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# On the asymptotic wavenumber of spiral waves in $\lambda - \omega$ systems

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## Abstract

In this paper we consider spiral wave solutions of a general class of $\lambda - \omega$ systems with a small twist parameter $q$ and we prove that the asymptotic wavenumber of the spirals is a $\mathcal{C}^\infty$-flat function of the perturbation parameter $q$.

Keywords: asymptotic methods, flat function, asymptotic wavenumber
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(Some figures may appear in colour only in the online journal)

## 1. Introduction

Rigidly rotating spiral waves are commonly found in many chemical systems and biological processes [Kur84, LJD91, Win72, ZZ70]. In particular they are most likely to occur in oscillatory models having a rotational symmetry, such as generic $\lambda - \omega$ systems [Kur84], [Sch98]. These can be derived as the normal form of oscillatory reaction-diffusion systems near a Hopf bifurcation and read:

$$u_t = \Delta u + \lambda(f)u - \omega(f)w, \tag{1}$$

$$w_t = \Delta w + \omega(f)u + \lambda(f)w, \tag{2}$$

where $u = u(x, y, t)$, $w = w(x, y, t)$, $\Delta$ denotes the Laplacian and $\lambda$ and $\omega$ are real functions of the modulus $f = \sqrt{u^2 + w^2}$. The conditions that $\lambda$ usually satisfies are: $\lambda(1) = 0$, to ensure

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that the system has a spatially independent limit solution and $\lambda' < 0$, to guarantee that this limit cycle is stable with respect to homogeneous (space independent) perturbations (see [NK81]). As for $\omega$, based on stability considerations, it is usually assumed that $|\omega'|$ is small.

Numerical computations reveal that the system (1) and (2) exhibits solutions in the shape of *n*-spirals (see for instance [BHO97, GB05]) and more precisely, in the shape of Archimedian spiral waves with a specific frequency $\Omega$. More concretely, these rigidly rotating solutions of (1) and (2) can then be written like

$$u(r, \phi, t) = f(r) \cos\left(\Omega t + n\phi - \int_0^r v(s)\, ds\right),$$

$$w(r, \phi, t) = f(r) \sin\left(\Omega t + n\phi - \int_0^r v(s)\, ds\right), \tag{3}$$

being $r = \sqrt{x^2 + y^2}$ the polar radius and $\phi$ the azimuth coordinate of the plane and thus the Laplacian can be expressed as $\Delta = \partial_{rr} + \partial_r/r + \partial_{\phi\phi}/r^2$. Therefore, since $f(r)$ plays the role of a modulus, $f(r) \geqslant 0 \, \forall \, r > 0$ and also $f(0) = 0$ in order for $u$ and $w$ to be regular at $r = 0$. Also, in order for these functions to have the shape of a spiral, the phase must increase or decrease monotonically as one moves away from the centre of the spiral and so $v(r)$, which is usually denoted as the *local wavenumber*, must have a constant sign for all $r \geqslant 0$. In the particular case where $n = 0$, the phase is purely radial (it only depends on $r$) and the solutions $u$, $w$ in (3) are usually denoted as *target patterns* since the lines of constant phase $\left(\int_0^r v(s) ds = c\right)$ become concentric rings of radius $r = r_c$, that is to say, along any radial line, the pattern is asymptotically that of a plane wave.

Substituting the particular expressions (3) in the partial differential equations (1) and (2) one obtains a set of ordinary differential equations in terms of the radial polar variable, $r \geqslant 0$, that reads

$$0 = f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + f(\lambda(f(r)) - v^2(r)), \tag{4}$$

$$0 = f(r)v'(r) + \frac{f(r)v(r)}{r} + 2f'(r)v(r) + f(r)(\Omega - \omega(f(r))). \tag{5}$$

We note that any arbitrary constant can be added to the phase of the sine and cosine functions of $u$ and $w$ in (3) and they would still yield the same equations (4) and (5).

Now we describe the boundary conditions that the solutions of (4) and (5) have to satisfy to give rise to an Archimedian spiral wave. We first deal with the function $v(r)$. We note that, using the identity

$$f(r)v'(r) + \frac{f(r)v(r)}{r} + 2f'(r)v(r) = \frac{(f^2(r)v(r)r)'}{rf(r)}, \tag{6}$$

along with the fact that $(f^2(r)v(r)r)|_{r=0} = 0$, equation (5) can be expressed in the integral form,

$$v(r) = (rf^2(r))^{-1} \int_0^r tf^2(t)(\omega(f(t)) - \Omega)\, dt,$$

and this yields $v(0) = 0$ using the Hôpital's rule. Archimedian spiral waves are characterized by the fact that the distance between two neighbouring fronts of the isophase lines tends to a constant, as $r \to \infty$. That is to say, if we consider two points of an isophase line

$(n\phi - \int_0^r v(s)\,\mathrm{d}s = C$, where $C \in \mathbb{R}$ is a constant value) one with coordinates $(\phi, r)$ and the following one on the same radial line with coordinates $(\phi + 2\pi. \, r + \delta(r))$, one obtains,

$$n\phi + \int_0^r v(s)\,\mathrm{d}s = n(\phi + 2\pi) + \int_0^{r+\delta(r)} v(s)\,\mathrm{d}s. \tag{7}$$

The separation between these two fronts is thus here represented by $\delta(r)$ and satisfies

$$\int_r^{r+\delta(r)} v(s)\,\mathrm{d}s = 2\pi n.$$

For Archimedian spiral waves to exist, it is expected that $\delta(r) \to D < \infty$ as $r \to \infty$. Using the mean value theorem in the last equality gives $v(r) \to v_\infty < \infty$ as $r \to \infty$ with $v_\infty = 2\pi n/D$, that is to say, $v_\infty$ is proportional to the inverse of the spirals' front separation $D$, and it is usually known as the *asymptotic wavenumber*.

As for the modulus, $f(r)$, the type of solutions that have been observed are such that $f(r)$ has a bounded limit and $f'(r) \to 0$ as $r \to \infty$. We will therefore focus on solutions of (4) and (5) such that $f(0) = 0$, $f(r)$ and $v(r)$ have bounded limits and $f'(r) \to 0$ as $r \to \infty$.

Summarizing, Archimedian spiral waves correspond to solutions of (4) and (5) with boundary conditions:

$$
\begin{aligned}
&f(0) = v(0) = 0, \\
&\lim_{r\to\infty} f(r) = f_\infty < \infty, \quad \lim_{r\to\infty} f'(r) = 0, \quad \lim_{r\to\infty} v(r) = v_\infty < \infty \\
&f(r) > 0 \text{ and } v(r) \text{ has constant sign } \forall\, r > 0.
\end{aligned}
\tag{8}
$$

These are too many restrictions to a singular third order system of differential equations which suggests that there exists a selection mechanism for the frequency $\Omega$, that is to say, $\Omega$ cannot be arbitrary.

As in previous works [Gre81, Hag82, NK81], in this paper we assume that $|\omega'| \ll 1$. Therefore we write $\omega(z) = \omega_0 + q\bar\omega(z)$, introducing the small parameter $0 \leqslant q \ll 1$. We also introduce a new parameter $\bar\Omega$ like $\Omega = \omega_0 + q\bar\Omega$. Dropping the bars to simplify the notation, equations (4) and (5) read

$$0 = f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + f(r)(\lambda(f(r)) - v^2(r)), \tag{9}$$

$$0 = f(r)v'(r) + \frac{f(r)v(r)}{r} + 2f(r)'v(r) + qf(r)(\Omega - \omega(f(r))), \tag{10}$$

where $0 < q \ll 1$ and $\Omega \in \mathbb{R}$ are the new parameters. As for the boundary conditions, we also consider the ones given in (8) which are also used in [NK81]. The parameter $q$ has indeed a physical meaning and it is usually denoted as the *twist parameter*. As we shall show (see remark 2.3), when $q = 0$, the solution is $v(r) = 0$. Therefore the isophase lines (7) become straight lines emanating from the origin and for this reason these solutions are often known as *radial hedgehog* solutions, specially in the context of liquid crystals. In this sense, the effect of $q > 0$ is that of 'twisting' the isophase lines to become spirals.

To illustrate the behavior of the solutions of system (9) and (10), along with the boundary conditions provided in (8) we integrate numerically the system for some particular functions $\lambda$ and $\omega$. As an example we consider the complex Ginzburg–Landau case, which corresponds to $\lambda(z) = 1 - z^2$ and $\omega(z) = z^2$ and in figure 1 we plot the solutions $(f(r; q), v(r; q))$ for different small values of $q$ ranging from $q = 0.05$ to $q = 0.45$ for $n = 1$.
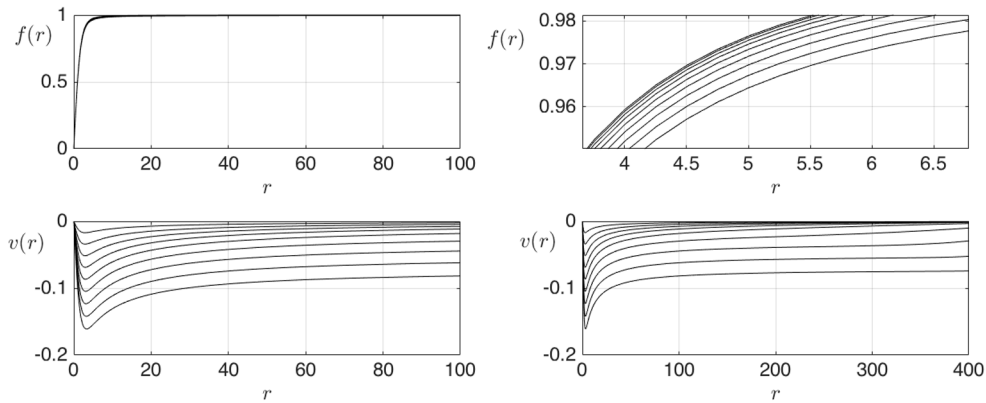
**Figure 1.** Solutions $(f(r; q), v(r; q))$ of the complex Ginzburg–Landau system (9) and (10) with boundary conditions (8) for different values of $q$ ranging from $q = 0.05$ to $q = 0.45$ with step $0.05$, for a winding number of $n = 1$. The curves are ordered in $q$ so that the lower ones correspond to the higher values of $q$.

Spiral wave solutions of systems of the type in (9) and (10) with boundary conditions (8) have been studied by numerous researchers. Kopell & Howard in [NK81], using singular perturbation theory, establish the existence of spiral wave solutions ($n \neq 0$) and target patterns ($n = 0$) under the hypotheses $\lambda(1) = 0, \lambda' < 0$ and $\omega' < 0$ for a particular value of $\Omega = \Omega(q)$ if $q$ is small enough. Hagan in [Hag82] considers the particular case of the complex Ginzburg–Landau equation where $\lambda(z) = 1 - z^2$ and $\omega(z) = z^2$. He uses the method of matching formal asymptotic expansions to construct spiral wave solutions for small values of the parameter $q$. In particular, he formally finds an asymptotic formula for the asymptotic wavenumber $v_\infty = \lim_{r \to \infty} v(r)$ and $\Omega + 1$ which are exponentially small in $q$. Also, Greenberg in [Gre81] uses a formal perturbation technique to construct solutions of (8)–(10) when $\lambda(z) = 1 - z$ and $\omega(z) = z - 1$.

Other than the results in [NK81] there is no proof of the existence of $\Omega = \Omega(q)$ which guarantees the existence of solutions of (8)–(10). Furthermore, the result in [NK81] does not give any quantitative information of $\Omega = \Omega(q)$ and $v_\infty = v_\infty(q)$ when $q$ is small.

In this paper we consider a general class of $\lambda - \omega$ systems (equations (9) and (10)) with the following conditions:

(A1) $\lambda$ and $\omega$ belong to $\mathcal{C}^\infty(\mathbb{R})$ and they are such that $\lambda(1) = 0$ and $\lambda'(1) < 0$. We remark that by suitably rescaling of the radius variable $r$ and the phase function $v$ a new function $\bar{\lambda}$ may be written such that $\bar{\lambda}(0)$ has any prescribed value. Therefore, and without loss of generality, we also assume that $\lambda(0) = 1$.

(A2) $z\lambda(z)$ is concave, that is to say, $\partial_z^2(z\lambda(z)) < 0$ if $z \in [0, 1]$.

Under the above assumptions, we prove that a necessary condition for the problem (8)–(10) to have a solution is that $\Omega = \Omega(q)$ has to be a $\mathcal{C}^\infty$ function of $q$ such that $\partial_q^k \Omega(0) = 0$, for all $k \geqslant 1$. To prove this result we provide a formal expansion in $q$ of the solutions $(f(r; q), v(r; q))$ of equations (9) and (10) with boundary conditions (8) and the parameter $\Omega(q)$, see the expansions in (12). We obtain an infinite set of differential equations with suitable boundary conditions, one for each order in $q$, and we rigorously prove that all these equations have a unique bounded solution if and only if $\Omega(q) - \Omega(0)$ is a $\mathcal{C}^\infty$-flat function of $q$. Therefore, equations (9) and (10) with boundary conditions (8) can be solved up to any order in $q$.

As a straightforward consequence of our results we can give the following quantitative information about the solutions obtained in [NK81]: the parameter $\Omega(q) - \Omega(0)$

and the wavenumber $v_\infty(q) = \lim_{r\to\infty} v(r;q)$ are $\mathcal{C}^\infty$-flat functions of $q$ at $q = 0$. That is $\Omega(q) - \Omega(0) = O(q^k)$ and $v_\infty(q) = O(q^k)$ for all $k \geqslant 1$. Results about the $\mathcal{C}^\infty$ flatness of relevant functions have been provided by many authors. For instance for smooth convex billiards tables, in [MM82] and [CdV84], it is proven that, under suitable conditions, the difference of lengths between two periodic billiard trajectories of the same period $T \to \infty$, $L_T$ is of $O(T^{-k})$ for all $k \geqslant 1$.

The results obtained in this paper are a first step towards a more challenging problem: to obtain rigorous asymptotic formula for the value of $\Omega(q) - \Omega(0)$ and $v_\infty(q)$ as $q \to 0$ which has to be exponentially small in $q$. In particular in the cases considered in [ACW10, Hag82], we plan to give a rigorous proof for the validity of the asymptotic formulas formally deduced in these works. This rigorous study will require different techniques, see for instance [BFGS12, Gel99, MRRTS16b] and references therein, and will be the goal of a forthcoming paper. In fact, in [MRRTS16b] it is proven that, under the assumptions in the previous work [MM82] above mentioned, the quantity $L_T$ is actually exponentially small in $1/T$.

The paper is organised as follows. We start in section 2 by posing the formal solution of (9) and (10) as a power series of $q$. We then introduce our main result, theorem 2.2, of existence and uniqueness of this formal solution provided the power series for $|\Omega - \Omega(0)|$ has vanishing terms. In section 3 we prove the main result as follows: we write an infinite set of differential equations and boundary conditions that each term in the asymptotic expansion satisfies and we then proceed by induction to prove that these equations have indeed solutions.

## 2. Main result: a formal solution

In this section we introduce and justify the expected particular form of the asymptotic expansion in $q$ for the solution of system (9) and (10) with boundary conditions (8). The first result, lemma 2.1 states that the frequency $\Omega$ is indeed a function of $q$, that is $\Omega = \Omega(q)$. We then formulate our main result which establishes the existence of a unique formal solution of our problem provided $\Omega(q) - \omega(1)$ is a flat function in $q$.

**Lemma 2.1.** *If system* (9) *and* (10) *with boundary conditions* (8) *has a solution* $(f(r), v(r))$ *then*

$$v_\infty^2 - \lambda(f_\infty) = 0, \quad \omega(f_\infty) - \Omega = 0, \tag{11}$$

*where* $v_\infty = \lim_{r\to+\infty} v(r)$ *and* $f_\infty = \lim_{r\to+\infty} f(r)$. *In addition,* $\lim_{r\to\infty} v'(r) = 0$ *and the parameter* $\Omega$ *has to be a suitable function of* $q$, *i.e.:* $\Omega = \Omega(q)$.

**Proof.** Let $(f(r), v(r)) = (f(r;q,\Omega), v(r;q,\Omega))$ be a solution of system (9) and (10) with boundary conditions (8). We will omit the dependence on the parameters if there is no danger of confusion.

Note that when $q = 0$, for any value of $\Omega$ equation (10) becomes

$$0 = f(r)v'(r) + \frac{f(r)v(r)}{r} + 2f'(r)v(r) = \frac{(f^2(r)v(r)r)'}{f(r)r},$$

so $f^2(r)v(r)r \equiv c$, which, upon evaluating at $r = 0$ gives $c = 0$. Since we are interested in non trivial solutions for $f(r)$, we obtain that $v(r) \equiv 0$. As a consequence, $v_\infty = 0$ when $q = 0$ which implies $v_\infty = O(q)$ if $q \neq 0$.

Now we check that $f_\infty \neq 0$ for any value of $q$. Assume that $f_\infty = 0$. In this case, since $v_\infty = O(q)$, we have that $v_\infty^2 \ll 1 = \lambda(0) = \lambda(f_\infty)$ taking $q$ small enough. It follows that, for $r_0$

sufficiently large, $n^2/r^2 < \lambda(f(r)) - v^2(r)$, for all $r \geqslant r_0$ and using that $f$ is a positive solution of (9):

$$(rf'(r))' = rf''(r) + f'(r) < 0 \Longrightarrow f'(r) < \frac{r_0}{r}f'(r_0) \Longrightarrow f(r) - f(r_0) < r_0 f'(r_0) \log(r_0^{-1}r).$$

Since $f_\infty = 0$ and $f(r) > 0$, one can take $r_0$ sufficiently large such that $f'(r) < 0$ for $r \geqslant r_0$. Taking $r \to \infty$ in the above inequality, we find a contradiction with the fact that $f$ is a bounded function.

According to equation (9) and using that $\lim_{r \to +\infty} f'(r) = 0$, we obtain that $\lim_{r \to \infty} f''(r) = -f_\infty(\lambda(f_\infty) - v_\infty^2)$, while equation (10) provides that $\lim_{r \to \infty} v'(r)f_\infty = -q f_\infty(\Omega - \omega(f_\infty))$. Then, taking into account that $f_\infty \neq 0$ and the simple fact that:

if $h(r)$ is bounded for all $r > 0$ and $\lim_{r \to +\infty} h'(r) = b$, then $b = 0$,

which is immediate by Hôpital's rule, it is found that $\lim_{r \to +\infty} f''(r) = 0$ and $\lim_{r \to +\infty} v'(r) = 0$. Therefore, one is left with the couple of equations for the boundary values at infinity:

$$\lambda(f_\infty) - v_\infty^2 = 0, \quad \Omega - \omega(f_\infty) = 0.$$

Finally, if we explicitly write the dependence on $q, \Omega$, we have that, in particular, to have solutions of our problem it is required that:

$$\chi(q, \Omega) := \Omega - \omega(f_\infty(q, \Omega)) = 0.$$

Note that, when $q = 0$, for any value of $\Omega$, $v(r; 0, \Omega) = 0$ and the modulus $f$ has to satisfy the equation

$$0 = f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + f(r)\lambda(f(r))$$

which is independent of $\Omega$ so that $f_\infty(0, \Omega) = f_\infty^0$ does not depend on $\Omega$ and thus $\partial_\Omega f_\infty(0, \Omega) = 0$. Therefore, differentiating $\chi(q, \Omega)$ with respect to $\Omega$ one is left with $\partial_\Omega \chi(0, \omega(f_\infty^0)) = 1 \neq 0$, which along with the fact that $\chi(0, \omega(f_\infty^0)) = 0$, the implicit function theorem defines a function $\Omega(q)$ such that $\chi(q, \Omega(q)) = 0$ if $|q|$ is small enough. $\qquad\square$

Lemma 2.1 above implies that the solution of system (8)–(10) only depends on the small parameter $q$. We will call it $(f(r; q), v(r; q))$. Analogously we will write:

$$v_\infty = v_\infty(q) = \lim_{r \to +\infty} v(r; q), \qquad f_\infty = f_\infty(q) = \lim_{r \to +\infty} f(r; q).$$

By inspecting equations (9) and (10), one deduces that the modulus $f(r; q)$, as well as the unknown frequency $\Omega(q)$, are even functions of $q$, that is $f(r; q) = f(r; -q)$ and $\Omega(q) = \Omega(-q)$, while $v$ is an odd function of $q$, and so $v(r; q) = -v(r; -q)$. We can thus restrict our attention to positive values of $q$ without lost of generality. Moreover, using this even and odd character of the functions with respect to $q$ we shall formally find the solutions to (9) and (10) as power series in $q$ of the form:

$$f(r; q) = \sum_{k \geqslant 0} f_k(r)q^{2k}, \qquad v(r; q) = q\sum_{k \geqslant 0} v_k(r), \qquad \Omega(q) = \sum_{k \geqslant 0} \Omega_k q^{2k}. \tag{12}$$

Since we will deal with the behaviour as $r \to 0$ and $r \to +\infty$, we also introduce the notation

$$\psi \in \mathrm{O}_m^l \Longleftrightarrow \psi(r) = \mathrm{O}(r^m), \ r \to 0 \quad \text{and} \quad \psi(r) = \mathrm{O}(r^{-l}), \ r \to +\infty, \tag{13}$$

and

$$\psi \in \mathrm{O}_m^{l,j} \Longleftrightarrow \psi(r) = \mathrm{O}(r^m), \ r \to 0 \quad \text{and} \quad \psi(r) = \mathrm{O}(r^{-l} \log^j r), \ r \to +\infty,$$

which will be used throughout this paper without further special mention.

The main result in this paper is:

**Theorem 2.2.** *Assume hypotheses* (A1)–(A2) *hold. Then the system* (9) *and* (10) *with boundary conditions* (8) *has a unique formal solution of the form* (12) *with* $\lim_{r \to +\infty} f_0(r) = 1$ *and satisfying that for all* $k \geqslant 0$*:*

$$f_k(0) = v_k(0) = 0 \quad \text{and} \quad f_k(r), v_k(r) \quad \text{are bounded as } r \to \infty$$

*if and only if*

$$\Omega_0 = \omega(1), \ \text{and} \ \Omega_k = 0, \ \forall k \geqslant 1.$$

*Moreover, the functions* $f_k(r), v_k(r)$ *also satisfy that*

$$\lim_{r \to \infty} v_0'(r) = 0, \quad \lim_{r \to \infty} v_0(r) = 0,$$

*and*

$$\lim_{r \to \infty} f_k(r) = \lim_{r \to \infty} v_k(r) = 0, \quad \text{for all } k > 0.$$

*Furthermore,*

$$f_0(r) = \mathrm{O}(r^n) \ \text{as} \ r \to 0, \quad 1 - f_0(r) = \mathrm{O}(r^{-2}) \ \text{as} \ r \to +\infty.$$
$$f_0' \in \mathrm{O}_{n-1}^3, \ f_0'' \in \mathrm{O}_{\max\{n-2,0\}}^4$$
$$v_0 \in \mathrm{O}_1^{1,1}, \ v_0' \in \mathrm{O}_0^{2,1}, \ v_0'' \in \mathrm{O}_0^{3,1}$$

*and for* $k \geqslant 1$,

$$f_k \in \mathrm{O}_n^{2,2k}, \ f_k' \in \mathrm{O}_{n-1}^{3,2k}, \ f_k'' \in \mathrm{O}_{\max\{n-2,0\}}^3$$
$$v_k \in \mathrm{O}_1^{1,2k+1}, \ v_k' \in \mathrm{O}_0^{2,2k+1}, \ v_k'' \in \mathrm{O}_0^{3,2k+1}.$$

*Finally if* $\omega$ *is a monotone function,* $v_0$ *has constant sign.*

From this theorem we conclude that, if system (9) and (10) with boundary conditions (8) has a solution $(f(r; q), v(r; q))$ then $v_\infty(q) = \lim_{r \to \infty} v(r; q) = O(q^k), \ \forall k \geqslant 0$, that is, $v_\infty(q)$ is $\mathcal{C}^\infty$-flat in $q$. Therefore, the value of $v_\infty(q)$, which is beyond all orders, cannot be captured by any of the terms of the power expansion of $v(r; q)$ in (12).

In fact, numerical computations reveal that $v(r; q)$ is indeed not zero at infinity. As an example we have considered a Ginzburg–Landau system with $n = 1$, which corresponds to system (9) and (10) with $\lambda(z) = 1 - z^2$ and $\omega(z) = z^2$, that is:

$$0 = f''(r) + \frac{f'(r)}{r} - f(r)\frac{1}{r^2} + f(r)(1 - f(r)^2 - v^2(r)), \tag{14}$$
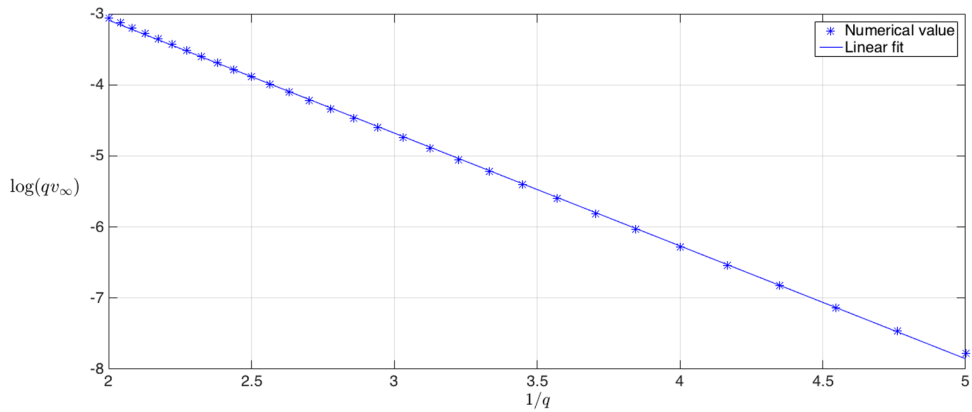
**Figure 2.** Value of $\log(q v_\infty)$ as a function of $1/q$, with $q \in [0.2, 0.5]$ for the system (14)–(15).

$$0 = f(r)v'(r) + \frac{f(r)v(r)}{r} + 2f'(r)v(r) + qf(r)(\Omega - f^2(r)). \tag{15}$$

with boundary conditions (8). To solve (14), (15), (8) we have used a MATLAB routine to obtain $v_\infty(q)$ which uses a finite difference scheme implementing the three-stage Lobatto Illa formula. This routine provides a $\mathcal{C}^1$-continuous solution that is fourth-order accurate uniformly in the interval of integration. We have computed $v_\infty(q)$ and we compare our results with the expression $v_\infty \sim A e^{-B/q} q^{-1}$ formally obtained in [Hag82] and [ACW10]. Performing a linear fit of $\log(q v_\infty(q))$ with 95% confidence we obtain $B = 1.588\,191\,499\,224\,517$ using moderate values of $q \in [0.2, 0.5]$ (see figure 2), which agrees with their predicted value $B = \pi/2$.

A rigorous numerical computation of $A$ and $B$ would require working with multiprecision and it is beyond the scope of this paper, see for instance [MRRTS16a] where a exponentially small quantity in the billiard tables setting, is numerically computed. Moreover, from the analytic point of view, the natural and significantly more difficult question is to prove an asymptotic formula for $v_\infty$ as a function of $q$ in terms of some universal constants, usually called Stokes' constants. As we pointed out in the introduction, other different techniques that the ones used in this work are required to deal with this beyond all order phenomenon.

**Remark 2.3.** As we saw in the proof of lemma 2.1 $v_\infty(0) = \lim_{r \to +\infty} v(r; 0) = 0$ which implies that $v_\infty(q) = \mathrm{O}(q)$. Henceforth, equations (11) imply that $\lambda(f_\infty(0)) = 0$ and $\Omega(0) = \omega(f_\infty(0))$. Since by assumption (A1) $\lambda(1) = 0$ it seems natural to choose $f_\infty(0) = 1$. In fact, if one assumes $\lambda'(z) < 0$, then $\lambda(z) \neq 0$ for $0 < z < 1$ and the only choice for $f_\infty(0)$ is to be 1, see [AB11].

Moreover, since $\lambda'(1) < 0$ and $v_\infty(q) = \mathrm{O}(q)$, by the implicit function theorem the equation $\lambda(f_\infty(q)) = v_\infty^2(q)$ has a solution $f_\infty(q)$ satisfying that $|f_\infty(q) - 1| = \mathrm{O}(q^2)$, if $|q|$ is small enough.

We finally point out that if $\lambda(z)$ has another zero $z_0 < 1$ satisfying $\lambda'(z_0) < 0$, theorem 2.2 can also be applied in this case by rescaling $f = \overline{f} z_0$. In conclusion, the condition $\lim_{r \to +\infty} f_0(r) = 1$ is not restrictive.

In what follows, in section 3, we will find the differential equations and the boundary conditions that $f_k(r)$ and $v_k(r)$ have to satisfy. To solve these differential equations we will find that $\Omega_0 = \omega(1)$ and that all the following terms in the expansion of the frequency $\Omega_k$, for $k \geqslant 1$, must vanish.

## 3. Power series expansions: proof of theorem **2.2**

The idea of the proof is as follows: we first start by describing the system of equations for $f_k, v_k, k \geqslant 0$ introduced in (12). We then deduce the boundary conditions that $f_k, v_k$ must satisfy in order to have bounded solutions of these equations. This is done in section 3.1. As we will see in proposition 3.1, it turns out that the leading order terms $f_0$ and $v_0$ satisfy nonlinear differential equations while $f_k$ and $v_k$, for $k \geqslant 1$, satisfy non-homogeneous linear equations with the same homogeneous linear part. We then prove in proposition 3.2, section 3.2, that $\Omega_0 = \omega(1)$ and some useful properties of $f_0, v_0$. Finally, in section 3.3, we prove the existence of $f_k, v_k$, for $k \geqslant 1$, provided $\Omega_k = 0$ using an induction procedure along with a suitable fixed point equation.

We emphasize that, in this work, we are not interested in the convergence of the expansions of $f(r; q)$, $v(r; q)$ and $\Omega(q)$ in (12), that is to say, we focus on the formal procedure and, in particular, we do not pay special attention to some constants which, of course, could grow with respect to $k$ at any formal step. For this reason we sometimes avoid the exact computation of some of these constants and we indeed may use the same name to denote different constants.

To avoid cumbersome notation, we shall in general omit the dependence on the parameter $q$ and the independent variable $r$ unless such omission leads to confusion.

### 3.1. Differential equations for $f_k, v_k$

We are going to describe the equations that $\{f_k, v_k\}_{k \geqslant 0}$ have to satisfy. As usually, the equations for the leading order terms $f_0$ and $v_0$ will be nonlinear while the equations for $f_k, v_k$ will be found to be non-homogeneous linear equations. To shorten the notation we introduce $F(z) = z\lambda(z)$ and $\tilde{\omega}(z) = z\omega(z)$ and we use $DF(z)$ and $D\tilde{\omega}(z)$ to denote the derivatives with respect to $z$ of these functions.

With this notation and omitting the dependence on $r$ and $q$ of $f, v$, equations (9) and (10) read:

$$0 = f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + F(f) - fv^2, \tag{16}$$

$$0 = fv' + \frac{fv}{r} + 2f'v + q(f\Omega - \tilde{\omega}(f)), \tag{17}$$

and we consider the formal expansions defined in (12):

$$f(r; q) = \sum_{k \geqslant 0} f_k(r)q^{2k}, \quad v(r; q) = q\sum_{k \geqslant 0} v_k(r)q^{2k}, \quad \Omega(q) = \sum_{k \geqslant 0} \Omega_k q^{2k}.$$

**Proposition 3.1.**  *The leading order terms $f_0$ and $v_0$, satisfy the equations*

$$0 = f_0'' + \frac{f_0'}{r} - n^2\frac{f_0}{r^2} + F(f_0), \tag{18}$$

$$0 = f_0 v_0' + \frac{f_0 v_0}{r} + 2f_0'v_0 + f_0\Omega_0 - \tilde{\omega}(f_0). \tag{19}$$

*For $k \geqslant 1$, $f_k$ and $v_k$ satisfy the linear nonhomogeneous equations:*

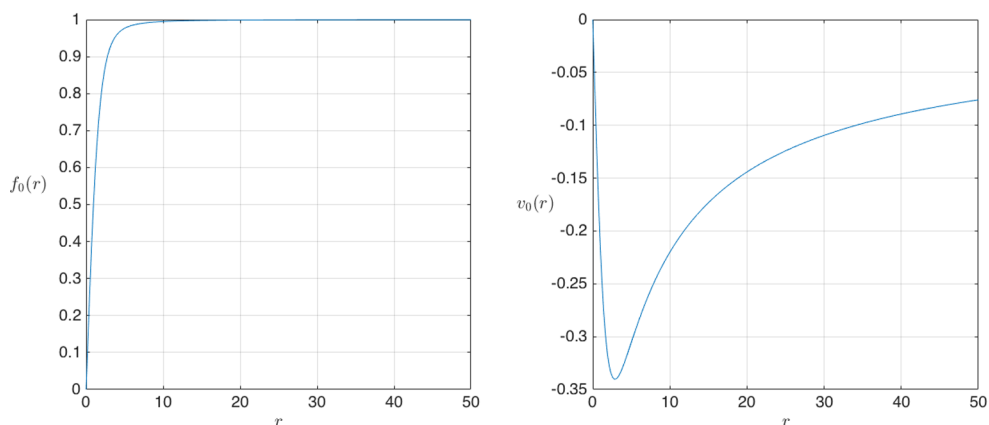$$f_k'' + \frac{f_k'}{r} - n^2\frac{f_k}{r^2} + DF(f_0)f_k = b_k(r), \tag{20}$$

**Figure 3.** Leading order terms: solutions of (18) and (19).

$$f_0 v_k' + \frac{f_0 v_k}{r} + 2f_0' v_k + f_0 \, \Omega_k = c_k(r), \tag{21}$$

*where*

$$b_k(r) = -\sum_{i=1}^{k} D^{2i} F(f_0(r)) \sum_{\substack{k_1+\cdots+k_i=k \\ 1\leqslant k_j \leqslant k-1}} f_{k_1}(r)\cdots f_{k_i}(r) + \sum_{i=1}^{k} \sum_{l=1}^{i} f_{k-i}(r) v_{l-1}(r) v_{i-l}(r) \tag{22}$$

$$c_k(r) = \sum_{i=0}^{k-1} (f_{k-i}(r)(v_i'(r) + r^{-1} v_i(r)) + 2f_{k-i}'(r) v_i(r)) + \sum_{i=0}^{k-1} f_{k-i}(r) \, \Omega_i$$
$$- \sum_{i=1}^{2k} D^{2i} \tilde{\omega}(f_0(r)) \sum_{\substack{k_1+\cdots+k_i=k \\ 1\leqslant k_j}} f_{k_1}(r)\cdots f_{k_i}(r) \tag{23}$$

*with $F(z) = z\lambda(z)$ and $\tilde{\omega}(z) = z\omega(z)$. In particular, $b_k$ is independent of $f_k$ and $v_k$ and $c_k$ is independent of $v_k$.*

**Proof.** By substituting expression (12) in (16), one obtains equation (18) for $f_0$. As for $v$, equation (17), gives to leading order equation (19) for $v_0$. The solutions $(f_0(r), v_0(r))$ are represented in figure 3.

We now deal with $f_k, v_k$, for $k \geqslant 1$. To illustrate the procedure we start by obtaining the particular equations for $f_1, v_1$. Expanding equation (16) in powers of $q$, the order $\mathrm{O}(q^2)$ provides an equation for $f_1$ in terms of $v_0$ and $f_0$, which reads,

$$f_1'' + \frac{f_1'}{r} - n^2 \frac{f_1}{r^2} + DF(f_0)f_1 = f_0 v_0^2, \tag{24}$$

giving $b_1(r) = f_0(r) v_0(r)^2$.

Expanding equation (17) in powers of $q$, the order $\mathrm{O}(q^3)$ provides an equation for $v_1$ in terms of $f_0, f_1$ and $v_0$:

$$f_0 v_1' + \frac{f_0 v_1}{r} + 2f_0' v_1 + f_0(\Omega_1 - f_1 D\omega(f_0)) = c_1(r)$$

with

$$c_1(r) = -f_1(r)(\Omega_0 - \omega(f_0(r))) - f_1(r)v_0'(r) - r^{-1}f_1(r)v_0(r) - 2v_0(r)f_1'(r). \tag{25}$$

To deal with the general case we first observe that the ansatz (12) may also be expressed in terms of a Taylor expansion of $f$, $v$ and $\Omega$ with respect to $q$. Therefore,

$$f_k(r) = \frac{\partial_q^{2k} f(r;0)}{(2k)!}, \quad v_k(r) = \frac{\partial_q^{2k+1} v(r;0)}{(2k+1)!} \quad \text{and} \quad \Omega_k = \frac{\partial_q^{2k} \Omega(0)}{(2k)!}. \tag{26}$$

As a consequence, in order to obtain the equations for $f_k, v_k$ and a general expression for $b_k, c_k$ it is enough to differentiate equations (16) and (17) with respect to $q$. We shall use Leibnitz's rule along with Faa di Bruno formula for $\partial_q^N(F \circ f)(r;0)$, which we recall here for $N = 2k$:

$$\frac{\partial_q^{2k}(F \circ f)(r;0)}{(2k)!} = \sum_{i=1}^{2k} D^i F(f(r;0)) \sum_{\substack{k_1 + \cdots + k_i = 2k \\ 1 \leqslant k_j}} \frac{\partial_q^{k_1} f(r;0)}{k_1!} \cdots \frac{\partial_q^{k_i} f(r;0)}{k_i!}. \tag{27}$$

We first deal with the differential equation (16). We must compute the $2k$- derivative with respect to $q$ of the nonlinear term $F(f) - fv^2$ and then evaluate at $q = 0$. Using Faa di Bruno's formula in (27) and the identity (26) gives

$$\frac{\partial_q^{2k}(F \circ f)(r;0)}{(2k)!} = DF(f_0(r))f_k(r) + b_k^1(r),$$

where, upon using once more identity (26) along with $\partial_q^{2l+1} f(r;0) = 0$, $b_k^1$ is found to read:

$$b_k^1(r) := \sum_{i=1}^{k} D^{2i} F(f_0(r))) \sum_{\substack{k_1 + \cdots + k_i = k \\ 1 \leqslant k_j \leqslant k-1}} f_{k_1}(r) \cdots f_{k_i}(r).$$

We note that the last sum does only depend on $f_l$ with $0 < l < k$.

We now proceed likewise with $\partial_q^{2k}(fv^2)(r;0)$. Here we also note that $\partial_q^{2l} v(r;0) = 0$. Then, using Leibnitz rule:

$$
\begin{aligned}
b_k^2(r) :&= \partial_q^{2k}(fv^2)(r;0) = \sum_{i=2}^{2k} \binom{2k}{i} \partial_q^{2k-i} f(r;0) \partial_q^i v^2(r;0) \\
&= \sum_{j=1}^{k} \sum_{m=1}^{2j-1} \binom{2k}{2j}\binom{2j}{m} \partial_q^{2k-2j} f(r;0) \partial_q^m v(r;0) \partial_q^{2j-m} v(r;0) \\
&= (2k)! \sum_{j=1}^{k} \sum_{l=1}^{j} f_{k-j}(r) v_{l-1}(r) v_{j-l}(r),
\end{aligned} \tag{28}
$$

so $b_k^2$ only depends on $f_l$, $v_l$ with $0 \leqslant l < k$.

Using the above expressions for $\partial_q^{2k}(F(f) - fv^2)$, (subtracting (27) and (28)) we compute the $2k$-derivative of equation (16) with respect to $q$ and, evaluating at $q = 0$, one finds that $f_k(r) = \partial_q^{2k} f(r; 0)/(2k)!$ is a solution of the linear equation

$$f_k''(r) + \frac{f_k'(r)}{r} - f_k(r)\frac{n^2}{r^2} + DF(f_0(r))f_k(r) + b_k^1(r) - \frac{b_k^2(r)}{(2k)!} = 0.$$

Therefore, $f_k$ satisfies equation (20) with $b_k = -b_k^1 + b_k^2/(2k)!$ having the form (22).

We now deal with equation (17). The procedure is exactly analogous to the one for equation (16). First, we observe that, using the Leibnitz's rule, as well as identity (26),

$$\partial_q^{2k+1}\left(f(r; 0)v'(r; 0) + \frac{f(r; 0)v(r; 0)}{r} + 2f'(r; 0)v(r; 0)\right)$$
$$= (2k + 1)!\left[f_0(r)\left(v_k'(r) + \frac{v_k(r)}{r}\right) + 2f_0'(r)v_k(r)\right] + c_k^1(r),$$

where,

$$c_k^1(r) = (2k + 1)! \sum_{i=0}^{k-1}\left(f_{k-i}(r)\left(v_i'(r) + \frac{v_i(r)}{r}\right) + 2f_{k-i}'(r)v_i(r)\right).$$

It only remains to compute the $2k + 1$-derivative with respect to $q$ of the nonlinear term $qf(\Omega(q) - \omega(f))$. First, we define $\widetilde{\Omega}(q) = q\,\Omega(q)$ and we compute $\partial_q^{2k+1}(f(r; 0)\widetilde{\Omega}(0))$. We obtain, using Leibnitz's rule,

$$\partial_q^{2k+1}\left(f(r; 0)\widetilde{\Omega}(0)\right) = \sum_{i=0}^{2k+1}\binom{2k + 1}{i}\partial_q^{2k+1-i}f(r; 0)\partial_q^i\widetilde{\Omega}(0)$$
$$= (2k + 1)!f_0\,\Omega_k + c_k^2(r),$$

where

$$c_k^2(r) = (2k + 1)! \sum_{i=0}^{k-1} f_{k-i}(r)\,\Omega_i.$$

We now introduce $\tilde{\omega}(z) = z\omega(z)$ and compute $c_k^3(r) := \partial_q^{2k+1}(q\tilde{\omega}(f(r; q)))|_{q=0}$:

$$\frac{c_k^3(r)}{(2k + 1)!} = \frac{1}{(2k)!}\partial_q^{2k}\left(\tilde{\omega}(f(r; 0))\right)$$
$$= \sum_{i=1}^{2k} D^i\tilde{\omega}(f(r; 0)) \sum_{\substack{k_1 + \cdots + k_i = 2k \\ 1 \leqslant k_j}} \frac{\partial_q^{k_1}f(r; 0)}{k_1!} \cdots \frac{\partial_q^{k_i}f(r; 0)}{k_i!}$$
$$= \sum_{i=1}^{2k} D^i\tilde{\omega}(f_0(r)) \sum_{\substack{k_1 + \cdots + k_i = k \\ 1 \leqslant k_j}} f_{k_1}(r) \cdots f_{k_i}(r)$$

Finally we compute the $2k + 1$-derivative with respect to $q$ of equation (17) and we obtain that $v_k(r) = \partial_q^{2k+1} v(r; 0)/(2k + 1)!$ satisfies equation (21) with $c_k$ defined as in (23). $\qquad\square$

### 3.2. The leading order term

We have already proved that the leading order terms $f_0$ and $v_0$, have to be solutions of the boundary problems:

$$
\begin{aligned}
0 &= f_0'' + \frac{f_0'}{r} - n^2 \frac{f_0}{r^2} + f_0 \lambda(f_0) \\
f_0(0) &= 0, \quad \lim_{r \to \infty} f_0(r) = 1,
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
0 &= f_0 v_0' + \frac{f_0 v_0}{r} + 2 f_0' v_0 + f_0 (\Omega_0 - \omega(f_0)) \\
v_0(0) &= 0, \quad \lim_{r \to \infty} v_0(r) < +\infty.
\end{aligned}
\tag{30}
$$

It is clear that the nonlinear equation for $f_0$ is qualitatively different to the ones for $f_k$ with $k \geqslant 1$, which are all of them nonhomogeneous linear equations. Moreover, in order to begin an induction procedure (which will be our strategy to prove theorem 2.2) we also need to prove the existence and some properties of $v_0$. For this reason we study the leading order terms separately. The following proposition proves the part of theorem 2.2 related to $f_0$ and $v_0$.

**Proposition 3.2.** *The boundary problem* (29) *has a bounded solution $f_0 > 0$. Moreover, $f_0$ satisfies the following inequalities*

$$
0 < r f_0'(r) \leqslant n^2 f_0(r), \qquad r > 0,
$$

*and it has the asymptotic expansions,*

$$
f_0(r) = \alpha r^n + \mathrm{O}(r^{n+1}), \quad \text{as} \ \ r \to 0 \quad \text{and} \quad f_0(r) = 1 - \frac{n^2}{\beta r^2} + \mathrm{O}(r^{-4}), \quad \text{as} \ \ r \to +\infty.
$$

*with $\beta = -\lambda'(1)$. We also have that $f_0' \in \mathrm{O}_{n-1}^3$ with $\lim_{r \to +\infty} r^3 f_0'(r) = 2 \frac{n^2}{\beta}$ and $f_0'' \in \mathrm{O}_{\min\{0, n-2\}}^4$.*
*The problem* (30) *has a bounded solution $v_0$ if and only if $\Omega_0 = \omega(1)$. Moreover,*

$$
v_0(r) = (r f_0^2(r))^{-1} \int_0^r t f_0(t)^2 (\omega(f_0(t)) - \Omega_0) \, \mathrm{d}t,
\tag{31}
$$

*and it satisfies the asymptotic expansions*

$$
v_0(r) = Cr + \mathrm{O}(r^2) \ \text{as} \ r \to 0, \quad v_0(r) = -\frac{n^2 \omega'(1) \log(r)}{\beta r} + \mathrm{O}(1/r) \ \text{as} \ r \to +\infty.
\tag{32}
$$

*We also have that $v_0' \in \mathrm{O}_0^{2,1}$, $v_0'' \in \mathrm{O}_0^{3,1}$.*
*When $\omega$ is a monotone function the solution $v_0$ has constant sign.*

**Proof.**  As it is shown in [AB11], the boundary problem (29) has a unique bounded solution. The inequalities and the expansions for $f_0(r)$ were rigourously proven in [AB11] for the case that $\lambda$ is an analytic function (section 2 as $r \to 0$ and sections 4 and 5 as $r \to +\infty$). The expansion for $r$ small enough is also true in the case $\lambda \in \mathcal{C}^\infty$ and the behaviour as $r \to +\infty$ can be straightforwardly deduced from lemma 2.14 and remark 2.15 in [AB11]. Then, re-writting equation (29) we obtain the identity $(rf_0'(r))' = n^2 r^{-1} f_0(r) - f_0(r)\lambda(f_0(r))$. From this identity we deduce the asymptotic expansions for $f_0'$ and $f_0''$. In [AB11] it is also shown that $\lim_{r \to +\infty} r^3 f_0'(r)$ exists. To compute this limit, we use L'Hôpital's rule, and the asymptotic expansion for $f_0$:

$$\frac{n^2}{\beta} = \lim_{r \to +\infty} r^2 (1 - f_0(r)) = \lim_{r \to +\infty} r^2 \int_r^{+\infty} f'(\xi)\, d\xi = \lim_{r \to +\infty} \frac{r^3 f'(r)}{2}$$

and so the results for $f_0$ are proven.

As for $v_0$, since $v_0(0) = 0$ and it satisfies equation (30), using identity (6), gives to leading order expression (31). Now, using the asymptotic behaviour of $f_0(r)$ as $r \to \infty$ in (31), gives

$$v_0(r) = (r - 2n^2/(\beta^2 r) + o(r^{-1}))^{-1} \left( \int_0^{r_0} tf_0(t)^2 (\omega(f_0(t)) - \Omega_0)\, dt \right.$$
$$\left. + \int_{r_0}^r t \left( \omega(1) - \Omega_0 - \frac{n^2}{\beta t^2}(2(\omega(1) - \Omega_0) + \omega'(1)) + o(t^{-2}) \right) dt \right),$$

provided $r \geqslant r_0$ and $r_0$ is sufficiently large. This last expression shows that in order for $v_0$ to be bounded at infinity, we have to impose $\omega(1) = \Omega_0$ and so this gives the asymptotic behaviour of $v_0(r)$ as $r \to \infty$ presented in (32).

Also, the asymptotic behaviour of $v_0(r)$ as $r \to 0$ is easily obtained by using the asymptotic expression of $f_0$ in equation (31),

$$v_0(r) \sim \frac{\int_0^r t\alpha^2 t^{2n}(\omega(\alpha t^n) - \omega(1))\, dt}{\alpha^2 r^{2n+1}} = \frac{\omega(0) - \omega(1)}{2n + 2} r + O(r^2).$$

The asymptotic behaviour of both $v_0'$ and $v_0''$ follows from the fact that $v_0 \in O_1^{1,1}$ is a solution of equation (30) along with the asymptotic behaviour of $f_0, f_0'$.

It only remains to check that $v_0(r)$ has constant sign when $\omega(z)$ is a monotone function. For instance, according to (31) if $\omega(z)$ is decreasing, since $\Omega_0 = \omega(1)$, $\omega(0) - \Omega_0 \geqslant \omega(f_0(r)) - \Omega_0 \geqslant \omega(1) - \Omega_0 = 0$, and hence $v_0(r) \geqslant 0$ for all $r \geqslant 0$. Likewise, if $\omega(z)$ is increasing, $v_0(r) \leqslant 0$ for all $r \geqslant 0$.  ☐

### 3.3. Existence and properties of $f_k$. An induction procedure

In this section we are going to prove the results of theorem 2.2 related to $f_k, v_k$ for $k \geqslant 1$. We will use the notation and results from proposition 3.1. More precisely, we will prove that the problems:

$$f_k'' + \frac{f_k'}{r} - n^2 \frac{f_k}{r^2} + DF(f_0)f_k = b_k(r)$$
$$f_k(0) = 0, \qquad f_k(r) \text{ bounded } r \geqslant 0 \tag{33}$$

with $F(z) = z\lambda(z)$, and

$$f_0 v_k' + \frac{f_0 v_k}{r} + 2f_0' v_k + f_0 \, \Omega_k = c_k(r),$$

$$v_k(0) = 0, \qquad v_k(r) \text{ bounded } r \geqslant 0$$

(34)

have solutions $f_k$ and $v_k$ provided

$$\Omega_k = 0.$$

Recall that $b_k, c_k$, were defined in proposition 3.1, for $k \geqslant 1$. To prove this result we will use an induction procedure.

We first recall that, if $f_0, f_1, \ldots, f_{k-1}$ and $v_0, v_1, \ldots, v_{k-1}$ are known, then, the independent term $b_k$ of (33) is determined and henceforth $f_k$ satisfies a linear non-homogeneous equation. If we are able to prove the existence of such a solution, then, by property (6) and taking into account that $v_k(0) = 0$, we will have an explicit expression for $v_k$ which depends on $\Omega_k$ and $c_k$:

$$v_k(r) = (rf_0^2(r))^{-1} \int_0^r tf_0(t)(c_k(t) - f_0(t)\,\Omega_k)\,\mathrm{d}t$$

(35)

Recall here that $c_k$ depends only on $f_0, \ldots, f_k$ and $v_0, \ldots, v_{k-1}$.

Therefore, once one knows how to solve the equation for $f_k$, the function $v_k$ is totally determined. Since all the equations for $f_k$ have the same shape, it is necessary to study the existence of solutions of linear equations of the form

$$g''(r) + \frac{g'(r)}{r} - n^2 \frac{g(r)}{r^2} + DF(f_0(r))g(r) = h(r)$$

$$g(0) = 0, \qquad g(r) \text{ bounded } r \geqslant 0.$$

(36)

We state the following technical lemma which will be proven in section 3.4 by using the fixed point theorem in a suitable Banach space.

**Lemma 3.3.** *Let $h : [0, +\infty) \to \mathbb{R}$ be a $C^2$ function. We define*

$$\mathcal{E}[h](r) := h''(r) + \frac{h'(r)}{r} - h(r)\frac{n^2}{r^2} + [DF(f_0(r)) + \beta]h(r)$$

*with $F(z) = z\lambda(z)$ and $\beta = -\lambda'(1)$. Assume that $\mathcal{E}[h] \in \mathrm{O}_{n-1}^3$, that is:*

$$\mathcal{E}[h](r) = \mathrm{O}(r^{n-1}), \quad r \to 0, \qquad \mathcal{E}[h](r)(r) = \mathrm{O}(r^{-3}), \quad r \to +\infty.$$

(37)

*Then there exists a unique bounded solution $g$ of the boundary problem (36). Moreover, if $\delta g := g + h\beta^{-1}$, we have that*

$$\delta g \in \mathrm{O}_n^3, \qquad \delta g' \in \mathrm{O}_{n-1}^3, \qquad \delta g'' \in \mathrm{O}_{\max\{n-2,0\}}^3.$$

*In particular, $\lim_{r \to +\infty} g(r) = 0$.*

Now we begin our induction scheme. We begin with $f_1$ which satisfies equation (24), that is:

$$f_1'' + \frac{f_1'}{r} - n^2\frac{f_1}{r^2} + DF(f_0)f_1 = b_1(r)$$

with $b_1(r) = f_0(r)v_0^2(r)$. We want to apply lemma 3.3 and so we check that $\mathcal{E}[b_1] \in O_{n-1}^3$. We point out that, by proposition 3.2, $b_1 = f_0 v_0^2 \in O_{n+2}^{2,2}$, $b_1' \in O_{n+1}^{3,2}$, $b_1'' \in O_n^{4,2}$ and, consequently:

$$b_1''(r) + \frac{b_1'(r)}{r} - b_1(r)\frac{n^2}{r^2} \in O_n^{4,2}.$$

In addition, $[DF(f_0(r)) + \beta]b_1(r) = O(r^{n+2})$ as $r \to 0$ and, since $DF(1) = \lambda'(1) = -\beta$, and using proposition 3.2 for the asymptotics of $f_0$ as $r \to \infty$ yields

$$\left[DF(f_0(r)) + \beta\right] = O\big(f_0(r) - 1\big) = O(r^{-2}) \tag{38}$$

and this gives $[DF(f_0(r)) + \beta]b_1(r) = O(r^{-4}\log^2 r)$. Therefore we conclude that $\mathcal{E}[b_1] \in O_n^{4,2} \subset O_{n-1}^3$. Then, lemma 3.3 gives the existence of a solution $f_1$ of problem (33) for $k = 1$ with $\delta f_1 = f_1 + \beta^{-1}b_1$ satisfying

$$\delta f_1 \in O_n^3, \qquad \delta f_1' \in O_{n-1}^3, \qquad \delta f_1'' \in O_{\max\{n-2,0\}}^3,$$

which gives:

$$f_1 \in O_n^{2,2}, \qquad f_1' \in O_{n-1}^{3,2}, \qquad f_1'' \in O_{\max\{n-2,0\}}^3.$$

Now we deal with $v_1$ and $\Omega_1$. As we state in (35),

$$v_1(r) = (rf_0^2(r))^{-1} \int_0^r tf_0(t)(c_1(t) - f_0(t)\,\Omega_1)\,\mathrm{d}t,$$

with $c_1$ defined in proposition 3.1, formula (25). Using that $f_1 \in O_n^{2,2}$, $f_1' \in O_{n-1}^{3,2}$, along with $v_0 \in O_1^{1,1}$ and $v_0' \in O_0^{2,1}$, we have that $c_1 \in O_n^{2,2}$. Therefore, $v_1$ will be a bounded solution if and only if $rf_0(r)(c_1(r) - f_0(r)\,\Omega_1)$ is a bounded function. This implies that

$$0 = \lim_{r \to +\infty} c_1(r) - f_0(r)\,\Omega_1 = \Omega_1.$$

Hence we actually have that

$$v_1(r) = (rf_0^2(r))^{-1} \int_0^r tf_0(t)c_1(t)\,\mathrm{d}t.$$

Now we need to compute the asymptotic behaviour of $v_1$. Clearly, for $r \to 0$, since $c_1 \in O_n^{2,2}$, $v_1(r) = O(r)$. Now we deal with $r \to +\infty$. We notice that, if $r_0$ is sufficiently large,

$$\left| \int_{r_0}^r tf_0(t)c_1(t)\,\mathrm{d}t \right| \leqslant C \int_{r_0}^r \frac{\log^2 t}{t}\,\mathrm{d}t \leqslant C\log^3 r.$$

Then

$$|v_1(r)| \leqslant C\frac{\log^3 r}{r}, \qquad \text{as } r \to +\infty.$$

Summarizing, $v_1 \in O_1^{1,3}$. Moreover, from (34) with $k = 1$:

$$f_0(r)v_1'(r) = c_1(r) - 2f_0'(r)v_1(r) - r^{-1}f_0(r)v_1(r)$$

which implies that $v_1' \in O_0^{2,3}$. We can also deduce that $v_1'' \in O_0^{3,3}$.

Now we state the induction hypothesis: the unique bounded solution $f_{k-1}$, $k \geqslant 2$, of problem (33) satisfies

$$f_{k-1} \in O_n^{2,2(k-1)}, \qquad f_{k-1}' \in O_{n-1}^{3,2(k-1)}, \qquad f_{k-1}'' \in O_{\max\{n-2,0\}}^3. \tag{39}$$

Moreover, problem (34) has bounded solution $v_{k-1}$ if and only if $\Omega_{k-1} = 0$ and in this case,

$$v_{k-1} \in \mathrm{O}_1^{1,2k-1}, \qquad v_{k-1}' \in \mathrm{O}_0^{2,2k-1}, \qquad v_{k-1}'' \in \mathrm{O}_0^{3,2k-1}. \tag{40}$$

We begin first by checking that $b_k \in \mathrm{O}_{n+1}^{2,2k}$ and $c_k \in \mathrm{O}_n^{2,2k}$. Indeed, by the induction hypothesis (39) and (40) and formula (22) for $b_k$, we have that

$$b_k \in \mathrm{O}_{2n}^{2,2k} \cap \mathrm{O}_{n+2}^{2,2k} \subset \mathrm{O}_{n+1}^{2,2k}.$$

We emphasize that if $n = 1$, $2n < n + 2$, but if $n \geqslant 2$, $2n \geqslant n + 2$. To unify both cases we have considered $b_k \in \mathrm{O}_{n+1}^{2,2k}$. Analogously one sees that $c_k \in \mathrm{O}_n^{2,2k}$.

In order to compute the orders for $b_k'$ and $c_k'$ we take into account that, by the induction hypothesis if $l \leqslant k - 1$, the functions $f_l'(r), v_l'(r), v_l''(r)$ are of order of $f_l(r)r^{-1}$, $v_l(r)r^{-1}$ and $v_l(r)r^{-2}$ respectively, so the same happens for the products of these functions. Moreover, $f_l'' \in \mathrm{O}_{\max\{n-2,0\}}^3$, for $l \leqslant k - 1$. Then, tedious but easy computations yield:

$$b_k \in \mathrm{O}_{n+1}^{2,2k}, \qquad b_k' \in \mathrm{O}_n^{3,2k}, \qquad b_k'' \in \mathrm{O}_{n-1}^3 \tag{41}$$

and

$$c_k \in \mathrm{O}_n^{2,2k}, \qquad c_k' \in \mathrm{O}_{n-1}^{3,2k}. \tag{42}$$

The first consequence is that $\mathcal{E}[b_k] \in \mathrm{O}_{n-1}^3$ and hence by lemma 3.3 there exists a unique solution $f_k$ of problem (33) satisfying that

$$f_k + \beta^{-1}b_k \in \mathrm{O}_n^3, \qquad f_k' + \beta^{-1}b_k' \in \mathrm{O}_{n-1}^3, \qquad f_k'' + \beta^{-1}b_k'' \in \mathrm{O}_{\max\{n-2,0\}}^3$$

and taking into account the expansions of $b_k$ in (41), the induction hypothesis (39) is fullfilled for $f_k$.

Finally we deal with $v_k$. We proceed likewise as $v_1$. From identity (35),

$$v_k(r) = (rf_0^2(r))^{-1} \int_0^r tf_0(t)(c_k(t) - f_0(t)\,\Omega_k)\,\mathrm{d}t,$$

$v_k$ will be a bounded solution if and only if $rf_0(r)(c_k(r) - f_0(r)\,\Omega_k)$ is a bounded function and consequently, since $c_k \in \mathrm{O}_n^{2,2k}$,

$$0 = \lim_{r \to +\infty} c_k(r) - f_0(r)\,\Omega_k = \Omega_k\,.$$

Therefore the induction hypothesis for $\Omega_k$ is also satisfied. We rewrite $v_k$ as

$$v_k(r) = (rf_0^2(r))^{-1} \int_0^r tf_0(t)c_k(t)\,\mathrm{d}t,$$

and compute the asymptotic behaviour of $v_k$. Since $c_k(r), f_0(r) = \mathrm{O}(r^n)$ as $r \to 0$, one deduces that $v_k(r) = \mathrm{O}(r)$. As in the case $k = 1$, if $r_0$ is sufficiently large,

$$\left| \int_{r_0}^r tf_0(t)c_k(t)\,\mathrm{d}t \right| \leqslant C \int_{r_0}^r \frac{\log^{2k} t}{t}\,\mathrm{d}t \leqslant C \log^{2k+1} r$$

and hence

$$|v_k(r)| \leqslant C \frac{\log^{2k+1} r}{r}, \qquad \text{as } r \to +\infty.$$

Summarizing, $v_k \in O_1^{1,2k+1}$. Moreover,

$$f_0(r)v_k'(r) = c_k(r) - 2f_0'(r)v_k(r) - r^{-1}f_0(r)v_k(r)$$

implies that $v_k' \in O_0^{2,2k+1}$ and we finally deduce that $v_k'' \in O_0^{3,2k+1}$ by using (42).

　　This ends the proof of theorem 2.2.

### 3.4. Proof of lemma 3.3

We first write equation (36) in a more suitable way, i.e. as a fixed point equation. Adding and subtracting the term $\beta g$, where $\beta = -\lambda'(1)$, which is positive since $\lambda'(1) < 0$, performing the change of variables $s = \sqrt{\beta}\, r$ and denoting by $\tilde{g}(s) = g(s/\sqrt{\beta})$, $\tilde{h}(s) = \beta^{-1}h(s/\sqrt{\beta})$ yields

$$\tilde{g}''(s) + \frac{\tilde{g}'(s)}{s} - \tilde{g}(s)\left(\frac{n^2}{s^2} + 1\right) = \tilde{h}(s) - \tilde{g}(s)\left[\frac{DF(\tilde{f}_0(s))}{\beta} + 1\right] \tag{43}$$

$$\tilde{g}(0) = 0, \qquad \tilde{g}(s) \ \text{bounded} \ s \geqslant 0,$$

where we call $\tilde{f}_0(s) = f_0(s/\sqrt{\beta})$.

　　As we showed in (38)

$$\beta^{-1}DF(\tilde{f}_0(s)) + 1 = O(s^{-2}), \quad \text{as} \quad s \to +\infty. \tag{44}$$

This implies that the dominant term of equation (43) as $s \to \infty$ is the singular equation $-\tilde{g}(s) = \tilde{h}(s)$, therefore, it is natural to write $\tilde{g} = -\tilde{h} + \delta g$ with $\delta g$ being a solution of

$$\delta g'' + \frac{\delta g'}{s} - \delta g\left(\frac{n^2}{s^2} + 1\right) = -\tilde{\mathcal{E}}[\tilde{h}](s) - \delta g\left(\frac{DF(\tilde{f}_0(s))}{\beta} + 1\right), \tag{45}$$

and $\tilde{\mathcal{E}}[\tilde{h}]$ is defined by

$$\tilde{\mathcal{E}}[\tilde{h}](s) = \frac{1}{\beta^2}\mathcal{E}[h](s/\sqrt{\beta}) = \tilde{h}''(s) + \frac{\tilde{h}'(s)}{s} - \tilde{h}(s)\frac{n^2}{s^2} + \left[\frac{DF(\tilde{f}_0(s))}{\beta} + 1\right]\tilde{h}(s).$$

Recall that the operator $\mathcal{E}$ is defined in the statement of lemma 3.3. The boundary conditions are $\delta g(0) = 0$ and $\delta g(s)$ bounded for $s \geqslant 0$.

　　Our goal now is to write equation (45) as a fixed point equation. We emphasize that the dominant part of this equation is the left hand side. Indeed, on the one hand, using (44), one sees that the linear term in the right hand side of equation (45), contributes a small quantity to the equation for large values of $s$, being the left hand side of equation (45) the dominant part as $s \to \infty$. On the other hand, as $s \to 0$, even if this linear term is of order one, the dominant part of equation (43) is provided by the first three terms of the left hand side, that is $\delta g''(s) + \delta g'(s)/s - \delta g(s)n^2/s^2$, and so the right hand side in (43) is also relatively small for small values of $s$.

　　To obtain a fixed point equation we note that the homogeneous modified Bessel equation

$$\varphi''(s) + \frac{\varphi'(s)}{s} - \varphi(s)\left(\frac{n^2}{s^2} + 1\right) = 0$$

has two well-known linearly independent solutions, namely $I_n(s)$ and $K_n(s)$ known as the modified Bessel functions of the first and second kind respectively (see [AS64]). Hence, a

fundamental matrix of solutions of the homogeneous equation corresponding to having zero at the left hand side in (43) reads,

$$M(s) = \begin{pmatrix} K_n(s) & I_n(s) \\ K_n'(s) & I_n'(s) \end{pmatrix},$$

whose Wronskian is known to be $W(K_n(s), I_n(s)) = 1/s$. We denote by

$$\mathcal{R}[\delta g](s) = \tilde{\mathcal{E}}[\tilde{h}](s) + \delta g(s) \left( \frac{DF(\tilde{f}_0(s))}{\beta} + 1 \right). \tag{46}$$

We recall here that $\delta g$ has to be a bounded solution of problem (45) with boundary condition $\delta g(0) = 0$. Therefore, using the variation of parameters formula, equation (45) becomes a fixed point equation:

$$\delta g(s) = \mathcal{F}[\delta g](s) := K_n(s) \int_0^s \xi I_n(\xi) \mathcal{R}[\delta g](\xi) \, d\xi + I_n(s) \int_s^\infty \xi K_n(\xi) \mathcal{R}[\delta g](\xi) \, d\xi. \tag{47}$$

In order to prove the existence of the solution of (47) (and consequently of problem (43)), we will prove that the linear operator $\mathcal{F}$ is contractive in some appropriate Banach space $\mathcal{X}$. However to guarantee the uniqueness of this solution in the space of bounded functions, we need to carefully study the following linear operator:

$$\mathcal{T}[\psi](s) = K_n(s) \int_0^s \xi I_n(\xi) \psi(\xi) \, d\xi + I_n(s) \int_s^\infty \xi K_n(\xi) \psi(\xi) \, d\xi, \tag{48}$$

where $\psi$ is a function defined on $J = [0, +\infty)$. We notice that $\mathcal{F} = \mathcal{T} \circ \mathcal{R}$.

The operators $\mathcal{T}$ and $\mathcal{F}$ are studied in the lemmas 3.4 and 3.5 whose proofs are deferred to the end of this section.

**Lemma 3.4.** *Let $\mathcal{T}$ be the linear operator defined in (48). Let $\psi$ be a function defined on $J = [0, +\infty)$. We take $0 \leqslant m < n - 1$ and $l \geqslant 0$. Then*

$$\psi \in \mathrm{O}_m^l \Longrightarrow \mathcal{T}[\psi] \in \mathrm{O}_{m+2}^l,$$

*where the notation $\mathrm{O}_m^l$ was introduced in (13). In particular, if $\psi$ is bounded, then $\mathcal{T}(\psi) \in \mathrm{O}_2^0$.*

*In the cases $\psi \in \mathrm{O}_{n-1}^l$ or $\psi \in \mathrm{O}_n^l$ we can only conclude that $\mathcal{T}[\psi] \in \mathrm{O}_n^l$.*

*In addition, if $\psi \in \mathcal{C}^i(J)$, then $\mathcal{T}[\psi] \in \mathcal{C}^{i+1}(J)$ and*

$$\psi \in \mathrm{O}_m^l \Longrightarrow \mathcal{T}[\psi]' \in \mathrm{O}_{m+1}^l.$$

*In the cases $\psi \in \mathrm{O}_{n-1}^l$ or $\psi \in \mathrm{O}_n^l$ we conclude $\mathcal{T}[\psi]' \in \mathrm{O}_{n-1}^l$.*

We now define the Banach space where the solution $\delta g$ will belong. We consider the weight function

$$w(s) = f_0'(s/\sqrt{\beta}) \tag{49}$$

and the functional space

$$\mathcal{X} = \left\{ \varphi : J \to \mathbb{R}, \quad \varphi \in \mathcal{C}^0(J), \quad \left| \frac{\varphi(s)}{w(s)} \right| < +\infty \right\}.$$

We endow $\mathcal{X}$ with the norm

$$\|\varphi\|_w = \sup_{s \geqslant 0} \left| \frac{\varphi(s)}{w(s)} \right|,$$

and it becomes a Banach space. In addition, since by proposition 3.2, $w \in \mathrm{O}_{n-1}^3$,

$$\mathcal{X} = \mathrm{O}_{n-1}^3 \cap \mathcal{C}^0(J), \tag{50}$$

that is to say, $\psi \in \mathcal{X}$ if and only if $\psi \in \mathrm{O}_{n-1}^3$ and $\psi \in \mathcal{C}^0(J)$.

**Lemma 3.5.** *For any given $\varphi \in \mathcal{X}$, let $\mathcal{F}_\varphi$ be the linear operator defined by* (47)*:*

$$\mathcal{F}_\varphi[g](s) = K_n(s) \int_0^s \xi I_n(\xi) \mathcal{R}_\varphi[g](\xi) \, \mathrm{d}\xi + I_n(s) \int_s^\infty \xi K_n(\xi) \mathcal{R}_\varphi[g](\xi) \, \mathrm{d}\xi$$

*where, analogously to* (46)*, we denote $\mathcal{R}_\varphi$ by*

$$\mathcal{R}_\varphi[g](s) = \varphi(s) + g(s) \left( \frac{DF(\tilde{f}_0(s))}{\beta} + 1 \right).$$

*Then,*

(i) *If $g \in \mathcal{X}$, then $\mathcal{F}_\varphi[g] \in \mathcal{C}^1(J)$ and $\mathcal{F}_\varphi[g], \mathcal{F}_\varphi[g]' \in \mathcal{X}$. In fact $\mathcal{F}_\varphi[g](s) \in \mathrm{O}_n^3$.*
(ii) *$\mathcal{F}_\varphi$ is contractive in $\mathcal{X}$.*

**End of the proof of lemma 3.3.**     We have to deal with both, existence and uniqueness of solutions of problem (43). We recall that we look for $\tilde{g}$ as $\tilde{g} = -\tilde{h} + \delta g$, being $\delta g$ a solution of the fixed point equation $\delta g = \mathcal{F}[\delta g] = \mathcal{T}[\mathcal{R}[\delta g]]$ given in (47). For the existence we will use mainly lemma 3.5 where $\varphi(s) = \tilde{\mathcal{E}}[\tilde{h}](s)$. Then, hypothesis (37) of lemma 3.3 and the fact that $\tilde{f}_0' \in \mathrm{O}_{n-1}^3$, guarantees that $\tilde{\mathcal{E}}[\tilde{h}]$ belongs to $\mathcal{X}$ and henceforth lemma 3.5 provides a solution $\delta g \in \mathcal{X}$ such that $\delta g \in \mathrm{O}_n^3$, $\delta g' \in \mathrm{O}_{n-1}^3$. In addition, since $\delta g$ is a solution of the differential equation (45), $\delta g'' \in \mathrm{O}_{\max\{n-2,0\}}^3$.

Now it only remains to check that $\tilde{g} = -\tilde{h} + \delta g$ is the unique bounded solution of our problem or equivalently, we shall see that $\delta g$ is the only bounded solution of (45). Let $\delta \bar{g}$ be a bounded solution of equation (45). Then it has to a be solution of the fixed point equation $\delta \bar{g} = \mathcal{F}[\delta \bar{g}] = (\mathcal{T} \circ \mathcal{R})[\delta \bar{g}]$. We note that

$$\mathcal{R}[\delta \bar{g}](s) \leqslant C |\delta \bar{g}(s)| + |\tilde{\mathcal{E}}[\tilde{h}](s)|.$$

Therefore, since at least $\delta \bar{g}$ is bounded and $\tilde{\mathcal{E}}[\tilde{h}] \in \mathcal{X}$, lemma 3.4 with $l = m = 0$ implies that $\delta \bar{g}(s) = \mathrm{O}(s^2)$ as $s \to 0$, then applying iteratively this lemma, we obtain that $\delta \bar{g}(s) = \mathrm{O}(s^n)$ as $s \to 0$. In particular, since $w(s) \in \mathrm{O}_{n-1}^3$:

$$|\delta \bar{g}(s)| \leqslant C w(s), \qquad \text{as} \qquad s \to 0.$$

Now we study the behaviour of $\delta \bar{g}$ as $s \to +\infty$. We first recall that, according to (44) $\beta^{-1} DF(\tilde{f}_0(s)) + 1 = \mathrm{O}(s^{-2})$. Then, since $\tilde{\mathcal{E}}[\tilde{h}] \in \mathcal{X} \subset \mathrm{O}_{n-1}^3$ and $\delta \bar{g}(s) \in \mathrm{O}_n^0$, we conclude that $\mathcal{R}[\delta \bar{g}] \in \mathrm{O}_{n-1}^2$. Now we apply lemma 3.4 and we obtain that $\delta \bar{g} = \mathcal{F}[\delta \bar{g}] = \mathcal{T}[\mathcal{R}[\delta \bar{g}]] \in \mathrm{O}_n^2$. Therefore, repeating the previous argumentation, since $\tilde{\mathcal{E}}[\tilde{h}] \in \mathrm{O}_{n-1}^3$ and $\delta \bar{g} \in \mathrm{O}_n^2$ we obtain that $\delta \bar{g} \in \mathrm{O}_n^3$. Hence, as $\delta \bar{g} \in \mathcal{X}$ and $\mathcal{F}$ is a contractive operator over $\mathcal{X}$, $\delta \bar{g} = \delta g$. $\qquad\square$

The remaining part of this section is devoted to prove the technical lemmas 3.4 and 3.5.

*3.4.1. Proof of lemma 3.4.* Let $m < n - 1$, $l \geqslant 0$ and $\psi : J = [0, +\infty) \to \mathbb{R}$ be a function in $O_m^l$. To study the behavior of $\mathcal{T}[\psi]$ (see (48)) as $s \to 0$ and $s \to \infty$, we recall the asymptotic expansions of the modified Bessel functions $K_n$, $I_n$ and their derivatives.

When $s \to 0$ one has:

$$K_n(s) \sim \frac{\Gamma(n)}{2}(s/2)^{-n} \quad I_n(s) \sim \frac{1}{\Gamma(n+1)}(s/2)^n,$$

$$K_n'(s) \sim -\frac{n\Gamma(n)}{4}(s/2)^{-n-1}, \quad I_n'(s) \sim \frac{n}{2\Gamma(n+1)}(s/2)^{n-1}.$$

And when $s \to \infty$:

$$K_n(s) \sim \mathrm{e}^{-s}\sqrt{\pi/2s}, \quad I_n(s) \sim \mathrm{e}^s/\sqrt{2\pi s}$$

$$K_n'(s) \sim -\mathrm{e}^{-s}\sqrt{\pi/2s}, \quad I_n'(s) \sim \mathrm{e}^s/\sqrt{2\pi s}.$$

From now on we will use the expansions of the Bessel functions without explicit mention.

We start by proving the behaviour of $\mathcal{T}[\psi]$ as $s \to 0$. Since $\psi \in O_m^l$, there exists $C > 0$ such that $|\psi(s)| \leqslant Cs^m$ for any $s \in J$. Let $s_0 > 0$ be such that the above expansions for $s \to 0$ are true for $0 \leqslant s < s_0$. We have that

$$
\begin{aligned}
|\mathcal{T}[\psi](s)| &\leqslant CK_n(s)\int_0^s \xi I_n(\xi)\xi^m\,\mathrm{d}\xi + CI_n(s)\int_s^{+\infty} \xi K_n(\xi)\xi^m\,\mathrm{d}\xi \\
&\leqslant c\left(s^{-n}\int_0^s \xi^{n+1+m}\,\mathrm{d}\xi + s^n\int_s^{s_0}\xi^{-n+1+m}\,\mathrm{d}\xi + s^n\int_{s_0}^{+\infty}\xi^{1+m}K_n(\xi)\,\mathrm{d}\xi\right) \\
&\leqslant \bar{c}\max\{s^{m+2}, s^n\} = \bar{c}s^{m+2}.
\end{aligned}
$$

where $c, \bar{c}$ are generic constants depending only on $n$ and, $s_0$. We have used that, by hypothesis, $m < n - 1$ and that $\int_{s_0}^{+\infty}\xi^{1+m}K_n(\xi)\,\mathrm{d}\xi$ is bounded.

We proceed likewise with the behavior of $\mathcal{T}[\psi](s)$ as $s \to \infty$. We take $s_1 > 0$ be such that the expansions of the Bessel functions for $s \to \infty$ are true for $s > s_1$. As $\psi \in O_m^l$, there exists $\overline{C}$ such that $|\psi(s)| \leqslant \overline{C}s^{-l}$ for $s > s_1$ and $|\psi(s)| \leqslant \overline{C}$ for any $s \in J$. We obtain

$$
\begin{aligned}
|\mathcal{T}[\psi](s)| &\leqslant \overline{C}K_n(s)\int_0^s \xi I_n(\xi)\xi^{-l}\,\mathrm{d}\xi + \bar{C}I_n(s)\int_s^{+\infty}\xi K_n(\xi)\xi^l\,\mathrm{d}\xi \\
&\leqslant c\mathrm{e}^{-s}s^{-1/2}\left(\int_0^{s_1}\xi I_n(\xi)\,\mathrm{d}\xi + \int_{s_1}^s \xi^{-l+1/2}\mathrm{e}^\xi\,\mathrm{d}\xi + \int_s^{+\infty}\xi^{-l+1/2}\mathrm{e}^{-\xi}\,\mathrm{d}\xi\right) \\
&\leqslant \bar{c}s^{-l},
\end{aligned}
$$

where, as before, the values of $c, \bar{c}$ only depend on $n$ and $s_1$. Therefore $\mathcal{T}[\psi] \in O_{m+2}^l$. In particular, applying the above inequalities for $m = l = 0$, that is $\psi$ bounded, we have that $\mathcal{T}[\psi] \in O_2^0$.

In addition, if $\psi$ is continuous, and since every integral in the definition of $\mathcal{T}$ is uniformly convergent, we have that

$$\mathcal{T}[\psi]'(s) = K_n'(s)\int_0^s \xi I_n(\xi)\psi(\xi)\,\mathrm{d}\xi + I_n'(s)\int_s^\infty \xi K_n(\xi)\psi(\xi)\,\mathrm{d}\xi.$$

is also a continuous and bounded function. We proceed as above to check the asymptotic expansions for $T[\psi]'$ and lemma 3.4 is proven.

*3.4.2. Proof of lemma 3.5.* We notice that $\mathcal{F}_\varphi = \mathcal{T} \circ \mathcal{R}_\varphi$. Let $g \in \mathcal{X}$. It is clear that $\mathcal{R}_\varphi[g] \in \mathcal{X}$ and, by (50), $g, \mathcal{R}_\varphi[g] \in O^3_{n-1}$. In consequence, the first item is a straightforward consequence of lemma 3.4.

To prove (ii) we need to show that there exists a constant $0 < K < 1$ such that, for any $g_1, g_2 \in \mathcal{X}, \|\mathcal{F}_\varphi[g_1] - \mathcal{F}_\varphi[g_2]\|_w \leqslant K\|g_1 - g_2\|_w$. We first point out that, since $0 \leqslant \tilde{f}_0(s) \leqslant 1$ and by hypothesis (A2), $\partial_z^2(z\lambda(z)) < 0$, the function $DF(z) = z\lambda'(z) + \lambda(z)$ is decreasing. Therefore, using that by hypothesis (A1), $\lambda(1) = 0$:

$$-\beta = \lambda'(1) + \lambda(1) = DF(1) < DF(\tilde{f}_0(s)) \leqslant DF(0) = \lambda(0) = 1,$$

which gives

$$0 < \frac{DF(\tilde{f}_0(s))}{\beta} + 1 < \frac{1}{\beta} + 1. \tag{51}$$

Now we find that

$$
\left|\mathcal{F}_\varphi[g_1](s) - \mathcal{F}_\varphi[g_2](s)\right| \leqslant K_n(s) \int_0^s \xi I_n(\xi)\left(\frac{DF(\tilde{f}_0(\xi))}{\beta} + 1\right)|g_1(\xi) - g_2(\xi)|\,\mathrm{d}\xi
$$
$$
+ I_n(s) \int_s^\infty \xi K_n(\xi)\left(\frac{DF(\tilde{f}_0(\xi))}{\beta} + 1\right)|g_1(\xi) - g_2(\xi)|\,\mathrm{d}\xi
$$
$$
\leqslant \|g_1 - g_2\|_w\, T(s)
$$

where the function $T$ is defined by

$$
T(s) := K_n(s) \int_0^s \xi I_n(\xi)\left(\frac{DF(\tilde{f}_0(\xi))}{\beta} + 1\right)w(\xi)\,\mathrm{d}\xi
$$
$$
+ I_n(s) \int_s^\infty \xi K_n(\xi)\left(\frac{DF(\tilde{f}_0(\xi))}{\beta} + 1\right)w(\xi)\,\mathrm{d}\xi. \tag{52}
$$

We first observe that $T(s) > 0$ if $s \geqslant 0$ since both $K_n(s), I_n(s)$ are positive, the weight function (see (49)) $w(s) > 0$ and inequality (51).

We now want to show that $\|T\|_w < 1$. We begin by rewriting $T$ in a more appropriate way. Concretely, we will check that

$$
T(s) = w(s) - \mathcal{T}[h_0](s)
$$
$$
= w(s) - K_n(s) \int_0^s \xi I_n(\xi)h_0(\xi)\,\mathrm{d}\xi - I_n(s) \int_s^{+\infty} \xi K_n(\xi)h_0(\xi)\,\mathrm{d}\xi \tag{53}
$$

being $\mathcal{T}$ the linear operator defined in (48) and

$$
h_0(s) = \frac{\sqrt{\beta}}{s^3}\left[2n^2\tilde{f}_0(s) - \frac{s}{\sqrt{\beta}}f_0'\left(\frac{s}{\sqrt{\beta}}\right)\right]. \tag{54}
$$

To prove expression (53) we deal with the differential equation that $f_0'(r)$ satisfies. Indeed, since $f_0(r)$ is a solution of equation (18), $f_0'(r)$ is a solution of the nonhomogeneous linear equation:

$$\varphi''(r) + \frac{\varphi'(r)}{r} - \varphi(r)\frac{n^2}{r^2} + DF(f_0(r))\varphi(r) = -\frac{2n^2}{r^3}f_0(r) + \frac{1}{r^2}f_0'(r).$$

Performing the change $\psi(s) = \varphi(s/\sqrt{\beta})$ to this equation and taking into account that $\tilde{f}_0(s) = f_0(s/\sqrt{\beta})$, we get that $w(s) = f_0'(s/\sqrt{\beta})$ is a solution of

$$\psi''(s) + \frac{\psi'(s)}{s} - \psi(s)\frac{n^2}{s^2} + \frac{DF(\tilde{f}_0(s))}{\beta}\psi(s) = -\frac{2n^2\sqrt{\beta}}{s^3}\tilde{f}_0(s) + \frac{1}{s^2}f_0'(s/\sqrt{\beta}). \tag{55}$$

We define

$$\mathcal{L}[\psi](s) = \psi''(s) + \frac{\psi'(s)}{s} - \psi(s)\left(\frac{n^2}{s^2} + 1\right).$$

We notice that $\mathcal{L}[K_n] = \mathcal{L}[I_n] = 0$ and that equation (55), for $w$, can be rewritten as

$$\left(\frac{DF(\tilde{f}_0(s))}{\beta} + 1\right)w(s) = -\mathcal{L}[w](s) - h_0(s). \tag{56}$$

The linear differential operator $\mathcal{L}$ satisfies that, upon integrating by parts,

$$-\int_a^b \xi B_n(\xi)\mathcal{L}[\psi](\xi)\,\mathrm{d}\xi = -\xi\psi'(\xi)B_n(\xi)\Big|_a^b + \xi\psi(\xi)B_n'(\xi)\Big|_a^b, \tag{57}$$

being either $B_n = K_n$ or $B_n = I_n$. This property was strongly used in [AB11]. Using that $w$ satisfies equation (56), property (57) and that $s(I_n'(s)K_n(s) - K_n'(s)I_n(s)) = 1$, we have that definition (52) of $T$ becomes

$$T(s) = -K_n(s)\int_0^s \xi I_n(\xi)\mathcal{L}[w](\xi)\,\mathrm{d}\xi - I_n(s)\int_s^\infty \xi K_n(\xi)\mathcal{L}[w](\xi)\,\mathrm{d}\xi.$$
$$- K_n(s)\int_0^s \xi I_n(\xi)h_0(\xi)\,\mathrm{d}\xi - I_n(s)\int_s^{+\infty} \xi K_n(\xi)h_0(\xi)\,\mathrm{d}\xi$$
$$= w(s) - \mathcal{T}[h_0](s)$$

and (53) is proven.

Since by proposition 3.2, for any $r \geqslant 0$, $rf_0'(r) \leqslant n^2 f_0(r)$, we have that the function $h_0$, defined in (54), satisfies that $h_0(s) > 0$ if $s > 0$, and therefore $\mathcal{T}[h_0](s) > 0$, if $s > 0$. Consequently:

$$0 \leqslant T(s) < w(s), \qquad s > 0.$$

Now, in order to check that $\|T\|_w < 1$ it only remains to see that

$$\lim_{s\to 0}\frac{\mathcal{T}[h_0](s)}{w(s)} \neq 0, \qquad \lim_{s\to +\infty}\frac{\mathcal{T}[h_0](s)}{w(s)} \neq 0.$$

Indeed, recalling again definition (49) of $w$, and using proposition 3.2, we have that

$$\lim_{s\to 0} w(s)s^{-n+1} = n\alpha\beta^{-(n-1)/2}, \qquad \lim_{s\to +\infty} s^3 w(s) = 2n^2\sqrt{\beta},$$

$$\lim_{s\to 0} h_0(s)s^{-n+3} = \frac{n(2n-1)\alpha}{\beta^{(n-1)/2}}, \qquad \lim_{s\to +\infty} s^3 h_0(s) = 2n^2\sqrt{\beta}.$$

Let $s_0 > 0$ be small enough. By applying Hôpital's rule

$$\lim_{s \to 0} \frac{\mathcal{T}[h_0](s)}{w(s)} = \frac{\beta^{(n-1)/2}}{2n^2\alpha} \lim_{s \to 0} \left( s^{-2n+1} \int_0^s \xi^{n+1} h_0(\xi) + s \int_s^{s_0} \xi^{-n+1} h_0(\xi) \right)$$

$$= \frac{\beta^{(n-1)/2}}{2n^2\alpha} \lim_{s \to 0} \left( \frac{s^{n+1} h_0(s)}{(2n-1)s^{2n-2}} + s^{-n+3} h_0(s) \right)$$

$$= \frac{1}{2n} + \frac{2n-1}{2n} = 1.$$

Now we deal with $s \to +\infty$. Let then $s_0 > 0$ be sufficiently large. Then,

$$\lim_{s \to +\infty} \frac{\mathcal{T}[h_0](s)}{w(s)} = \frac{1}{4n^2\sqrt{\beta}} \lim_{s \to +\infty} \left( s^{5/2} e^{-s} \int_{s_0}^s e^{\xi} \xi^{1/2} h_0(\xi)\, d\xi + s^{5/2} e^s \int_s^{+\infty} e^{-\xi} \xi^{1/2} h_0(\xi)\, d\xi \right)$$

$$= \frac{1}{4n^2\sqrt{\beta}} \lim_{s \to +\infty} 2s^3 h_0(s) = 1.$$

This ends the proof of lemma 3.5.

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