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Structure and Gevrey asymptotic of solutions representing topological defects to some partial differential equations

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Abstract
We consider the positive real-valued solutions of a particular type of ordinary differential equations that arise when considering defect solutions to semilinear partial differential equations. We provide sufficient conditions on the nonlinear term to ensure the existence, uniqueness and monotonicity of solutions to the ordinary differential equation with the prescribed boundary conditions. We then focus on the behaviour of such solutions at infinity and we prove that there is a unique formal expansion at infinity of the Gevrey type, i.e. the coefficients of the expansion grow as a power of a factorial. Moreover, we show that the actual solution is indeed asymptotic 1-Gevrey to this formal expansion. We also present a numerical algorithm to compute the solution for arbitrary values of the degree n in the particular case of the Ginzburg–Landau equation. In particular, we address the difficulty in the numerical computations when n is relatively large due to the fact that the shooting parameter becomes exponentially small for the whole class of nonlinearities considered in this work.

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1. Introduction

In this work we analyse the positive real-valued solutions to the following problem:

$$f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + F(f(r)) = 0, \tag{1}$$

$$f(0) = 0, \qquad \lim_{r\to\infty} f(r) = 1, \tag{2}$$

where n may be taken, by symmetry, as a positive real number and F is a smooth function satisfying $F(0) = F(1) = 0$.

These types of equations arise as a representation of the so-called vortex or point-defect solutions in $\mathbb{R}^2$ of semilinear equations of the form

$$-\Delta u = F(u), \tag{3}$$

u being a complex-valued scalar field subject to the condition at infinity $|u(x)| \to 1$ as $|x| \to +\infty$. Vortex solutions are characterized by having a non-vanishing total winding number or degree at infinity, which is given by

$$\deg_\infty u = \deg\left(\frac{u}{|u|}, \partial B_R\right),$$

for large enough values of R, with $B_R = \{z \in \mathbb{C} : |z| < R\}$. In particular, solutions of the form

$$u(x) = \left(\frac{x}{|x|}\right)^n f(|x|), \qquad x \in \mathbb{C}, \tag{4}$$

represent vortices with a degree at infinity of $n \in \mathbb{Z}$, and moreover, f is readily found to satisfy the ordinary differential equation (1) along with the boundary conditions (2).

As we shall see, we will be dealing with quite general nonlinear terms F. Essentially we will consider smooth functions (continuously differentiable) that vanish at least at zero and one. Examples of this sort of nonlinearities often arise in the continuum description of problems involving phase transitions, as it happens for instance in superconductivity models, liquid crystals or relativistic strings (see [33]). In these contexts, the so-called *order parameter*, u, has two preferred homogeneous states or phases, namely $u = 0$ or $u = 1$, corresponding to two equilibrium solutions of the partial differential equation. In this sense, vortex solutions are non-trivial patterns connecting these two stable states, where the solution does vanish only at a single point and remains close to one elsewhere. The main result in this paper will hence provide a set of conditions on F for such type of functions to exist as solutions to the ordinary differential equation (1) with the boundary conditions given by (2).

When $F(x) = x(1 - x^2)$, and n is an integer, f represents the modulus of single-vortex solutions of the celebrated Ginzburg–Landau equation. This particular case has been widely studied from both the point of view of the partial differential equation and also from the equation satisfied by its modulus $|u| = f$. For instance, in the work by Bethuel *et al* [9], the authors focus on the structure of the energy associated with equation (3) to prove that there exist solutions $u : \mathbb{C} \to \mathbb{R}^2$ with a non-zero degree in bounded domains. They further show that such solutions may be expressed in terms of a function $u(x)$ as described in (4), but they leave, as an open problem, the interesting question of whether this solution u is unique and hence may be just represented by a function of the form (4). Also in [10] the authors deal with the same problem for solutions in the whole $\mathbb{R}^2$, but again they leave unanswered the question of the uniqueness of solutions like the one in (4). Later on, Mironescu in [30] finds that $u(x)$ defined as in (4) is indeed unique when considering solutions in the plane, Millot and Pisante in [29] also affirmatively answer this same question in dimension three and Farina in [19] finally provides uniqueness for any dimension greater than or equal to three.

The fact that the solutions to (3) with a non-vanishing degree are uniquely defined by (4) strongly motivates the analysis of the ordinary differential problem (1)–(2) by itself. In this case, existence and uniqueness of such solutions have also been determined by some authors such as Chen *et al* in [15] or Hervé and Hervé in [23]. In these works the authors use shooting arguments which also allow us to prove the monotonicity and positivity of the solution $f(r)$. However, these shooting arguments are quite restrictive since they strongly rely on the precise structure of the nonlinearity $F(x) = x(1-x^2)$, so they prove to be not so useful for more general nonlinear terms. In this work we tackle the problem from another perspective that is more commonly found in the dynamical system approaches, that is turning (1)–(2) into a suitable fixed point equation. Moreover, this fixed point equation allows us to elucidate in some depth some features of the structure of the solutions to this general problem. Needless to say that the techniques and results in this work also apply to the well-known Ginzburg–Landau case.

Another related interesting problem is obtained with $F(z) = (1 + \mathrm{i}\alpha)z - (1 + \mathrm{i}\beta)z|z|^2$, where now $z \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$. It is easy to check that when $\alpha = \beta$ equation (3) can be reduced to the standard Ginzburg–Landau equation (see for instance [2]), so the general situation where the parameters differ is usually known as the *complex* Ginzburg–Landau equation. In this case it seems that there also exist vortex solutions that may also be expressed in terms of ordinary differential equations. In particular, such vortex solutions are also of the form $u(x) = (x/|x|)^n f(|x|)$, but now $f(r) \in \mathbb{C}$, so it yields a system of two coupled ordinary differential equations for the real and imaginary part of the modulus of the solution u. The existence of these solutions in the whole plane remains unsolved. One may regard this general problem as a perturbation of the problem we have studied in this work. However, this type of perturbation makes the equation entirely different from the unperturbed one since the unperturbed equation has an associated energy that allows us to use the techniques of the calculus of variations that are of no use in the case of the complex Ginzburg–Landau equation. In this respect, we want to emphasize that the functional analysis setting in this work is independent of the variational structure of the equation since it is based on a suitable fixed point equation, so our aim is to apply this new approach to tackle the more difficult case of the complex Ginzburg–Landau system.

This paper is hence organized in two main parts. The first part is concerned with the general problem given by (1)–(2), where our main result, theorem 2.1, characterizes some nonlinearities F for which there exists a unique solution. We note that equation (1) intuitively seems to have very different behaviours at zero and at infinity; the dynamics of the solution at the origin is dominated essentially by the linear part in (1), the one that comes from the Laplacian in (3), while as $r \to \infty$, the dominating part stops being an ordinary differential equation and it is the algebraic equation $F(f) - n^2/r^2 = 0$ that seems to govern the behaviour of f. However, we have succeeded in developing a scheme in terms of a fixed point equation which enables us to prove the existence and uniqueness of the solution to (1)–(2). Moreover, we provide an upper bound for the leading coefficient of this solution f around $r \sim 0$ ($\alpha_n = \lim_{r\to 0} f(r)/r^n$) which is exponentially small in degree n. In the second part, we focus on some properties of the solutions in the generic case,

$$f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + F(f(r)) = 0, \qquad \frac{\partial F}{\partial x}(1) < 0 \tag{5}$$

$$f(0) = 0, \qquad \lim_{r\to\infty} f(r) = 1, \tag{6}$$

where we perform a rigorous study of the behaviour of these solutions at infinity. We note that the Ginzburg–Landau equation, which corresponds to $F(x) = x(1 - x^2)$, is also included. In particular, we derive a unique formal expansion of 1-Gevrey type and show that the solution to (5)–(6) is real analytic at infinity (that is, for $r \geqslant r_0 > 0$ with r_0 large enough) and it is Gevrey asymptotic to the formal expansion. This type of result has not been found before and it is very interesting from the point of view of the physics being modelled by this family of equations since it provides a framework to use the formal asymptotic expansion as a representation of the solution at infinity. A similar regularity result was stated by Duan *et al* [18] regarding the so-called generalized Ginzburg–Landau equation in one dimension. We remark that the requirement on the derivative of F at $x = 1$, although it restricts the family of possible nonlinear terms under consideration, is also necessary in order to provide stability of the vortex pattern when such a solution is regarded as an equilibrium solution of the evolution partial differential equation $u_t - \Delta u = F(u)$, so this requirement is often satisfied.

We also derive a numerical scheme to compute $\alpha_n = \lim_{r\to 0} f(r)/r^n$ for some values of n. As we shall show, these numerical computations are in general non-trivial due to the fact that the shooting parameter becomes exponentially small in n.

The outline of this paper is as follows. In section 2, we first deal with the general problem given by (1)–(2) and introduce theorem 2.1, which provides sufficient conditions on the nonlinearities F for which there exists a unique solution of (1)–(2). This is achieved by, first, posing the problem in terms of a fixed point equation (10) which serves to construct a monotone solution satisfying the required boundary conditions, and second, by using a sliding method, which was first introduced in [5] and has also been used in [12], to prove uniqueness. In section 3 we focus on the behaviour of the solution as $r \to 0$. In particular, it is found that, although there exist infinitely many solutions departing from $r = 0$ like $f(r) \sim \alpha r^n$ with a continuous set of initial parameters α, the precise value for the parameter that ensures that $f(r) \to 1$ as $r \to +\infty$ is exponentially small in n. This explains the difficulties encountered in obtaining numerical computations for $f(r)$ already for moderate values of n. In the rest of this paper we study the generic case where F is analytic and $F(x) \sim b(1-x)+\mathcal{O}((1-x)^2)$, with $b > 0$, around $x \sim 1$, whose solution is readily found to be $\mathcal{C}^\infty$ by means of theorem 2.1. We start, in sections 4 and 5, by describing some asymptotic properties of the solution at infinity. In particular, in section 4 we prove that there is a unique formal expansion at infinity of the Gevrey type, i.e., the coefficients of the expansion grow as a power of a factorial. We then analyse the actual solution of problem (5)–(6) and show, in section 5, that it is indeed asymptotic 1-Gevrey to the previously found formal expansion. Some numerical computations, obtained with a multiple shooting technique, are presented in the last section 6 for the Ginzburg–Landau equation: $F(x) = x(1-x^2)$. These show that the profile of $f(r)$ flattens proportionally as n increases, enlarging the area where $f(r)$ is close to the initial value zero. This, as we shall also explain in section 6, can be regarded as an expansion of the vortex core as n increases.

2. Existence and uniqueness: a fixed point equation

In what follows we will be denoting $\partial F(x) = \frac{\partial}{\partial x}F(x)$. In this section we will prove the following result:

Theorem 2.1. *Assume that $F : [0,1] \to [0,+\infty)$ belongs to $\mathcal{C}^m([0,1])$, $m \geqslant 1$ (the case $m = +\infty$ is also included) and it satisfies*

(i) $F(x) \geqslant 0$ if $x \in [0,1]$ and $F(0) = F(1) = 0$.

(ii) F is injective in a neighbourhood of $x = 1$. In particular, we have that $\partial F(x) \leqslant 0$ if $x \sim 1$.

Then, for any given real $n > 0$, problem (1)–(2) admits a unique monotone solution $f \in [0,1)$ with $f \in \mathcal{C}^{m+2}([0,+\infty))$.

In addition, when $\partial F(1) = -b < 0$, $f(r) = 1 - \frac{n^2}{br^2} + o(r^{-2})$ as $r \to +\infty$.

Remark 2.2. The functions $F(x) = x^p(1-x^2)^q$, $p, q \geqslant 1$, satisfy the hypotheses of our result. In fact, our result is also true for the more general nonlinearities:

$$F(x) = F_0(x)M(x)(1-x^2)^q, \qquad F(x) = F_0(x)M(x)\exp\left(-\frac{1}{(1-x)^\nu}\right), \quad \nu > 0$$

with $q \geqslant 1$, $F_0(x) = x^p, \mathrm{e}^{-1/x^\beta}$, $p \geqslant 1$, $\beta > 0$ and $M(x) > 0$ if $x \in [0,1]$.

We would also like to stress that F could vanish at some points different from $x = 0, 1$.

Concerning the uniqueness of the solution to problem (1)–(2), we note that theorem 2.1 states that if there is a solution satisfying the boundary conditions in (2) such that it always remains bounded with values between zero and one, then such a solution is unique. However, the possibility of obtaining a solution satisfying the boundary conditions in (2) reaching some values outside the region [0, 1] remains. In particular, it looks plausible to have more than one solution to (1)–(2) for some nonlinearities. Indeed, if we consider for instance $F(x) = x(1-x^2)^q$, q being an even number, it is not difficult to guess that there seems to be two different asymptotic expansions approaching one as $r \to +\infty$. Nevertheless, the following corollary provides a condition on the nonlinearity which ensures that there will only be one solution to (1)–(2).

Corollary 2.3. *Let $F : [0, a] \to \mathbb{R}$ be such that $F \in \mathcal{C}^m([0, a])$, $m \geqslant 1$, for some $a > 1$ (the case $a = +\infty$ is also included), satisfying the hypothesis in theorem 2.1. If there exists some $x_0 \in (0, 1)$ such that $\partial F(x) \leqslant 0$ for $x \in (x_0, a)$, then problem (1)–(2) has a unique monotone solution.*

Proof. If the nonlinear term F is such that it is defined in $[0, a]$, with $a > 1$, and also $\partial F(x) \leqslant 0$ if $x \in [x_0, a)$ for some $0 < x_0 < 1$, it is not difficult to see that the solutions to problem (1)–(2) are actually smaller than one, and hence the solution is unique. Indeed, let us assume that there exists a solution reaching values greater than one. In such a case, the smoothness of f along with the fact that the solution goes to one at infinity implies that f should achieve a maximum at some point r_* where $f(r_*) > 1$, $f'(r_*) = 0$ and $f''(r_*) \leqslant 0$. On the other hand, $F(f(r_*)) \leqslant 0$ and thus,

$$f''(r_*) + \frac{f'(r_*)}{r_*} - f(r_*)\frac{n^2}{r_*^2} + F(f(r_*)) < 0,$$

which yields a contradiction. □

Proof of theorem 2.1. The proof is separated into six steps.

Step 1. Derivation of an integral expression equivalent to (1). First of all we note that, since $F \in \mathcal{C}^1([0, 1])$, $F(x) \geqslant 0$ and $F(1) = 0$, by the mean value theorem,

$$0 \leqslant F(1-x) \leqslant \sup_{z\in[0,1]} |\partial F(z)|x := d \cdot x. \tag{7}$$

We start by performing the change of function $f = 1 - g$ and the change of variable $s = \sqrt{d}\, r$ so the equation for g turns out to be

$$g''(s) + \frac{g'(s)}{s} - g(s)\left(\frac{n^2}{s^2} + 1\right) = -\frac{n^2}{s^2} - g(s) + \frac{1}{d}F(1 - g(s)). \tag{8}$$

To obtain a fixed point equation we note that the homogeneous equation corresponding to keeping just the left-hand side in (8) corresponds to a modified Bessel equation which has two well-known linearly independent solutions, namely I_n and K_n known as the modified Bessel functions of the first and second kind, respectively. Hence, a fundamental matrix of solutions reads

$$M = \begin{pmatrix} K_n(s) & I_n(s) \\ K_n'(s) & I_n'(s) \end{pmatrix},$$

whose Wronskian is known to be $W(K_n(s), I_n(s)) = 1/s$ (see [1]). We denote the nonlinear term by

$$\mathcal{R}[g](s) = \frac{n^2}{s^2} + g(s) - \frac{1}{d}F(1 - g(s)). \tag{9}$$

The variation of the parameters' formula, along with the condition on $g(0) = 1 - f(0) = 1$ and imposing g to be bounded at infinity, yields

$$g(s) = K_n(s)\int_0^s \xi I_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi, \qquad (10)$$

$$g'(s) = K_n'(s)\int_0^s \xi I_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi + I_n'(s)\int_s^{+\infty} \xi K_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi. \qquad (11)$$

These equations are actually decoupled, so one can focus only on the first one to use it as a fixed point equation of the form $g(s) = \mathcal{F}[g](s)$ to prove the existence of solutions.

Lemma 2.4. *Let $\mathcal{F}[g]$ be the nonlinear operator defined by*

$$\mathcal{F}[g](s) = K_n(s)\int_0^s \xi I_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi, \qquad (12)$$

with $\mathcal{R}[g]$ given by (9), and let $\mathcal{X}$ be the Banach space defined by

$$\mathcal{X} = \{g : J = [0, +\infty) \to \mathbb{R}^+,\ g \in \mathcal{C}^0(J),\ \lim_{s\to+\infty} g(s) = 0\},$$

with the usual supremum norm.

Let $B_1 \in \mathcal{X}$ be the convex set given by

$$B_1 = \{g \in \mathcal{X},\ \text{such that } 0 \leqslant g(s) \leqslant 1\ \forall s \in J\}.$$

Then,

(i) If $g \in \mathcal{X}$, $\mathcal{F}[g] \in \mathcal{C}^2(J)$.
(ii) The operator $\mathcal{F}$ sends B_1 to itself: $\mathcal{F}[B_1] \subset B_1$.
(iii) If $g \in B_1$, then $\mathcal{F}[g](0) = 1$.

Proof. It is clear that $\mathcal{F}[g] \in \mathcal{C}^2(J)$ if $g \in \mathcal{C}^0(J)$. We now need to check that $\mathcal{F}(g)(s) \to 0$ as $s \to +\infty$. This is seen upon using Hôpital's rule and the fact that the modified Bessel functions at infinity behave like $K_n(s) \sim \mathrm{e}^{-s}\sqrt{\pi/2s}$, $I_n(s) \sim \mathrm{e}^{s}/\sqrt{2\pi s}$ and thus $K_n'(s) \sim -\mathrm{e}^{-s}\sqrt{\pi/2s}$ and $I_n'(s) \sim \mathrm{e}^{s}/\sqrt{2\pi s}$:

$$\begin{aligned}\lim_{s\to+\infty}\mathcal{F}[g](s) &= -\lim_{s\to+\infty}\left(\frac{sI_n(s)\mathcal{R}[g](s)}{K_n'(s)/K_n^2(s)} - \frac{sK_n(s)\mathcal{R}[g](s)}{I_n'(s)/I_n^2(s)}\right)\\ &= 2\lim_{s\to+\infty} s\mathcal{R}[g](s)K_n(s)I_n(s) = \lim_{s\to+\infty}\mathcal{R}[g](s)\\ &= \lim_{s\to+\infty}\left(g(s) - \frac{1}{d}F(1-g(s))\right) = 0,\end{aligned}$$

provided $g(s) \to 0$ as $s \to +\infty$.

Now we consider $g \in \mathcal{C}^0(J)$ such that $0 \leqslant g \leqslant 1$. Then, in view of the hypothesis of theorem 2.1, $0 \leqslant F(1-x) \leqslant d\cdot x$, so the functional $\mathcal{R}[g](s) = n^2/s^2 + g(s) - d^{-1}F(1-g(s))$ satisfies

$$0 < \frac{n^2}{s^2} \leqslant \mathcal{R}[g](s) \leqslant 1 + \frac{n^2}{s^2},$$

provided $0 \leqslant g \leqslant 1$. Therefore,

$$\begin{aligned}0 < \mathcal{F}[g](s) &= K_n(s)\int_0^s \xi I_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\mathcal{R}[g](\xi)\,\mathrm{d}\xi\\ &\leqslant K_n(s)\int_0^s \xi I_n(\xi)\left(1+\frac{n^2}{\xi^2}\right)\mathrm{d}\xi + I_n(s)\int_s^{+\infty}\xi K_n(\xi)\left(1+\frac{n^2}{\xi^2}\right)\mathrm{d}\xi\\ &= sW(K_n(s), I_n(s)) = 1,\end{aligned}$$

where we have seen that the modified Bessel functions K_n and I_n are positive $\forall s \in J$ and satisfy

$$\begin{aligned} sI_n' &= \int_0^s \xi I_n(\xi)\left(\frac{n^2}{\xi^2}+1\right)\mathrm{d}\xi, \\ sK_n' &= -\int_s^{+\infty} \xi K_n(\xi)\left(\frac{n^2}{\xi^2}+1\right)\mathrm{d}\xi. \end{aligned} \tag{13}$$

This proves that $\mathcal{F}[B_1] \subset B_1$.

We now show that the limit as $s \to 0^+$ of $\mathcal{F}[g](s)$ is indeed 1, provided only that g is continuous at $s = 0^+$. In effect, if we use that $K_n(s) \sim 1/2\,\Gamma(n)(s/2)^{-n}$, $I_n(s) \sim (s/2)^n\,1/\Gamma(n+1)$ and hence $K_n'(s) \sim -(n/2s)\,\Gamma(n)(s/2)^{-n}$ and $I_n'(s) \sim (n/s)(s/2)^n\,1/\Gamma(n+1)$, along with Hôpital's rule, this gives

$$\begin{aligned} \lim_{s\to 0} \mathcal{F}[g](s) &= -\lim_{s\to 0}\left(\frac{sI_n(s)\mathcal{R}[g](s)}{K_n'(s)/K_n^2(s)} - \frac{sK_n(s)\mathcal{R}[g](s)}{I_n'(s)/I_n^2(s)}\right) \\ &= \lim_{s\to 0}\left(s^2\mathcal{R}[g](s)\frac{\Gamma(n)}{n\Gamma(n+1)}\right) = 1. \end{aligned}$$

Therefore, $\mathcal{F}$ maps functions $g \in \mathcal{C}^0(J)$ satisfying the boundary conditions $g(0) = 1$ and $g(s) \to 0$ as $s \to +\infty$ into continuous functions in J satisfying precisely the same boundary conditions. □

The following lemma states that the types of nonlinearities that are being considered in theorem 2.1 may be bounded by a $\mathcal{C}^2$ function with a set of properties that will be useful to prove the existence of a solution to problem (1)–(2).

Lemma 2.5. *Let F be such that it satisfies the hypotheses of theorem 2.1. Then there exist $x_0 > 0$ and a function G with the following properties:*

(i) $G(0) = 0$ and $F(1-x) \geqslant G(x)$ if $x \in [0, x_0]$.
(ii) $G \in \mathcal{C}^2([0, x_0])$, $\partial G(x_0) = 0$ and $\partial G(x) > 0$ if $x \in (0, x_0)$.
(iii) For any $\hat{x}_0 \in (0, x_0)$ there exists a constant $B > 0$ such that if $x \in [0, \hat{x}_0]$,

$$G(x) \leqslant Bx\partial G(x), \qquad -x\partial^2 G(x) \leqslant B\partial G(x).$$

In addition, if $F(1-x) \geqslant cx^q$ for $x \sim 0$, the function G satisfying the above conditions may be chosen to be of the form

$$G(x) = x^q\left(\frac{c}{2} - x\right).$$

Proof. We notice that if F satisfies the hypotheses of theorem 2.1, then there exists $x_1 \in (0, 1)$ such that $F(1-x) > 0$ and $\partial F(1-x) \leqslant 0$ if $x \in (0, x_1]$. We define now

$$G(x) = (1 - bx)\int_0^x F(1-\xi)\,\mathrm{d}\xi,$$

with $b = 1/x_1 > 1$. We note that $G(0) = 0$ and $G(x) > 0$ if $x \in (0, b^{-1})$. $G \in \mathcal{C}^2([0, 1])$ provided $F \in \mathcal{C}^1([0, 1])$. Moreover, as we have already pointed out, $F(1-x)$ is an increasing function if $x \in [0, b^{-1}]$, and thus

$$G(x) \leqslant (1 - bx)xF(1-x) \leqslant F(1-x),$$

provided $x \in [0, b^{-1}]$, which proves item (i).

Now we deal with (ii). Note that

$$\partial G(x) = (1 - bx)F(1-x) - b\int_0^x F(1-\xi)\,\mathrm{d}\xi \geqslant (1 - 2bx)F(1-x) > 0,$$

if $x \in (0, 1/(2b))$. Moreover $\partial G(b^{-1}) < 0$ which implies that there exists $x_0 \in (1/(2b), 1/b)$ satisfying item (ii).

Finally we check (iii). As we have seen $\partial G(x) \geqslant (1-2bx)F(1-x)$, therefore,

$$G(x) \leqslant (1-bx)xF(1-x) \leqslant 2x(1-2bx)F(1-x) \leqslant 2x\partial G(x),$$

if $x \in [0, 1/(3b)]$. Moreover, since $\partial F(1-x) \leqslant 0$,

$$\begin{aligned} -x\partial^2 G(x) &= (1-bx)x\partial F(1-x) + 2bxF(1-x) \leqslant 2bxF(1-x) \\ &\leqslant (1-2bx)F(1-x) \leqslant \partial G(x), \end{aligned}$$

provided $x \in [0, 1/(4b)]$. Item (iii) follows from the fact that both $G(x)(x\partial G(x))^{-1}$ and $-x\partial^2 G(x)(\partial G(x))^{-1}$ are continuous at $[1/(4b), \hat{x}_0]$ provided $\hat{x}_0 \in [1/(4b), x_0)$. □

Step 2. Construction of a suitable sequence $\{g_k\} \subset \mathcal{C}^0(J)$ satisfying $g_{k+1} = \mathcal{F}[g_k]$. We will prove the following proposition:

Proposition 2.6. *There exists $g_{k+1} = \mathcal{F}[g_k]$ satisfying the following conditions:*

(i) $g_0 \in \mathcal{C}^0(J)$*, it is decreasing monotone and* g_0 *tends to* 0 *as* $s \to +\infty$.
(ii) $g_k \in B_1$ *and* $0 < g_{k+1} < g_k \leqslant 1$ *for all* k.
(iii) The family given by the sequence $\mathcal{S} = \{g_k : k \in \mathbb{N}\} \subset \mathcal{C}^0(J)$ *is equicontinuous at* $J = [0, +\infty) \cup \{+\infty\}$. *That is to say, it is equicontinuous for any* $s \in [0, +\infty)$ *and furthermore, it is equicontinuous at* $+\infty$ *which means that for any* $\varepsilon > 0$*, there exists* $s_0 > 0$ *such that* $0 < g_k(s) < \varepsilon$ *for all* $s \geqslant s_0$ *and for all* $k \geqslant 0$.

We start by constructing a suitable function g_0. By hypothesis (i) and (ii) of lemma 2.5, there exists $x_0 > 0$ such that $F(1-x) \geqslant G(x)$ for $x \in [0, x_0]$ and the function G satisfies that $G \in \mathcal{C}^2([0, x_0])$, $G(0) = 0$, $\partial G(x_0) = 0$ and $\partial G(x) > 0$ if $0 < x < x_0$. We note that hence $G : [0, x_0] \to [0, +\infty)$ is an injective function. Let $\hat{x}_0 \in (0, x_0)$ be such that it satisfies the following two conditions:

$$G(\hat{x}_0) > \frac{G(x_0)}{2}, \qquad \partial G(\hat{x}_0) < \frac{G(x_0)}{n^2}. \tag{14}$$

Note that such a number, $\hat{x}_0$, exists because of the continuity of both G, ∂G and from the fact that $\partial G(x_0) = 0$.

Next we state a technical lemma which is a consequence of the hypotheses on G stated in lemma 2.5.

Lemma 2.7. *Let* $a = G(\hat{x}_0)$. *For any* $C > 0$*, there exists a decreasing function* $\hat{g}_0 : [\sqrt{C/a}, +\infty) \to [0, \hat{x}_0]$ *satisfying that*

$$\frac{C}{s^2} = G(\hat{g}_0(s)). \tag{15}$$

Moreover,

(i) $\hat{g}_0 \in \mathcal{C}^2((\sqrt{C/a}, +\infty))$ *and if* $s \geqslant \sqrt{C/a}$*, then* $\hat{g}_0''(s) \leqslant As^{-2}\hat{g}_0(s)$ *for some constant* $A > 0$*;*
(ii) $n^2(1 - \hat{g}_0(\sqrt{C/a})) < \sqrt{C/a}\,|\hat{g}_0'(\sqrt{C/a})|$.

Furthermore, in the special case that $G(x) = O(x^q)$ *as* $x \sim 0$, $\hat{g}_0 = O(s^{-2/q})$.

Proof. Let $C > 0$. Since $G : [0, \hat{x}_0] \to [0, a]$ is a bijective function, the existence of $\hat{g}_0$ is guaranteed. Moreover differentiating the functions involved in expression (15),

$$\begin{aligned} -\frac{2C}{s^3} &= \partial G(\hat{g}_0(s))\hat{g}_0'(s), \\ \frac{6C}{s^4} &= \partial^2 G(\hat{g}_0(s))(\hat{g}_0'(s))^2 + \partial G(\hat{g}_0(s))\hat{g}_0''(s), \end{aligned} \tag{16}$$

which implies that $\hat{g}_0'(s) < 0$ if $s \geqslant \sqrt{C/a}$ due to the fact that $\partial G(x) > 0$ if $x \in (0, \hat{x}_0]$. Also, using that $C = s^2 G(\hat{g}_0(s))$, one has

$$\hat{g}_0''(s) = \frac{1}{\partial G(\hat{g}_0(s))}\left(\frac{6G(\hat{g}_0(s))}{s^2} - \partial^2 G(\hat{g}_0(s))(\hat{g}_0'(s))^2\right). \tag{17}$$

We next define $s_+ = \sqrt{C/a}$ and observe that $\hat{g}_0(s_+) = \hat{x}_0$ with $\hat{x}_0$ defined by conditions (14). Now we are going to check that $s^2\hat{g}_0''(s) \leqslant A\hat{g}_0$ if $s \geqslant s_+$. We recall that, according to item (iii) in lemma 2.5, if $x \in [0, \hat{x}_0]$ then there exists some constant B such that $Bx\partial G(x) \geqslant G(x)$ and we observe that, from (16),

$$|\hat{g}_0'(s)s| = \frac{2G(\hat{g}_0(s))}{\partial G(\hat{g}_0(s))} \leqslant 2B\hat{g}_0(s). \tag{18}$$

Indeed, again by item (iii) of lemma 2.5 one has that there exists a constant $B > 0$ such that $-x\partial^2 G(x) \leqslant B\partial G(x)$ if $x \in [0, \hat{x}_0]$. Hence, using expression (17) for $\hat{g}_0''$, along with $Bx\partial G(x) \geqslant G(x)$ and (18), yields

$$\hat{g}_0''(s) \leqslant \frac{6B}{s^2}\hat{g}_0(s) + \frac{B}{\hat{g}_0(s)}(\hat{g}_0'(s))^2 \leqslant \frac{6B}{s^2}\hat{g}_0(s) + \frac{4B^3}{s^2}\hat{g}_0(s) = A\frac{\hat{g}_0(s)}{s^2},$$

which concludes the proof.

We now prove item (ii) in the lemma. Since $\hat{x}_0 = \hat{g}_0(s_+)$, with $\hat{x}_0$ satisfying conditions (14),

$$|s_+\hat{g}_0'(s_+)| = \frac{2G(\hat{g}_0(s_+))}{\partial G(\hat{g}_0(s_+))} = \frac{2G(\hat{x}_0)}{\partial G(\hat{x}_0)} \geqslant n^2 \geqslant n^2(1 - \hat{g}_0(s_+)).$$

Finally, it is now straightforward that if $G(x) = O(x^q)$ as $x \to 0$, then $\hat{g}_0(s) = O(s^{-2/q})$ as $s \to +\infty$. □

Lemma 2.8. *There exists a decreasing function $g_0 \in B_1$ such that $g_1 = \mathcal{F}[g_0] < g_0$ and $g_0(s) \to 0$ as $s \to +\infty$.*

Moreover, if $G(x) = O(x^q)$ as $x \sim 0$, then $g_0(s) = O(s^{-2/q})$.

Proof. In order to shorten the notation we introduce the operator

$$\mathcal{H}[h](s) = \frac{h'(s)}{s} - h(s)\left(\frac{n^2}{s^2} + 1\right) + \frac{n^2}{s^2} + h(s) - \frac{1}{d}F(1 - h(s)). \tag{19}$$

We take $C > 0$ large enough and $\hat{g}_0$ as in lemma 2.7, satisfying $Cs^{-2} = G(\hat{g}_0(s))$. First we are going to check that if $s \geqslant \sqrt{C/a}$ then

$$\hat{g}_0''(s) + \mathcal{H}[\hat{g}_0](s) \leqslant 0. \tag{20}$$

Indeed, we recall that by hypothesis (i) of lemma 2.5, $F(1 - x) \geqslant G(x)$ if $x \in [0, x_0]$. By using lemma 2.7, we find that

$$\begin{aligned} \hat{g}_0''(s) + \mathcal{H}[\hat{g}_0](s) &\leqslant A\frac{\hat{g}_0(s)}{s^2} + \frac{n^2}{s^2}s(1 - \hat{g}_0(s)) - \frac{1}{d}G(\hat{g}_0(s)) \\ &= s^{-2}d^{-1}[(A + n^2)d - C] \leqslant 0, \end{aligned} \tag{21}$$

upon taking $C \geqslant (A + n^2)d$.

We now define the straight line

$$g_+(s) = \hat{g}_0(s_+) + (s - s_+)\hat{g}_0'(s_+), \tag{22}$$

where $s_+ = \sqrt{C/a}$. It is clear that $g_+(s_+) = \hat{g}_0(s_+)$ and $g_+'(s_+) = \hat{g}_0'(s_+)$ and that $g_+(s_-) = 1$ with

$$s_- = s_+ - \frac{1 - \hat{g}_0(s_+)}{|\hat{g}_0'(s_+)|}, \tag{23}$$

and it is straightforward to check that $s_- \in (0, s_+)$ simply using item (ii) of lemma 2.7. Finally, if $s \in [s_-, s_+]$, using again item (ii) of lemma 2.7 along with the fact that $1 = g_+(s_-) \geqslant g_+(s) \geqslant g_+(s_+) = \hat{g}_0(s_+)$, we have that

$$\begin{aligned} &g_+''(s) + \frac{g_+'(s)}{s} - g_+(s)\left(\frac{n^2}{s^2} + 1\right) + \frac{n^2}{s^2} + g_+(s) - \frac{1}{d}F(1 - g_+(s)) \\ &\leqslant \frac{\hat{g}_0'(s_+)}{s} + \frac{n^2}{s^2}(1 - g_+(s)) \leqslant 0. \end{aligned} \tag{24}$$

Here we have also used that, by means of hypothesis (i) of theorem 2.1, $F(1 - x) \geqslant 0$.

Finally we define $g_0 \in \mathcal{C}^1(0, +\infty)$ as

$$g_0(s) = \begin{cases} 1 & s \leqslant s_-, \\ g_+(s) & s_- \leqslant s \leqslant s_+, \\ \hat{g}_0(s) & s \geqslant s_+. \end{cases} \tag{25}$$

We note that g_0 is decreasing and actually, by construction, $g_0 \in \mathcal{C}^1((s_-, +\infty))$, $g_0(s_-) = 1$, $\lim_{s\to+\infty} g_0(s) = 0$ and by (20) and (24), if $s > s_-$ and $s \neq s_+$,

$$g_0''(s) + \frac{g_0'(s)}{s} - g_0(s)\left(\frac{n^2}{s^2} + 1\right) + \frac{n^2}{s^2} + g_0(s) - \frac{1}{d}F(1 - g_0(s)) \leqslant 0. \tag{26}$$

We claim that $g_1 = \mathcal{F}[g_0] \leqslant g_0$. Indeed, it is clear that if $s \leqslant s_-$, then $g_1 \leqslant 1 = g_0(s)$. Therefore, we can restrict ourselves to prove the inequality when $s > s_-$. We introduce the linear differential operator

$$\mathcal{L}[h](t) = h''(t) + \frac{h'(t)}{t} - h(t)\left(\frac{n^2}{t^2} + 1\right).$$

First we observe that upon integrating by parts,

$$-\int_a^b \xi B_n(\xi)\mathcal{L}[h](\xi)\,\mathrm{d}\xi = -\xi h'(\xi)B_n(\xi)|_a^b + \xi h(\xi)B_n'(\xi)|_a^b,$$

with either $B_n = K_n$ or $B_n = I_n$. Thence, using definition (25) of g_0 and the fact that, by property (26), $\mathcal{R}[g_0](s) \leqslant -\mathcal{L}[g_0](s)$ if $s \neq s_-, s_+$ one obtains for $s_- < s < s_+$,

$$\begin{aligned} g_1(s) = \mathcal{F}[g_0](s) &\leqslant K_n(s)\int_0^{s_-} \xi I_n(\xi)\mathcal{R}[1](\xi)\,\mathrm{d}\xi - K_n(s)\int_{s_-}^{s} \xi I_n(\xi)\mathcal{L}[g_+](\xi)\,\mathrm{d}\xi \\ &\quad - I_n(s)\int_s^{s_+} \xi K_n(\xi)\mathcal{L}[g_+](\xi)\,\mathrm{d}\xi - I_n(s)\int_{s_+}^{+\infty} \xi K_n(\xi)\mathcal{L}[\hat{g}_0](\xi)\,\mathrm{d}\xi \\ &\leqslant g_+(s) + K_n(s)I_n(s_-)s_-g_+'(s_-) \\ &< g_0(s). \end{aligned}$$

Analogously one can deal with the case $s \geqslant s_+$. □

Thus far the first result in proposition 2.6 is proved. The following lemma proves the second item in proposition 2.6.

Lemma 2.9. *Let g_0 be the function obtained in lemma 2.8. Then, the sequence $g_{k+1} = \mathcal{F}[g_k] \subset B_1$ and it is decreasing, i.e. $g_{k+1}(s) < g_k(s)$ for all $s \in J$.*

Proof. We observe that since $g_0 \in B_1$, then by lemma 2.4, $g_k \in B_1$. We define the smooth function $H(x) = x - d^{-1}F(1-x)$. For d large enough, $H'(x) > 0$ and hence, since $0 < g_1 < g_0 < 1$, $\mathcal{R}[g_1](s) \leqslant \mathcal{R}[g_2](s)$. Therefore,

$$g_2(s) = \mathcal{F}[g_1](s) \leqslant \mathcal{F}[g_0](s) = g_1(s),$$

and the result follows for all k by induction. □

Finally, we prove the equicontinuity of the family $\mathcal{S} = \{g_k\}$ for $s \in J = [0, +\infty)$ and also the equicontinuity at $+\infty$. If $s_0 \in (0, +\infty)$, the equicontinuity of the family $\mathcal{S}$ at s_0 follows straightforwardly from the fact that K_n and I_n are continuous along with the definition of the operator $\mathcal{F}$.

The equicontinuity at $+\infty$ is also a straightforward consequence of the fact that $0 < g_k < g_0$ and that $g_0 \to 0$ as $s \to +\infty$.

It only remains to ensure the equicontinuity at $s_0 = 0$. Let $\varepsilon > 0$, since $\mathcal{F}[g_k](0) = 1$, one just needs to check that $0 < 1 - \mathcal{F}[g_k](s) < \varepsilon$, if $0 < s < \delta$. Indeed, using (13) and the fact that $0 \leqslant g_k(s) \leqslant 1$,

$$\begin{aligned} 1 - \mathcal{F}[g_k](s) &= K_n(s) \int_0^s \xi I_n(\xi) \left(1 - g_k(\xi) + \frac{1}{d} F(1 - g_k(\xi))\right) \mathrm{d}\xi \\ &\quad + I_n(s) \int_s^{+\infty} \xi K_n(\xi) \left(1 - g_k(\xi) + \frac{1}{d} F(1 - g_k(\xi))\right) \mathrm{d}\xi \\ &\leqslant K_n(s) \int_0^s \xi I_n(\xi)\, \mathrm{d}\xi + I_n(s) \int_s^{+\infty} \xi K_n(\xi)\, \mathrm{d}\xi \end{aligned}$$

provided that, by definition (7) of d, $d \cdot x \geqslant F(1-x)$. Equicontinuity of the family $\mathcal{S} = \{g_k\}$ at $s = 0$ follows now immediately upon the fact that

$$\lim_{s\to 0} \left(K_n(s) \int_0^s \xi I_n(\xi)\, \mathrm{d}\xi + I_n(s) \int_s^{+\infty} \xi K_n(\xi)\, \mathrm{d}\xi\right) = 0.$$

Step 3. Existence of a $\mathcal{C}^{m+2}(J)$ solution of the fixed point equation $g = \mathcal{F}(g)$. We will use the Ascoli–Arzelà theorem. However, in our case, the family of functions is not defined on a compact set since the domain is $J = [0, +\infty)$. We can get around this by performing the bijective change of variables $s = \varphi(t) = 1/(1-t)$ which maps $[0, +\infty)$ into $[0, 1]$ and considering instead the sequence $h_k(t) = g_k(\frac{1}{1-t})$. Since the family $\{g_k\}$ satisfies the equicontinuity condition given in item (iii) of proposition 2.6, the sequence $\{h_k\}$ is equicontinuous at $[0, 1]$ (and uniformly bounded) and hence we can apply the Ascoli–Arzelà Theorem to $\{h_k\}$ and get a partial subsequence which is uniformly convergent at $[0, 1]$. Therefore, the sequence $\{g_k\}$ has a partial subsequence uniformly convergent at $J = [0, +\infty)$. Let $g^- = \lim_{k\to+\infty} g_{n_k}$, it is thus clear that $g^- = \mathcal{F}(g^-)$ and therefore this is the solution to the fixed point problem that we were searching.

So far we have only proved the existence of at least one continuous solution to problem (1)–(2). However, by writing this solution in terms of the fixed point equation (10), we have also proved that it is actually $\mathcal{C}^{m+2}(J)$. Therefore, if $m \geqslant n$, f may be expressed as a Taylor series around $r = 0$ which is readily found to be of the form

$$f(r) = \alpha_n r^n + o(r^n), \tag{27}$$

where α_n is *a priori* unknown, and, as we shall show, depends on degree n.

Step 4. Monotonicity of the solution $g^- = \mathcal{F}[g^-] = \lim_{k\to+\infty} g_k$. We recall that, by construction, g_0 is a decreasing function. Hence, in order to prove that our solution is also decreasing, we just have to check that the operator $\mathcal{F}$ conserves the monotonicity. To this end we introduce the functions

$$\varphi_1(s) = K_n(s)\int_0^s I_n(\xi)\frac{n^2}{\xi}\,\mathrm{d}\xi + I_n(s)\int_s^{+\infty} K_n(\xi)\frac{n^2}{\xi}\,\mathrm{d}\xi,$$
$$\varphi_2(s) = 1-\varphi_1(s) = K_n(s)\int_0^s \xi I_n(\xi)\,\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\,\mathrm{d}\xi.$$

Lemma 2.10. *The function* φ_1 *is decreasing, and hence* φ_2 *is increasing and both of them satisfy* $0 \leqslant \varphi_1(s), \varphi_2(s) \leqslant 1$.

Proof. We deal first with the statement for φ_1. It is clear that $0 \leqslant \varphi_1(s) \leqslant 1$. We now consider the second-order differential equation:

$$\varphi''(s) + \frac{\varphi'(s)}{s} - \varphi(s)\left(\frac{n^2}{s^2}+1\right) + \frac{n^2}{s^2} = 0.$$

We observe that φ_1 is the unique solution to the above equation, bounded in $[0, +\infty)$. We further note that $\varphi_1'(s)$ satisfies the second-order equation:

$$\varphi(s)'' + \frac{\varphi'(s)}{s} - \varphi(s)\left(\frac{n^2+1}{s^2}+1\right) - \frac{2n^2}{s^3}(1-\varphi_1(s)) = 0,$$

which, upon writing $\nu = \sqrt{n^2+1}$, may be expressed as

$$\varphi'(s) = -K_\nu(s)\int_0^s \xi I_\nu(\xi)\frac{2n^2}{\xi^3}(1-\varphi_1(\xi))\,\mathrm{d}\xi - I_\nu(s)\int_s^{+\infty} \xi K_\nu(\xi)\frac{2n^2}{\xi^3}(1-\varphi_1(\xi))\,\mathrm{d}\xi,$$

from where it becomes clear that it is negative provided $\varphi_1(s) \in [0, 1]$. This proves the statement for φ_1, so the fact that $\varphi_2 = 1 - \varphi_1$ completes the proof. □

Now we will check that if $h \in B_1$ is a decreasing function, $\mathcal{F}[h]$ is also decreasing. Indeed, it is clear that if $h \in B_1$ is a decreasing function, then the function $\mathcal{T}[h](s) := h(s) - d^{-1}F(1-h(s))$ is also decreasing. Moreover, $0 \leqslant \mathcal{T}[h] \leqslant 1$ provided $0 \leqslant h \leqslant 1$ and according to hypothesis (i) of theorem 2.1. Hence, since $K_n' \leqslant 0$ and $I_n' \geqslant 0$, we have that

$$\begin{aligned}\mathcal{F}'[h] &= K_n'(s)\int_0^s \xi I_n(\xi)\left(\frac{n^2}{\xi^2}+\mathcal{T}[h](\xi)\right)\mathrm{d}\xi \\ &\quad + I_n'(s)\int_s^{+\infty} \xi K_n(\xi)\left(\frac{n^2}{\xi^2}+\mathcal{T}[h](\xi)\right)\mathrm{d}\xi \\ &\leqslant \varphi_1'(s) + \mathcal{T}(h)(s)\left(K_n'(s)\int_0^s \xi I_n(\xi)\,\mathrm{d}\xi + I_n'(s)\int_s^{+\infty}\xi K_n(\xi)\,\mathrm{d}\xi\right) \\ &= \varphi_1'(s) + \mathcal{T}(h)(s)\varphi_2'(s) \leqslant \varphi_1'(s) + \varphi_2'(s) = 0.\end{aligned}$$

This shows that the operator $\mathcal{F}$ conserves the monotonicity. Thence, since g_0 is a decreasing function, the rest of the functions in the sequence, g_k, which are obtained by $g_{k+1} = \mathcal{F}[g_k]$, are also decreasing and consequently their limit function $g = \mathcal{F}[g]$ is indeed a decreasing function.

So far it has only been used that $F \in \mathcal{C}^0([0, 1])$ and that it is a Lipschitz function. However, in what follows we will use that at least $F \in \mathcal{C}^1([0, 1])$ to prove uniqueness.

Step 5. Uniqueness of the solution. To prove the uniqueness of the solution we will use the following comparison lemma for ODEs several times, which can be found in [34]:

Lemma 2.11 ([34]). *Let (a, b) be an interval in $\mathbb{R}$, let $\Omega = \mathbb{R}^2 \times (a, b)$ and let $\mathcal{H} \in \mathcal{C}^1(\Omega, \mathbb{R})$. Suppose $h \in \mathcal{C}^2((a, b))$ satisfies $h''(r) + \mathcal{H}(h(r), h'(r), r) = 0$. If $\partial_h \mathcal{H} \leqslant 0$ on Ω and if there exist functions $M, m \in \mathcal{C}^2((a, b))$ satisfying $M''(r) + \mathcal{H}(M(r), M'(r), r) \leqslant 0$ and $m''(r) + \mathcal{H}(m(r), m'(r), r) \geqslant 0$, as well as the boundary conditions $m(a) \leqslant h(a) \leqslant M(a)$ and $m(b) \leqslant h(b) \leqslant M(b)$, then for all $r \in (a, b)$ we have $m(r) \leqslant h(r) \leqslant M(r)$.*

Lemma 2.12. *Let f be a solution to*

$$f''(r) + \frac{f'(r)}{r} - f(r)\frac{n^2}{r^2} + F(f(r)) = 0, \tag{28}$$

satisfying that $f(r) \in [0, 1)$ for $r \geqslant 0$ and the boundary conditions $f(0) = 0$, $\lim_{s\to+\infty} f(r) = 1$.

Then:

(i) There exists $r_0 > 0$ such that f is strictly increasing at $[r_0, +\infty)$, in particular $f'(r) \geqslant 0$ if $r \geqslant r_0$. Consequently $\lim_{r\to+\infty} rf'(r) = 0$.
(ii) For any $r > 0$, $rf'(r) - n^2 f(r) < 0$.

Proof. We start by proving (i). We note that f' is a solution of the linear equation

$$\varphi''(r) + \frac{\varphi'(r)}{r} - \varphi(r)\frac{n^2+1}{r^2} + \frac{2n^2}{r^3} f(r) + \partial F(f(r))\varphi(r) = 0,$$

satisfying the boundary conditions $\lim_{s\to+\infty} \varphi(s) = 0$, $\varphi(0) = 0$ if $n \geqslant 2$ and bounded at the origin if $n = 1$. Since $f(0) = 0$, $\lim_{s\to+\infty} f(r) = 1$ and $0 \leqslant f(r) < 1$, it is clear that, for all $R > 0$, there exist $r_R \geqslant R$ such that $f'(r_R) \geqslant 0$. Let R be such that $\partial F(f(r)) \leqslant 0$ if $r \geqslant R$ (such $R > 0$ exists because $f(r) \to 1$ as $r \to +\infty$ and thanks to hypothesis (ii) of theorem 2.1). We may apply lemma 2.11 with $(a, b) = (r_R, +\infty)$,

$$\mathcal{H}(\varphi(r), \varphi'(r), r) = \frac{\varphi'(r)}{r} - \varphi(r)\frac{n^2+1}{r^2} + \frac{2n^2}{r^3} f(r) + \partial F(f(r))\varphi(r),$$

and $m(r) = 0$ to obtain that $f'(r) \geqslant 0$ if $r \geqslant r_R$. Now we are going to prove that f is strictly increasing. Indeed, assume that there exists $r_R \leqslant r_1 < r_2$ such that $f(r_1) = f(r_2)$ then, since $f'(r) \geqslant 0$ if $r \geqslant r_R$, we have that $f'(r) \equiv 0$ if $r \in [r_1, r_2]$ which implies that f is constant in $[r_1, r_2]$. In such a case, since f is a solution of (28), $f \equiv 0$ which gives a contradiction.

Now we check that the second property holds. We define $h(r) = rf'(r) - n^2 f(r)$. On the one hand, it is clear that $h(0) = 0$, and hence if there exists $r_1 > 0$ such that $h(r_1) > 0$, we can assume that $h'(r_1) > 0$. On the other hand, since f is a solution of (28) we have that

$$\begin{aligned} h'(r) &= rf''(r) + f'(r) - n^2 f'(r) = f(r)\frac{n^2}{r} - n^2 f'(r) - rF(f(r)) \\ &= -\frac{h(r)}{r} - (n^2 - 1) f'(r) - rF(f(r)). \end{aligned}$$

Finally, evaluating at $r = r_1$ one obtains $h'(r_1) < 0$ which gives a contradiction.

If $h(r) \leqslant 0$ for $r > 0$ and there exists $r_1 > 0$ such that $h(r_1) = 0$, it is easy to check that $h'(r_1) = 0$. In this case, we have that $f(r_1) = f'(r_1) = 0$ which implies $f \equiv 0$ provided f is a solution of (28) and also because of the uniqueness of solutions of the Cauchy problem for $r_1 > 0$. □

Lemma 2.13. *Let f be a solution of (28) satisfying $f(0) = 0$ and* $\lim_{r\to+\infty} f(r) = 1$. *Then the translated function for $a > 0$ defined by $f_a(r) = f(a + r)$ satisfies*

$$f_a''(r) + \frac{f_a'(r)}{r} - f_a(r)\frac{n^2}{r^2} + F(f_a(r)) < 0, \qquad r \in [0, +\infty).$$

Proof. We note that f_a is a solution of the second-order differential equation

$$f_a''(r) + \frac{f_a'(r)}{r+a} - f_a(r)\frac{n^2}{(r+a)^2} + F(f_a(r)) = 0.$$

Then,

$$\begin{aligned} f_a''(r) + \frac{f_a'(r)}{r} - f_a(r)\frac{n^2}{r^2} + F(f_a(r)) &= f_a'(r)\left(\frac{1}{r} - \frac{1}{r+a}\right) - f_a(r)\left(\frac{n^2}{r^2} - \frac{n^2}{(r+a)^2}\right) \\ &= \frac{1}{r^2(r+a)^2}[af_a'(r)r(r+a) - an^2 f_a(r)(2r+a)] \\ &= \frac{a}{r^2(r+a)^2}[r[(r+a)f_a'(r) - 2n^2 f_a(r)] - an^2 f_a(r)], \end{aligned}$$

and the lemma is proved using item (ii) in lemma 2.12. □

Let now f^+, f^- be two solutions of (28) satisfying the boundary conditions $f^+(0) = f^-(0) = 0$, $\lim_{r\to+\infty} f^+(r) = \lim_{r\to+\infty} f^-(r) = 1$.

To prove uniqueness of the solution we will explode a very useful technique, namely the sliding method, which was introduced in [5] and it is also used in [12]. We start then by considering the translated function

$$f_a(r) = f(a+r), \qquad a > 0, \tag{29}$$

and we consider the sets

$$\begin{aligned} U^- &= \{a > 0 : f_a^-(r) \geqslant f^+(r),\ r \geqslant 0\} \subset (0, +\infty), \\ U^+ &= \{a > 0 : f_a^+(r) \geqslant f^-(r),\ r \geqslant 0\} \subset (0, +\infty). \end{aligned}$$

We will see that the sets $U^\pm$ are (a) non-empty, (b) open and closed and therefore $U^\pm = (0, +\infty)$. We note that this implies the uniqueness of the solution.

We begin by proving that U is non-empty. Concretely we will prove the following:

$$\begin{aligned} &\exists a > 0 \qquad \text{such that } a \in U \iff \\ &\exists a > 0 \qquad \text{such that } f_a^-(r) > f^+(r), \quad r \geqslant 0. \end{aligned} \tag{30}$$

Since $\lim_{r\to+\infty} f^+(r) = 1$ and $\partial F(x) \leqslant 0$ if $x \sim 1$ (hypothesis (ii) of theorem 2.1), there exists $r_1 > 0$ such that $\partial F(f^+(r)) \leqslant 0$ if $r \geqslant r_1$. Moreover, since $\lim_{a\to+\infty} f_a^-(r) = 1$ and by item (i) of lemma 2.12 we can choose $a > 0$ such that f_a^- is a strictly increasing function, $\partial F(f_a^-(r)) \leqslant 0$ and $f_a^-(r) > f^+(r)$ if $r \in [0, r_1]$. Now we are going to prove that $f_a^-(r) \geqslant f^+(r)$ if $r \geqslant r_1$. In effect, we define $\Delta f = f_a^- - f^+$ and we note that, by lemma 2.13,

$$\Delta f''(r) + \frac{\Delta f'(r)}{r} - \Delta f(r)\frac{n^2}{r^2} - \frac{F(f^+(r)) - F(f_a^-(r))}{f_a^-(r) - f^+(r)}\Delta f(r) < 0. \tag{31}$$

We next introduce

$$D(r) = \frac{F(f^+(r)) - F(f_a^-(r))}{f_a^-(r) - f^+(r)} = -\int_0^1 \partial F(f_a^-(r) + \lambda(f^+(r) - f_a^-(r)))\,\mathrm{d}\lambda,$$

and we note that, with the choice of a and r_1, $D(r) \geqslant 0$ if $r \geqslant r_1$. We consider then the linear differential equation

$$\varphi''(r) + \frac{\varphi'(r)}{r} - \varphi(r)\frac{n^2}{r^2} - D(r)\varphi(r) = 0,$$

and we observe that $\Delta f(r_1) = f_a^-(r_1) - f^+(r_1) > 0$ and $\lim_{r\to+\infty} \Delta f(r) = 0$. Hence, by (31), we can apply lemma 2.11 to the solution $\varphi \equiv 0$ and we obtain $\Delta f \geqslant 0$ if $r \geqslant r_1$. Now we have already proved that $f_a^- \geqslant f^+$, but we note that if there exists $r_0 \geqslant 0$ such that $f_a^-(r_0) = f^+(r_0)$, then for any $\varepsilon > 0$, we have that $f_{a+\varepsilon}^-(r_0) = f^-(a+\varepsilon+r_0) > f^-(a+r_0) = f^+(r_0)$ provided that f^- is strictly increasing in $[a, +\infty)$, which proves (30).

The following step is to prove that, for $a > 0$,

$$f_a^-(r) \geqslant f^+(r), \qquad r \geqslant 0 \Longrightarrow f_a^-(r) > f^+(r), \; r \geqslant 0. \tag{32}$$

We note that f_a^- and f^+ cannot be identically equal simply because $f_a^-(0) > 0 = f^+(0)$. Let us then assume that there exists r_0 such that $f_a^-(r_0) = f^+(r_0)$. Then, $(f_a^-)'(r_0) = (f^+)'(r_0)$ provided $f_a^- \geqslant f^+$. Therefore, we have that, on the one hand, by Taylor's theorem, for $r \sim r_0$,

$$f_a^-(r) - f^+(r) = ((f_a^-)''(r_0) - (f^+)''(r_0))\frac{(r-r_0)^2}{2} + O((r-r_0)^3 \geqslant 0,$$

which implies that $(f_a^-)''(r_0) - (f^+)''(r_0) \geqslant 0$. And on the other hand, assuming $f_a^-(r_0) = f^+(r_0)$ and $(f_a^-)'(r_0) = (f^+)'(r_0)$ and taking into account lemma 2.13 one obtains

$$0 > (f_a^-)''(r_0) + \frac{(f_a^-)'(r_0)}{r_0} - f_a^-(r_0)\frac{n^2}{r_0^2} + F(f_a^-(r_0)) = (f_a^-)''(r_0) - (f^+)''(r_0),$$

which gives a contradiction.

We now prove the last step. That is we will check that U is open (U closed is immediate):

$$f_a^- \geqslant f^+ \Longrightarrow f_{a+\varepsilon}^- \geqslant f^+, \qquad \text{if } |\varepsilon| \text{ is small enough.} \tag{33}$$

Assume that $f_a^- \geqslant f$ for some $a > 0$. By (32) we have in fact that $f_a^- > f^+$. Let $r_1 > 0$ be such that $\partial F(f^+(r)) \leqslant 0$ if $r \geqslant r_1$ (as usual we have used that F is decreasing in a neighbourhood of $x = 1$). It is clear that, since $f_a^- \geqslant f^+$, we also have that $\partial F(f_a^-(r)) \leqslant 0$ if $r \geqslant r_1$. We introduce the positive quantity

$$b = \min_{r\in[0,r_1]} f_a^-(r) - f^+(r) > 0,$$

and let $|\varepsilon| < a/2$ small enough such that

$$\max_{r\in[0,r_1]} |f_{a+\varepsilon}^-(r) - f_a^-(r)| \leqslant \frac{b}{2}.$$

Then we have that, if $r \in [0, r_1]$,

$$f_{a+\varepsilon}^-(r) - f^+(r) = f_{a+\varepsilon}^-(r) - f_a^-(r) + f_a^-(r) - f^+(r) \geqslant \frac{b}{2} > 0.$$

We have already proved that $f_{a+\varepsilon}^-(r) > f^+(r)$ if $r \in [0, r_1]$. It only remains to check that $f_{a+\varepsilon}^-(r) \geqslant f^+(r)$ if $r \geqslant r_1$, but the proof of this fact is completely analogous to the one used to prove (30).

Step 6. Behaviour at infinity of f for generic nonlinearities $F \in \mathcal{C}^1([0, 1])$.

Lemma 2.14. *In the generic case $\partial F(1) = -b < 0$, the solution $f(r) = 1 - \frac{n^2}{br^2} + o(r^{-2})$ as $r \to +\infty$.*

Proof. We write $\widetilde{f}(t) = 1 - f(t/\sqrt{b})$ and we just have to check that $\widetilde{f}(t)$ satisfies

$$\lim_{t\to+\infty} t^2 \widetilde{f}(t) = n^2.$$

We first point out that, when $\partial F(1) < 0$, then by lemma 2.5, one can take G of the form $G(x) = \frac{b}{4}x - x^2$. Therefore, since $g(s) = 1 - f(s/\sqrt{d})$, by lemma 2.8 and *step 3*,

$$\widetilde{f}(t) = g(t\sqrt{d/b}) = \lim_{k\to+\infty} g_k(t\sqrt{d/b}) < g_0(t\sqrt{d/b}) = O(t^{-2}).$$

We also have that $\widetilde{f}(t)$ satisfies the differential equation

$$\widetilde{f}''(t) + \frac{\widetilde{f}'(t)}{t} - \widetilde{f}(t)\left(1 + \frac{n^2}{t^2}\right) + \frac{n^2}{t^2} - \frac{1}{b}[F(1-\widetilde{f}(t)) - b\widetilde{f}(t)] = 0,$$

and (following *step 1*) may be written in terms of the fixed point equation

$$\widetilde{f}(t) = \varphi_1(t) + K_n(t)\int_0^t \xi I_n(\xi)\mathcal{R}[\widetilde{f}](\xi)\,\mathrm{d}\xi + I_n(t)\int_t^{+\infty} \xi K_n(\xi)\mathcal{R}[\widetilde{f}][\xi]\,\mathrm{d}\xi,$$

where $\varphi_1(t)$ was introduced in lemma 2.10 and

$$\mathcal{R}[\widetilde{f}](t) = -\frac{1}{b}[F(1-\widetilde{f}(t)) - b\widetilde{f}(t)] = o(\widetilde{f}(t)) = o(t^{-2}).$$

It is easy to check, using Hôpital's rule along with the behaviour of the modified Bessel functions I_n and K_n at infinity, that

$$\lim_{t\to+\infty} t^2\varphi_1(t) = n^2.$$

As for the remaining part, since $\mathcal{R}[\widetilde{f}](t) = o(t^{-2})$, it is also straightforward that

$$\lim_{t\to+\infty}\left(t^2K_n(t)\int_0^t \xi I_n(\xi)\mathcal{R}[\widetilde{f}](\xi)\,\mathrm{d}\xi + t^2I_n(t)\int_t^{+\infty}\xi K_n(\xi)\mathcal{R}[\widetilde{f}](\xi)\,\mathrm{d}s\right) = 0.$$

Again one must use here a combination of Hôpital's rule and that the modified Bessel functions at infinity behave like $K_n(s) \sim \mathrm{e}^{-s}\sqrt{\pi/2s}$, $I_n(s) \sim \mathrm{e}^s/\sqrt{2\pi s}$. After having proved this, it becomes clear that $f(r) = 1 - n^2/(br^2) + o(r^{-2})$.

Remark 2.15. Assuming the hypotheses of the previous lemma hold, we note that if $F \in \mathcal{C}^2([0,1])$ and one applies Taylor's theorem, then

$$f(r)\frac{n^2}{r^2} - F(f(r)) = \frac{n^2}{r^2} + b(f(r)-1) + \mathcal{O}((1-f(r))^2) + \mathcal{O}(r^{-2}(1-f(r))).$$

Therefore, by the previous lemma, $f(r)n^2r^{-1} - rF(f(r)) = \mathcal{O}(r^{-3})$. Moreover, since $(rf'(r))' = n^2r^{-1}f(r) - rF(f(r))$ and item (i) of lemma 2.12 provides that $rf'(r) \to 0$ as $r \to +\infty$, one obtains

$$rf'(r) = \int_r^{+\infty} f(\xi)\frac{n^2}{\xi} - \xi F(f(\xi))\,\mathrm{d}\xi = \mathcal{O}(r^{-4}).$$

This concludes the proof of theorem 2.1 □

Some remarks on the hypothesis. To complete this section we give some comments regarding the optimality of the hypothesis:

(i) Lemma 2.5 also holds provided $F \in \mathcal{C}^0([0,1])$ and $F(1-x) > 0$ for $x \sim 0$. Indeed, this is clear by choosing

$$\int_0^x (1-bu)\int_0^u F(1-\xi)\,\mathrm{d}\xi\,\mathrm{d}u,$$

where $b = 1/x_1$ with x_1 such that $F(1-x) > 0$ when $0 < x < x_1$.

(ii) To prove the existence, monotonicity and regularity of the solution f it is just required that $F \in \mathcal{C}^0([0,1])$, $F(1-x) > 0$ if $x \sim 0$ and that F is Lipschitz.

(iii) The full set of hypotheses is only necessary in order to obtain the uniqueness of f.

(iv) $F \in \mathcal{C}^0([0,1])$ seems not to be a sufficient condition to guarantee even the existence of a solution. For example, let us consider $F(x) = x^{1/\ell}(1-x^2)$ and equation

$$f''(r) + \frac{f'(r)}{r} - \frac{n^2}{r^2} f(r) + F(f(r)) = 0,$$

for $n > 2$ and $\ell \geqslant n/(n-2)$. Formally, if f is a solution to this equation satisfying $f(0) = 0$, then $f(r) \sim \alpha r^m + o(r^m)$, since $f \in \mathcal{C}^2([0,+\infty))$, for some α and m. If one now substitutes this expression into the equation, one gets

$$\alpha(m^2 - n^2) r^{m-2} + \alpha^{1/\ell} r^{m/\ell} = o(r^{m-2}) + o(r^{m/\ell}).$$

It is now clear that either $m - 2 < m/\ell$, in which case $m = n$ yielding a contradiction, or $m - 2 = m/\ell$ and $m^2 - n^2 < 0$ (recall that $\alpha > 0$), in which case one finds $\ell < n/(n-2)$ giving again a contradiction.

3. Exponential smallness of α_n

In this section we focus on the behaviour at the origin of the solution f to

$$f''(r) + \frac{f'(r)}{r} - \frac{n^2}{r^2} f(r) + F(f(r)) = 0,$$

with boundary conditions (2). We are actually going to prove that $\lim_{r\to 0} r^{-n} f(r) = \alpha_n$ exists and we will show that indeed α_n happens to be exponentially small in parameter n.

Lemma 3.1. *Under the hypotheses of theorem 2.1,* $\alpha_n = \lim_{r\to 0} f(r) r^{-n}$ *satisfies that*

$$0 \leqslant \alpha_n \leqslant \frac{n^{1/3} d^{n/2}}{2^n n!} \frac{3^{2/3}\Gamma(2/3)}{(d+1)2^{1/3}} \left(1 + 0\left(\frac{1}{n^{4/3}}\right)\right),$$

where $d := \sup_{z\in[0,1]} |\partial F(z)|$ *as defined in (7).*

Proof. First we note that, since $F(0) = 0$, by the mean value theorem,

$$0 \leqslant F(x) \leqslant \sup_{z\in[0,1]} |\partial F(z)| \cdot x = d \cdot x.$$

We perform the change $f(s/\sqrt{d}) = h(s)$ and we recall that h satisfies the equation

$$h''(s) + \frac{h'(s)}{s} - \frac{n^2}{s^2} h(s) + d^{-1} F(h(s)) = 0.$$

We shall apply lemma 2.11 in order to bind our solution h. Thus, we define

$$\mathcal{H}(h'(s), h(s), s) = \frac{h'(s)}{s} - \frac{n^2}{s^2} h(s) + d^{-1} F(h(s)).$$

Let $(a, b) = (0, n)$ and $M(s) = C J_n(s)$ with $C = h(n)/J_n(n)$ and J_n the Bessel function. We have that $\partial_h \mathcal{H}(h'(s), h(s), s) \leqslant 0$ if $s \leqslant n$ and M satisfies $h(0) = M(0) = 0$ and $h(n) = M(n)$. Moreover

$$M''(s) + \mathcal{H}(M'(s), M(s), s) = -C J_n(s) + d^{-1} F(C J_n(s)) \leqslant 0,$$

provided $0 \leqslant C J_n(s) \leqslant h(n) \leqslant 1$ if $s \leqslant n$ (see [1] and definition of d). Hence by lemma 2.11, $h(s) \leqslant C J_n(s)$ if $s \leqslant n$. Therefore,

$$\alpha_n = \lim_{r\to 0} \frac{f(r)}{r^n} = \lim_{r\to 0} \frac{h(\sqrt{d}\, r)}{r^n} \leqslant C \lim_{r\to 0} \frac{J_n(\sqrt{d}\, r)}{r^n} = C \frac{d^{n/2}}{2^n n!}.$$

It only remains to estimate $C = h(n)/J_n(n)$. It is clear that $h(n) \leqslant 1$ and, moreover, it is also known [1] that

$$J_n(n) = \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)n^{1/3}}\left(1 + 0\left(\frac{1}{n^{4/3}}\right)\right).$$

Using these estimates we obtain

$$C \leqslant \frac{n^{1/3}3^{2/3}\Gamma(2/3)}{2^{1/3}}\left(1 + 0\left(\frac{1}{n^{4/3}}\right)\right),$$

and therefore,

$$\alpha_n \leqslant \frac{n^{1/3}d^{n/2}}{2^n n!}\frac{3^{2/3}\Gamma(2/3)}{2^{1/3}}\left(1 + 0\left(\frac{1}{n^{4/3}}\right)\right). \qquad \square$$

4. Formal solution at infinity and Gevrey estimates

In the rest of this work we focus on the generic case $\partial F(1) < 0$ and F an analytic function in a neighbourhood of $x = 1$. As a particular case, the well-known Ginzburg–Landau equation, which corresponds to $F = f(1 - f^2)$, is included. This equation first arose as a model for phase transition problems in superconductivity, and later on was also proved to be a good model in many other systems such as in superfluidity or nematic liquid crystals among others (see [21, 22, 28]).

There are a number of works in the literature dealing with this equation that prove the existence, uniqueness and monotonicity of solutions. However, the structure of such solutions at infinity has not been carefully studied before, despite the fact that on many occasions it was necessary to use some estimates on, for instance, the rate of growth of f at infinity. The goal in this section is hence to derive a formal expansion of the solutions of equation (1) at infinity and in particular to show that there exists a unique formal solution that tends to one at infinity. Furthermore, we shall show that this formal solution is 1-Gevrey (see the appendix).

Proposition 4.1. *Let $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ be an analytic function satisfying the hypotheses of theorem 2.1. Assume that there exists $\delta > 0$ such that $\overline{B}_\delta = \{z \in \mathbb{C} : |1 - z| \leqslant \delta\} \subset \mathcal{U}$. If $\partial F(1) = -b < 0$, given equation*

$$f''(r) + \frac{f'(r)}{r} - \frac{n^2}{r^2}f(r) + F(f(r)) = 0, \tag{34}$$

there is a unique formal solution of the form

$$\hat{f}(r) = 1 + \sum_{k\geqslant 1}\frac{a_k}{r^{2k}},$$

such that $\hat{f}$ is an asymptotic expansion of 1-Gevrey type, that is

$$|a_k| \leqslant C^{2k}(2k)!,$$

with $C > 0$ and adequate constant.

Proof. To prove this proposition we first construct recursively the asymptotic expansion, afterwards we show that all the odd terms in the expansion vanish and finally we show that the series is of 1-Gevrey type.

To clarify the exposition, we will deal with $h(t) = 1 - f(t^{-1})$. If f is a solution of (34) then h satisfies

$$t^4h''(t) + t^3h'(t) - n^2t^2h(t) + n^2t^2 - F(1 - h(t)) = 0. \tag{35}$$

We emphasize that the formal series $\hat{f}(r) = \sum_{k\geqslant 0} \frac{a_k}{r^k}$ is a formal solution of (34) if and only if $\hat{h}(t) = 1 - \sum_{k\geqslant 0} a_k t^k$ is a formal solution of (35). Moreover, $\hat{f}$ is 1-Gevrey at infinity if and only if $\hat{h}$ is 1-Gevrey at the origin (see the appendix).

Construction of the formal asymptotic expansion. We first start by posing the expansion at the origin given by

$$\hat{h}(t) = \sum_{k\geqslant 0} a_k t^k,$$

and we define the truncated series

$$h_N(t) = \sum_{k=0}^{N} a_k t^k.$$

We have kept the same notation for the coefficients a_k. We also introduce $F_k = (-1)^k (k!)^{-1} \partial^k F(1)$ and we note that $F_0 = 0$ and

$$F(1-z) = \sum_{k=1}^{+\infty} F_k z^k.$$

Finally we consider the truncated error

$$E_N(t) = t^4 h_N''(t) + t^3 h_N'(t) - n^2 t^2 h_N(t) + n^2 t^2 - F(1 - h_N(t)). \tag{36}$$

In the case $N = 2$ we obtain $E_2(t) = \mathcal{O}(t^4)$ simply by taking

$$a_0 = a_1 = 0, \qquad a_2 = \frac{n^2}{F_1}.$$

Let $e_N = \partial_t^N E_{N-1}(0)/N!$. We claim that if $a_N = e_N/F_1$ for $N \geqslant 3$, then $E_N(t) = \mathcal{O}(t^{N+1})$. Indeed, we proceed by induction. Assume that by taking $a_0 = a_1 = 0, a_2 = n^2/F_1$, $a_3, \ldots, a_{N-1}$ as $a_k = e_k/F_1$, the truncated error of order $N-1$ satisfies $E_{N-1}(t) = e_N t^N + \mathcal{O}(t^{N+1})$. Then, writing $h_N(t) = h_{N-1}(t) + a_N t^N$,

$$E_N(t) = E_{N-1}(t) + t^{N+2} a_N (N^2 - n^2) + F(1 - h_{N-1}(t)) - F(1 - h_N(t)),$$

and using

$$\begin{aligned} F(1 - h_N(t)) - F(1 - h_{N-1}(t)) &= \sum_{k\geqslant 1} F_k [h_{N-1}^k(t) - (h_{N-1}(t) + a_N t^N)^k] \\ &= -F_1 a_N t^N + \mathcal{O}(t^{N+2}), \end{aligned}$$

we obtain that

$$E_N(t) = (e_N - F_1 a_N) t^N + \mathcal{O}(t^{N+1}),$$

which implies that taking $a_N = e_N/F_1$, the truncated error, which has been defined in (36), satisfies $E_N(t) = \mathcal{O}(t^{N+1})$.

It only remains to obtain an iterative formula for $e_N = \partial_t^N E_{N-1}(0)$. Such formula is a straightforward application of the Faa di Bruno formula which we recall here: let f and g be two $\mathcal{C}^{+\infty}$ composable functions, then

$$\frac{D^k (f\circ g)(z)}{k!} = \sum_{l=1}^{k} \sum_{\substack{k_1+\cdots+k_l = k \\ 1\leqslant k_i}} \frac{D^l f(g(z))}{l!} \frac{[D^{k_1} g(z), \ldots, D^{k_l} g(z)]}{k_1! \cdots k_l!}. \tag{37}$$

Applying Faa di Bruno to obtain $\partial_t^N(F\circ(1-h_{N-1}))$, taking into account that $h_{N-1}(t) = a_2t^2+\cdots+a_{N-1}t^{N-1}$ and the definition of F_k, we obtain

$$e_N = a_{N-2}((N-2)^2-n^2) - \sum_{k=2}^{N} \sum_{\substack{l_1+\cdots+l_k=N\\ 2\leqslant l_i\leqslant N-1}} F_k\cdot(a_{l_1}\cdot\cdots\cdot a_{l_k}). \tag{38}$$

Now we are going to show that the asymptotic expansion has only non-vanishing even terms which is equivalent to seeing that $e_{2N+1}=0$. Recalling that $a_1=e_1=0$, we shall again proceed by induction and start by assuming that $e_{2\ell+1}=0$ for $\ell\leqslant N$. Therefore, (38) yields

$$e_{2N+1} = -\sum_{k=2}^{2N+1}\sum_{\substack{l_1+\cdots+l_k=2N+1\\ 2\leqslant l_i\leqslant 2N}} F_k\cdot(a_{l_1}\cdot\cdots\cdot a_{l_k}).$$

If $l_1+\cdots+l_k=2N+1$, then some l_i must be necessarily odd, therefore, by induction, $a_{l_i}=0$ which implies $e_{2k+1}=a_{2k+1}=0$.

The formal series is 1*-Gevrey.* First we introduce some notation. We denote $A = \sup_{z\in\overline{B}_\delta}|F(z)|$. Since F is analytic in $\overline{B}_\delta$, we have

$$|\partial^N F(1)| = |F_N|N! \leqslant A\cdot\delta^{-N}N!. \tag{39}$$

We now fix $N_0\geqslant n$ large enough such that

$$\frac{Ae^{2/\delta}}{N_0}\leqslant\frac{|F_1|}{2}. \tag{40}$$

To show that this expansion is 1-Gevrey, we must now prove that $|a_{2N}|\leqslant B\cdot D^{2N}\cdot(2N)!$ for some constants $B, D>0$. Recall that $a_{2N+1}=0$. It is clear that a_i, for $i=1,\ldots,N_0$, satisfy this growing condition upon taking appropriate constants. Moreover, it is also clear that this growing condition is equivalent to

$$|a_{2N}|\leqslant C^{2N}(2N)!, \qquad \forall N\geqslant 1, \tag{41}$$

with $C\geqslant\max\{BD, B^{-1}D, \sqrt{2/|F_1|}\}$ (this definition of C will be used later on). As for $N\geqslant N_0$, we proceed by induction, we assume that the terms $a_2,\ldots,a_{2(N-1)}$ satisfy property (41) and we show that so does the term $a_{2N}=e_{2N}/F_1$. We will follow the strategy in lemma 3.6 in [3]. We use the simple fact that if $b\leqslant c$, then $(a+b)!c!\leqslant(a+c)!b!$, henceforth if $m_1+\cdots+m_k=2N$ and $l_i\geqslant 2$, then we deduce

$$m_1!\cdots m_k!\leqslant 2^{k-1}(2N-2(k-1))!.$$

Moreover, it is also known that

$$\sharp\{l_1+\cdots+l_k=N : l_i\geqslant 1\} = \binom{N-1}{k-1}.$$

Therefore, using bound (39) of F_k, formula (38) for e_{2N} and taking into account that $a_{2k+1}=0$,

$$|F_1a_{2N}|\leqslant C^{2N-2}(2N-2)![(2N-2)^2-n^2] + AC^{2N}\sum_{k=2}^{N}\frac{1}{\delta^k}\sum_{\substack{l_1+\cdots+l_k=N\\ 1\leqslant l_i\leqslant N-1}}(2l_1)!\cdots(2l_k)!$$

$$\leqslant C^{2N-2}(2N-2)!(2N-2)^2 + \frac{A\,C^{2N}}{2}\sum_{k=2}^{N}(2\delta^{-1})^k(2N-2(k-1))!\binom{N-1}{k-1}$$

$$\leqslant C^{2N-2}(2N-2)!(2N-2)^2 + A\,C^{2N}(2N!)\frac{1}{N}e^{2/\delta},$$

where in the last inequality we have used that

$$(2N-2(k-1))!\binom{N-1}{k-1} \leqslant 4\frac{(2N-1)!}{k!}.$$

Then, by definition of C and N_0 and taking into account that $N \geqslant N_0$,

$$|a_{2N}| \leqslant C^{2N}(2N)!\left(\frac{(2N-2)^2}{C^2|F_1|2N(2N-1)} + \frac{Ae^{2/\delta}}{|F_1|N}\right) \leqslant C^{2N}(2N)!.$$

and thus (41) holds. □

5. Solution at infinity, Gevrey asymptotic

In this section we analyse the solution of equation (34) with boundary conditions $f(0) = 0$, $\lim_{r\to+\infty} f(r) = 1$, for large values of the radius. We shall also show, in what follows, that the solution f is 1-Gevrey asymptotic to the formal solution $\hat{f}(r) = \sum_{k\geqslant 0} a_k r^{-2k}$ obtained in the previous section in some sectors of the complex plane. That is, if $r \to +\infty$ belongs to some appropriate sectors in the complex plane, then

$$\left| r^{2N}\left(f(r) - \sum_{k=0}^{N-1}\frac{a_k}{r^{2k}}\right)\right| \leqslant BA^{2N}(2N)!,$$

for some constants A, B independent of N. We refer the reader to the appendix where the Gevrey-asymptotic concepts are explained. As a consequence, we now show that this solution is not only real analytic for large enough values of r, but can further be analytically extended to some sectors of the complex plane.

There are some related results regarding the regularity of the solutions of partial differential equations in a more general setting in open (even not bounded) sets [17, 20]. Such results, applied to our problem, prove the real analyticity of the solution f in $(r_0, +\infty)$ for some $r_0 > 0$. However, to our knowledge, the Gevrey asymptotic at infinity for the problem considered in this work, has not been studied before in the sense described above.

We want to mention here that there are many works tackling the global Gevrey regularity and decay at infinity of solitary waves of semilinear equations of the form $\Delta u = F(u)$, see for instance [7, 8, 13, 14]. In addition, the results in [36, 37] deal with the Gevrey asymptotic (with respect to the perturbation parameter) for singularly perturbed analytic ordinary equations.

We first start by giving a previous definition and then state the main result in this section.

Definition 5.1. *Given $\beta, \rho > 0$, the sector of radius ρ and opening $\beta \in [0, \pi]$ is given by*

$$S(\beta, \rho) = \{z \in \mathbb{C} : |z| > \rho, |arg(z-\rho)| < \beta\}.$$

Theorem 5.2. *Let $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ be a function satisfying the hypotheses of theorem 2.1. Assume that there exists $\delta > 0$ such that $\overline{B}_\delta := \{z \in \mathbb{C} : |1-z| \leqslant \delta\} \subset \mathcal{U}$, that F is analytic in $\overline{B}_\delta$ and that*

$$\partial F(1) = -b < 0.$$

Then, for all $\beta < \pi/2$ there exists $r_0 > 0$ large enough such that the unique solution f of equation

$$f''(r) + \frac{f'(r)}{r} - \frac{n^2}{r^2}f(r) + F(f(r)) = 0 \tag{42}$$

with boundary conditions $f(0) = 0$ *y* $\lim_{r\to+\infty} f(r) = 1$

(i) is real analytic in $[r_0, +\infty)$ *and can be analytically extended to the sector* $S(\beta, r_0)$ *and it satisfies*

$$\sup_{r\in S(\beta,r_0)} |r^2(1 - f(r))| \leqslant A$$

for some positive constant A.

(ii) Moreover, f *is asymptotic* 1*-Gevrey to the formal solution* $\hat{f}$ *obtained in proposition 4.1 in the sector* $S(\beta, r_0)$.

In what follows we will assume, without explicitly mentioning it, that the hypotheses of theorem 5.2 hold.

In order to prove this theorem, we need some previous results. As we did in *step 6*, let $g(s) = 1 - f(s/\sqrt{b})$ be the solution of

$$g''(s) + \frac{g'(s)}{s} - g(s) = g(s)\frac{n^2}{s^2} - \frac{n^2}{s^2} + \frac{1}{b}[F(1 - g(s)) - bg(s)], \qquad (43)$$

such that $g(0) = 1$ and $\lim_{s\to+\infty} g(s) = 0$. According to proposition 4.1, there exists a unique formal solution $\hat{g} = \sum_{k\geqslant 2} b_k s^{-2k}$ of (43) that is of 1-Gevrey type. This formal solution, as we show in what follows, is the asymptotic expansion of a set of functions defined not only on the real line, but also on a sector in the complex plane, that satisfy equation (43) up to an exponentially small quantity.

Lemma 5.3. *For all* $\beta < \pi/2$*, there exists* $\rho > 0$ *large enough and a function* $\tilde{g} : S(\beta, \rho) \to \mathbb{C}$ *such that*

- $\tilde{g}$ *is asymptotic* 1*-Gevrey to the formal expansion* $\hat{g}(s) = \sum_{k\geqslant 2} b_k s^{-2k}$.
- $\tilde{g}$ *satisfies*

$$E(s) = \tilde{g}''(s) + \frac{\tilde{g}'(s)}{s} - \tilde{g}(s) - \tilde{g}(s)\frac{n^2}{s^2} + \frac{n^2}{s^2} - \frac{1}{b}[F(1 - \tilde{g}(s)) - b\tilde{g}(s)], \qquad (44)$$

where $E : S(\beta, \rho) \to \mathbb{C}$ *is such that*

$$\sup_{s\in S(\beta,\rho)} |E(s)\exp(c|s|)| < \kappa,$$

for some constants $c, \kappa > 0$.

Proof. Let $0 < \beta < \beta' < \pi/2$ and $\rho > \rho' > 0$. It is clear that $\overline{S(\beta, \rho)} \subset S_\infty(\beta', \rho')$ (see the appendix for the definition of S_∞ sectors). The first part of the lemma follows from the Borel–Ritt–Gevrey theorem (see item (i) of proposition A.5), which states that, given a 1-Gevrey asymptotic expansion $\hat{g}$ and a sector $S_\infty(\beta', \rho')$, there exists a function, $\tilde{g}$, analytic in $S_\infty(\beta', \rho')$ that is 1-Gevrey to this formal expansion $\hat{g}$.

As for the residue, $E(s)$, we will prove that $E \cong_1 \hat{0}$ (see the appendix for the definition). In that case, by item (ii) of proposition A.5, there exist $c, \kappa > 0$ such that,

$$\sup_{s\in S(\beta,\rho)} |E(s)\exp(c|s|)| < \kappa.$$

We observe that, there exists a closed sector $\overline{S^1_\infty}$ of $S_\infty(\beta', \rho')$ such that $S(\beta, \rho) \subset \overline{S^1_\infty}$.

Now we are going to check that $E \cong_1 \hat{0}$. First, we introduce $h(t) = g(t^{-1})$ and we notice that h satisfies the differential equation

$$t^4h''(t) + t^3h'(t) - h(t) = n^2t^2h(t) - n^2t^2 + \frac{1}{b}[F(1 - h(t)) - bh(t)]. \qquad (45)$$

As we pointed out in the proof of proposition 4.1, the formal power series $\hat{h}(t) = \sum_{k\geqslant 1} b_k t^{2k}$ is a formal solution of equation (45). Recall that $\hat{g}(s) = \sum_{k\geqslant 1} b_k s^{-2k}$. Moreover, $\widetilde{h}(t) = \widetilde{g}(t^{-1})$ satisfies the equation

$$t^4\widetilde{h}''(t) + t^3\widetilde{h}'(t) - \widetilde{h}(t) = n^2t^2\widetilde{h}(t) - n^2t^2 + \frac{1}{b}[F(1-h(t)) - bh(t)] + E(t^{-1}),$$

where E is the residue, and by proposition A.1, $\widetilde{h} \cong_1 \hat{h}$. We also have that by item (ii) of proposition A.2,

$$\lim_{\substack{t\to 0\\ t\in S_0(\beta',1/\rho')}} \partial_t^{2k}\widetilde{h}(t) = (2k!)b_k, \qquad \lim_{\substack{t\to 0\\ t\in S_0(\beta',1/\rho')}} \partial_t^{2k+1}\widetilde{h}(t) = 0, \tag{46}$$

for all $k \in \mathbb{N}$. Let us write $\overline{E}(t) = E(t^{-1})$. On the one hand, we have that the function $\overline{E}$ belongs to $G_1(S_0(\beta', 1/\rho'))$ (see the appendix) and on the other hand, by the formal construction and (46)

$$\lim_{\substack{t\to 0\\ t\in S_0(\beta',1/\rho')}} \partial_t^{k}\overline{E}(t) = 0,$$

and again by proposition A.2, $\overline{E} \cong_1 \hat{0}$. Finally applying proposition A.1, one obtains that $E \cong_1 \hat{0}$. □

Now we define $\Delta g = g - \tilde{g}$ and

$$\mathcal{D}(s) = \frac{n^2}{s^2} - \frac{1}{b}\int_0^1 [b + \partial F(1 - \lambda(g(s) - \tilde{g}(s)))]\,\mathrm{d}\lambda. \tag{47}$$

Since g satisfies equation (43) and $\tilde{g}$ satisfies equation (44) Δg satisfies the linear equation

$$\Delta g''(s) + \frac{\Delta g'(s)}{s} - \Delta g(s) = \mathcal{D}(s)\Delta g(s) - E(s). \tag{48}$$

Here we have used that both g and $\tilde{g}$ can be seen either as known or unknown functions.

Lemma 5.4. *For any $s_0 > 0$, and for any solution h_1 and h_2 of (43) and (48) respectively, satisfying $\lim_{s\to+\infty} h_1(s) = \lim_{s\to+\infty} h_2(s) = 0$ there exist two unique constants $C_{h_1}(s_0)$ and $C_{h_2}(s_0)$ such that h_1, h_2 are solutions of the following fixed point equations:*

$$h_1(s) = K_0(s)C_{h_1}(s_0) + K_0(s)\int_{s_0}^{s} \xi I_0(\xi)\mathcal{T}[h_1](\xi)\,\mathrm{d}\xi + I_0(s)\int_s^{+\infty} \xi K_0(\xi)\mathcal{T}[h_1](\xi)\,\mathrm{d}\xi,$$

with $\mathcal{T}[h](s) = -g(s)\frac{n^2}{s^2} + \frac{n^2}{s^2} - \frac{1}{b}[F(1-g(s)) - bg(s)]$ and

$$h_2(s) = K_0(s)C_{h_2}(s_0) + K_0(s)\int_{s_0}^{s} \xi I_0(\xi)\mathcal{R}[h_2](\xi)\,\mathrm{d}\xi + I_0(s)\int_s^{+\infty} \xi K_0(\xi)\mathcal{R}[h_2](\xi)\,\mathrm{d}\xi,$$

where $\mathcal{R}[h](s) = -\mathcal{D}(s)h(s) + E(s)$.

Proof. We choose $s_0 > 0$. If h is a solution to (48), then, for all $s_1 > 0$,

$$h(s) = K_0(s)\left(C_1 + \int_{s_0}^{s} \xi I_0(\xi)\mathcal{R}[h](\xi)\right)\mathrm{d}\xi + I_0(s)\left(C_2 - \int_{s_1}^{s} \xi K_0(\xi)\mathcal{R}[h](\xi)\,\mathrm{d}\xi\right).$$

Upon imposing the condition at infinity $\lim_{s\to+\infty} h(s) = 0$, it turns out that $C_2 = \int_{s_1}^{+\infty} \xi K_0(\xi)\mathcal{R}[h](\xi)\,\mathrm{d}\xi$ and C_1 is also determined. The other case is completely analogous. □

Let $c_1 = C_g(s_0)$ and $c_2 = C_{g-\tilde{g}}(s_0)$ be the constants in previous lemma corresponding to g and $g - \tilde{g}$, respectively. We note that

$$(g-\tilde{g})(s) = K_0(s)c_2 + K_0(s)\int_{s_0}^{s} \xi I_0(\xi)\mathcal{R}[g-\tilde{g}](\xi)\,\mathrm{d}\xi + I_0(s)\int_{s}^{+\infty} \xi K_0(\xi)\mathcal{R}[g-\tilde{g}](\xi)\,\mathrm{d}\xi,$$

$$(g'-\tilde{g}')(s) = K_0'(s)c_2 + K_0'(s)\int_{s_0}^{s} \xi I_0(\xi)\mathcal{R}[g-\tilde{g}](\xi)\,\mathrm{d}\xi + I_0'(s)\int_{s}^{+\infty} \xi K_0(\xi)\mathcal{R}[g-\tilde{g}](\xi)\,\mathrm{d}\xi.$$

Therefore,

$$c_2 = s_0((g-\tilde{g})(s_0)I_0'(s_0) - (g'-\tilde{g}')(s_0)I_0(s_0)). \tag{49}$$

Analogously one can see that

$$c_1 = s_0(g(s_0)I_0'(s_0) - g'(s_0)I_0(s_0)). \tag{50}$$

Lemma 5.5. *There exists a constant A such that for any $s_0 > 0$ sufficiently large,*

$$|c_1|, |c_2| \leqslant A\frac{\mathrm{e}^{s_0}}{s_0^{3/2}}.$$

Proof. We only check the above bound for c_2, the other case being analogous. Along this proof we will denote by C any arbitrary constant. We note that, if ρ is sufficiently large,

$$|g(s)s^2|, \quad |\tilde{g}(s)s^2|, \quad |g'(s)s^3|, \quad |\tilde{g}'(s)s^3| \leqslant C, \qquad \text{for all } s \geqslant \rho. \tag{51}$$

As we proved in theorem 2.1 $|g(s)| < C/s^2$. This result is well known, see for instance [15]. Moreover, we also have that, as we pointed out in remark 2.15, $|g'(s)s^3| < C$. On the other hand, since $\tilde{g}$ is 1-Gevrey asymptotic to the formal expansion $\hat{g}$, it follows that, if $|s|$ is sufficiently large,

$$\left|\tilde{g}(s) - b_1\frac{1}{s^2}\right| \leqslant C\frac{1}{|s|^4},$$

and hence the inequality for $\tilde{g}$ does also hold. In the same way, $\tilde{g}'$ is 1-Gevrey asymptotic to $\hat{g}'$ and therefore $|\tilde{g}'(s)s^3| \leqslant C$.

Furthermore, as is well known, there exists a constant C such that $0 < I_0(s), I_0'(s) \leqslant C\mathrm{e}^s s^{-1/2}$ for all $s \geqslant \rho$, provided ρ is large enough. Therefore, upon using condition (49),

$$|c_2| \leqslant C s_0^{1/2}\mathrm{e}^{s_0}(s_0^{-2} + s_0^{-3}) \leqslant C\mathrm{e}^{s_0}s_0^{-3/2},$$

which proves the inequality. □

We define the linear operators

$$\mathcal{L}[h](s) = K_0(s)\int_{s_0}^{s} \xi I_0(\xi)h(\xi)\,\mathrm{d}\xi + I_0(s)\int_{s}^{+\infty} \xi K_0(\xi)h(\xi)\,\mathrm{d}\xi, \tag{52}$$

$$\mathcal{G}[h] = \mathcal{L}[\mathcal{D}\cdot h], \tag{53}$$

with $\mathcal{D}$ defined in (47). We observe that, by lemma 5.4, we can express the fixed point equation for the difference Δg as

$$(\mathrm{Id} - \mathcal{G})[\Delta g](s) = K_0(s)c_2 - \mathcal{L}[E](s). \tag{54}$$

As a consequence, in order to check item (ii) of theorem 5.2 it will be enough to study the linear operator $\mathrm{Id} - \mathcal{G}$ in some suitable Banach space. In addition, the fixed point equation for g can be written as

$$g = K_0(s)c_1 + \mathcal{L}[\mathcal{T}[g]]. \tag{55}$$

In order to prove item (i) of theorem 5.2 we will prove that the above fixed point has a unique solution which can be extended to an analytic solution in an appropriate sector. Thence, since our solution g satisfies the fixed point equation, we will get the result.

For each $\beta < \pi$, $s_0 > 0$ and $\gamma > 0$, we define the complex Banach spaces

$$\mathcal{X} = \{h : S(\beta, s_0) \to \mathbb{C};\, h \text{ real analytic},\ \|h\|_{\mathcal{X}} := \sup_{s\in S(\beta,s_0)} |h(s)\exp(\gamma|s - s_0|)| < +\infty\},$$

and

$$\mathcal{Y} = \{h : S(\beta, s_0) \to \mathbb{C};\, h \text{ real analytic},\ \|h\|_{\mathcal{Y}} := \sup_{s\in S(\beta,s_0)} |h(s)s^2| < +\infty\},$$

and define $B_{\mathcal{X}}(R) \subset \mathcal{X}$ like the open ball of radius R centred at the origin. Analogously we introduce $B_{\mathcal{Y}}(R) \subset \mathcal{Y}$. We also define the real Banach space

$$\mathcal{Y}_{\mathbb{R}} = \{h : [s_0, +\infty) \to \mathbb{R}; h \in \mathcal{Y}, h \in \mathcal{C}^0, \|h\|_{\mathcal{Y}_{\mathbb{R}}} = \sup_{s\in[s_0,+\infty)} |s^2 h(s)| < +\infty\}.$$

Lemma 5.6. *Let $\beta < \pi/2$ and let ρ_1 be such that $\tilde{g}$ and consequently E are defined on sector $S(\beta, \rho_1)$. If $s_0 > \rho_1$ is sufficiently large and $\gamma = \min\{(\cos\beta)/2, c\cos\beta\}$ with c, the constant defined in lemma 5.3, then the linear operator $\mathcal{L}$ defined in (52) satisfies that there exists a positive constant A independent of s_0 such that*

(i) $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ is well defined and

$$\|\mathcal{L}[h]\|_{\mathcal{X}} \leqslant A\|h\|_{\mathcal{X}}.$$

(ii) The operators $\mathcal{L} : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{L} : \mathcal{Y}_{\mathbb{R}} \to \mathcal{Y}_{\mathbb{R}}$ are well defined and moreover,

$$\|\mathcal{L}[h]\|_{\mathcal{Y}} \leqslant A\|h\|_{\mathcal{Y}}, \qquad \|\mathcal{L}[h]\|_{\mathcal{Y}_{\mathbb{R}}} \leqslant A\|h\|_{\mathcal{Y}_{\mathbb{R}}}.$$

Proof. In what follows we denote by C any arbitrary constant, so the value of C may change throughout the rest of the proof without explicit notice.

We recall that, s_0 being sufficiently large, for all $s \in S(s_0, \beta)$,

$$|K_0(s)| \leqslant C\frac{|\mathrm{e}^{-s}|}{|s|^{1/2}}, \qquad |I_0(s)| \leqslant C\frac{|\mathrm{e}^{s}|}{|s|^{1/2}}. \tag{56}$$

First we shall show item (i). Indeed, given $h \in \mathcal{X}$, using that $|h(r)| \leqslant \|h\|\mathrm{e}^{-\gamma|r-s_0|}$ and the inequalities provided in (56) one easily gets

$$\begin{aligned}|\mathrm{e}^{\gamma|s-s_0|}\mathcal{L}[h(s)]| &\leqslant C\|h\|\frac{\mathrm{e}^{-\mathrm{Re}(s)+\gamma|s-s_0|}}{|s|^{1/2}}\left|\int_{s_0}^{s}|\xi|^{1/2}\mathrm{e}^{\mathrm{Re}(\xi)-\gamma|\xi-s_0|}\,d\xi\right| \\ &\quad + C\|h\|\frac{\mathrm{e}^{\mathrm{Re}(s)+\gamma|s-s_0|}}{|s|^{1/2}}\left|\int_{s}^{+\infty}|\xi|^{1/2}\mathrm{e}^{-\mathrm{Re}(\xi)-\gamma|\xi-s_0|}\,\mathrm{d}\xi\right| \\ &:= (I_1 + I_2)\|h\|.\end{aligned} \tag{57}$$

We note that, given that $|s - s_0| \leqslant (\mathrm{Re}(s) - \mathrm{Re}(s_0))/\cos\beta$ and $\gamma \leqslant \cos\beta/2$,

$$\begin{aligned}I_1 &\leqslant C\mathrm{e}^{-\mathrm{Re}(s)+\gamma|s-s_0|}\mathrm{e}^{\mathrm{Re}(s_0)}|s - s_0|\int_0^1 \mathrm{e}^{\xi(\mathrm{Re}(s)-\mathrm{Re}(s_0))}\mathrm{e}^{-\xi\gamma|s-s_0|}\,\mathrm{d}\xi \\ &\leqslant C\mathrm{e}^{-\mathrm{Re}(s)+\gamma|s-s_0|}\mathrm{e}^{\mathrm{Re}(s_0)}|s - s_0|\frac{\mathrm{e}^{\mathrm{Re}(s)-\mathrm{Re}(s_0)-\gamma|s-s_0|}}{\mathrm{Re}(s) - \mathrm{Re}(s_0) + \gamma|s - s_0|} \leqslant C.\end{aligned}$$

We now search for a bound to I_2. By using the Cauchy theorem, we change the integration path in the integral involving the definition of I_2, and get

$$\begin{aligned} I_2 &\leqslant C\frac{e^{\mathrm{Re}(s)}}{|s|^{1/2}}\int_0^{+\infty}|s+\xi|^{1/2}e^{-\mathrm{Re}(s)-\xi}\,d\xi = C|s|^{-1/2}\int_0^{+\infty}|s+\xi|^{1/2}e^{-\xi}\,d\xi \\ &= C|s|^{-1/2}\left(|s|^{1/2}+\int_0^{+\infty}e^{-\xi}\frac{\mathrm{Re}(s)+\xi}{|s+\xi|^{3/2}}\,d\xi\right) \\ &\leqslant C + C|s|^{-1/2}\int_0^{+\infty}e^{-\xi}|s+\xi|^{-1/2}\,d\xi \leqslant C. \end{aligned}$$

Then, using the above obtained bounds for I_1, I_2 in (57), we find that $\mathcal{L}(h)\in\mathcal{X}$ and furthermore, $\|\mathcal{L}[h]\|\leqslant A\|h\|$ as we wished.

Now we prove (ii). Considering the inequalities (56), if $h\in\mathcal{Y}_{\mathbb{R}}$ and $s\geqslant s_0$,

$$|s^2\mathcal{L}[h](s)]\leqslant C\|h\|_{\mathcal{Y}_{\mathbb{R}}}\left(e^{-s}s^{3/2}\int_{s_0}^{s}\frac{e^{\xi}}{\xi^{3/2}}\,d\xi + e^{s}s^{3/2}\int_s^{+\infty}\frac{e^{-\xi}}{\xi^{3/2}}\,d\xi\right). \tag{58}$$

Upon integrating by parts it is readily found that

$$\begin{aligned} e^{-s}s^{3/2}\int_{s_0}^{s}\frac{e^{\xi}}{\xi^{3/2}}\,d\xi &\leqslant 1 + e^{-s}s^{3/2}\frac{3}{2s_0}\int_{s_0}^{s}\frac{e^{\xi}}{\xi^{3/2}}\,d\xi, \\ e^{s}s^{3/2}\int_s^{+\infty}\frac{e^{-\xi}}{\xi^{3/2}}\,d\xi &\leqslant 1, \end{aligned}$$

and hence, if $s_0>2$

$$|s^2\mathcal{L}[h](s)]\leqslant C\|h\|_{\mathcal{Y}_{\mathbb{R}}}((1-s_0^{-1})+1)\leqslant A\|h\|_{\mathcal{Y}_{\mathbb{R}}}.$$

Let $h\in\mathcal{Y}$. Again by inequalities (56), if $s\in S(\beta,s_0)$,

$$|s^2\mathcal{L}[h](s)|\leqslant C|s|^{3/2}e^{-\mathrm{Re}(s)}\|h\|_{\mathcal{Y}}\left|\int_{s_0}^{s}\frac{e^{\mathrm{Re}(\xi)}}{|\xi|^{3/2}}\,d\xi\right| + e^{\mathrm{Re}(s)}|s|^{3/2}\left|\int_s^{+\infty}\frac{e^{-\mathrm{Re}(\xi)}}{|\xi|^{3/2}}\,d\xi\right|.$$

Now we use that if $s\in S(\beta,s_0)$, then $|s-s_0|\leqslant(\mathrm{Re}(s)-\mathrm{Re}(s_0))/\cos\beta$ and $|s|\leqslant C\mathrm{Re}(s)$ with C a constant depending only on β. The previous bound along with this fact allows us to obtain

$$|s^2\mathcal{L}[h](s)|\leqslant C(\mathrm{Re}(s))^{3/2}e^{-\mathrm{Re}(s)}\|h\|_{\mathcal{Y}}\left(\int_{s_0}^{\mathrm{Re}(s)}\frac{e^{\xi}}{\xi^{3/2}}\,d\xi + e^{\mathrm{Re}(s)}(\mathrm{Re}(s))^{3/2}\int_{\mathrm{Re}s}^{+\infty}\frac{e^{-\xi}}{\xi^{3/2}}\,d\xi\right)$$

which only depends on $\mathrm{Re}(s)$ and can be bounded in the same way as bound (58). This ends the proof of this lemma. □

Finally we prove theorem 5.2.

Proof of theorem 5.2. Along this proof we will denote by C an arbitrary constant independent of s_0. First we recall that, by the results in [20], we already know that f is real analytic in $(s_0,+\infty)$. Moreover, we observe that, in the case that the interval $[0,1]$ is contained in the analyticity domain of the nonlinearity F, then again according to [20], f is real analytic in $(0,+\infty)$.

We will deal with $g(s)=1-f(s/\sqrt{b})$. We begin by proving (i). We recall that our solution g was a solution of the fixed point equation (55):

$$h=\mathcal{H}[h](s):=K_0(s)c_1+\mathcal{L}\left[\mathcal{T}[h]\right] \tag{59}$$

where $\mathcal{T}[h]$ was defined in lemma 5.4. The strategy to prove item (i) is the following: on the one hand we will prove that there is only one solution $g_{\mathcal{Y}}$ of the fixed point equation belonging to $\mathcal{Y}$, and on the other hand, we will check that there is a unique solution of the fixed point equation $g_{\mathcal{Y}_{\mathbb{R}}} \in \mathcal{Y}_{\mathbb{R}}$. Assuming these facts, since obviously $g_{\mathcal{Y}} \in \mathcal{Y}_{\mathbb{R}}$, we have that $g_{\mathcal{Y}}$ extends $g_{\mathcal{Y}_{\mathbb{R}}}$ to a complex sector. Finally, by theorem 2.1, $g \in \mathcal{Y}_{\mathbb{R}}$ and it is a solution of (59) and therefore $g = g_{\mathcal{Y}_{\mathbb{R}}}$.

Let $s_1 > 3$ be sufficiently large. We note that $K_0(s) \in \mathcal{Y}$ and moreover, if $s \in S(\beta, s_1)$,

$$|s^2 K_0(s) c_1| \leqslant C \frac{|s|^{3/2} \mathrm{e}^{s_0 - \mathrm{Re}(s)}}{s_0^{3/2}} \leqslant C \frac{(\mathrm{Re}(s))^{3/2} \mathrm{e}^{s_0 - \mathrm{Re}(s)}}{s_0^{3/2}} \leqslant B$$

for some adequate constant B. Moreover we also have that, for all $s \in \mathbb{C}$,

$$|s^2 \mathcal{T}[0](s)| = n^2.$$

Hence we have proved that $\mathcal{H}[0] \in \mathcal{Y}$. Let now $R = 8\|\mathcal{H}[0]\|_{\mathcal{Y}}$ (where here the definition domain is $S(\beta, s_1)$ independent of s_0). From now on we will take s_0 big enough such that $R|s_0|^{-2} < \delta$ in such a way that if $h \in B_{\mathcal{Y}}$ then $F(1 - h(s))$ is analytic in $S(\beta, s_0)$. We point out then that, if $h \in B_{\mathcal{Y}}(R)$, the function $\mathcal{H}[h]$ is real analytic and it is well defined on the sector $S(\beta, s_0)$.

We take $h_1, h_2 \in B_{\mathcal{Y}}(R)$. By the mean's value theorem and definition of $\mathcal{T}$ in lemma 5.4 it is easy to deduce that, if $s \in S(\beta, s_0)$,

$$\left| s^2 (\mathcal{T}[h_1](s) - \mathcal{T}[h_2](s)) \right| \leqslant C \frac{1 + R}{s_0^2} \|h_1 - h_2\|_{\mathcal{Y}},$$

C being a constant depending only on n, $\sup_{x \in B_\delta} |\partial^2 F(1 - x)|$ and b. Henceforth, by taking s_0 large enough,

$$\|\mathcal{H}[h_1] - \mathcal{H}[h_2]\|_{\mathcal{Y}} \leqslant s_0^{-1} \|h_1 - h_2\|_{\mathcal{Y}},$$

which implies that $\mathcal{H}$ is a Lipschitz operator. We emphasize that, if $h \in B_{\mathcal{Y}}(R)$,

$$\|\mathcal{H}[h]\|_{\mathcal{Y}} \leqslant \|\mathcal{H}[0]\|_{\mathcal{Y}} + \|\mathcal{H}[h] - \mathcal{H}[0]\|_{\mathcal{Y}} \leqslant \frac{R}{8} + \frac{R}{s_0} < R,$$

and consequently, applying the fixed point theorem, we conclude that there is only one solution of the fixed point equation (59) belonging to $B_{\mathcal{Y}}(R) \subset \mathcal{Y}$. The uniqueness of the solution in the full Banach space $\mathcal{Y}$ is straightforward from the fact that we can reduce the norm of a function belonging to $\mathcal{Y}$ by enlarging s_0.

Analogously (in fact, easily) one can prove that there is only one solution of the fixed point equation (59) belonging to $\mathcal{Y}_{\mathbb{R}}$. This concludes the proof of item (i).

Now we are going to prove item (ii). First we set $\varepsilon > 0$. By definition (47) we have that $\mathcal{D} : S(\beta, s_0) \to \mathbb{C}$ is real analytic and $\mathcal{D}(s) \to 0$ as $|s| \to +\infty$ provided $g, \tilde{g} \in \mathcal{Y}$ and $g(s), \tilde{g}(s) \to 0$ as $|s| \to +\infty$. Thus, there exists $s_0 \in \mathbb{R}$ large enough such that

$$\sup_{s \in S(\beta, s_0)} |\mathcal{D}(s)| \leqslant \varepsilon.$$

Finally, since $\mathcal{G}[h] = \mathcal{L}[\mathcal{D} \cdot h]$, by item (i) of lemma 5.6, we have that

$$\|\mathcal{G}[h]\| \leqslant A\|\mathcal{D} \cdot h\| \leqslant \sup_{s \in S(\beta, s_0)} |\mathcal{D}(s)| \|h\| \leqslant \varepsilon \|h\|,$$

if s_0 is large enough. This implies that the operator $\mathrm{Id} - \mathcal{G} : \mathcal{X} \to \mathcal{X}$ is invertible. Secondly, we point out that one can write the fixed point equation (54) as

$$\Delta g = (\mathrm{Id} - \mathcal{G})^{-1} [c_2 \cdot K_0 - \mathcal{L}[E]].$$

We already have that $K_0 \in \mathcal{X}$. Moreover, given that $|s - s_0| \leqslant (\mathrm{Re}(s) - s_0)/\cos\beta$ and $2\gamma \leqslant \cos\beta$, for all $s \in S(\beta, s_0)$,

$$|\mathrm{e}^{\gamma|s-s_0|} K_0(s) c_2| \leqslant C \frac{\mathrm{e}^{s_0}}{s_0^{3/2}} \frac{\mathrm{e}^{-\mathrm{Re}(s)}}{|s|^{1/2}} \mathrm{e}^{\gamma|s-s_0|} \leqslant C \frac{\mathrm{e}^{s_0}}{s_0^{3/2}} \frac{\mathrm{e}^{-\mathrm{Re}(s)}}{|s|^{1/2}} \mathrm{e}^{\gamma(\mathrm{Re}(s-s_0))/\cos\beta}$$
$$\leqslant C \mathrm{e}^{-(\mathrm{Re}(s-s_0))/2} |s_0|^{-3/2} |s|^{-1/2} \leqslant C |s_0|^{-2},$$

C being a suitable constant that is independent of s_0.

It is also clear that $E \in \mathcal{X}$ and, for any $s \in S(\beta, s_0)$,

$$|\mathrm{e}^{\gamma|s-s_0|} E(s)| = \mathrm{e}^{\gamma|s-s_0| - c|s|} |\mathrm{e}^{c|s|} E(s)| \leqslant \mathrm{e}^{\gamma(\mathrm{Re}(s-s_0))/\cos\beta - c\mathrm{Re}(s)} |\mathrm{e}^{c|s|} E(s)| \leqslant |\mathrm{e}^{c|s|} E(s)| \mathrm{e}^{-a s_0},$$

taking into account that $\gamma \leqslant c\cos\beta$ and $a = \min\{1/2, c\}$.

Therefore, taking $\varepsilon = 1/2$, by lemma 5.6, we conclude that $\Delta g \in \mathcal{X}$ and moreover

$$\|\Delta g\| \leqslant 2\big(\|c_2 \cdot K_0\| + \|\mathcal{L}[E]\|\big) \leqslant C(|s_0|^{-2} + C\mathrm{e}^{-a s_0}) < 1,$$

if s_0 is small enough. Here we have used that $\|(\mathrm{Id} - \mathcal{G})^{-1}\| \leqslant (1 - \|\mathcal{G}\|)^{-1} \leqslant 2$.

6. Numerical results

In this section we present some numerical computations based on the Ginzburg–Landau nonlinearity. In particular, we have computed the parameter α for degrees up to $n = 11$ and also the corresponding solutions $f(r)$. We then will deal only with the case $F(f) = f(1-f^2)$, but the method can be applied to other nonlinearities.

6.1. Numerical results for α_n

As we have already shown in the previous sections, all the solutions of

$$f''(r) + \frac{f'(r)}{r} - \frac{n^2 f(r)}{r^2} + f(r)(1 - f(r)^2) = 0, \tag{60}$$

where $f(0) = 0$, depart from $r = 0$ like $f(r) = \alpha r^n + \mathcal{O}(r^{n+2})$. However, there is a unique critical value of α_n, which depends on n, such that the corresponding solution tends to 1 as $r \to +\infty$. As is shown in [15], the idea is that one must catch the right value of α_n such that the cubic term f^3 balances with the linear one f. Otherwise, either f dominates, in which case it is easy to see that the solution behaves essentially like the Bessel function of the first kind, $J_n(r)$, and hence it oscillates and tends to zero at infinity, or f^3 dominates, in which case the solutions tend to infinity for some finite radius. This criticality, along with the fact that α_n becomes exponentially small in n, makes it difficult to compute α_n numerically already for moderate values of n.

In this section we present some numerical results for the shooting parameter α_n, when $n = 1, 2, \ldots, 11$ and describe the method that has been used to compute them, as well as the corresponding solution $f(r)$.

The values of α_n have been computed using two different approaches:

The first approach is the most standard. Let us denote by $\psi_n(\alpha_n, r)$ the solution of (60) such that it looks like $\psi_n(\alpha_n, r) = \alpha_n r^n (1 + \mathcal{O}(r^2))$ as $r \sim 0$. We take α_n as a result of numerically computing $\alpha(R)$ as the value that satisfies the equation

$$\psi_n(\alpha(R), R) = 1,$$

taking R sufficiently large. In particular, we have chosen $R = 150$ since previous numerical tests showed that, for all values of $n = 1, 2, \ldots, 11$, it satisfies that the difference

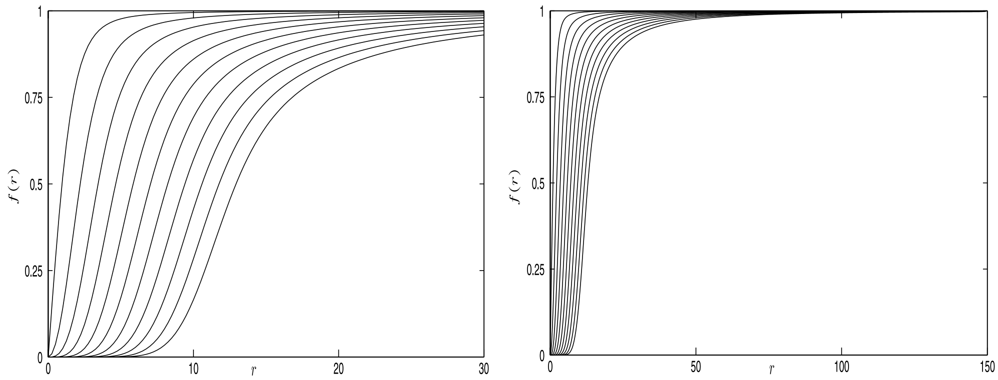

Figure 1. Solutions to (60) for increasing values of n up to $n = 11$.

between the corresponding values of α_n evaluated at radius slightly larger is small enough, $|\alpha(R) - \alpha(R + \Delta R)| \leqslant 10^{-14}$. To compute the value of $\alpha(150)$ we have used the method of multiple shooting in [11] due to the above-mentioned strong dependence of the solution with respect to the initial condition.

Along with the value of α_n, we have also obtained the solutions of (60) up to $R = 150$ for these same values of n. The results are presented in the plots in figure 1.

The second approach is based on the fact that one can obtain a formula for α_n in terms of the values of the whole orbit using the previously derived fixed point equation. Indeed, let us recall that $g(s) = 1 - f(s/\sqrt{2})$ satisfies the fixed point equation

$$g(s) = \mathcal{F}(g)(s) = K_n(s)\int_0^s \xi I_n(\xi)\left(\frac{n^2}{\xi^2} + \frac{3}{2}g^2(\xi) - \frac{1}{2}g^3(\xi)\right)\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\left(\frac{n^2}{\xi^2} + \frac{3}{2}g^2(\xi) - \frac{1}{2}g^3(\xi)\right)\mathrm{d}\xi,$$

then, using the equality

$$K_n(s)\int_0^s \xi I_n(\xi)\left(\frac{n^2}{\xi^2} + 1\right)\mathrm{d}\xi + I_n(s)\int_s^{+\infty} \xi K_n(\xi)\left(\frac{n^2}{\xi^2} + 1\right)\mathrm{d}\xi = 1$$

(which was used before), one obtains that the solution f equivalently satisfies the fixed point equation given by

$$f(r) = K_n(\sqrt{2}r)\int_0^r \xi I_n(\sqrt{2}\xi) f(\xi)(3 - f^2(\xi))\,\mathrm{d}\xi + I_n(\sqrt{2}r)\int_r^{+\infty} \xi K_n(\sqrt{2}\xi) f(\xi)(3 - f^2(\xi))\,\mathrm{d}\xi.$$

Finally, one only has to compute $\lim_{r\to 0} f(r)r^{-n}$ to obtain the following formula for α_n:

$$\alpha_n = \frac{1}{2^n n!}\int_0^{+\infty} \xi K_n(\sqrt{2}\xi) f(\xi)(3 - f^2(\xi))\,\mathrm{d}\xi. \tag{61}$$

We have thus used a trapezoidal rule followed by four steps of the method of extrapolation at the interval [0, 128]. Hence we have assumed that the remainder

$$\frac{1}{2^n n!}\int_{128}^{+\infty} s K_n(\sqrt{2}s) f(s)(3 - f^2(s))\,\mathrm{d}s \tag{62}$$

is small.

Table 1. Values of the initial parameter α_n computed with a multiple shooting approach and using (61).

n	α_{num}	α_{formula}	$\lvert\alpha_{\text{num}} - \alpha_{\text{formula}}\rvert/\lvert\alpha_{\text{num}}$
1	5.831 894 958 603e − 001	5.831 894 967 818e − 001	1.580 096 358 805e − 009
2	1.530 991 028 595e − 001	1.530 991 028 595e − 001	3.719 975 397 348e − 013
3	2.618 342 071 679e − 002	2.618 342 071 679e − 002	6.722 324 341 929e − 016
4	3.327 173 400 679e − 003	3.327 173 400 679e − 003	1.178 299 661 891e − 014
5	3.365 939 408 587e − 004	3.365 939 408 587e − 004	6.919 541 989 412e − 016
6	2.829 450 940 371e − 005	2.829 450 940 371e − 005	1.013 887 550 218e − 014
7	2.034 869 339 621e − 006	2.034 869 339 621e − 006	1.944 741 125 116e − 014
8	1.278 837 761 468e − 007	1.278 837 761 468e − 007	6.446 795 689 138e − 015
9	7.137 294 928 957e − 009	7.137 294 928 956e − 009	2.122 727 448 703e − 014
10	3.582 513 464 123e − 010	3.582 513 464 123e − 010	1.006 588 457 156e − 014
11	1.633 851 226 714e − 011	1.633 851 226 714e − 011	1.417 568 246 031e − 016

The results are shown in table 1, α_{num} being the value of α_n computed following the first alternative and α_{formula} the value using the closed formula (61).

We want to emphasize that the relative error between the two quantities is very small. Indeed, for some values of n we note that the comparison error is actually of the order of the double precision computation, so it might be even smaller. We also note that when obtaining this comparison error we have always neglected (62), and this quantity might be larger for smaller values of n, which would explain why one obtains less significant differences for $n = 1$.

To compute α_n for $n \geqslant 12$ we believe that multiprecision techniques are required, since, as was shown in the previous section, $\alpha_n \leqslant K2^{-n}n^{1/3}/n!$, becoming too small to be caught by just using methods implemented in double precision.

6.2. The vortex core

As we have already mentioned, in the case $F(x) = x(1-x^2)$, and when n is an integer number, $f(r)$ represents the modulus of single-vortex solutions of the two-dimensional complex-valued Ginzburg–Landau equation

$$u_t - \Delta u = u(1 - |u|^2), \tag{63}$$

which are stationary solutions of the form $f(r)\mathrm{e}^{\mathrm{i}n\phi}$, with $r = r(x)$ and $\phi = \phi(x)$ the polar coordinates of $x \in \mathbb{C}$. However, in general this equation is known to possess solutions with a non-zero degree, composed by a number of different separated vortices which locally have the form of a single-vortex solution. In that case, it is well known that the centres of these vortices, that is to say, the discrete points where the solution, u, vanishes, organize the dynamics of the whole solution. This way the dynamics of these patterns may be described in terms of a quite simple law of motion for the centres of the vortices (see [6, 24, 31, 32, 38] among others), which is only valid provided the vortices remain far enough. It is also known [40] that the most stable vortices are those with unitary degree. However, vortices with degrees greater than one do exist until they either split or annihilate, and the region of validity of these laws of motion strongly depends on the form of the solution locally close to each vortex (cf [16, 39]), which in turn, and as we have shown in this work, is determined by its degree n.

This law of motion for the centres of vortices was initially obtained using heuristic methods based on asymptotic analysis techniques (see [31]). These methods strongly rely on the fact that the modulus of vortex solutions are very close to 1 everywhere in space except in small areas around the vortex' centres where $|u|$ rapidly grows from 0 to 1, which were denoted as

Table 2. Core radius in terms of the degree n.

n	0.90	0.95	0.98
1	2.765 133e + 000	3.684 603e + 000	5.343 683e + 000
2	4.957 614e + 000	6.650 362e + 000	1.016 816e + 001
3	7.144 353e + 000	9.747 130e + 000	1.514 987e + 001
4	9.363 609e + 000	1.290 943e + 001	2.015 565e + 001
5	1.161 147e + 001	1.608 995e + 001	2.516 952e + 001
6	1.387 695e + 001	1.927 817e + 001	3.0187 24e + 001
7	1.615 225e + 001	2.247 141e + 001	3.520 708e + 001
8	1.843 306e + 001	2.566 754e + 001	4.022 824e + 001
9	2.071 714e + 001	2.886 485e + 001	4.525 028e + 001
10	2.300 340e + 001	3.206 297e + 001	5.027 325e + 001
11	2.529 215e + 001	3.526 170e + 001	5.529 661e + 001

the *core* of the vortex. The size of this core is very important in these approaches since it defines the range of validity of the law of motion, that is to say, the law of motion remains valid as long as the centres of the vortices do not reach any other vortex's core. Although the results in this heuristic approach were later on rigorously proved in [25, 26] for the case of finite domains, and [27, 38, 39] for the case of the entire plane where finer bounds for the velocities of the vortices were obtained, it is still of use in some physic's contexts such as, for instance, in the evolution of singularities in nematic liquid crystals (see [35]) or even in the derivation of a law of motion in a generalized version of (63) (see [2]).

We have thus computed the size of the vortex core for vortices of degrees from 1 to 11 by determining the radius at which the solution is 0.90, 0.95 and 0.98. The results are presented in table 2.

We note that the size of the core seems to grow linearly in n, at least for these values of the degree.

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Appendix A. Gevrey asymptotic

The aim of this section is to summarize basic definitions and results about Gevrey asymptotic and Gevrey regularity. Most of the features we state can be found in the literature (see for instance [4] and appendix A of [3]).

In this work we deal with Gevrey asymptotic at infinity (that is, for large values of z), but for technical reasons we need to relate the Gevrey asymptotic at infinity with the (classical) Gevrey asymptotic at the origin.

We define an open sector at the origin of radius ρ and opening $\gamma \in [0, \pi)$ as

$$S_0(\gamma, \rho) = \{z \in \mathbb{C} : 0 < |z| < \rho, |\arg(z)| < \gamma\},$$

and also we define a sector at infinity as

$$S_\infty(\gamma,\rho)=\{z\in\mathbb{C}:|z|>\rho,|\arg(z)|<\gamma\}.$$

We claim that if $z^{-1}\in S_\infty(\gamma,\rho^{-1})$, then $z\in S_0(\gamma,\rho)$. Analogously we define the closed sectors

$$\begin{aligned}\overline{S_0^1}(\gamma,\rho)&=\{z\in\mathbb{C}:0<|z|\leqslant\rho,|\arg(z)|\leqslant\gamma\},\\ \overline{S_\infty^1}(\gamma,\rho)&=\{z\in\mathbb{C}:|z|\geqslant\rho,|\arg(z)|\leqslant\gamma\}.\end{aligned}$$

Note that both S_0^1, S_∞^1 are non-compact sets.

In what follows $\alpha>0$. Let $\widehat{f_0}(z)=\sum_{n=0}^{+\infty}f_kz^k$ and $\widehat{f_\infty}(z)=\sum_{n=0}^{+\infty}f_kz^{-k}$ be two formal power series belonging to $\mathbb{C}[[z]]$ and $\mathbb{C}[[z^{-1}]]$, respectively. We will say that both are α-Gevrey if for any $k\in\mathbb{N}$,

$$|f_k|\leqslant BA^k(k!)^\alpha$$

for some constants $A,B>0$ independent of k.

An analytic function f_0 defined in a sector at the origin S_0 is α-Gevrey if and only if for every closed sector at the origin $\overline{S_0^1}\subset S_0^1$ there exist constants $A,B>0$ (depending only on $\overline{S_0^1}$) such that, for any $k\in\mathbb{N}$,

$$\sup_{z\in\overline{S_0^1}}|\partial_z^k f_0(z)|\leqslant BA^k(k!)^{1+\alpha}.$$

Analogously, we will say that an analytic function f_∞ in a sector at infinity S_∞ is α-Gevrey if and only if it satisfies that, $k\in\mathbb{N}$,

$$\sup_{z\in\overline{S_\infty^1}}|z^k\partial_z^k f_\infty(z)|\leqslant BA^k(k!)^{1+\alpha},$$

with $\overline{S_\infty^1}\subset S_\infty$ a closed sector at infinity and $A,B>0$ two positive constants independent of k. We will write $f_0\in G_\alpha(S_0)$ and $f_\infty\in G_\alpha(S_\infty)$, respectively. This definition tells us about Gevrey regularity.

Let f_0 be an analytic function in a sector S_0. We will say that f_0 is asymptotic α-Gevrey to a formal power series $\widehat{f_0}\in\mathbb{C}[[z]]$, or equivalently that $\widehat{f_0}$ is the α-Gevrey asymptotic expansion of f_0, if for any $\overline{S_0^1}\subset S_0$ closed sector at the origin and $k\in\mathbb{N}$

$$\sup_{z\in\overline{S_0^1}}\left|z^{-k}\left(f_0(z)-\sum_{l=0}^{k-1}f_lz^l\right)\right|\leqslant BA^k(k!)^\alpha,$$

for some positive constants A,B depending only on $\overline{S_0^1}$. We will write $f_0\simeq_\alpha\widehat{f_0}$.

Similarly, if f_∞ is an analytic function in a sector at infinity S_∞, f_∞ is asymptotic α-Gevrey to the formal series $\widehat{f_\infty}\in\mathbb{C}[[z^{-1}]]$ if and only if, for any $\overline{S_\infty^1}\subset S_\infty$ closed sector at infinity and $k\in\mathbb{N}$,

$$\sup_{z\in\overline{S_\infty^1}}\left|z^{k}\left(f_\infty(z)-\sum_{l=0}^{k-1}f_lz^{-l}\right)\right|\leqslant BA^k(k!)^\alpha,$$

for some constants $A,B>0$ depending only on $\overline{S_\infty^1}$. We will write $f_\infty\simeq_\alpha\widehat{f_\infty}$.

Proposition A.1. *Let $\gamma\in[0,\pi)$, $\rho>0$ and f_0 be an analytic function defined at $S_0(\gamma,\rho)$. We define $f_\infty(z)=f_0(z^{-1})$. We have that*

(i) f_0 is asymptotic α-Gevrey to the formal power series $\widehat{f_0}(z)=\sum_{k=0}^{+\infty}f_kz^k$ if and only if f_∞ is asymptotic α-Gevrey to $\widehat{f_\infty}(z)=\sum_{k=0}^{+\infty}f_kz^{-k}$.

(ii) The function f_0 belongs to $G_\alpha(S_0(\gamma,\rho))$ if and only if $f_\infty\in G_\alpha(S_\infty(\gamma,\rho^{-1}))$.

Proof. Item (i) is straightforward from definition. Moreover item (ii) is a consequence from the Faa di Bruno formula (37). Indeed, applying the Faa di Bruno formula to f_∞,

$$\partial_z^k f_\infty(z) = \frac{(-1)^k}{z^k}\sum_{l=1}^{k}\sum_{\substack{k_1+\cdots+k_l=k\\ 1\leqslant k_i}} \frac{\partial_z^l f_0(z^{-1})}{l!} = \frac{(-1)^k}{z^k}\sum_{l=1}^{k}\frac{\partial_z^l f_0(z^{-1})}{l!}\binom{k-1}{l-1},$$

for $z \in S_\infty(\gamma, \rho^{-1})$). We fix a closed sector $\overline{S_\infty^1} \subset S_\infty(\gamma, \rho^{-1})$. Since $f_0 \in G_\alpha(S_0(\gamma, \rho))$ we have that $|\partial_z^k f_0(z^{-1})| \leqslant BA^k(k!)^{1+\alpha}$ for any $z \in \overline{S_\infty^1}$. Then,

$$|\partial_z^k f_\infty(z)| \leqslant B\frac{1}{z^k}\sum_{l=1}^{k} A^l(l!)^\alpha \binom{k-1}{l-1} \leqslant B\frac{1}{z^k}(k!)^\alpha(1+A)^{k-1},$$

which implies that $f_\infty \in G_\alpha(S_\infty(\gamma, \rho^-))$. Finally, since $f_0(z) = f_\infty(z^{-1})$ we can proceed analogously to check the reciprocal. □

Now we are going to state (without proof) some useful properties about Gevrey asymptotic at the origin (see [3, 4]). Using proposition A.1 one can use such features also for Gevrey asymptotic at infinity.

Proposition A.2. *Let $\gamma \in [0, \pi)$, $\rho > 0$ and f_0 be an analytic function in $S_0(\gamma, \rho)$ and $\widehat{f_0}(z) = \sum_{k=0}^{+\infty} f_k z^k$ be a formal power series.*

(i) If f_0 is asymptotic α-Gevrey to the formal series $\widehat{f_0}$, then $\widehat{f_0}$ is α-Gevrey.
(ii) f_0 is asymptotic α-Gevrey to the formal series $\widehat{f_0}$ if and only if $f_0 \in G_\alpha(S_0(\gamma, \rho))$ and for all $k \in \mathbb{N}$,

$$\lim_{\substack{z \to 0\\ z \in S_0(\gamma, \rho)}} \partial_z^k f_0(z) = k! f_k.$$

If $f_0 \simeq_\alpha \widehat{f_0}$, then for any $k \in \mathbb{N}$, $\partial_z^k f_0 \simeq_\alpha \partial_z^k \widehat{f_0}$, where here $\partial_z^k \widehat{f_0}$ is the termwise k-derivative formal of $\widehat{f_0}$.

We remark that, as a consequence of item (ii) from proposition A.2, the asymptotic expansion of a α-Gevrey function is unique. Conversely, if we restrict the angle γ, we have the following result which is a generalization of the well-known Borel–Ritt theorem (see [4]).

Theorem A.3. *Let $\widehat{f_0} \in \mathbb{C}[[z]]$ be a α-Gevrey formal power series. We take $\gamma < \alpha\pi/2$ and $\rho > 0$. Then, there exists an analytic function in $S_0(\gamma, \rho)$, f_0, such that $f_0 \simeq_\alpha \widehat{f_0}$.*

Moreover, even when this analytic function f_0 is not unique, the difference between two functions asymptotic α-Gevrey to the same formal series is exponentially small.

Proposition A.4. *Let $\gamma < \alpha\pi/2$ and $\rho > 0$. Assume that f_0 is an analytic function in $S_0(\gamma, \rho)$ satisfying that $f_0 \simeq_\alpha \widehat{0}$, $\widehat{0}$ being the zero power series. Then, for every closed sector $\overline{S_0^1} \subset S_0(\gamma, \rho)$, there exist $c, \kappa > 0$ such that*

$$\sup_{z \in \overline{S_0^1}} |f_0(z)\exp(c|z|^{-1/\alpha})| \leqslant \kappa.$$

Theorem A.3 and proposition A.4 can be adapted to the case of Gevrey asymptotic at infinity simply using proposition A.1. Particularly we have:

Proposition A.5. *For any $\gamma < \alpha\pi/2$ and $\rho > 0$,*

(i) If $\widehat{f_\infty} \in \mathbb{C}[[z^{-1}]]$ is a α-Gevrey formal power series, there exists f_∞ an analytic function in $S_\infty(\gamma, \rho)$, such that $f_\infty \cong_\alpha \widehat{f_\infty}$.

(ii) If f_∞ is an analytic function in $S_\infty(\gamma, \rho)$ such that $f_\infty \cong_\alpha \widehat{0}$, then

$$\sup_{z\in \overline{S^1_\infty}} |f_\infty(z) \exp(c|z|^{1/\alpha})| \leqslant \kappa,$$

where $\overline{S^1_\infty}$ is any closed sector contained in $S_\infty(\gamma, \rho)$ and c, κ are two positive constants depending only on $\overline{S^1_\infty}$.

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