\nu_0 \kappa} + \frac{2\beta \eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\eta_3 \eta_7}{\nu_0 \kappa}, \\ \nu_5 &= \frac{\eta_9}{\nu_0} + \frac{\eta_2 \eta_9}{\nu_0 \kappa^2} + \frac{\beta \eta_7}{\nu_0 \kappa} + \frac{2\beta \eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\eta_3 \eta_7}{\nu_0 \kappa}. \end{aligned}$$

Moreover, for $\varrho_1 \in (1, 60)$, $\varrho_2 \in (1, 3)$ and $\kappa \geq 3$, the functions ν_1, \dots, ν_5 are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 .

Proof. Let us first look for a lower bound for the denominator $1 + \partial_W \mathcal{K}(U, Z^u)$. Notice that,

$$|\partial_W \mathcal{K}(U, Z^u)| \leq \frac{\eta_2}{\kappa^2} \Bigg|_{\kappa=3, \varrho_1=60, \varrho_2=3} < 1.$$

Then, by the triangular inequality,

$$|1 + \partial_W \mathcal{K}(U, Z^u)|^2 \geq \left(1 - \frac{\eta_2}{\kappa^2}\right)^2 := \nu_0.$$

We now analyze the derivatives of $\mathcal{R}_1[Z^u]$. Indeed, by the formulae in Appendix A, one has that

$$\begin{aligned} |\partial_W \mathcal{R}_1[Z^u](U)| &\leq \frac{\eta_4}{\nu_0 |U|^3} + \frac{\eta_2 \eta_4}{\nu_0 |U|^5} + \frac{\eta_1 \eta_6}{\nu_0 |U|^3} \\ &\leq \frac{1}{|U|^3} \left(\frac{\eta_4}{\nu_0} + \frac{\eta_2 \eta_4}{\nu_0 \kappa^2} + \frac{\eta_1 \eta_6}{\nu_0} \right) := \frac{\nu_1}{|U|^3}, \\ |\partial_X \mathcal{R}_1[Z^u](U)| &\leq \frac{\eta_5}{\nu_0 |U|^{\frac{7}{3}}} + \frac{\eta_2 \eta_5}{\nu_0 |U|^{\frac{13}{3}}} + \frac{\eta_1 \eta_7}{\nu_0 |U|^{\frac{16}{3}}} \\ &\leq \frac{1}{|U|^{\frac{7}{3}}} \left(\frac{\eta_5}{\nu_0} + \frac{\eta_2 \eta_5}{\nu_0 \kappa^2} + \frac{\eta_1 \eta_7}{\nu_0 \kappa^3} \right) := \frac{\nu_2}{|U|^{\frac{7}{3}}}, \end{aligned}$$

$$|\partial_Y \mathcal{R}_1[Z^u](U)| \leq \frac{\nu_2}{|U|^{\frac{7}{3}}}.$$

The derivatives of $\mathcal{R}_2[Z^u]$ satisfy

$$\begin{aligned} |\partial_W \mathcal{R}_2[Z^u](U)| &\leq \frac{1}{\nu_0} \left(\frac{\eta_7}{|U|^{\frac{5}{3}}} + \frac{\beta\eta_6}{|U|^{\frac{2}{3}}} \right) \left(1 + \frac{\eta_2}{|U|^2} \right) + \frac{1}{\nu_0} \left(\frac{\eta_3}{|U|^{\frac{4}{3}}} + \frac{\beta\eta_2}{|U|^{\frac{10}{3}}} \right) \eta_6 |U|^{\frac{2}{3}} \\ &\leq \frac{1}{|U|^{\frac{2}{3}}} \left(\frac{\eta_7}{\nu_0 \kappa} + \frac{\eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\beta\eta_6}{\nu_0} + \frac{2\beta\eta_2 \eta_6}{\nu_0 \kappa^2} + \frac{\eta_3 \eta_6}{\nu_0} \right) := \frac{\nu_3}{|U|^{\frac{2}{3}}}, \\ |\partial_X \mathcal{R}_2[Z^u](U)| &\leq \frac{1}{\nu_0} \left(\frac{\eta_8}{|U|^2} + \frac{\eta_2}{|U|^2} + \frac{\beta\eta_7}{|U|^3} \right) \left(1 + \frac{\eta_2}{|U|^2} \right) \\ &\quad + \frac{1}{\nu_0} \left(\frac{\eta_3}{|U|^{\frac{4}{3}}} + \frac{\beta\eta_2}{|U|^{\frac{10}{3}}} \right) \frac{\eta_7}{|U|^{\frac{5}{3}}} \\ &\leq \frac{1}{|U|^2} \left(\frac{\eta_8}{\nu_0} + \frac{\eta_2 \eta_8}{\nu_0 \kappa^2} + \frac{\eta_2}{\nu_0} + \frac{\eta_2^2}{\nu_0 \kappa^2} + \frac{\beta\eta_7}{\nu_0 \kappa} + \frac{2\beta\eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\eta_3 \eta_7}{\nu_0 \kappa} \right) \\ &:= \frac{\nu_4}{|U|^2}, \\ |\partial_Y \mathcal{R}_2[Z^u](U)| &\leq \frac{1}{\nu_0} \left(\frac{\eta_9}{|U|^2} + \frac{\beta\eta_7}{|U|^3} \right) \left(1 + \frac{\eta_2}{|U|^2} \right) + \frac{1}{\nu_0} \left(\frac{\eta_3}{|U|^{\frac{4}{3}}} + \frac{\beta\eta_2}{|U|^{\frac{10}{3}}} \right) \frac{\eta_7}{|U|^{\frac{5}{3}}} \\ &\leq \frac{1}{|U|^2} \left(\frac{\eta_9}{\nu_0} + \frac{\eta_2 \eta_9}{\nu_0 \kappa^2} + \frac{\beta\eta_7}{\nu_0 \kappa} + \frac{2\beta\eta_2 \eta_7}{\nu_0 \kappa^3} + \frac{\eta_3 \eta_7}{\nu_0 \kappa} \right) := \frac{\nu_5}{|U|^2}. \end{aligned}$$

An analogous procedure leading to the same estimates holds for the derivatives of \mathcal{R}_3 .

Finally, since η_1, \dots, η_9 are decreasing functions for κ and increasing for ϱ_1 and ϱ_2 , one can see that the functions ν_1, \dots, ν_5 are increasing for $\varrho_1 \in (1, 60)$ and $\varrho_2 \in (1, 3)$ and decreasing for $\kappa \geq 3$. \square

4.5. The Lipschitz constant of \mathcal{F} . The next lemma gives estimates for the Lipschitz constant of the operator \mathcal{F} in (16) with respect to each variable.

Proposition 4.12. *Assume $\gamma \in (0, \arctan \frac{\sqrt{3}}{2})$, $\kappa \geq 3$, $\varrho_1 \in (1, 60)$, $\varrho_2 \in (1, 3)$.*

Then, for any $Z^u, \tilde{Z}^u \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$, one has that

$$\begin{aligned} \|\mathcal{F}_1[Z^u] - \mathcal{F}_1[\tilde{Z}^u]\|_{\frac{8}{3}} &\leq \frac{\tilde{\nu}_1}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \tilde{\nu}_2 \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \tilde{\nu}_2 \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{F}_2[Z^u] - \mathcal{F}_2[\tilde{Z}^u]\|_{\frac{4}{3}} &\leq \frac{\tilde{\nu}_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\tilde{\nu}_4}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\tilde{\nu}_5}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{F}_3[Z^u] - \mathcal{F}_3[\tilde{Z}^u]\|_{\frac{4}{3}} &\leq \frac{\tilde{\nu}_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\tilde{\nu}_5}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\tilde{\nu}_4}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \end{aligned}$$

where

$$\tilde{\nu}_i = \nu_i \frac{\sqrt{\pi} \Gamma(\frac{4}{3})}{2\Gamma(\frac{11}{6})}, \quad \text{for } i = 1, 2, \quad \tilde{\nu}_j = \frac{\nu_j}{\sin \gamma (\cos \gamma)^{\frac{4}{3}}}, \quad \text{for } j = 3, 4, 5$$

and ν_j are the functions introduced in Lemma 4.11. Moreover, for $\kappa \geq 3$, $\varrho_1 \in (1, 60)$ and $\varrho_2 \in (1, 3)$, the functions $\tilde{\nu}_i$, $i = 1 \dots 5$, are positive functions decreasing in κ and increasing in ϱ_1 and ϱ_2 .

This implies that

$$\|\mathcal{F}[Z^u] - \mathcal{F}[\tilde{Z}^u]\|_\times \leq L\|Z^u - \tilde{Z}^u\|_\times,$$

where

$$L = \max \left\{ \frac{\tilde{\nu}_1}{\kappa^2} + 2\tilde{\nu}_2, \frac{\tilde{\nu}_3 + \tilde{\nu}_4 + \tilde{\nu}_5}{\kappa^2} \right\}. \quad (24)$$

Proof. In order to estimate the Lipschitz constant, we first estimate each component $\mathcal{R}_j[Z^u] - \mathcal{R}_j[\tilde{Z}^u]$ separately for $j = 1, 2, 3$. By the mean value theorem we have

$$\mathcal{R}_j[Z^u] - \mathcal{R}_j[\tilde{Z}^u] = \left[\int_0^1 D\mathcal{R}_j[sZ^u + (1-s)\tilde{Z}^u] ds \right] (Z^u - \tilde{Z}^u).$$

Then, for $j = 2, 3$,

$$\begin{aligned} & \|\mathcal{R}_1[Z^u] - \mathcal{R}_1[\tilde{Z}^u]\|_{\frac{11}{3}} \\ & \leq \|W^u - \tilde{W}^u\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_1[\varphi]\|_1 \\ & \quad + \|X^u - \tilde{X}^u\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_1[\varphi]\|_{\frac{7}{3}} + \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_1[\varphi]\|_{\frac{7}{3}}, \\ & \|\mathcal{R}_j[Z^u] - \mathcal{R}_j[\tilde{Z}^u]\|_{\frac{4}{3}} \\ & \leq \|W^u - \tilde{W}^u\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_j[\varphi]\|_{-\frac{4}{3}} \\ & \quad + \|X^u - \tilde{X}^u\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_j[\varphi]\|_0 + \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_j[\varphi]\|_0. \end{aligned}$$

Applying Lemma 4.11, we obtain

$$\begin{aligned} \|\mathcal{R}_1[Z^u] - \mathcal{R}_1[\tilde{Z}^u]\|_{\frac{11}{3}} & \leq \frac{\nu_1}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \nu_2 \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \nu_2 \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{R}_2[Z^u] - \mathcal{R}_2[\tilde{Z}^u]\|_{\frac{4}{3}} & \leq \frac{\nu_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\nu_4}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\nu_5}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{R}_3[Z^u] - \mathcal{R}_3[\tilde{Z}^u]\|_{\frac{4}{3}} & \leq \frac{\nu_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\nu_5}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\nu_4}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}. \end{aligned}$$

Finally, applying Lemma 4.4, for all $0 < \sigma \leq \gamma$, we obtain the following estimates

$$\begin{aligned} \|\mathcal{F}_1[Z^u] - \mathcal{F}_1[\tilde{Z}^u]\|_{\frac{8}{3}} & \leq \frac{\sqrt{\pi}\Gamma(\frac{4}{3})}{2\Gamma(\frac{11}{6})} \left[\frac{\nu_1}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} \right. \\ & \quad \left. + \nu_2 \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \nu_2 \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}} \right], \\ \|\mathcal{F}_2[Z^u] - \mathcal{F}_2[\tilde{Z}^u]\|_{\frac{4}{3}} & \leq \frac{1}{\sin \sigma (\cos \sigma)^{\frac{4}{3}}} \left[\frac{\nu_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} \right. \\ & \quad \left. + \frac{\nu_4}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\nu_5}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}} \right], \\ \|\mathcal{F}_3[Z^u] - \mathcal{F}_3[\tilde{Z}^u]\|_{\frac{4}{3}} & \leq \frac{1}{\sin \sigma (\cos \sigma)^{\frac{4}{3}}} \left[\frac{\nu_3}{\kappa^2} \|W^u - \tilde{W}^u\|_{\frac{8}{3}} \right. \\ & \quad \left. + \frac{\nu_5}{\kappa^2} \|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\nu_4}{\kappa^2} \|Y^u - \tilde{Y}^u\|_{\frac{4}{3}} \right]. \end{aligned}$$

Let us notice that $f(\sigma) = \frac{1}{\sin \sigma (\cos \sigma)^{\frac{4}{3}}}$ has a minimum at $\sigma_* = \arctan(\sqrt{3}/2)$ and is decreasing for $\sigma \in (0, \gamma)$ since we are assuming that $\gamma \in (0, \arctan(\sqrt{3}/2))$.

Then, defining

$$\begin{aligned}\tilde{\nu}_1 &= \frac{\sqrt{\pi}\Gamma(\frac{4}{3})}{2\Gamma(\frac{11}{6})}\nu_1, & \tilde{\nu}_2 &= \frac{\sqrt{\pi}\Gamma(\frac{4}{3})}{2\Gamma(\frac{11}{6})}\nu_2, \\ \tilde{\nu}_3 &= f(\gamma)\nu_3, & \tilde{\nu}_4 &= f(\gamma)\nu_4, & \tilde{\nu}_5 &= f(\gamma)\nu_5,\end{aligned}$$

one has that

$$\begin{aligned}\|\mathcal{F}_1[Z^u] - \mathcal{F}_1[\tilde{Z}^u]\|_{\frac{8}{3}} &\leq \frac{\tilde{\nu}_1}{\kappa^2}\|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \tilde{\nu}_2\|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \tilde{\nu}_2\|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{F}_2[Z^u] - \mathcal{F}_2[\tilde{Z}^u]\|_{\frac{4}{3}} &\leq \frac{\tilde{\nu}_3}{\kappa^2}\|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\tilde{\nu}_4}{\kappa^2}\|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\tilde{\nu}_5}{\kappa^2}\|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}, \\ \|\mathcal{F}_3[Z^u] - \mathcal{F}_3[\tilde{Z}^u]\|_{\frac{4}{3}} &\leq \frac{\tilde{\nu}_3}{\kappa^2}\|W^u - \tilde{W}^u\|_{\frac{8}{3}} + \frac{\tilde{\nu}_5}{\kappa^2}\|X^u - \tilde{X}^u\|_{\frac{4}{3}} + \frac{\tilde{\nu}_4}{\kappa^2}\|Y^u - \tilde{Y}^u\|_{\frac{4}{3}}.\end{aligned}$$

□

From now on, we emphasize that all the constants ξ_j, η_j, ν_j and $\tilde{\nu}_j$ are in fact functions of $\varrho_1, \varrho_2, \kappa$ and γ . From now on we will write this dependence explicitly.

To apply Proposition 4.12 we first need to impose that $\mathcal{F} : \mathbf{R}_\kappa(\varrho_1, \varrho_2) \rightarrow \mathbf{R}_\kappa(\varrho_1, \varrho_2)$ is well defined.

Proposition 4.13. *Assume $\varrho_1 \in (1, 60)$, $\varrho_2 \in (1, 3)$, $\kappa \geq 3$, $\gamma \in (0, \arctan(\frac{\sqrt{3}}{2}))$ and denote*

$$\begin{aligned}g_1(\kappa, \varrho_1, \varrho_2, \gamma) &= \left(\varrho_1 - 1 - \frac{\tilde{\nu}_1(\kappa, \varrho_1, \varrho_2)}{\kappa^2} \varrho_1 \right) \alpha_0(\kappa) - 2\tilde{\nu}_2(\kappa, \varrho_1, \varrho_2) \varrho_2 \beta_0(\kappa), \\ g_2(\kappa, \varrho_1, \varrho_2, \gamma) &= \left(\varrho_2 - 1 - \frac{\tilde{\nu}_4(\kappa, \varrho_1, \varrho_2, \gamma) + \tilde{\nu}_5(\kappa, \varrho_1, \varrho_2, \gamma)}{\kappa^2} \varrho_2 \right) \beta_0(\kappa) \\ &\quad - \frac{\tilde{\nu}_3(\kappa, \varrho_1, \varrho_2, \gamma)}{\kappa^2} \varrho_1 \alpha_0(\kappa).\end{aligned}$$

Then, $\mathcal{F} : \mathbf{R}_\kappa(\varrho_1, \varrho_2) \rightarrow \mathbf{R}_\kappa(\varrho_1, \varrho_2)$ is well-defined provided

$$g_1(\kappa, \varrho_1, \varrho_2, \gamma) \geq 0 \quad \text{and} \quad g_2(\kappa, \varrho_1, \varrho_2, \gamma) \geq 0. \quad (25)$$

Proof. Let $Z^u \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$. Then, by Propositions 4.6, 4.12 and for $j = 2, 3$,

$$\begin{aligned}\|\mathcal{F}_1[Z^u]\|_{\frac{8}{3}} &\leq \|\mathcal{F}_1[Z^u] - \mathcal{F}_1[0]\|_{\frac{8}{3}} + \|\mathcal{F}_1[0]\|_{\frac{8}{3}} \leq \left(1 + \frac{\tilde{\nu}_1}{\kappa^2} \varrho_1 \right) \alpha_0(\kappa) + 2\tilde{\nu}_2 \varrho_2 \beta_0(\kappa), \\ \|\mathcal{F}_j[Z^u]\|_{\frac{4}{3}} &\leq \|\mathcal{F}_j[Z^u] - \mathcal{F}_j[0]\|_{\frac{4}{3}} + \|\mathcal{F}_j[0]\|_{\frac{4}{3}} \\ &\leq \frac{\tilde{\nu}_3}{\kappa^2} \varrho_1 \alpha_0(\kappa) + \left(1 + \frac{\tilde{\nu}_4 + \tilde{\nu}_5}{\kappa^2} \varrho_2 \right) \beta_0(\kappa).\end{aligned}$$

We need to impose that $\mathcal{F}[Z^u] \in \mathbf{R}_\kappa(\varrho_1, \varrho_2)$, which leads to the conditions

$$\|\mathcal{F}_1[Z^u]\|_{\frac{8}{3}} \leq \varrho_1 \alpha_0(\kappa) \quad \text{and} \quad \|\mathcal{F}_j[Z^u]\|_{\frac{4}{3}} \leq \varrho_2 \beta_0(\kappa).$$

□

Remark 4.14. Let $\varrho_1 = 40$, $\varrho_2 = 2$ and $\gamma = 0.5$ and consider the functions g_1 and g_2 considered in Proposition 4.13. Then,

$$\begin{aligned}g_1(6, 40, 2, 0.5) &\approx -0.0626, & g_2(6, 40, 2, 0.5) &\approx -0.0665, \\ g_1(7, 40, 2, 0.5) &\approx 0.1613, & g_2(7, 40, 2, 0.5) &\approx 0.1836,\end{aligned}$$

$$g_1(8, 40, 2, 0.5) \approx 0.2851, \quad g_2(8, 40, 2, 0.5) \approx 0.3226.$$

Notice that we take $\varrho_1 = 20\varrho_2$. This ratio is considered because

$$\lim_{\kappa \rightarrow +\infty} \frac{\beta_0(\kappa)}{\alpha_0(\kappa)} = \frac{243}{8} \left(\frac{2}{9} + \frac{14\sqrt{\pi} \Gamma(\frac{2}{3})}{81 \Gamma(\frac{7}{6})} \right) \approx 20.3323.$$

However, other ratios may be considered.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. The first step to prove the theorem is to choose suitable constant so that we can apply Proposition 4.13.

Indeed, choosing $\kappa = \kappa^* = 6.24$ (see (13)), $\gamma = 0.5$, $\varrho_1 = 20\varrho_2$ and $\varrho_2 = 1.9$, one has that (25) is satisfied. Indeed

$$g_1(\kappa, \varrho_1, \varrho_2, \gamma) > 0.0371 \quad \text{and} \quad g_2(\kappa, \varrho_1, \varrho_2, \gamma) > 0.0047.$$

These values are chosen to obtain a small value for $\rho^*(\kappa^*, \gamma)$ (see (11)).

Moreover, the constant L in (24) satisfies

$$0 < L \leq 0.93 < 1$$

and therefore, by Proposition 4.12, the operator \mathcal{F} is contractive from $\mathbf{R}_\kappa(\varrho_1, \varrho_2)$ to itself. Thus, it has a unique fixed point. This completes the proof of Theorem 3.1. \square

5. Difference between the solutions of the inner equation. To prove Theorem 3.2, the first step is to provide for a good approximation of the solution of the inner equation “close to infinity”. To this end, we define the domains

$$\mathcal{D}_{\kappa, \eta}^u = \{U \in \mathcal{D}_\kappa^u : \operatorname{Re} U \leq -\eta\}, \quad \mathcal{D}_{\kappa, \eta}^s = -\mathcal{D}_{\kappa, \eta}^u,$$

where $\mathcal{D}_\kappa^u, \mathcal{D}_\kappa^s$ are the domains introduced in (10) and $\eta > \kappa$.

We provide an analogue of Theorem 3.1 in these smaller domains with large η . This will provide lower values for the constants b_1, b_2 .

Proposition 5.1. *The functions $Z^\diamond(U) = (W^\diamond(U), X^\diamond(U), Y^\diamond(U))^T$, $\diamond = u, s$, introduced in Theorem 2.2 are defined in $\mathcal{D}_{\kappa^*, \eta^*}^\diamond$ for*

$$\kappa^* = 6.24, \quad \gamma = \frac{1}{2} \quad \text{and} \quad \eta^* = 1000.$$

In addition, they satisfy that, for $U \in \mathcal{D}_{\kappa^, \eta^*}^\diamond$,*

$$|U^{\frac{8}{3}} W^\diamond(U)| \leq \tilde{b}_1, \quad |U^{\frac{4}{3}} X^\diamond(U)| \leq \tilde{b}_2, \quad |U^{\frac{4}{3}} Y^\diamond(U)| \leq \tilde{b}_2,$$

where

$$\tilde{b}_1 \leq 0.7, \quad \tilde{b}_2 \leq 0.71.$$

The proof of this proposition follows exactly the same lines as the proof of Theorem 3.1. Indeed, it is enough to point out that the only difference is that to prove the theorem we strongly used that

$$\operatorname{dist}(\mathcal{D}_\kappa^u, 0) \geq \kappa,$$

whereas now

$$\operatorname{dist}(\mathcal{D}_{\kappa, \eta}^u, 0) \geq \eta.$$

Taking this fact into account, the proof goes through verbatim replacing κ by η in the estimates.

To validate the condition (14) from Theorem 3.2 we use the bounds on Z^u and Z^s from Proposition 5.1 which are valid in the domains $\mathcal{D}_{\kappa^*, \eta^*}^u$ and $\mathcal{D}_{\kappa^*, \eta^*}^s$, respectively, and propagate them to the section $\{\operatorname{Re} U = 0\}$ by means of an interval arithmetic integrator, and establish that the distance between them is non zero. Our tool of choice for this task is the CAPD² library [12].

The CAPD integrator can work with vector fields defined in reals, so our first step is to rewrite the vector field for the inner equation (6) in real form. The method for doing so is to separate the real and imaginary parts of the equations. To achieve this aim we consider two additional complex variables

$$A = \frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} \quad \text{and} \quad B = U^{-\frac{1}{3}}.$$

With these variables we can introduce the following notation

$$\tilde{\mathcal{H}}(W, X, Y, A, B) = W + XY + \tilde{\mathcal{K}}(W, A, B)$$

and

$$\tilde{\mathcal{K}}(W, A, B) = -\frac{3}{4} \frac{1}{B^2} W^2 - \frac{1}{3} B^2 (A - 1). \quad (26)$$

Remark 5.2. Notice that, by introducing A and B , we have achieved that

$$\mathcal{H}(U, W, X, Y) = \tilde{\mathcal{H}}(W, X, Y, A(U, W, X, Y), B(U)).$$

with \mathcal{H} as given in (3). In addition, $\tilde{\mathcal{H}}$ is polynomial with the only exception of the term involving B^{-2} . This term will not present problems in the separation of the real parts from the imaginary parts of the vector field in the coordinates (U, W, X, Y, A, B) , since it is easy to separate complex numbers z^{-2} and z^{-3} into their real and imaginary parts.

We also write

$$\begin{aligned} \tilde{\mathcal{J}}(W, X, Y, B) &= \frac{4}{9} W^2 B^2 - \frac{16}{27} W B^4 + \frac{16}{81} B^6 + \frac{4}{9} (X + Y) B^3 \left(W - \frac{2}{3} B^2 \right) \\ &\quad - \frac{4}{3} i (X - Y) B^2 - \frac{1}{3} (X^2 + Y^2) B^4 + \frac{10}{9} X Y B^4. \end{aligned}$$

Note that

$$\mathcal{J}(U, W, X, Y) = \tilde{\mathcal{J}}(W, X, Y, B(U)),$$

where \mathcal{J} is given in (5)

To derive the formulae for the vector field with the two additional variables A and B , we first observe that

$$\begin{aligned} \frac{\partial A}{\partial x} &= -\frac{1}{2} \frac{1}{\left(1 + \tilde{\mathcal{J}}(W, X, Y, B)\right)^{\frac{3}{2}}} \frac{\partial \tilde{\mathcal{J}}}{\partial x}(W, X, Y, B) \\ &= -\frac{1}{2} A^3 \frac{\partial \tilde{\mathcal{J}}}{\partial x}(W, X, Y, B), \quad \text{for } x \in \{W, X, Y, B\}, \\ \frac{\partial A}{\partial U} &= \frac{\partial A}{\partial B}(W, X, Y, B) \frac{\partial B}{\partial U}(B), \quad \text{where} \quad \frac{\partial B}{\partial U}(B) = -\frac{1}{3} B^4. \end{aligned} \quad (27)$$

²<http://capd.ii.uj.edu.pl>

We can now write the ODE (6) in the new variables as

$$\begin{aligned}
\dot{U} &= \frac{\partial \mathcal{H}}{\partial W} = \frac{\partial \tilde{\mathcal{H}}}{\partial W} + \frac{\partial \tilde{\mathcal{H}}}{\partial A} \frac{\partial A}{\partial W}, \\
\dot{W} &= -\frac{\partial \mathcal{H}}{\partial \bar{U}} = -\left(\frac{\partial \tilde{\mathcal{H}}}{\partial \bar{B}} \frac{\partial B}{\partial \bar{U}} + \frac{\partial \tilde{\mathcal{H}}}{\partial A} \frac{\partial A}{\partial \bar{U}} \right), \\
\dot{X} &= i \frac{\partial \mathcal{H}}{\partial Y} = i \left(\frac{\partial \tilde{\mathcal{H}}}{\partial Y} + \frac{\partial \tilde{\mathcal{H}}}{\partial A} \frac{\partial A}{\partial Y} \right), \\
\dot{Y} &= -i \frac{\partial \mathcal{H}}{\partial \bar{X}} = -i \left(\frac{\partial \tilde{\mathcal{H}}}{\partial \bar{X}} + \frac{\partial \tilde{\mathcal{H}}}{\partial A} \frac{\partial A}{\partial \bar{X}} \right), \\
\dot{A} &= \frac{\partial A}{\partial U} \dot{U} + \frac{\partial A}{\partial W} \dot{W} + \frac{\partial A}{\partial X} \dot{X} + \frac{\partial A}{\partial Y} \dot{Y}, \\
\dot{B} &= \frac{\partial B}{\partial U} \dot{U}.
\end{aligned} \tag{28}$$

Before we discuss expressing (28) as a vector field on reals, let us sidetrack to make a useful comment. To do so, let us introduce a function $S : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ defined as

$$S(U, W, X, Y, A, B) := (-\bar{U}, \bar{W}, -\bar{X}, -\bar{Y}, \bar{A}, -\bar{B}).$$

(In the above, for $z \in \mathbb{C}$ we use \bar{z} for the complex conjugate.) It turns out that (28) has a time reversing symmetry with respect to S . This will be useful later on, and is expressed in the following lemma and a resulting corollary.

Lemma 5.3. *Let $F : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ stand for the right hand side of the vector field (28). Then*

$$S \circ F = -F \circ S.$$

Proof. The result follows from (lengthy but elementary) direct validation. \square

Corollary 5.4. *The manifolds $Z^u = (W^u, X^u, Y^u)$ and $Z^s = (W^s, X^s, Y^s)$ are symmetric in the following sense: for $U \in \mathcal{D}_{\kappa^*}^u$*

$$W^s(-\bar{U}) = \overline{W^u(U)}, \quad X^s(-\bar{U}) = -\overline{X^u(U)}, \quad Y^s(-\bar{U}) = -\overline{Y^u(U)}.$$

Let us also observe that from (27) it follows that the right hand side of the vector field (28) do not depend on U , which means that we can consider only the five coordinates W, X, Y, A, B obtaining an ODE in \mathbb{C}^5 ; instead of an ODE in \mathbb{C}^6 .

The right hand side of (28) is ‘almost’ polynomial, with the exception of the terms involving B^{-2} and B^{-3} , (these terms come from $\tilde{\mathcal{K}}$ and its derivatives, see (26)). Since for a complex number $z = a + ib$ we have explicit formulae for the real and imaginary parts of z^{-2} and z^{-3} , namely

$$\begin{aligned}
z^{-2} &= (a + ib)^{-2} = |z|^{-4} (a^2 - b^2 - i2ab), \\
z^{-3} &= (a + ib)^{-3} = |z|^{-6} (a(a^2 - 3b^2) + ib(b^2 - 3a^2)),
\end{aligned} \tag{29}$$

we see that a computation of elementary sums and products of complex numbers, combined with (29), leads to the separation of the real and complex parts on the right hand side of (28).

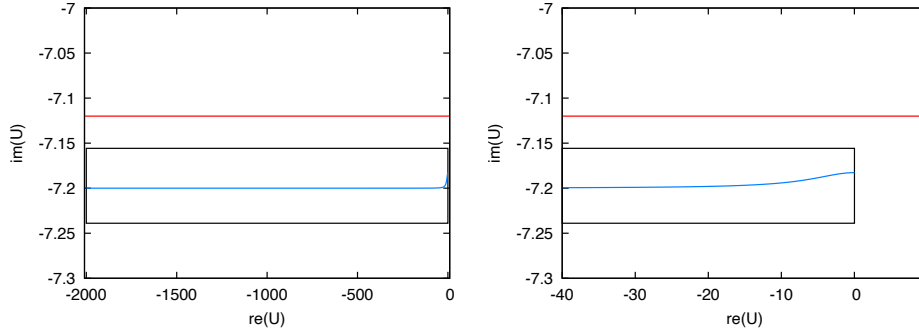


FIGURE 4. The bound on the domain on U within which our trajectory resides is depicted as the black box. The red line is $\text{Im } U = -\rho_0$. In blue we have a non-rigorous plot of the trajectory, which is added to the figure as a point of reference.

Such computation is laborious. (Especially for the formula for \dot{A} .) We have not performed it by hand, but have used Wolfram Mathematica [19] to perform these manipulations. We emphasise that this does not require any sophisticated computations apart from multiplying complex numbers and grouping the resulting terms into real and complex parts. We treat the results returned by Wolfram Mathematica as reliable; in fact more reliable than if they were performed by us by hand. We enclose a short Wolfram Mathematica script, together with our code, with which we have performed the symbolic derivation of the separation of (28) into the real and complex parts³. We use the resulting vector field for our CAPD interval arithmetic computations.

Our objective is to compute a bound on $\|\Delta Z(-i\rho)\|$ for some ρ satisfying that $\rho > \rho_0 = 7.12$. To do so, we first observe that by the S -symmetry of the system (see Corollary 5.4) we have

$$\|\Delta Z(-i\rho)\| \geq |\text{Re}(\Delta Y(-i\rho))| = 2|\text{Re}(Y^u(-i\rho))|. \quad (30)$$

So, it is enough to show that

$$|\text{Re}(Y^u(-i\rho))| > 0, \quad (31)$$

for some $\rho > \rho_0 = 7.12$.

To compute a bound on $Y^u(-i\rho)$ we proceed as follows. First, we choose an initial point

$$U_0 := -2000 - i\rho_0.$$

Then, from Proposition 5.1 we know that $Z^u(U_0)$ is inside of the set

$$Z^u(U_0) \in \mathbf{Z}_0 := \left\{ (W, X, Y) : |W| \leq \tilde{b}_1 |U_0|^{-\frac{8}{3}}, |X| \leq \tilde{b}_2 |U_0|^{-\frac{4}{3}}, |Y| \leq \tilde{b}_2 |U_0|^{-\frac{4}{3}} \right\}.$$

Let us write $\Gamma = \{\Gamma(t)\}$ for the trajectory starting from $(U_0, Z^u(U_0))$. Such trajectory is contained in the unstable manifold. We have obtained a bound on Γ by integrating the ODE (28) in interval arithmetic, with the initial condition chosen

³The code for the computer assisted part of the proof is available on the personal web page of MJC.

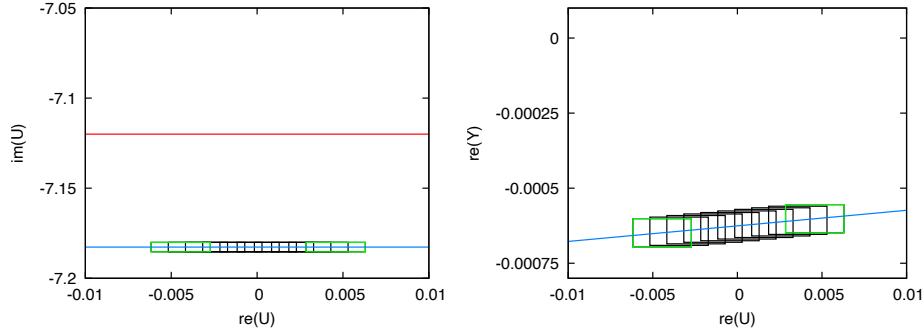


FIGURE 5. A closeup of the crossing of the trajectory through the section $\{\text{Re } U = 0\}$ projected onto U on the left (compare with Figure 4), and onto coordinates $(\text{Re } U, \text{Re } Y)$ on the right. In black we have the interval arithmetic bounds. In green, we have singled out the bounds on the trajectory for two disjoint time intervals, to demonstrate that it indeed does cross $\{\text{Re } U = 0\}$. In blue we have a non-rigorous plot of the trajectory, which is added to the figure as a point of reference.

as the set $\mathbf{Z}_0 \times \mathbf{A}_0 \times \mathbf{B}_0$, where

$$\mathbf{A}_0 := \left\{ \frac{1}{\sqrt{1 + \mathcal{J}(U_0, Z)}} \mid Z \in \mathbf{Z}_0 \right\} \quad \text{and} \quad \mathbf{B}_0 := \{U_0^{-1/3}\}.$$

We make sure that the interval arithmetic bound on Γ is always in $\{\text{Im } U < \rho_0\}$ (see Figure 4) and that it passes through⁴ $\{\text{Re } U = 0, \text{Im } U \in [-7.186, -7.18]\}$ (see Figure 5, left). We also obtain the following bound (see Figure 5, right)

$$\text{Re } \pi_Y(\Gamma \cap \{\text{Re } U = 0\}) \in [-0.00075, -0.0005],$$

where π_Y denotes the projection into the Y coordinate, which means that

$$\text{Re}(Y^u(-i\rho)) \in [-0.00075, -0.0005], \quad \text{for } \rho \in [-7.186, -7.18].$$

This implies (31) and we thus obtain (30). This concludes the proof of Theorem 3.2.

The computer assisted computation took a minute on a standard laptop. The vast majority of this time was spent to integrate in interval arithmetic from U_0 to reach the section $\{\text{Re } U = 0\}$. (Such integration requires to move along the flow for a time roughly equal to 2000; equal to the distance between U_0 and the section).

Appendix A. Explicit expressions for the remainder \mathcal{R} . We devote this appendix to provide formulas for the derivatives of the function \mathcal{R} introduced in (8).

⁴In our computer program we have validated that we cross the section $\{\text{Re } U = 0\}$ by using the Bolzano type argument, which is visualised in Figure 5. The CAPD library does have a built in method for obtaining bounds for a flow reaching a prescribed section, which is transverse to the flow [13], but these have failed in the case of our problem. This is why we have obtained the bound for crossing of $\{\text{Re } U = 0\}$ without their use. With our present technology the integration to the chosen section is close to the limit of what is achievable for us. This in particular means that the bounds obtained in Proposition 5.1 can not easily be extended beyond the chosen surface of section.

We denote $Z = (W, X, Y)$ and assume $U \in \mathcal{D}_\kappa^u$ (see (10)). The function \mathcal{R} is defined as

$$\mathcal{R}[Z](U) = \left(\frac{f_1(U, Z)}{1 + g(U, Z)}, \frac{\tilde{f}_2(U, Z)}{1 + g(U, Z)}, \frac{\tilde{f}_3(U, Z)}{1 + g(U, Z)} \right), \quad (32)$$

where

$$\tilde{f}_2(U, Z) = f_2(U, Z) - iXg(U, Z), \quad \tilde{f}_3(U, Z) = f_3(U, Z) + iYg(U, Z)$$

and

$$f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T, \quad g = \partial_W \mathcal{K}$$

and \mathcal{K} is the Hamiltonian given in (4) in terms of the function \mathcal{J} (see (5)).

To give formulas for the derivatives of \mathcal{R} , we first compute the second derivatives of \mathcal{J} and \mathcal{K} .

Formulae for $\mathcal{J}(U, Z)$. The function \mathcal{J} given in (5) is defined as

$$\begin{aligned} \mathcal{J}(U, Z) &= \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U} \left(W - \frac{2}{3U^{\frac{2}{3}}} \right) \\ &\quad - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2 + Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}. \end{aligned}$$

Then, its first derivatives are given by

$$\begin{aligned} \partial_U \mathcal{J}(U, Z) &= -\frac{8W^2}{27U^{\frac{5}{3}}} + \frac{64W}{81U^{\frac{7}{3}}} - \frac{32}{81U^3} - \frac{4(X+Y)W}{9U^2} \\ &\quad + \frac{40(X+Y)}{81U^{\frac{8}{3}}} + \frac{8i(X-Y)}{9U^{\frac{5}{3}}} + \frac{4(X^2 + Y^2)}{9U^{\frac{7}{3}}} - \frac{40XY}{27U^{\frac{7}{3}}}, \\ \partial_W \mathcal{J}(U, Z) &= \frac{8W}{9U^{\frac{2}{3}}} - \frac{16}{27U^{\frac{4}{3}}} + \frac{4(X+Y)}{9U}, \\ \partial_X \mathcal{J}(U, Z) &= \frac{4W}{9U} - \frac{8}{27U^{\frac{5}{3}}} - \frac{4i}{3U^{\frac{2}{3}}} - \frac{2X}{3U^{\frac{4}{3}}} + \frac{10Y}{9U^{\frac{4}{3}}}, \\ \partial_Y \mathcal{J}(U, Z) &= \frac{4W}{9U} - \frac{8}{27U^{\frac{5}{3}}} + \frac{4i}{3U^{\frac{2}{3}}} - \frac{2Y}{3U^{\frac{4}{3}}} + \frac{10X}{9U^{\frac{4}{3}}} \end{aligned}$$

and the second derivatives are given by

$$\begin{aligned} \partial_{UW} \mathcal{J}(U, Z) &= -\frac{16W}{27U^{\frac{5}{3}}} + \frac{64}{81U^{\frac{7}{3}}} - \frac{4(X+Y)}{9U^2}, \\ \partial_{UX} \mathcal{J}(U, Z) &= -\frac{4W}{9U^2} + \frac{40}{81U^{\frac{8}{3}}} + \frac{8i}{9U^{\frac{5}{3}}} + \frac{8X}{9U^{\frac{7}{3}}} - \frac{40Y}{27U^{\frac{7}{3}}}, \\ \partial_{UY} \mathcal{J}(U, Z) &= -\frac{4W}{9U^2} + \frac{40}{81U^{\frac{8}{3}}} - \frac{8i}{9U^{\frac{5}{3}}} + \frac{8Y}{9U^{\frac{7}{3}}} - \frac{40X}{27U^{\frac{7}{3}}}, \\ \partial_W^2 \mathcal{J}(U, Z) &= \frac{8}{9U^{\frac{2}{3}}}, \quad \partial_{WX} \mathcal{J}(U, Z) = \frac{4}{9U}, \quad \partial_{WY} \mathcal{J}(U, Z) = \frac{4}{9U}, \\ \partial_X^2 \mathcal{J}(U, Z) &= -\frac{2}{3U^{\frac{4}{3}}}, \quad \partial_{XY} \mathcal{J}(U, Z) = \frac{10}{9U^{\frac{4}{3}}}, \quad \partial_Y^2 \mathcal{J}(U, Z) = -\frac{2}{3U^{\frac{4}{3}}}. \end{aligned}$$

Formulae for \mathcal{K} . The Hamiltonian \mathcal{K} introduced in (4) is given by

$$\mathcal{K}(U, Z) = -\frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}}\left(\frac{1}{\sqrt{1+\mathcal{J}(U, Z)}} - 1\right).$$

Then, its first derivatives are

$$\begin{aligned}\partial_U\mathcal{K}(U, Z) &= -\frac{W^2}{2U^{\frac{1}{3}}} + \frac{2}{9U^{\frac{5}{3}}}\left(\frac{1}{\sqrt{1+\mathcal{J}}} - 1\right) + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_U\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}}, \\ &= -\frac{W^2}{2U^{\frac{1}{3}}} - \frac{2}{9U^{\frac{5}{3}}}\frac{\mathcal{J}}{\sqrt{1+\mathcal{J}}(1+\sqrt{1+\mathcal{J}})} + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_U\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}}, \\ \partial_W\mathcal{K}(U, Z) &= -\frac{3}{2}U^{\frac{2}{3}}W + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_W\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}}, \\ \partial_X\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_X\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}}, \\ \partial_Y\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_Y\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}}\end{aligned}$$

and its second derivatives are

$$\begin{aligned}\partial_{UW}\mathcal{K}(U, Z) &= -\frac{W}{U^{\frac{1}{3}}} - \frac{1}{9U^{\frac{5}{3}}}\frac{\partial_W\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{UW}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_U\mathcal{J} \cdot \partial_W\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_{UX}\mathcal{K}(U, Z) &= -\frac{1}{9U^{\frac{5}{3}}}\frac{\partial_X\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{UX}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_U\mathcal{J} \cdot \partial_X\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_{UY}\mathcal{K}(U, Z) &= -\frac{1}{9U^{\frac{5}{3}}}\frac{\partial_Y\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{UY}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_U\mathcal{J} \cdot \partial_Y\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_W^2\mathcal{K}(U, Z) &= -\frac{3}{2}U^{\frac{2}{3}} + \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_W^2\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{(\partial_W\mathcal{J})^2}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_{WX}\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{WX}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_W\mathcal{J} \cdot \partial_X\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_{WY}\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{WY}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_W\mathcal{J} \cdot \partial_Y\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_X^2\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_X^2\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{(\partial_X\mathcal{J})^2}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_{XY}\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_{XY}\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{\partial_X\mathcal{J} \cdot \partial_Y\mathcal{J}}{(1+\mathcal{J})^{\frac{5}{2}}}, \\ \partial_Y^2\mathcal{K}(U, Z) &= \frac{1}{6U^{\frac{2}{3}}}\frac{\partial_Y^2\mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - \frac{1}{4U^{\frac{2}{3}}}\frac{(\partial_Y\mathcal{J})^2}{(1+\mathcal{J})^{\frac{5}{2}}}.\end{aligned}$$

Formulae for the derivatives of \mathcal{R}

By the expression of $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ in (32), one obtains

$$\partial_W \mathcal{R}_1[Z](U) = -\frac{\partial_{UW}\mathcal{K}(1 + \partial_W\mathcal{K}) - \partial_U\mathcal{K} \cdot \partial_W^2\mathcal{K}}{(1 + \partial_W\mathcal{K})^2},$$

$$\partial_X \mathcal{R}_1[Z](U) = -\frac{\partial_{UX}\mathcal{K}(1 + \partial_W\mathcal{K}) - \partial_U\mathcal{K} \cdot \partial_{WX}\mathcal{K}}{(1 + \partial_W\mathcal{K})^2},$$

$$\partial_Y \mathcal{R}_1[Z](U) = -\frac{\partial_{UY}\mathcal{K}(1 + \partial_W\mathcal{K}) - \partial_U\mathcal{K} \cdot \partial_{WY}\mathcal{K}}{(1 + \partial_W\mathcal{K})^2}.$$

Analogously, for \mathcal{R}_2 and \mathcal{R}_3

$$\partial_W \mathcal{R}_2[Z](U) = i\frac{(\partial_{WY}\mathcal{K} - X \cdot \partial_W^2\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_Y\mathcal{K} - X\partial_W\mathcal{K})\partial_W^2\mathcal{K}}{(1 + \partial_W\mathcal{K})^2}$$

$$\partial_X \mathcal{R}_2[Z](U) = i\frac{(\partial_{XY}\mathcal{K} - \partial_W\mathcal{K} - X \cdot \partial_{WX}\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_Y\mathcal{K} - X\partial_W\mathcal{K})\partial_{WX}\mathcal{K}}{(1 + \partial_W\mathcal{K})^2}$$

$$\partial_Y \mathcal{R}_2[Z](U) = i\frac{(\partial_Y^2\mathcal{K} - X \cdot \partial_{WY}\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_Y\mathcal{K} - X\partial_W\mathcal{K})\partial_{WY}\mathcal{K}}{(1 + \partial_W\mathcal{K})^2}$$

$$\partial_W \mathcal{R}_3[Z](U) = \frac{(\partial_{WX}\mathcal{K} - Y \cdot \partial_W^2\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_X\mathcal{K} - Y\partial_W\mathcal{K})\partial_W^2\mathcal{K}}{i(1 + \partial_W\mathcal{K})^2}$$

$$\partial_X \mathcal{R}_3[Z](U) = \frac{(\partial_X^2\mathcal{K} - Y \cdot \partial_{WX}\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_X\mathcal{K} - Y\partial_W\mathcal{K})\partial_{WX}\mathcal{K}}{i(1 + \partial_W\mathcal{K})^2}$$

$$\partial_Y \mathcal{R}_3[Z](U) = \frac{(\partial_{XY}\mathcal{K} - \partial_W\mathcal{K} - Y \cdot \partial_{WY}\mathcal{K})(1 + \partial_W\mathcal{K}) - (\partial_X\mathcal{K} - Y\partial_W\mathcal{K})\partial_{WY}\mathcal{K}}{i(1 + \partial_W\mathcal{K})^2}.$$

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