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Invariant manifolds of parabolic fixed points (II). Approximations by sums of homogeneous functions

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Abstract

We study the computation of local approximations of invariant manifolds of parabolic fixed points and parabolic periodic orbits of periodic vector fields. If the dimension of these manifolds is two or greater, in general, it is not possible to obtain polynomial approximations. Here we develop an algorithm to obtain them as sums of homogeneous functions by solving suitable cohomological equations. We deal with both the differentiable and analytic cases. We also study the dependence on parameters. In the companion paper [4] these approximations are used to obtain the existence of true invariant manifolds close by. Examples are provided.

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1. Introduction

This paper is the second part of our study on the invariant manifolds of parabolic points for $\mathcal{C}^r$ and analytic maps started in [4]. We refer to that paper for the motivation and references concerning such setting.

In this set of two papers we provide conditions that guarantee the existence of stable invariant manifolds associated of such points. We use the parametrization method [5–7,9–11]. The operators involved in this method are more regular than the graph transform, which is an advantage in the present situation, where only finite differentiability is assumed. Also, it often provides efficient algorithms to compute explicitly approximations of the invariant manifolds. In fact, this is the main purpose of the present paper. To apply this method we need a minimum regularity to be able to have a polynomial approximation of the map.

We consider maps $F : U \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, with $(0, 0) \in U$ such that $F(0, 0) = (0, 0)$, $DF(0, 0) = \mathrm{Id}$. We assume some hypotheses, to be specified later, on the first non-vanishing nonlinear terms which imply the existence of some "weak contraction" in the $(x, 0)$-directions, as well as some hypotheses concerning the $(0, y)$-directions that may imply "weak expansion" in these directions (but not always). The parametrization method consists of looking for the invariant stable manifold W^s of the origin as an immersion $K : V \subset \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, with $K(0) = (0, 0)$, $DK(0) = (\mathrm{Id}, 0)^\top$, and satisfying the *invariance equation*

$$F \circ K = K \circ R, \tag{1.1}$$

where $R : V \to V$ is a reparametrization of the dynamics of F on W^s.

The procedure to find such K and R has two steps. First, to find functions $K^{\leq}$ and R solving approximately the invariance equation, that is, satisfying

$$F \circ K^{\leq}(x) - K^{\leq} \circ R(x) = o(\|x\|^{\ell}), \tag{1.2}$$

to a high enough order which depends on the first non-vanishing nonlinear terms of F.

Second, with the reparametrization R obtained so far to look for K as a perturbation of $K^{\leq}$. This second step is carried out in [4] where, assuming that R and a sufficiently good approximation $K^{\leq}$ are known, an "a posteriori" type result is obtained.

In this paper we obtain approximate solutions of (1.1). This is accomplished by solving a set of cohomological equations. In the case that the fixed point is hyperbolic instead of parabolic, it is possible to find solutions of the cohomological equations in the ring of polynomials, both for K and R (see [5–7]). The same happens when one looks for one dimensional invariant manifolds associated to parabolic fixed points [3].

However, when the parabolic invariant manifolds have dimension two or more, a simple computation shows that generically there are no polynomial approximate solutions of the invariance equation. The reason is simple: when looking for polynomial solutions, since the terms of order k are determined in order to kill the terms of order $k + j$ of some error expression, where $j \geq 1$ is related to the degree of the first non-vanishing monomials in the expansion of F around the origin, the number of conditions on the coefficients corresponding to monomials of degree k is larger than the number of coefficients if the dimension of the manifold is at least 2. In fact, the number of obstructions increases with the order k. Of course, it may happen that these obstructions vanish in some particular examples (like several instances of the three body problem, see [4]), but generically they are unavoidable.

The cohomological equations for the terms of the approximate solutions of (1.1) can be written as a linear PDE of the form

$$Dh(x)p(x) - Q(x)h(x) = w(x), \qquad x \in V \subset \mathbb{R}^n,$$

where p, Q are fixed homogeneous functions that depend on the first non-vanishing nonlinear terms of the Taylor expansion of F and w is an arbitrary homogeneous function. Of course, the problem lies in finding global solutions of this PDE. In this work we prove that, under suitable hypotheses (see H1, H2, H3 and (2.3) in Section 2.1), the cohomological equations have homogeneous solutions defined in the whole domain under consideration. Their order is related to the order of w. This result allows us to find the approximate solutions of (1.1) as a sum of homogenous functions of increasing order. In general, these functions are not polynomials, not even rational functions. We deal with both the differentiable and analytic cases. In the differentiable case there may be a loss of regularity. It is also worth mentioning that the regularity assumption needed for obtaining R and the approximation are sufficient to deal with the second stage of the procedure. We remark that our conditions allow several characteristic directions in the domain under consideration (see [1,8]).

The structure of the paper is as follows. In Section 2 we present the hypotheses and main results of the paper. In Section 6 we show that our hypotheses are indeed necessary, that the loss of differentiability can take place and remark the differences between the case of one-dimensional and multidimensional parabolic manifolds. In sections 3 and 4 we prove the main theorems.

Section 3 contains the study of the cohomological equations used in the actual proof of the main theorems in Section 4. Section 5 is devoted to the dependence with respect to parameters.

2. Main result

The main result of this work deals with the computation of approximations of stable manifolds of parabolic points, expressed as the range of a function K, in such a way that the invariance condition (1.1), $F \circ K - K \circ R = 0$, is satisfied up to a prefixed order (see equation (1.2)). We will look for K and R as a finite sum of homogeneous functions not necessarily polynomials. Each term of these sums is a homogeneous solution of a so called cohomological equation. We are forced to look for homogeneous solutions of the cohomological equations because, in this multidimensional case $x \in \mathbb{R}^n$ with $n > 1$, as we will see in Section 4, in general these equations do not admit polynomial solutions. We also refer to the reader to Section 6 where several examples are studied.

In addition, we also study the dependence on parameters of the solutions of the cohomological equations (see Section 2.3).

At the end of this section, we present the result about approximate solutions of the invariance equation in the vector field case.

2.1. Set up and general hypotheses

The context we present here is the same as the one in [4], which we reproduce for the convenience of the reader.

Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set such that $(0, 0) \in U$ and let $F : U \to \mathbb{R}^{n+m}$ be a map of the form

$$F(x, y) = \begin{pmatrix} x + p(x, y) + f(x, y) \\ y + q(x, y) + g(x, y) \end{pmatrix}, \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m, \tag{2.1}$$

where p and q are homogeneous polynomials of degrees $N \geq 2$ and $M \geq 2$ respectively, $D^l f(x, y) = \mathcal{O}(\|(x, y)\|^{N+1-l})$ and $D^l g(x, y) = \mathcal{O}(\|(x, y)\|^{M+1-l})$ for $l = 0, 1$. Clearly $(0, 0)$ is a fixed point of F and $DF(0, 0) = \mathrm{Id}$.

Since the degrees of p and q, N and M, respectively, need not to be the same, we introduce

$$L = \min\{M, N\}.$$

We denote by $\pi_x(x, y) = x$ and $\pi_y(x, y) = y$ the natural projections and by B_ϱ the open ball centered at the origin of radius $\varrho > 0$. However, to simplify notation, we will often denote the projection onto a variable as a subscript, i.e., $X_x := \pi_x X$.

Now we state the minimum hypotheses to guarantee that the cohomological equations we encounter can be solved and consequently, we are able to find approximate solutions up to the required order.

Given $V \subset \mathbb{R}^n$ such that $0 \in \partial V$ and $\varrho > 0$, we introduce

$$V_\varrho = V \cap B_\varrho. \tag{2.2}$$

In this paper we will say that $V \subset \mathbb{R}^n$ is star-shaped with respect to 0 if $0 \in \partial V$ and for all $x \in v$ and $\lambda \in (0, 1]$, $\lambda x \in V$.

Take $\varrho > 0$, norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and consider the following constants:

$$\begin{aligned}
a_p &= -\sup_{x\in V_\varrho} \frac{\|x+p(x,0)\|-\|x\|}{\|x\|^N}, & b_p &= \sup_{x\in V_\varrho} \frac{\|p(x,0)\|}{\|x\|^N},\\
A_p &= -\sup_{x\in V_\varrho} \frac{\|\mathrm{Id}+D_x p(x,0)\|-1}{\|x\|^{N-1}}, & B_p &= \sup_{x\in V_\varrho} \frac{\|\mathrm{Id}-D_x p(x,0)\|-1}{\|x\|^{N-1}},\\
B_q &= -\sup_{x\in V_\varrho} \frac{\|\mathrm{Id}-D_y q(x,0)\|-1}{\|x\|^{M-1}}, & &\\
c_p &= \begin{cases} a_p, & \text{if } B_q \le 0,\\ b_p, & \text{otherwise}\end{cases}, & d_p &= \begin{cases} a_p, & \text{if } A_p \le 0,\\ b_p, & \text{otherwise,}\end{cases}
\end{aligned} \tag{2.3}$$

where the norms of linear maps are the corresponding operator norms. We emphasize that all these constants depend on ϱ

We assume that there exist an open set $V \subset \mathbb{R}^n$, V star-shaped with respect to 0, and appropriate norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ satisfying, taking ϱ small enough,

H1 The homogenous polynomial p satisfies that

$$a_p > 0.$$

If $M > N$, we further ask $A_p/d_p > -1$.

H2 The homogenous polynomial q satisfies $q(x,0) = 0$ for $x \in V_\varrho$, and

$$\begin{aligned}
&D_y q(x,0) \text{ is invertible } \forall x \in \overline{V_\varrho}\setminus\{0\}, && \text{if } M < N,\\
&2 + \frac{B_q}{c_p} > \max\left\{1 - \frac{A_p}{d_p}, 0\right\}, && \text{if } M = N.
\end{aligned}$$

H3 There exists a constant $a_V > 0$ such that, for all $x \in V_\varrho$,

$$\operatorname{dist}(x + p(x,0), (V_\varrho)^c) \ge a_V \|x\|^N.$$

We emphasize that H1–H3 are asked to be satisfied not in a neighborhood of the origin but in V_ϱ. As usual in the parabolic case, a stable invariant manifold is defined over a subset V such that $0 \in \partial V$. It may happen that the manifold is not defined in a neighborhood of the origin. However, some regularity at the origin may be retained. For this reason we introduce the following definition.

Definition 2.1. Let $V \subset \mathbb{R}^l$ be an open set, $x_0 \in \overline{V}$ and $f : V \cup \{x_0\} \subset \mathbb{R}^l \to \mathbb{R}^k$. We say that f is C^1 at x_0 if f is C^1 in $V \cap (B_\epsilon(x_0) \setminus \{x_0\})$, for some $\varepsilon > 0$ and $\lim_{x\to x_0,\, x\in V} Df(x)$ exists.

Finally we introduce some notation. Given $l, k, \ell \in \mathbb{N}$ and an open set $\mathcal{U} \subset \mathbb{R}^l$ such that $0 \in \partial\mathcal{U} \cup \mathcal{U}$, we define

$$\mathcal{H}^{\geq \ell} = \{h \in \mathcal{C}^0(\mathcal{U}, \mathbb{R}^k) \ : \ \text{for } u \in \mathcal{U}, \ \ \|h(u)\| = \mathcal{O}\big(\|u\|^\ell\big)\},$$
$$\mathcal{H}^{> \ell} = \{h \in \mathcal{C}^0(\mathcal{U}, \mathbb{R}^k) \ : \ \text{for } u \in \mathcal{U}, \ \ \|h(u)\| = o\big(\|u\|^\ell\big)\},$$
$$\mathcal{H}^{\ell} = \{h \in \mathcal{C}^0(\mathcal{U}, \mathbb{R}^k) \ : \ \forall \lambda \in \mathbb{R}, \ \forall u \in \mathcal{U}, \ \text{s.t. } \lambda u \in \mathcal{U}, \ \ h(\lambda u) = \lambda^\ell h(u)\}.$$

To simplify notation, we skip the reference to l, k and $\mathcal{U}$, which will be fixed and clearly understood from the context.

2.2. Approximate solutions of the invariance equation for maps

In this section we present two results. The first one is about the existence of approximate solutions having the “simplest” form. The other one (which can be useful in some applications) is about the freedom we have for solving the cohomological equations.

As we will prove in an algorithmic way, even when F is an analytic function, we cannot, in general, obtain $\mathcal{C}^\infty$ approximations of the stable manifold, unlike the hyperbolic case. For instance, if $A_p < d_p$ and $M \geq N$, we obtain $\mathcal{C}^{r_*}$-regularity of these approximations, where r_* is given by

$$r_* = \begin{cases} \max\left\{k \in \mathbb{N} : \left(1 - \dfrac{A_p}{d_p}\right)k < 2 + \dfrac{B_q}{c_p}\right\}, & \text{if } M = N, \\ \max\left\{k \in \mathbb{N} : \left(1 - \dfrac{A_p}{d_p}\right)k < 2\right\}, & \text{if } M > N. \end{cases} \tag{2.4}$$

Theorem 2.2. *Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ be defined in a neighborhood of the origin and having the form (2.1). Assume that $F \in \mathcal{C}^r$, with $r \geq N$, and satisfies hypotheses H1, H2 and H3 for some $\varrho_0 > 0$. Then, for any $N \leq \ell \leq r$ there exist $0 < \varrho \leq \varrho_0$ and $K : V_\varrho \to U$ and $R : V_\varrho \to V_\varrho$ such that*

$$F \circ K - K \circ R \in \mathcal{H}^{>\ell}. \tag{2.5}$$

In addition, we can choose K and R as a finite sum of homogeneous functions $K^j \in \mathcal{H}^j$ and $R^j \in \mathcal{H}^j$ (not necessarily polynomials), of the form

$$K_x(x) = x + \sum_{l=2}^{\ell-N+1} K_x^l(x), \qquad K_y(x) = \sum_{l=2}^{\ell-L+1} K_y^l(x),$$
$$R(x) = x + \sum_{l=N}^{\min\{\ell,\ell_*\}} R^l(x) \tag{2.6}$$

with $R^N(x) = p(x,0)$, $L = \min\{N, M\}$ and ℓ_* defined by

$$\ell_* = \begin{cases} N - 1 + \left[\frac{B_p}{a_p} + r_*\left(1 - \frac{A_p}{d_p}\right)\right], & \text{if } A_p < b_p \text{ and } M \geq N, \\ N - 1 + \left[\frac{B_p}{a_p}\right], & \text{if } A_p \geq b_p \text{ and } M \geq N, \\ \ell, & M < N. \end{cases} \tag{2.7}$$

Moreover, K and R extend to V by homogeneity of their terms. The functions $K_x^l(x)$, with $l = 2, \cdots, \ell_* - N + 1$, can be chosen arbitrarily, in particular, equal to 0.

Concerning the regularity of the approximation of the parametrization we have that K and R are $\mathcal{C}^1$ at the origin in the sense of Definition 2.1. Finally,

(1) *if either $A_p > d_p$ or $M < N$, K, R are analytic in a complex neighborhood of V,*
(2) *if $A_p = d_p$, K, R are $\mathcal{C}^\infty$ functions on V,*
(3) *if $A_p < d_p$ and $M \ge N$, K, R are $\mathcal{C}^{r_*}$ functions on V where r_* is defined in* (2.4).

Remark 2.3. We will see in Lemma 3.6 that $B_p/a_p \ge N$. Indeed, $B_{D\mathbf{p}}$ in that lemma corresponds to $-B_p$ in (2.3).

Remark 2.4. In [4] it is proven that under the hypotheses H1, H2 and H3, there exists an exact solution $\tilde{K}, R$ of the invariance equation (1.1). In addition, if $K^{\le}$ is the function provided by Theorem 2.2 for some ℓ big enough, then K has the form $K^{\le} + K^{>}$ with $K^{>} \in \mathcal{H}^{>\ell-N+1}$. Even more, assuming that $A_p, B_q > 0$ and the hypotheses of the theorem, the stable set is a manifold which is the graph of a differentiable function φ which can be approximated by $\pi_y K \circ (\pi_x K)^{-1}$ in the sense that $\varphi - \pi_y K \circ (\pi_x K)^{-1} \in \mathcal{H}^{>\ell-L+1}$.

Remark 2.5. As we will see in the proof of Theorem 2.2 in Section 4.4, we can choose different strategies in order to get R as a sum of homogeneous functions of degree less than ℓ_*. However, not for all strategies the obtained regularity will be optimal.

Remark 2.6. The results stated in Theorem 2.2 hold also true if, instead of assuming that F is a $\mathcal{C}^r$ function in an open neighborhood of the origin, we assume that F can be written as a sum of homogeneous functions which are $\mathcal{C}^r$ in V, that is F has the form:

$$
\begin{aligned}
F_x(x,y) &= x + p(x,y) + F_x^{N+1}(x,y) + \cdots + F_x^r(x,y) + F_x^{>r}(x,y),\\
F_y(x,y) &= y + q(x,y) + F_y^{M+1}(x,y) + \cdots + F_y^r(x,y) + F_y^{>r}(x,y),
\end{aligned}
$$

where all the functions are $\mathcal{C}^r$ in V, $p \in \mathcal{H}^N$, $q \in \mathcal{H}^M$, $F_x^j, F_y^j \in \mathcal{H}^j$ and $F_x^{>r}, F_y^{>r} \in \mathcal{H}^{>r}$.

An alternative point of view is the following result:

Theorem 2.7. *Assume the same hypotheses of Theorem 2.2. Let $N \le \ell \le r$ and $K_x^l \in \mathcal{H}^l$ for $l = 2, \cdots, \ell - N + 1$. Then for any function $K_x : V \to \mathbb{R}^n$ such that*

$$
K_x(x) - x - \sum_{l=2}^{\ell-N+1} K_x^l(x) \in \mathcal{H}^{>\ell-N+1}
$$

satisfying the regularity statements for K of Theorem 2.2, there exist $0 \le \varrho \le \varrho_0$ and $R : V_\varrho \to V_\varrho$ and $K_y : V_\varrho \to \mathbb{R}^m$ of the form

$$
R(x) = x + p(x,0) + \sum_{l=N+1}^{\ell} R^l(x), \qquad K_y(x) = \sum_{l=2}^{\ell-L+1} K_y^l(x)
$$

with $R^l \in \mathcal{H}^l$, $K_y^l \in \mathcal{H}^l$, such that $F \circ K - K \circ R \in \mathcal{H}^{>\ell}$ with $K = (K_x, K_y)$. Moreover the regularity statements are the same as the ones in Theorem 2.2 and K and R can be extended to V.

2.3. Dependence on parameters

Let $\Lambda \subset \mathbb{R}^{n'}$ be an open set of parameters, $U \subset \mathbb{R}^{n+m}$ be an open set and V as in Section 2.1. Assume that $F : U \times \Lambda \to \mathbb{R}^{n+m}$ are maps having the form (2.1) for any $\lambda \in \Lambda$, i.e.:

$$F(x, y, \lambda) = \begin{pmatrix} x + p(x, y, \lambda) + f(x, y, \lambda) \\ y + q(x, y, \lambda) + g(x, y, \lambda) \end{pmatrix}. \tag{2.8}$$

For any fixed $\lambda \in \Lambda$ the constants in (2.3) are well defined and depend on λ. We denote this dependence with a superindex. As we did in [4], we redefine the constants A_p, a_p, etc. by taking the supremum over $V_\varrho \times \Lambda$ instead of V_ϱ. For instance,

$$A_p = \inf_{\lambda \in \Lambda} A_p^\lambda = - \sup_{(x,\lambda) \in V_\varrho \times \Lambda} \frac{\|\mathrm{Id} + D_x p(x, 0, \lambda)\| - 1}{\|x\|^{N-1}}.$$

We note that, assuming H1, H2 and H3 for any $\lambda \in \Lambda$ we already have the existence of approximate solutions K_λ. To obtain uniform bounds, and therefore continuity and differentiability, with respect to $\lambda \in \Lambda$ we need to assume

Hλ Hypotheses H1, H2 and H3 hold true uniformly with respect to λ, namely, all the conditions involving the constants $a_p, b_p, A_p, B_p, d_p, c_p, B_q, a_V$ hold true with the new definition of these constants.

From now on we will abuse notation and we will write that a function h depending on a parameter μ, belongs to $\mathcal{H}^{\geq \ell}$ if $h(z, \mu) = \mathcal{O}(\|z\|^\ell)$ uniformly in μ. Analogously if $h \in \mathcal{H}^{>\ell}$. Moreover, $h \in \mathcal{H}^\ell$ will mean that h is homogeneous of degree ℓ for any fixed μ.

The differentiability class we work with was introduced in [6] and is also used in [4]. For any $s, r \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, we define the set

$$\Sigma_{s,r} = \left\{(i, j) \in (\mathbb{Z}^+)^2 : i + j \leq r + s,\ i \leq s\right\}$$

and for $\mathcal{U} \subset \mathbb{R}^l \times \mathbb{R}^{n'}$, the function space

$$\begin{aligned} \mathcal{C}^{\Sigma_{s,r}} = \big\{ f : \mathcal{U} \to \mathbb{R}^k\ : \forall (i, j) \in \Sigma_{s,r}, \\ D_\mu^i D_z^j f \text{ exists, is continuous and bounded} \big\}. \end{aligned} \tag{2.9}$$

Theorem 2.8. *Let $F \in \mathcal{C}^{\Sigma_{s,r}}$ be of the form (2.8) with $r \geq N$ satisfying Hλ for $\varrho_0 > 0$. Let $\ell \in \mathbb{N}$ be $\ell \leq r$ as in Theorem 2.2.*

Then the functions $K : V \times \Lambda \to \mathbb{R}^{n+m}$ and $R : V \times \Lambda \to \mathbb{R}^n$ given by Theorem 2.2 satisfy:

(1) *If either $A_p > d_p$ or $M < N$, K, R are $\mathcal{C}^s$ with respect to $\lambda \in \Lambda$ and real analytic with respect to $x \in V$. In addition, if F depends analytically on $\lambda \in \Lambda$, the functions K, R are real analytic in $V \times \Lambda$.*

(2) *If* $A_p = d_p$ *then* $K, R \in \mathcal{C}^{\Sigma_{s,\infty}}$ *in* $V \times \Lambda$.
(3) *If* $A_p < d_p$ *and* $M \geq N$, *then* $K, R \in \mathcal{C}^{\Sigma_{s_*, r_* - s_*}}$ *in* $V \times \Lambda$ *where* r_* *is defined in* (2.4) *and* $s_* = \min\{s, r_*\}$.

If $K_x : V \times \Lambda \to \mathbb{R}^n$ *is of the form given in Theorem 2.7 and satisfies the above regularity statements, then the functions* $R : V \times \Lambda \to \mathbb{R}^n$ *and* $K_y : V \times \Lambda \to \mathbb{R}^m$ *provided by Theorem 2.7 satisfy the same statements.*

2.4. Approximate solutions of the invariance equation for flows

We deduce the analogous results to Theorems 2.2 and 2.8 in the case of time periodic flows. It is worth to mention that we could deduce some results for flows from the previous ones using the Poincaré map. Nevertheless we prefer to give explicit results because, as we will see in Section 4, we can construct the approximate solutions without computing neither the Poincaré map nor the flow, which turns out to be very useful in applications.

In the case of flows, to shorten the exposition, we deal with the parametric case, being the free parameter case a straightforward consequence.

Let $U \subset \mathbb{R}^{n+m}$ be a neighborhood of the origin, $\Lambda \subset \mathbb{R}^{n'}$ and $X : U \times \mathbb{R} \times \mathbb{R}^{n'} \to \mathbb{R}^{n+m}$ a T-periodic vector field

$$\dot{z} = X(z, t, \lambda), \qquad X(z, t + T, \lambda) = X(z, t, \lambda) \tag{2.10}$$

such that

$$X(z, t, \lambda) = X(x, y, t, \lambda) = \begin{pmatrix} p(x, y, \lambda) + f(x, y, t, \lambda) \\ q(x, y, \lambda) + g(x, y, t, \lambda) \end{pmatrix}, \tag{2.11}$$

where p and q are homogeneous polynomials of degrees $N \geq 2$ and $M \geq 2$ respectively with respect to (x, y), and $f(x, y, t, \lambda) = \mathcal{O}(\|(x, y)\|^{N+1})$ and $g(x, y, t, \lambda) = \mathcal{O}(\|(x, y)\|^{M+1})$ uniformly in $(t, \lambda) \in \mathbb{R} \times \Lambda$.

If we want to deal with the invariant manifolds of parabolic periodic orbits, we translate the orbit to the origin and we get a vector field of the form (2.11).

From now on, in the case of flows, the spaces $\mathcal{H}^{>\ell}, \mathcal{H}^{\geq\ell}, \mathcal{H}^{\ell}$ will be the analogous to the ones in Section 2.1, respectively Section 2.3, with a T-periodic dependence on t and with uniform bounds with respect to $\lambda \in \Lambda$.

Let $\varphi(s; t_0, x, y, \lambda)$ be the flow of (2.10). The condition that the range of a function K, depending on (x, t, λ), is invariant by the flow of the vector field (2.11), analogous to (1.1) for maps, is

$$\varphi(s; t, K(x, t, \lambda), \lambda) = K(\psi(s; t, x, \lambda), s, \lambda), \tag{2.12}$$

for some function ψ. In the above equation the unknowns are K and ψ. However, if $\psi(s; t, x, \lambda)$ is the flow associated to some vector field $Y(x, t, \lambda)$, the invariance equation (2.12) is equivalent to its infinitesimal version

$$X(K(x, t, \lambda), t, \lambda) = D_x K(x, t, \lambda) Y(x, t, \lambda) + \partial_t K(x, t, \lambda), \tag{2.13}$$

where D_x denotes the derivative with respect to x.

Next theorem asserts that equation (2.13) can be solved up to a certain order using functions belonging to $\mathcal{C}^{\Sigma_{s',r'}}$ for some s' and r'. For technical reasons we will consider separately the differentiability with respect to (x, y) and (t, λ). That is, in the definition (2.9) of $\mathcal{C}^{\Sigma_{s,r}}$ we take $z = (x, y)$ and $\mu = (t, \lambda)$.

Theorem 2.9. *Let $X : U \times \mathbb{R} \times \Lambda \to \mathbb{R}^{n+m}$ be a vector field of the form (2.11) with U an open neighborhood of the origin. Assume that $X \in \mathcal{C}^{\Sigma_{s,r}}$ and it satisfies Hypothesis Hλ for some $\varrho_0 > 0$ and V as in Section 2.1.*

Then, for any $N \le \ell \le r$ there exist $0 < \varrho \le \varrho_0$, $K : V_\varrho \times \mathbb{R} \times \Lambda \to U$, T-periodic with respect to t, and $Y : V_\varrho \times \Lambda \to \mathbb{R}^n$ such that

$$X(K(x,t,\lambda), t, \lambda) - D_x K(x,t,\lambda) Y(x,\lambda) - \partial_t K(x,t,\lambda) \in \mathcal{H}^{>\ell}. \tag{2.14}$$

In addition, we can choose K and Y as a finite sum of homogeneous functions $K^j \in \mathcal{H}^j$ and $Y^j \in \mathcal{H}^j$ with respect to x (not necessarily polynomials), of the form

$$K_x(x,t,\lambda) = x + \sum_{l=2}^{\ell} K_x^l(x,t,\lambda), \qquad K_y(x,t,\lambda) = \sum_{l=2}^{\ell} K_y^l(x,t,\lambda),$$
$$Y(x,\lambda) = \sum_{l=N}^{\min\{\ell,\ell_*\}} Y^l(x,\lambda)$$

with $Y^N(x,\lambda) = p(x,0,\lambda)$, $L = \min\{N, M\}$ and ℓ_* defined in (2.7). The functions $K_x^l(x,\lambda)$, with $l = 2, \cdots, \ell_* - N + 1$, can be chosen arbitrarily, in particular, equal to 0. Moreover K and Y can be extended to V by homogeneity.

Concerning regularity we have that K and Y are $\mathcal{C}^1$ at the origin in the sense of Definition 2.1. Finally,

(1) *If either $A_p > d_p$ or $M < N$, K, Y are real analytic with respect to x and $\mathcal{C}^s$ with respect to (t, λ). In addition, if X depends analytically on $(t, \lambda) \in \mathbb{R} \times \Lambda$, then K, Y are real analytic in $V \times \mathbb{R} \times \Lambda$,*
(2) *If $A_p = d_p$, K, Y are $\mathcal{C}^\infty$ with respect to x and $\mathcal{C}^s$ with respect to (t, λ). Moreover, if $X \in \mathcal{C}^\infty$, then also $K, Y \in \mathcal{C}^\infty$.*
(3) *If $A_p < d_p$ and $M \ge N$, K, Y belong to $\mathcal{C}^{\Sigma_{s_*, r_* - s_*}}$ with $s_* = \min\{s, r_*\}$ and r_* defined in (2.4).*

Remark 2.10. Notice that the vector field Y can be chosen as a *finite* sum of homogeneous functions independent of t.

The rest of this paper is devoted to prove all these results. We first deal with the map case in the non parametric setting. In Section 3 we study the existence and regularity of global homogeneous solutions of a partial differential equation which is a model for all the cohomological equation we need to solve. Then, we prove Theorems 2.2, 2.7 and 2.9 by following an induction procedure with respect to the degree of differentiability. After that we deal with the dependence with respect to parameters. Finally we provide several examples to illustrate that our hypotheses are necessary to obtain approximate solutions and our results are (in some sense) optimal.

3. The cohomological equation

Let $V \subset \mathbb{R}^n$ be an open set, star-shaped with respect to 0 and $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^k$, $\mathbf{Q} : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ and $\mathbf{w} : V \to \mathbb{R}^k$ be such that $\mathbf{p} \in \mathcal{H}^N$, $\mathbf{Q} \in \mathcal{H}^{N-1}$, $\mathbf{w} \in \mathcal{H}^{\mathfrak{m}+N}$ with $N \geq 2$ and $\mathfrak{m} \geq 1$.

Note that $\mathbf{p}$, $\mathbf{Q}$ are determined by their restriction to an arbitrary small neighborhood U of the origin. In particular if they have some degree of regularity in U they have the same regularity in the whole space.

The linear partial differential equation

$$Dh(x) \cdot \mathbf{p}(x) - \mathbf{Q}(x) \cdot h(x) = \mathbf{w}(x) \tag{3.1}$$

for $h : V \to \mathbb{R}^k$ appears when we try to find approximations of K and R as sums of homogeneous functions. We are interested in solutions $h \in \mathcal{H}^{\mathfrak{m}+1}$.

Let V_{ϱ_0} be defined as in (2.2). Along this section we assume the following conditions for some $\varrho_0 > 0$:

HP1 $\mathbf{p}$ is $\mathcal{C}^1$ in V_{ϱ_0} and

$$a_{\mathbf{p}} = -\sup_{x \in V_{\varrho_0}} \frac{\|x + \mathbf{p}(x)\| - \|x\|}{\|x\|^N} > 0. \tag{3.2}$$

HP2 There exists a constant $a_V^{\mathbf{p}} > 0$ such that

$$\operatorname{dist}\big(x + \mathbf{p}(x), \big(V_{\varrho_0}\big)^c\big) \geq a_V^{\mathbf{p}} \|x\|^N, \qquad \forall x \in V_{\varrho_0}.$$

In the applications in this paper, $\mathbf{p}$ and $\mathbf{Q}$ will be polynomial functions.

Remark 3.1. If hypotheses HP1 and HP2 are satisfied for some ϱ_0, then they also hold for $0 < \varrho < \varrho_0$. As a consequence, we are always allowed to consider ϱ small enough (see Lemma 3.7).

We define the constants $b_{\mathbf{p}}$, $A_{\mathbf{p}}$, $B_{\mathbf{Q}}$, $A_{\mathbf{Q}}$, $c_{\mathbf{p}}$ and $d_{\mathbf{p}}$ by,

$$\begin{aligned}
&b_{\mathbf{p}} = \sup_{x \in V_{\varrho_0}} \frac{\|\mathbf{p}(x)\|}{\|x\|^N}, && A_{\mathbf{p}} = -\sup_{x \in V_{\varrho_0}} \frac{\|\mathrm{Id} + D\mathbf{p}(x)\| - 1}{\|x\|^{N-1}}, \\
&B_{\mathbf{Q}} = -\sup_{x \in V_{\varrho_0}} \frac{\|\mathrm{Id} - \mathbf{Q}(x)\| - 1}{\|x\|^{N-1}}, && A_{\mathbf{Q}} = \sup_{x \in V_{\varrho_0}} \frac{\|\mathrm{Id} + \mathbf{Q}(x)\| - 1}{\|x\|^{N-1}}, \\
&c_{\mathbf{p}} = \begin{cases} a_{\mathbf{p}}, & \text{if } B_{\mathbf{Q}} \leq 0, \\ b_{\mathbf{p}}, & \text{otherwise.} \end{cases} && d_{\mathbf{p}} = \begin{cases} a_{\mathbf{p}}, & \text{if } A_{\mathbf{p}} < 0, \\ b_{\mathbf{p}}, & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.3}$$

Next we introduce two ordinary differential equations which will play a key role in the proof of the results of this section. The first one is

$$\frac{dx}{dt} = \mathbf{p}(x). \tag{3.4}$$

We denote by $\varphi(t,x)$ its flow. The second one is the homogeneous linear equation

$$\frac{d\psi}{dt}(t,x) = \mathbf{Q}(\varphi(t,x))\psi(t,x) \tag{3.5}$$

and we denote by $M(t,x)$ its fundamental matrix such that $M(0,x) = \mathrm{Id}$.

Using uniqueness of solutions of (3.4) and homogenity,

$$\varphi(t,\lambda x) = \lambda\varphi(\lambda^{N-1}t,x), \qquad M(t,\lambda x) = M(\lambda^{N-1}t,x) \tag{3.6}$$

wherever they are defined.

In order to deal with the analytic case, we define the norm $\|\cdot\|$ in $\mathbb{C}^n$ as

$$\|x\| = \max\{\|\mathrm{Re}\,x\|, \|\mathrm{Im}\,x\|\}.$$

We define complex extensions of V and V_ϱ:

$$\Omega(\gamma) := \{x \in \mathbb{C}^n : \mathrm{Re}\,x \in V,\ \ \|\mathrm{Im}\,x\| < \gamma\|\mathrm{Re}\,x\|\},$$
$$\Omega(\varrho,\gamma) := \{x \in \mathbb{C}^n : \mathrm{Re}\,x \in V_\varrho,\ \ \|\mathrm{Im}\,x\| < \gamma\|\mathrm{Re}\,x\|\}.$$

Our analyticity results will be over solutions defined on a complex set $\Omega(\gamma)$ with a suitable choice of γ. We note that, if $x \in \Omega(\gamma)$ with $\gamma \le 1$, then $\|x\| = \|\mathrm{Re}\,x\|$. We will use this fact along this work without explicit mention.

Theorem 3.2. *Let* $\mathbf{p} \in \mathcal{H}^N$ *and* $\mathbf{Q} \in \mathcal{H}^{N-1}$ *be defined in an open set* U *of* $\mathbb{R}^n$ *and* $\mathbf{w} \in \mathcal{H}^{\mathfrak{m}+N}$ *defined on an open set* V *star-shaped with respect to* 0*, with* $N \ge 2$ *and* $\mathfrak{m} \ge 1$*. Assume that* $\mathbf{p}$ *satisfies hypotheses HP1 and HP2, for some* $\varrho_0 > 0$*, that* $\mathbf{p}, \mathbf{Q}$ *are* $\mathcal{C}^r$*,* $r \ge 1$*, in* U *and* $\mathbf{w}$ *is a* $\mathcal{C}^r$ *function in* V*.*

Then, if

$$\mathfrak{m} + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\left\{1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}, 0\right\}, \tag{3.7}$$

there exists a unique solution $h \in \mathcal{H}^{\mathfrak{m}+1}$ *of equation* (3.1) *which is given by:*

$$h(x) = \int_\infty^0 M^{-1}(t,x)\mathbf{w}(\varphi(t,x))\,dt, \qquad x \in V. \tag{3.8}$$

Moreover it is of class $\mathcal{C}^1$ *on* V*.*

Concerning its regularity we have the following cases:

(1) $A_{\mathbf{p}} \ge d_{\mathbf{p}}$*. If* $1 \le r \le \infty$*, then* h *is* $\mathcal{C}^r$ *in* V*.*

(2) $A_{\mathbf{p}} < d_{\mathbf{p}}$. *Let* r_0 *be the maximum of* $1 \le i \le r$ *such that*

$$\mathfrak{m} + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} - i\left(1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right) > 0. \tag{3.9}$$

Then h *is* $\mathcal{C}^{r_0}$ *in* V.

(3) $A_{\mathbf{p}} > d_{\mathbf{p}}$. *If* $\mathbf{p}, \mathbf{Q}, \mathbf{w}$ *are real analytic functions in* $\Omega(\gamma_0)$ *for some* γ_0 *then* h *is analytic in* $\Omega(\gamma)$ *for* γ *small enough. In particular it is real analytic in* V.

Remark 3.3. As we will see in Lemma 3.9 below, by Hypothesis HP2, V_{ϱ_0} is positively invariant by the flow φ, but it may happen that V is not. However since V is star-shaped with respect to the origin, $V \subset V^{\mathrm{e}}_{\varrho_0} = \{tx \,:\, t > 0,\ x \in V_{\varrho_0}\}$, $V^{\mathrm{e}}_{\varrho_0}$ is positively invariant by φ and the formula (3.8) makes sense with $\mathbf{w}$ understood as the unique extension of $\mathbf{w}$ to $V^{\mathrm{e}}_{\varrho_0}$ by homogeneity.

Remark 3.4. We notice that the condition $\mathfrak{m} + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\left\{1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}, 0\right\}$ is automatically satisfied if $B_{\mathbf{Q}}, A_{\mathbf{p}} \ge 0$.

Corollary 3.5. *Assume the conditions of Theorem 3.2. Let* $\nu \in \mathbb{N}$. *If* $\nu + B_{\mathbf{Q}}/c_{\mathbf{p}} \ge 0$, *then equation* (3.1) *has a solution* $h : V \to \mathbb{R}^k$ *belonging to* $\mathcal{H}^\nu$, *if and only if the integral*

$$\int_\infty^0 M^{-1}(t,x)\mathbf{w}(\varphi(t,x))\,dt$$

is convergent for $x \in V$.

We postpone the proof of these results to Section 3.3. First we establish some preliminary estimates.

3.1. Preliminary facts

This section deals with some basic facts that will be used henceforth without mention.

Lemma 3.6. *The constants* $A_{\mathbf{p}}$, $B_{\mathbf{Q}}$, $a_{\mathbf{p}}$, $b_{\mathbf{p}}$ *and* $A_{\mathbf{Q}}$ *are finite. They satisfy* $|a_{\mathbf{p}}| \le b_{\mathbf{p}}$, $a_{\mathbf{p}} \ge A_{\mathbf{p}}/N$, $B_{\mathbf{Q}} \le A_{\mathbf{Q}}$ *and* $-B_{D\mathbf{p}} \ge N a_{\mathbf{p}} > 0$.

Proof. The triangular inequality and the homogeneous character of $\mathbf{p}$ and $\mathbf{Q}$ imply that the constants are finite. Relation $|a_{\mathbf{p}}| \le b_{\mathbf{p}}$ is also a consequence of the triangular inequality.

From the definition of $A_{\mathbf{p}}$, we have that

$$\begin{aligned}\|x + \mathbf{p}(x)\| &\le \|x\| \int_0^1 \|\mathrm{Id} + D\mathbf{p}(\lambda x)\|\,d\lambda \le \|x\| \int_0^1 \left(1 - A_{\mathbf{p}}\lambda^{N-1}\|x\|^{N-1}\right) d\lambda \\ &= \|x\|\left(1 - \frac{A_{\mathbf{p}}}{N}\|x\|^{N-1}\right),\end{aligned} \tag{3.10}$$

therefore $a_{\mathbf{p}} \geq A_{\mathbf{p}}/N$.

As for $A_{\mathbf{Q}}$ and $B_{\mathbf{Q}}$, we notice that

$$\|\mathrm{Id} - \mathbf{Q}(x)^2\| \leq \|\mathrm{Id} + \mathbf{Q}(x)\| \cdot \|\mathrm{Id} - \mathbf{Q}(x)\| \leq \big(1 + A_{\mathbf{Q}}\|x\|^{N-1}\big)\big(1 - B_{\mathbf{Q}}\|x\|^{N-1}\big).$$

Since $\|\mathrm{Id} - \mathbf{Q}(x)^2\| \geq 1 - \|\mathbf{Q}(x)\|^2$, there exists some constant $K > 0$ such that

$$1 - K\|x\|^{2(N-1)} \leq 1 - (B_{\mathbf{Q}} - A_{\mathbf{Q}})\|x\|^{N-1} - A_{\mathbf{Q}}B_{\mathbf{Q}}\|x\|^{2(N-1)}.$$

Then, $B_{\mathbf{Q}} - A_{\mathbf{Q}} \leq (K - A_{\mathbf{Q}}B_{\mathbf{Q}})\|x\|^{N-1}$ and we get $B_{\mathbf{Q}} - A_{\mathbf{Q}} \leq 0$ taking $x \to 0$.

For the last claim, we note that, as we prove in (3.10),

$$\|x - \mathbf{p}(x)\| \leq \|x\| \left(1 - \frac{B_{D\mathbf{p}}}{N}\|x\|^{N-1}\right).$$

Since V_{ϱ_0} is invariant we apply the above inequality to $x + \mathbf{p}(x)$ and we obtain:

$$\|x + \mathbf{p}(x) - \mathbf{p}(x + \mathbf{p}(x))\| \leq \|x + \mathbf{p}(x)\| \left(1 - \frac{B_{D\mathbf{p}}}{N}\|x + \mathbf{p}(x)\|^{N-1}\right). \tag{3.11}$$

We note that $\|x + \mathbf{p}(x)\| \leq \|x\|\big(1 - a_{\mathbf{p}}\|x\|^{N-1}\big)$. Therefore, by (3.11)

$$\|x + \mathbf{p}(x) - \mathbf{p}(x + \mathbf{p}(x))\| \leq \|x\| \left(1 - \left(a_{\mathbf{p}} + \frac{B_{D\mathbf{p}}}{N}\right)\|x\|^{N-1} + K_2\|x\|^{2N-2}\right).$$

In addition

$$\|x + \mathbf{p}(x) - \mathbf{p}(x + \mathbf{p}(x))\| = \left\| x - \int_0^1 D\mathbf{p}(x + s\mathbf{p}(x))\mathbf{p}(x)\, ds \right\| \geq \|x\|\big(1 - K_1\|x\|^{2N-2}\big).$$

Then, again from (3.11), taking $K = K_1 + K_2$ we obtain

$$-K\|x\|^{N-1} \leq -a_{\mathbf{p}} - \frac{B_{D\mathbf{p}}}{N}$$

which gives the result taking $x \to 0$. □

The following lemma ensures that we can take ϱ as small as we need.

Lemma 3.7. *Let $0 < \overline{\varrho} < \varrho$. Denoting by $\overline{A_{\mathbf{p}}}, \overline{a_{\mathbf{p}}}, \overline{b_{\mathbf{p}}}, \overline{A_{\mathbf{Q}}}, \overline{B_{\mathbf{Q}}}$ the values of the constants $A_{\mathbf{p}}, a_{\mathbf{p}}, b_{\mathbf{p}}, A_{\mathbf{Q}}, B_{\mathbf{Q}}$ corresponding to $\overline{\varrho}$, we have that*

$$\overline{A_{\mathbf{p}}} \geq A_{\mathbf{p}}, \quad \overline{a_{\mathbf{p}}} \geq a_{\mathbf{p}}, \quad \overline{b_{\mathbf{p}}} = b_{\mathbf{p}}, \quad \overline{A_{\mathbf{Q}}} \leq A_{\mathbf{Q}}, \quad \overline{B_{\mathbf{Q}}} \geq B_{\mathbf{Q}}.$$

Then, for $x \in V_{\overline{\varrho}}$,

$$\|\mathrm{Id} + D\mathbf{p}(x)\| \le 1 - A_{\mathbf{p}}\|x\|^{N-1}, \qquad \|x + \mathbf{p}(x)\| \le \|x\|(1 - a_{\mathbf{p}}\|x\|^{N-1}),$$
$$\|\mathrm{Id} + \mathbf{Q}(x)\| \le 1 + A_{\mathbf{Q}}\|x\|^{N-1}, \qquad \|\mathrm{Id} - \mathbf{Q}(x)\| \le 1 - B_{\mathbf{Q}}\|x\|^{N-1}.$$

In addition, if HP1 and HP2 are satisfied for $\varrho > 0$*, they are also satisfied for all* $0 < \overline{\varrho} < \varrho$.

Proof. Indeed, let $\overline{\varrho} < \varrho$. The relations among the constants follow from the fact that $V_{\overline{\varrho}} \subset V_\varrho$ and (only for $b_{\mathbf{p}}$) $\mathbf{p}$ is a homogeneous function. Since $b_{\mathbf{p}}$ does not depend on ϱ, HP1 is satisfied for $\overline{\varrho}$. Now we deal with HP2. Let $x \in V_{\overline{\varrho}}$ and let $z \in \partial V_{\overline{\varrho}}$ be such that

$$\mathrm{dist}\big(x + \mathbf{p}(x), (V_{\overline{\varrho}})^c\big) = \|x + \mathbf{p}(x) - z\|.$$

We have two possibilities: either $z \in \partial V_\varrho$ or $z \in V_\varrho$ and $\|z\| = \overline{\varrho}$. If $z \in \partial V_\varrho$, then since $x \in V_\varrho$, by HP2 we have $\|x + \mathbf{p}(x) - z\| \ge a_V^{\mathbf{p}}\|x\|^N$. Finally, if $\|z\| = \overline{\varrho}$ we have that $z = \lambda(x + \mathbf{p}(x))$ with $\lambda = \overline{\varrho}\|x + \mathbf{p}(x)\|^{-1}$ and by HP1 and the definition of $a_{\mathbf{p}}$ in (3.2),

$$\|x + \mathbf{p}(x) - z\| = \overline{\varrho} - \|x + \mathbf{p}(x)\| \ge \|x\| - \|x + \mathbf{p}(x)\| \ge a_{\mathbf{p}}\|x\|^N. \quad \square$$

Next lemma will be used in the analytical case.

Lemma 3.8. *Let* $\varrho, \gamma > 0$.

(1) *If* $x \in \Omega(\varrho, \gamma)$ *and* $\chi : \Omega(\varrho, \gamma) \to \mathbb{C}^n$ *is a real analytic function belonging to* $\mathcal{H}^\ell$ *then*

$$\chi(x) = \chi(\mathrm{Re}\, x) + i D_x \chi(\mathrm{Re}\, x)\mathrm{Im}\, x + \gamma^2 \mathcal{O}(\|x\|^\ell).$$

(2) *If HP2 is satisfied and* $A_{\mathbf{p}} > b_{\mathbf{p}}$*, then there exists* $\gamma_0 \in (0, 1)$ *such that for any* $0 < \gamma \le \gamma_0$*, the complex set* $\Omega(\varrho_0, \gamma)$ *is an invariant set for the map* $x \mapsto x + \mathbf{p}(x)$.

Proof. Item (1) follows from Taylor's theorem, Cauchy-Riemann equations and the fact that χ is a real analytic function.

A property similar to (2) was proven in [2]. From (1), if $x \in \Omega(\varrho, \gamma)$,

$$x + \mathbf{p}(x) = x + \mathbf{p}(\mathrm{Re}\, x) + i D\mathbf{p}(\mathrm{Re}\, x)\mathrm{Im}\, x + \gamma^2 \mathcal{O}(\|x\|^N).$$

On the one hand we have that, by hypothesis HP2,

$$\mathrm{dist}(\mathrm{Re}\,\big(x + \mathbf{p}(x)\big), V_{\varrho_0}^c) \ge a_V^{\mathbf{p}}\|x\|^N - \gamma^2 \mathcal{O}(\|x\|^N) > 0 \tag{3.12}$$

which implies that $\mathrm{Re}\,\big(x + \mathbf{p}(x)\big) \in V_{\varrho_0}$ and on the other hand, using (3.12) and the definitions of $A_{\mathbf{p}}$ and $b_{\mathbf{p}}$, we have

$$\|\mathrm{Im}\,\big(x + \mathbf{p}(x)\big)\| - \gamma\|\mathrm{Re}\,\big(x + \mathbf{p}(x)\big)\| \le \gamma(b_{\mathbf{p}} - A_{\mathbf{p}} + \mathcal{O}(\gamma))\|\mathrm{Re}\, x\|^N < 0$$

provided γ is small enough. $\square$

3.2. Properties of $\varphi(t,x)$ and $M(t,x)$

In this section we describe some properties of the solutions of equations (3.4) and (3.5). We will denote by K a generic positive constant, which may take different values at different places. Also let

$$\alpha = \frac{1}{N-1}.$$

Lemma 3.9. *Assume hypotheses HP1 and HP2 for $\varrho_0 > 0$. Then:*

(1) *There exists $\varrho_1 \le \varrho_0$ such that for all $0 < \varrho \le \varrho_1$, V_ϱ is positively invariant by the flow φ.*
(2) *Assume that $A_{\mathbf{p}} > b_{\mathbf{p}}$ and that $\mathbf{p}$ has an analytic extension to $\Omega(\gamma_0)$ for some $0 < \gamma_0 \le 1$. Then there exist $0 < \varrho_1 \le \varrho_0$ and $0 < \gamma_1 \le \gamma_0$ such that for any $0 < \varrho \le \varrho_1$ and $0 \le \gamma \le \gamma_1$, the set $\Omega(\varrho, \gamma)$ is invariant by the complexified flow, i.e. $\varphi(t,x) \in \Omega(\varrho,\gamma)$, for $t > 0$ and $x \in \Omega(\varrho,\gamma)$.*

Proof. We first prove item (2). Since $\varphi(t,0) \equiv 0$ for all t and φ is $\mathcal{C}^1$, we have that, for some $\gamma \ge 0$ and ϱ small enough

$$\|\varphi(t,x)\| \le K\|x\|, \qquad t \in [0,1],\ x \in \Omega(\varrho,\gamma). \tag{3.13}$$

By Taylor's theorem,

$$\varphi(t,x) = x + t\mathbf{p}(x) + \int_0^t (t-s)D\mathbf{p}(\varphi(s,x))\mathbf{p}(\varphi(s,x))\,ds \tag{3.14}$$

and using that $\mathbf{p} \in \mathcal{H}^N$, (3.13) and (1) of Lemma 3.8 for $\chi = \mathbf{p}$, we get for $0 \le t \le 1$

$$\|\mathrm{Re}\,\varphi(t,x) - \big(\mathrm{Re}\,x + t\mathbf{p}(\mathrm{Re}\,x)\big)\| \le \gamma^2 K\|x\|^N t + K\|x\|^{2N-1}t^2. \tag{3.15}$$

Let $x \in \Omega(\varrho,\gamma)$. The fact that $\mathrm{Re}\,x \in V_\varrho$, (3.15) and HP2 imply that

$$\begin{aligned}
\mathrm{dist}\big(\mathrm{Re}\,\varphi(1,x), (V_\varrho)^c\big) \ge& \mathrm{dist}\big(\mathrm{Re}\,x + \mathbf{p}(\mathrm{Re}\,x), (V_\varrho)^c\big) \\
&- \|\mathrm{Re}\,x + \mathbf{p}(\mathrm{Re}\,x) - \mathrm{Re}\,\varphi(1,x)\| \\
\ge& \mathrm{dist}\big(\mathrm{Re}\,x + \mathbf{p}(\mathrm{Re}\,x), (V_\varrho)^c\big) - \gamma^2 K\|x\|^N - K\|x\|^{2N-1} \\
\ge& a_V^{\mathbf{p}}\|x\|^N - \gamma^2 K\|x\|^N - K\|x\|^{2N-1} \ge \frac{a_V^{\mathbf{p}}}{2}\|x\|^N
\end{aligned}$$

if ϱ, γ are small enough. We have proven that if $x \in \Omega(\varrho,\gamma)$ then $\mathrm{Re}\,\varphi(1,x) \in V_\varrho$.

In equality (3.6), take the values $t = 1$, $\lambda = t^\alpha$ with $t \in (0,1]$ and $x \in \Omega(\varrho,\gamma)$. Then

$$\varphi(t,x) = t^{-\alpha}\varphi(1, t^\alpha x).$$

Since $t^\alpha x \in \Omega(\varrho, \gamma)$ if $x \in \Omega(\varrho, \gamma)$, we already know that $\operatorname{Re}\varphi(t, x) \in V$. Moreover, by (3.15), taking ϱ, γ small enough and using that $\|\operatorname{Re} x\| = \|x\|$,

$$\|\operatorname{Re}\varphi(1, t^\alpha x)\| \le \|t^\alpha x\| \big(1 - t a_{\mathbf{p}} \|x\|^{N-1} + K t \gamma^2 \|x\|^{N-1} + K t^2 \|x\|^{2(N-1)}\big) \le t^\alpha \|x\|,$$

and consequently $\|\operatorname{Re}\varphi(t, x)\| \le \|x\| = \|\operatorname{Re} x\| \le \varrho$. This implies that $\operatorname{Re}\varphi(t, x) \in V_\varrho$ if $t \in [0, 1]$. Now, from identity (3.14), using (1) of Lemma 3.8 and the definitions of $b_{\mathbf{p}}$ and $A_{\mathbf{p}}$, we deduce that

$$\begin{aligned} \|\operatorname{Re}\varphi(t, x)\| &\ge \|\big(\operatorname{Re} x + t\mathbf{p}(\operatorname{Re} x)\big)\| - \gamma^2 K \|x\|^N t - K \|x\|^{2N-1} t^2 \\ &\ge \|\operatorname{Re} x\| (1 - t b_{\mathbf{p}} \|\operatorname{Re} x\|^{N-1}) - \gamma^2 K \|x\|^N t - K \|x\|^{2N-1} t^2, \qquad (3.16) \\ \|\operatorname{Im}\varphi(t, x)\| &\le \|\big(\mathrm{Id} + t D_x \mathbf{p}(\operatorname{Re} x)\big) \operatorname{Im} x\| + \gamma^2 K \|x\|^N t + K \|x\|^{2N-1} t^2 \\ &\le \|\operatorname{Im} x\| (1 - t A_{\mathbf{p}} \|\operatorname{Re} x\|^{N-1}) + \gamma^2 K \|x\|^N t + K \|x\|^{2N-1} t^2. \end{aligned}$$

Therefore, since $A_{\mathbf{p}} > b_{\mathbf{p}}$, taking ϱ, γ small enough,

$$\gamma \|\operatorname{Re}\varphi(t, x)\| - \|\operatorname{Im}\varphi(t, x)\| \ge 0.$$

As a consequence $\varphi(t, x) \in \Omega(\varrho, \gamma)$ for all $t \in [0, 1]$. Finally we extend this property to $t > 1$ by using inductively that $\varphi(t, x) = \varphi(1, \varphi(t-1, x))$. Note that in this part we have not to reduce the values of ϱ, γ.

A shorter but completely analogous argument proves (1) assuming neither that $\mathbf{p}$ is analytic nor $A_{\mathbf{p}} > b_{\mathbf{p}}$. $\square$

Lemma 3.10. *Assume that HP1 and HP2 are satisfied for some $\varrho_0 > 0$. Let $0 < a \le a_{\mathbf{p}}$ and $b \ge b_{\mathbf{p}}$. Then, for any $t \ge 0$ and $x \in V$,*

$$\frac{\|x\|}{\big(1 + (N-1) b t \|x\|^{N-1}\big)^\alpha} \le \|\varphi(t, x)\| \le \frac{\|x\|}{\big(1 + (N-1) a t \|x\|^{N-1}\big)^\alpha}.$$

If $A_{\mathbf{p}} > b_{\mathbf{p}}$ and $\mathbf{p}$ has an analytic extension to $\Omega(\gamma_0)$ for some $\gamma_0 \le 1$, for any $0 < a < a_{\mathbf{p}}$ and $b > b_{\mathbf{p}}$ there exists $\gamma \le \gamma_0$ such that for $t \ge 0$, φ is analytic in $\Omega(\gamma)$ and the previous bounds are true for $x \in \Omega(\gamma)$.

Proof. The definitions of $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ in (3.2) and (3.3), respectively, imply that for any $x \in V_\varrho$ and $t \in [0, 1]$,

$$\|x\| \big(1 - t b_{\mathbf{p}} \|x\|^{N-1}\big) \le \|x + t\mathbf{p}(x)\| \le \|x\| \big(1 - t a_{\mathbf{p}} \|x\|^{N-1}\big). \qquad (3.17)$$

Indeed, the inequality involving $b_{\mathbf{p}}$ follows from the triangular inequality. For the right hand side inequality, let $x \in V_\varrho$. Since V_ϱ is a star-shaped set, for any $t \in (0, 1]$, $t^\alpha x \in V_\varrho$ and hence,

$$-a_{\mathbf{p}} \ge \frac{\|t^\alpha x + \mathbf{p}(t^\alpha x)\| - \|t^\alpha x\|}{\|t^\alpha x\|^N} = \frac{\|x + t^{\alpha(N-1)} \mathbf{p}(x)\| - \|x\|}{t^{\alpha(N-1)} \|x\|^N}.$$

The result follows because $\alpha(N-1)=1$.

Let now $x \in \Omega(\varrho,\gamma)$, with ρ and γ given by Lemma 3.9. The real case, $x \in V_\varrho$, is obtained taking $\gamma = 0$. By Lemma 3.9, $\varphi(t,x) \in \Omega(\varrho,\gamma)$ and hence $\|\varphi(t,x)\| = \|\mathrm{Re}\,\varphi(t,x)\|$. Then from (3.16),

$$\|\varphi(t,x)\| \geq \|x\|\big(1 - b_{\mathbf{p}} t\|x\|^{N-1} - t\gamma^2 K\|x\|^{N-1} - t^2 K\|x\|^{2N-2}\big)$$

and from (3.15) and (3.17)

$$\|\varphi(t,x)\| \leq \|x\|\big(1 - a_{\mathbf{p}} t\|x\|^{N-1} + t\gamma^2 K\|x\|^{N-1} + t^2 K\|x\|^{2N-2}\big).$$

To obtain the bound for $\|\varphi(t,x)\|$, $t \in [0,1]$, we only have to take into account that, since $a_{\mathbf{p}} > a$ and $b_{\mathbf{p}} < b$, if ϱ,γ are small enough,

$$\|x\|\big(1 - a_{\mathbf{p}} t\|x\|^{N-1} + t\gamma^2 K\|x\|^{N-1} + t^2 K\|x\|^{2N-2}\big) \leq \frac{\|x\|}{\big(1 + a(N-1)t\|x\|^{N-1}\big)^{\alpha}},$$
$$\|x\|\big(1 - b_{\mathbf{p}} t\|x\|^{N-1} - t\gamma^2 K\|x\|^{N-1} - t^2 K\|x\|^{2N-2}\big) \geq \frac{\|x\|}{\big(1 + b(N-1)t\|x\|^{N-1}\big)^{\alpha}}.$$

Finally we are going to check that the results follow for any $t \geq 0$ and $x \in \Omega(\varrho,\gamma)$. In fact we will check the inequality involving a, being the other one analogous. We have already seen that if $t \in [0,1]$ the inequalities are true so we can proceed by induction assuming that the result is true for $t \in [0,l]$ with $l \in \mathbb{N}$. We introduce the auxiliary differential equation $\dot\chi = -a\chi^N$, $\chi \in \mathbb{R}$, and its flow $\chi(t,\xi)$, $\xi \in \mathbb{R}$. By induction hypothesis $\|\varphi(t,x)\| \leq \chi(t,\|x\|)$ if $t \in [0,l]$. Moreover, by Picard's theorem, if $\xi_1 < \xi_2$ then for all $t \geq 0$, $\chi(t,\xi_1) < \chi(t,\xi_2)$. Consequently, by using that $\Omega(\varrho,\gamma)$ is invariant by the flow φ, for any $s \in [0,1]$ and $t \in [0,l]$, we have that

$$\|\varphi(t+s,x)\| = \|\varphi(t,\varphi(s,x))\| \leq \chi\big(t,\|\varphi(s,x)\|\big) \leq \chi\big(t,\chi(s,\|x\|)\big) = \chi(t+s,\|x\|)$$

and the induction is completed.

Let $x \in \Omega(\gamma)$ and $\lambda > 0$ small enough such that $\lambda x \in \Omega(\varrho,\gamma)$. From (3.6),

$$\varphi(t,x) = \frac{1}{\lambda}\varphi\left(\frac{t}{\lambda^{N-1}}, \lambda x\right)$$

and from this expression, the bounds for $\|\varphi(s,\cdot)\|$ in $\Omega(\varrho,\gamma)$ extend to $\Omega(\gamma)$.

In the real case since $\gamma = 0$, the result is valid for any $0 < a < a_{\mathbf{p}}$ and $b > b_{\mathbf{p}}$ and we obtain the same bounds with $a = a_{\mathbf{p}}$ and $b = b_{\mathbf{p}}$. $\square$

Lemma 3.11. *Assume that HP1 and HP2 are fulfilled for some $\varrho_0 > 0$. Let $0 < a \leq a_{\mathbf{p}}$, $b \geq b_{\mathbf{p}}$, $A \geq A_{\mathbf{Q}}$ and $B \leq B_{\mathbf{Q}}$. Then, for all $x \in V$ and $t \geq 0$, we have the following bounds*

$$\big(1 + c(N-1)t\|x\|^{N-1}\big)^{\alpha\frac{B}{c}} \leq \|M(t,x)\| \leq \big(1 + \delta(N-1)t\|x\|^{N-1}\big)^{\alpha\frac{A}{\delta}}$$
$$\big(1 + \delta(N-1)t\|x\|^{N-1}\big)^{-\alpha\frac{A}{\delta}} \leq \|M^{-1}(t,x)\| \leq \big(1 + c(N-1)t\|x\|^{N-1}\big)^{-\alpha\frac{B}{c}}$$

with

$$c = \begin{cases} a, & \text{if } B \le 0, \\ b, & \text{otherwise.} \end{cases} \qquad \delta = \begin{cases} a, & \text{if } A \ge 0, \\ b, & \text{otherwise.} \end{cases} \tag{3.18}$$

If $\mathbf{p}$ *and* $\mathbf{Q}$ *have an analytic extension to* $\Omega(\gamma_0)$ *for some* $\gamma_0 \le 1$, *and* $A_{\mathbf{p}} > b_{\mathbf{p}}$, *then for any* $0 < a < a_{\mathbf{p}}$, $b > b_{\mathbf{p}}$, $A > A_{\mathbf{Q}}$ *and* $B < B_{\mathbf{Q}}$ *there exists* $\gamma \le \gamma_0$ *such that, for* $t \ge 0$, $M(t, x)$ *is analytic in* $\Omega(\gamma)$ *and the previous bounds are also true for* $x \in \Omega(\gamma)$.

Proof. By Lemma 3.9, the condition $A_{\mathbf{p}} > b_{\mathbf{p}}$ implies that there exist $\varrho > 0$ and $\gamma > 0$ such that the set $\Omega(\varrho, \gamma)$ is invariant by φ if γ is small enough provided that $\mathbf{p}$ has an analytic extension to $\Omega(\varrho, \gamma_0)$. This will be the only place where we use the condition $A_{\mathbf{p}} > b_{\mathbf{p}}$. For that reason we will perform our computations in the analytic case, the real case being just a direct consequence by taking $\gamma = 0$.

Let $x \in \Omega(\varrho, \gamma)$. First consider the auxiliary differential equation

$$\dot{\zeta} = \big(\mathrm{Id} + \mathbf{Q}(\varphi(t, x))\big)\zeta$$

and denote by $\chi(t, x)$ its fundamental matrix satisfying $\chi(0, x) = \mathrm{Id}$. We notice that $\chi(t, x) = \mathrm{e}^t M(t, x)$. Moreover,

$$\chi(t, x) = \mathrm{Id} + \int_0^t \big(\mathrm{Id} + \mathbf{Q}(\varphi(s, x))\big)\chi(s, x)\, ds.$$

Hence, by the definition of $A_{\mathbf{Q}}$ and Lemma 3.8, we have that

$$\begin{aligned} \|\chi(t, x)\| &\le 1 + \int_0^t \|\mathrm{Id} + \mathbf{Q}(\varphi(s, x))\|\, \|\chi(s, x)\|\, ds \\ &\le 1 + \int_0^t (1 + (A_{\mathbf{Q}} + K\gamma)\|\varphi(s, x)\|^{N-1})\|\chi(s, x)\|\, ds. \end{aligned}$$

Writing $A = A_{\mathbf{Q}} + K\gamma$ and using Gronwall's Lemma,

$$\|\chi(t, x)\| \le \exp\left(\int_0^t \big(1 + A\|\varphi(s, x)\|^{N-1}\big)\, ds\right) = \mathrm{e}^t \exp\left(A\int_0^t \|\varphi(s, x)\|^{N-1}\, ds\right).$$

By using that $\chi(t, x) = \mathrm{e}^t M(t, x)$, we obtain that

$$\|M(t, x)\| \le \exp\left(A\int_0^t \|\varphi(u, x)\|^{N-1}\, du\right). \tag{3.19}$$

In the real case, i.e. when $x \in V_\varrho = \Omega(\varrho, 0)$, we can take $A = A_{\mathbf{Q}}$.

Let us consider the differential equation

$$\dot{\zeta} = \big(\mathrm{Id} - \mathbf{Q}^\top(\varphi(t,x))\big)\zeta.$$

We have that its fundamental matrix $\psi(t,x)$ such that $\psi(0,x) = \mathrm{Id}$ is $\psi(t,x) = \mathrm{e}^t M^{-\top}(t,x)$, where here we have written $M^{-\top} = [M^{-1}]^\top$. Indeed,

$$\dot{\psi}(t,x) = \mathrm{e}^t M^{-\top}(t,x) + \mathrm{e}^t \dot{M}^{-\top}(t,x) = \psi(t,x) - \mathbf{Q}^\top(\varphi(t,x))\psi(t,x).$$

Now we have that

$$\psi(t,x) = \mathrm{Id} + \int_0^t \big(\mathrm{Id} - \mathbf{Q}^\top(\varphi(s,x))\big)\psi(s,x)\, ds.$$

We transpose the above equality and take norms to obtain

$$\|\psi^\top(t,x)\| \le 1 + \int_0^t \|\mathrm{Id} - \mathbf{Q}(\varphi(s,x))\| \|\psi^\top(s,x)\|\, ds.$$

Finally using the definition of $B_{\mathbf{Q}}$, Lemma 3.8 and Gronwall's Lemma we conclude that

$$\begin{aligned} \|\psi^\top(t,x)\| &\le \exp\left(\int_0^t 1 - (B_{\mathbf{Q}} - K\gamma)\|\varphi(s,x)\|^{N-1}\, ds\right) \\ &= \mathrm{e}^t \exp\left(-(B_{\mathbf{Q}} - K\gamma)\int_0^t \|\varphi(s,x)\|^{N-1}\, ds\right) \end{aligned}$$

and, as a consequence, since $\psi^\top(t,x) = \mathrm{e}^t M^{-1}(t,x)$ we have that

$$\|M^{-1}(t,x)\| \le \exp\left(-B\int_0^t \|\varphi(u,x)\|^{N-1}\, du\right), \tag{3.20}$$

where we have taken $B = B_{\mathbf{Q}} - K\gamma$. In order to bound $\int_0^t \|\varphi(u,x)\|^{N-1}\, du$ we use the bounds in Lemma 3.10 obtaining

$$\begin{aligned} \int_0^t \|\varphi(u,x)\|^{N-1}\, du &\le \|x\|^{N-1}\int_0^t \frac{1}{1 + a(N-1)u\|x\|^{N-1}}\, du \\ &= \frac{1}{a(N-1)} \log\big(1 + a(N-1)t\|x\|^{N-1}\big), \end{aligned}$$

$$\begin{aligned}\int_0^t \|\varphi(u,x)\|^{N-1}\,du &\geq \|x\|^{N-1}\int_0^t \frac{1}{1+b(N-1)u\|x\|^{N-1}}\,du \\ &= \frac{1}{b(N-1)}\log\left(1+b(N-1)t\|x\|^{N-1}\right).\end{aligned}$$

We recall that by Lemma 3.6, $B_{\mathbf{Q}} \leq A_{\mathbf{Q}}$. To obtain the inequalities in the statement from (3.19) and (3.20) we distinguish three cases according to the signs of $A_{\mathbf{Q}}$, $B_{\mathbf{Q}}$. The first case is $B_{\mathbf{Q}} > 0$. Let $0 < B < B_{\mathbf{Q}}$ and $A > A_{\mathbf{Q}}$. We take $0 < \gamma_1 \leq \gamma_0$ such that $0 < B \leq B_{\mathbf{Q}} - K\gamma_1$ and $A \geq A_{\mathbf{Q}} + K\gamma_1$. Then, if $0 \leq \gamma \leq \gamma_1$,

$$\begin{aligned}\|M(t,x)\| &\leq \left(1+a(N-1)t\|x\|^{N-1}\right)^{\frac{A}{a(N-1)}}, \\ \|M^{-1}(t,x)\| &\leq \left(1+b(N-1)t\|x\|^{N-1}\right)^{\frac{-B}{b(N-1)}}.\end{aligned}$$

The remaining inequalities follow from $\|M^{-1}(t,x)\| \geq \|M(t,x)\|^{-1}$. The other two cases, $A_{\mathbf{Q}} < 0$ and $B_{\mathbf{Q}} \leq 0 \leq A_{\mathbf{Q}}$, follow analogously.

Using the identity (3.6) $M(t,x) = M\left(\lambda^{-N+1}t, \lambda x\right)$, the inequalities extend to $\Omega(\gamma)$. Note that in the real case we can take $A = A_{\mathbf{Q}}$, $B = B_{\mathbf{Q}}$, $a = a_{\mathbf{p}}$ and $b = b_{\mathbf{p}}$. □

3.3. Proof of Theorem 3.2

We begin by checking that if $h : V \to \mathbb{R}^k$ is a differentiable solution of (3.1) in $\mathcal{H}^{\mathrm{m}+1}$, it has to be given by formula (3.8) given in Theorem 3.2, i.e.

$$h(x) = \int_\infty^0 M^{-1}(t,x)\mathbf{w}(\varphi(t,x))\,dt.$$

Indeed, let $h \in \mathcal{H}^{\mathrm{m}+1}$ be such that

$$Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x).$$

We define $\mu(t,x) = h(\varphi(t,x))$ and we have that

$$\dot{\mu}(t,x) = Dh(\varphi(t,x))\mathbf{p}(\varphi(t,x)) = \mathbf{Q}(\varphi(t,x))\mu(t,x) + \mathbf{w}(\varphi(t,x))$$

and then, since $\mu(0,x) = h(x)$,

$$\mu(t,x) = M(t,x)\left(h(x) + \int_0^t M^{-1}(s,x)\mathbf{w}(\varphi(s,x))\,ds\right).$$

Note that, with ϱ given by Lemma 3.9, if $x \in V_\varrho$, $\varphi(s,x) \in V_\varrho$ for all $s \geq 0$. The hypothesis (3.7), Lemmas 3.10 and 3.11 and the fact that $\|h(x)\| \leq K\|x\|^{\mathrm{m}+1}$, imply that $M^{-1}(t,x)\mu(t,x) = M^{-1}(t,x)h(\varphi(t,x)) \to 0$ as $t \to \infty$. Therefore we obtain the desired expression for h.

This provides the uniqueness statement in V_ϱ. The fact that h belongs to $\mathcal{H}^{\mathrm{m}+1}$ will be proven in the next lemma in a slightly more general setting. The homogeneity of h determines uniquely the extension of h to V which satisfies (3.1) in V. Then it remains to prove that actually h is well defined, it is a solution and its regularity. Our strategy to prove the regularity stated in Theorem 3.2 follows three steps. The first one deals with the continuity (resp. analyticity) of functions defined by integrals of the form

$$g(x) := \int_{\infty}^{0} \chi^{-1}(t,x)\omega(\varphi(t,x))\,dt \tag{3.21}$$

with χ and ω satisfying appropriate conditions. Note that definition (3.8) of h fits in this setting. This is done in Lemma 3.12 below.

Secondly, we deal with the $\mathcal{C}^1$ regularity, proving both: i) that $g \in \mathcal{C}^1$ and ii) that Dg can be expressed as

$$\int_{\infty}^{0} (\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\,dt$$

with χ^1 and ω^1 having the conditions required in the previous step for g to be a continuous function. This is proven in Lemma 3.14.

Finally, the third step consists of an inductive procedure with respect to the degree of differentiability.

In what follows we will use the constants introduced at the beginning of Section 3 depending on the homogeneous functions indicated in their subscripts without further notice.

Lemma 3.12. *Let $\mathbf{p} \in \mathcal{H}^N$ be defined on V and satisfying hypotheses HP1 and HP2 for ϱ_0, $\mathcal{Q} \in \mathcal{H}^{N-1}$ and $\omega \in \mathcal{H}^{\nu+N}$ on V, with $\nu \geq 1$. We denote by χ the fundamental matrix of*

$$\frac{d}{dt}\psi(t,x) = \mathcal{Q}(\varphi(t,x))\psi(t,x), \qquad \text{such that} \qquad \chi(0,x) = \mathrm{Id}\,.$$

If $\nu + 1 + \frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}} > 0$, with $c_{\mathbf{p}}$ defined in (3.3) taking $\mathbf{Q} = \mathcal{Q}$, then the function $g : V \to \mathbb{R}^k$ defined by (3.21) belongs to $\mathcal{H}^{\nu+1}$ being, in particular, a $\mathcal{C}^0$ function on V.

Moreover, if we also have $A_{\mathbf{p}} > b_{\mathbf{p}}$, then, there exists $\gamma > 0$ small enough such that the function g is analytic in $\Omega(\gamma)$ provided $\mathbf{p}$, $\mathcal{Q}$ and ω have analytic extensions to $\Omega(\gamma_0)$ for some $\gamma_0 > \gamma$.

Proof. If $\mathbf{p}$, $\mathcal{Q}$ and ω have analytic extensions to $\Omega(\gamma_0)$, let $0 < a < a_{\mathbf{p}}$, $b > b_{\mathbf{p}}$ and $B < B_{\mathcal{Q}}$ be such that $\nu + 1 + \frac{B}{c} > 0$ where c is defined in (3.18). We fix ϱ and γ satisfying the conditions of Lemmas 3.9, 3.10 and 3.11. In this case we have that $\Omega(\varrho, \gamma)$ is invariant by φ provided $A_{\mathbf{p}} > b_{\mathbf{p}}$. Since $V_\varrho = \Omega(\varrho, 0)$, we make the convention that in the real case, we take $\gamma = 0$. This allows us to deal with both cases (real and complex) at the same time. If ω is a $\mathcal{C}^0$ function on V we take $U = V$ and if ω has an analytic extension to $\Omega(\varrho, \gamma)$ for some $\gamma > 0$, we take $U = \Omega(\varrho, \gamma)$. With this convention, we define

$$\|\omega\| = \sup_{x\in U} \frac{\|\omega(x)\|}{\|x\|^{\nu+N}}.$$

We begin by proving that the function g is well defined and $\mathcal{C}^0$ in $\Omega(\varrho,\gamma)$. Indeed, we only need to check that the integral in the definition of g is convergent. For that we use Lemmas 3.10 and 3.11 applied to $\mathcal{Q}$. Let $x \in \Omega(\varrho,\gamma)$

$$\begin{aligned}\|\chi^{-1}(t,x)\omega(\varphi(t,x))\| &\le \|\omega\|\,\|\varphi(t,x)\|^{\nu+N}\|\chi^{-1}(t,x)\| \\ &\le \|\omega\| \frac{\|x\|^{\nu+N}}{\left(1+a(N-1)t\|x\|^{N-1}\right)^{\alpha(\nu+N+\frac{B}{c})}}\end{aligned}$$

because $c \ge a$ and $\kappa := \alpha\left(\nu+N+\frac{B}{c}\right) = \alpha(N-1)+\alpha\left(\nu+1+\frac{B}{c}\right) > 1$ by hypothesis. Therefore,

$$\|\chi^{-1}(t,x)\omega(\varphi(t,x))\| \le \|\omega\|\,\|x\|^{\nu+N}\left(1+a(N-1)t\|x\|^{N-1}\right)^{-\kappa} \tag{3.22}$$

which implies that

$$\|g(x)\| \le \|\omega\|\,\|x\|^{\nu+N}\int_0^\infty \frac{dt}{\left(1+a(N-1)t\|x\|^{N-1}\right)^{\kappa}} \le K\|\omega\|\,\|x\|^{\nu+1}.$$

Now we prove that g belongs to $\mathcal{H}^{\nu+1}$. As we mentioned in (3.6), for any $\lambda > 0$, one has that $\varphi(t,\lambda x) = \lambda\varphi(\lambda^{N-1}t,x)$ and $\chi^{-1}(t,\lambda x) = \chi^{-1}(\lambda^{N-1}t,x)$. Then,

$$\begin{aligned}g(\lambda x) &= \int_\infty^0 \chi^{-1}(t,\lambda x)\omega(\varphi(t,\lambda x))\,dt = \int_\infty^0 \chi^{-1}(\lambda^{N-1}t,x)\omega(\lambda\varphi(\lambda^{N-1}t,x))\,dt \\ &= \lambda^{1-N}\int_\infty^0 \chi^{-1}(t,x)\omega(\lambda\varphi(t,x))\,dt = \lambda^{1-N}\lambda^{\nu+N}\int_\infty^0 \chi^{-1}(t,x)\omega(\varphi(t,x))\,dt \\ &= \lambda^{\nu+1}g(x).\end{aligned}$$

Finally we check the regularity. We first check that g is analytic if ω, $\mathcal{Q}$ and $\mathbf{p}$ have analytic extensions to $\Omega(\varrho,\gamma)$. Let $x_0 \in \Omega(\varrho,\gamma)$ be a given point. Since $\Omega(\varrho,\gamma)$ is an open set, there exists $0 < r < \|x_0\|$ such that the open ball $B_r(x_0)$ is contained in $\Omega(\varrho,\gamma)$. Then, if $x \in B_r(x_0)$, $\|x\| \ge \|x_0\| - r$ and consequently, using (3.22),

$$\begin{aligned}\|\chi^{-1}(t,x)\omega(\varphi(t,x))\| &\le \|\omega\| \frac{\|x\|^{\nu+N}}{\left(1+a(N-1)t\|x\|^{N-1}\right)^{\kappa}} \\ &\le \|\omega\| \frac{\left(\|x_0\|+r_0\right)^{\nu+N}}{\left(1+a(N-1)t(\|x_0\|-r)^{N-1}\right)^{\kappa}}\end{aligned}$$

and the analyticity follows from the dominated convergence theorem because the right hand side of the above bound does not depend on x and it is integrable.

Since g is homogeneous we can extend it uniquely to an analytic homogeneous function in $\Omega(\gamma)$. Considering ω extended by homogeneity as indicated in Remark 3.3 the extension of g has the same expression (3.21).

In the real case, when $\mathbf{p}$ is $\mathcal{C}^1$ and ω, $\mathcal{Q}$ are continuous homogeneous functions, the same argument as the one given in the analytic case, leads to the proof that g is a continuous function. □

Now we are going to deal with the differentiable case. If g is a solution of

$$Dg(x)\mathbf{p}(x) - \mathcal{Q}(x)g(x) = \omega(x), \tag{3.23}$$

then Dg, if it is $\mathcal{C}^1$, should satisfy

$$D^2g(x)\mathbf{p}(x) - [\mathcal{Q}(x)Dg(x) - Dg(x)D\mathbf{p}(x)] = D\omega(x) + D\mathcal{Q}(x)g(x)$$

which is an equation for Dg analogous to (3.23) except that the second term, due to the lack of commutativity is more involved. Continuing in this way would imply to consider linear equations of the form

$$\dot{\chi} = \mathcal{Q}(\varphi(t, \lambda x))\chi - \chi D\mathbf{p}(\varphi(t, \lambda x)).$$

However we have chosen to consider the equivalent equation for a vector which contains all elements $D_{ij}g$ ordered one column after the other. This forces the introduction of the following notation.

We denote by D_j the derivative with respect to the variable x_j. We define the linear operator $\mathcal{S} : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k) \to \mathbb{R}^{n \cdot k}$:

$$\mathcal{S}(A) = \big((Ae_1)^\top, \cdots, (Ae_n)^\top\big)^\top, \qquad \text{being } \{e_1, \cdots, e_n\} \text{ the canonical basis,} \tag{3.24}$$

and the functions $\mathcal{B}_{\mathcal{Q}}, \mathcal{I}^k_{D\mathbf{p}}, \mathcal{Q}^1 : V_\varrho \to \mathcal{L}(\mathbb{R}^{n \cdot k}, \mathbb{R}^{n \cdot k})$:

$$\mathcal{B}_{\mathcal{Q}}(x) = \operatorname{diag}\big(\mathcal{Q}(x), \cdots, \mathcal{Q}(x)\big), \tag{3.25}$$

$$\mathcal{I}^k_{D\mathbf{p}}(x) = \begin{pmatrix} D_1\mathbf{p}_1(x)\mathrm{Id}_k & \cdots & D_1\mathbf{p}_n(x)\mathrm{Id}_k \\ \vdots & \vdots & \vdots \\ D_n\mathbf{p}_1(x)\mathrm{Id}_k & \cdots & D_n\mathbf{p}_n(x)\mathrm{Id}_k \end{pmatrix}, \tag{3.26}$$

$$\mathcal{Q}^1(x) = \mathcal{B}_{\mathcal{Q}}(x) - \mathcal{I}^k_{D\mathbf{p}}(x),$$

with $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_n)^\top$ and Id_k the identity in $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$.

For any $w \in \mathbb{R}^{n \cdot k}$, we also write

$$w = (w_1, \cdots, w_n), \quad \text{with} \quad w_i \in \mathbb{R}^k.$$

Finally we define the norm in $\mathbb{R}^{n \cdot k}$

$$\|w\| = \sup_{u\in\mathbb{R}^n\setminus\{0\}} \frac{\|u_1w_1+\cdots+u_nw_n\|}{\|u\|} = \sup_{\|u\|=1} \|u_1w_1+\cdots+u_nw_n\|,$$

where the norms in $\mathbb{R}^n$ and $\mathbb{R}^k$ are such that HP1 and HP2 hold.

Let $\chi^1(t,x)$ be the fundamental solution of

$$\frac{d\psi}{dt}(t,x) = \mathcal{Q}^1(\varphi(t,x))\psi(t,x) \qquad \text{such that} \quad \chi^1(0,x) = \mathrm{Id}\,. \tag{3.27}$$

Lemma 3.13. *Let* $0<\varrho\le\varrho_0$.

(1) *We have that*

$$B_{\mathcal{Q}^1} := -\sup_{x\in V_\varrho} \frac{\|\mathrm{Id}-\mathcal{Q}^1(x)\|-1}{\|x\|^{N-1}} \ge B_{\mathcal{Q}} + A_{\mathbf{p}}. \tag{3.28}$$

(2) *The fundamental matrix* χ^1 *of* (3.27) *satisfies*

$$(\chi^1)^{-1}(t,x) = \mathcal{I}^k_{D\varphi}(t,x)\cdot\mathcal{B}_{\chi^{-1}}(t,x)$$

with

$$\mathcal{B}_{\chi^{-1}}(t,x) = \mathrm{diag}\left(\chi^{-1}(t,x),\cdots,\chi^{-1}(t,x)\right),$$
$$\mathcal{I}^k_{D\varphi}(t,x) = \begin{pmatrix} D_1\varphi_1(t,x)\mathrm{Id}_k & \cdots & D_1\varphi_n(t,x)\mathrm{Id}_k \\ \vdots & \vdots & \vdots \\ D_n\varphi_1(t,x)\mathrm{Id}_k & \cdots & D_n\varphi_n(t,x)\mathrm{Id}_k \end{pmatrix}.$$

Proof. Let $w\in\mathbb{R}^{n\cdot k}$, $w=(w_1,\cdots,w_n)$ with $\|w\|=1$. We have that

$$\begin{aligned}\left\|\left(\frac12\mathrm{Id}-\mathcal{B}_{\mathcal{Q}}(x)\right)w\right\| &= \sup_{\|u\|=1}\left\|\left(\frac12\mathrm{Id}-\mathcal{Q}(x)\right)\left(w_1u_1+\cdots+w_nu_n\right)\right\| \\ &\le \left\|\frac12\mathrm{Id}-\mathcal{Q}(x)\right\|\sup_{\|u\|=1}\|w_1u_1+\cdots+w_nu_n\| \\ &= \left\|\frac12\mathrm{Id}-\mathcal{Q}(x)\right\| \le \frac12 - B_{\mathcal{Q}}\|x\|^{N-1}, \end{aligned} \tag{3.29}$$

where we have used that

$$\left\|\frac12\mathrm{Id}-\mathcal{Q}(x)\right\| = \left\|\frac12\left(\mathrm{Id}-\mathcal{Q}(2^{1/(N-1)}x)\right)\right\| \le \frac12\left(1-B_{\mathcal{Q}}\left(2^{1/(N-1)}\|x\|\right)^{N-1}\right).$$

In addition, we can decompose $\left(\frac12\mathrm{Id}+\mathcal{I}^k_{D\mathbf{p}}(x)\right)w = (\bar w_1,\cdots,\bar w_n)^\top$, with $\bar w_i\in\mathbb{R}^k$ and

$$\bar w_i - \frac12 w_i = D_i\mathbf{p}_1(x)w_1+\cdots+D_i\mathbf{p}_n(x)w_n.$$

Given $u=(u_1,\dots,u_n)\in\mathbb{R}^n$, letting $\bar u=\big(\frac{1}{2}\mathrm{Id}+D\mathbf{p}(x)\big)u$ we have

$$u_1\bar w_1+\cdots+u_n\bar w_n=\bar u_1 w_1+\cdots+\bar u_n w_n.$$

As a consequence,

$$\begin{aligned}\sup_{u\in\mathbb{R}^n\setminus\{0\}}\frac{\|u_1\bar w_1+\cdots+u_n\bar w_n\|}{\|u\|}&\le\left\|\left(\frac{1}{2}\mathrm{Id}+D\mathbf{p}(x)\right)\right\|\sup_{\bar u\in\mathbb{R}^n\setminus\{0\}}\frac{\|\bar u_1 w_1+\cdots+\bar u_n w_n\|}{\|\bar u\|}\\&=\left\|\left(\frac{1}{2}\mathrm{Id}+D\mathbf{p}(x)\right)\right\|\le\frac{1}{2}-A_{\mathbf{p}}\|x\|^{N-1}.\end{aligned}$$

The above bound jointly with (3.29) gives that

$$\|\mathrm{Id}-\mathcal{Q}^1(x)\|\le\left\|\frac{1}{2}\mathrm{Id}-\mathcal{B}_{\mathcal{Q}}(x)\right\|+\left\|\frac{1}{2}\mathrm{Id}+\mathcal{I}^k_{D\mathbf{p}}(x)\right\|\le 1-(B_{\mathcal{Q}}+A_{\mathbf{p}})\|x\|^{N-1}$$

and (3.28) is proven.

To obtain the expression for $(\chi^1)^{-1}(t,x)$ is a straightforward computation. □

Lemma 3.14. *Assume that* $\mathbf{p}$, $\mathcal{Q}$ *and* ω *are* $\mathcal{C}^1$ *functions on* V. *Let* χ *be the fundamental matrix of* $\frac{d}{dt}\psi(t,x)=\mathcal{Q}(\varphi(t,x))\psi(t,x)$ *satisfying* $\chi(0,x)=\mathrm{Id}$.

If hypotheses HP1 and HP2 are satisfied for ϱ_0 *and*

$$\nu+1+\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}>\max\left\{1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}},0\right\},\tag{3.30}$$

with $c_{\mathbf{p}}, d_{\mathbf{p}}$ *defined in* (3.3) *taking* $\mathbf{Q}=\mathcal{Q}$, *then the function* $g:V\to\mathbb{R}^k$ *defined in* (3.21) *belongs to* $\mathcal{H}^{\nu+1}$ *and is a* $\mathcal{C}^1$ *function on* V.

Moreover

$$\mathcal{S}(Dg(x))=\int_\infty^0(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\,dt,\tag{3.31}$$

where χ^1 *is the fundamental matrix of* (3.27) *such that* $\chi^1(0,x)=\mathrm{Id}$ *and*

$$\omega^1(x)=\mathcal{S}(D\omega(x))+\left((D_1\mathcal{Q}(x)g(x))^\top,\cdots,(D_n\mathcal{Q}(x)g(x))^\top\right)^\top.\tag{3.32}$$

Proof. Let $\varrho>0$ satisfying Lemma 3.9. We claim that for any $\tau\ge0$ and $x\in V_\varrho$,

$$\begin{aligned}\int_\tau^0 D_j\big[\chi^{-1}(t,x)\omega(\varphi(t,x))\big]\,dt=&-D_j\chi^{-1}(\tau,x)g(\varphi(\tau,x))\\&+\int_\tau^0\big[(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\big]_j\,dt.\end{aligned}\tag{3.33}$$

We recall here that the subscript in a vector in $\mathbb{R}^{n\cdot k}$ identifies a vector in $\mathbb{R}^k$.

We will use the following properties related to χ:

$$\frac{d}{dt}\big(\chi^{-1}(t,x)D_j\chi(t,x)\big) = \chi^{-1}(t,x)D_j\big(\mathcal{Q}(\varphi(t,x))\big)\chi(t,x), \tag{3.34}$$

$$\chi(u+v,x) = \chi(u,\varphi(v,x))\chi(v,x), \tag{3.35}$$

$$\chi^{-1}(t,x)D_j\chi(t,x) = -D_j\chi^{-1}(t,x)\chi(t,x). \tag{3.36}$$

Expression (3.34) follows by using the variational equation for χ. The second one follows from the uniqueness of solutions of $\dot\psi(t,x) = \mathcal{Q}(\varphi(t,x))\psi(t,x)$ and the last one taking derivatives in $\chi^{-1}(t,x)\chi(t,x) = \mathrm{Id}$.

From Lemma 3.13 and definition (3.32) of ω^1 we obtain that

$$\begin{aligned}&\big[(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\big]_j\\ &\qquad = \chi^{-1}(t,x)\big[D_j\big(\omega(\varphi(t,x))\big) + D_j\big(\mathcal{Q}(\varphi(t,x))\big)g(\varphi(t,x))\big].\end{aligned} \tag{3.37}$$

Using properties (3.35) in the definition of g, we obtain that

$$\begin{aligned} g(\varphi(t,x)) &= \int_{\infty}^{0} \chi^{-1}(s,\varphi(t,x))\omega(\varphi(s,\varphi(t,x)))\,ds \\ &= \chi(t,x)\int_{\infty}^{t} \chi^{-1}(s,x)\omega(\varphi(s,x))\,ds, \end{aligned} \tag{3.38}$$

and by (3.37), (3.38) and (3.34) we get

$$\begin{aligned}\big[(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\big]_j &= \chi^{-1}(t,x)D_j\big(\omega(\varphi(t,x))\big)\\ &\quad + \frac{d}{dt}\big(\chi^{-1}(t,x)D_j\chi(t,x)\big)\int_{\infty}^{t}\chi^{-1}(s,x)\omega(\varphi(s,x))\,ds.\end{aligned}$$

Integrating by parts and using $D_j\chi^{-1}(0,x) = 0$:

$$\begin{aligned}\int_{\tau}^{0}\big[(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\big]_j\,dt &= \int_{\tau}^{0}\chi^{-1}(t,x)D_j\big(\omega(\varphi(t,x))\big)\,dt\\ &\quad - \chi^{-1}(\tau,x)D_j\chi(\tau,x)\int_{\infty}^{\tau}\chi^{-1}(t,x)\omega(\varphi(t,x))\,dt\\ &\quad - \int_{\tau}^{0}\chi^{-1}(t,x)D_j\chi(t,x)\chi^{-1}(t,x)\omega(\varphi(t,x))\,dt.\end{aligned}$$

Finally, using (3.36) and expression (3.38), we obtain

$$\int_\tau^0 \left[(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\right]_j dt = D_j\chi^{-1}(\tau,x)g(\varphi(\tau,x)) + \int_\tau^0 \left[\chi^{-1}(t,x)D_j\big(\omega(\varphi(t,x))\big) + D_j\chi^{-1}(t,x)\omega(\varphi(t,x))\right]dt$$

from which (3.33) follows immediately.

We notice that, from (3.38) we get that $g(\varphi(\tau,x))$ is differentiable with respect to τ even if g is not. Moreover, let $\tilde{G}$ be the first term in the right hand side of (3.33). Then

$$\begin{aligned}\tilde{g}(\tau,x) &:= -\frac{d}{d\tau}\tilde{G}(\tau,x) = \frac{d}{d\tau}\left[D_j\chi^{-1}(\tau,x)g(\varphi(\tau,x))\right] \\ &= -\chi^{-1}(\tau,x)D_j\big(\mathcal{Q}(\varphi(\tau,x))\big)g(\varphi(\tau,x)) + D_j\chi^{-1}(\tau,x)\omega(\varphi(\tau,x)).\end{aligned} \tag{3.39}$$

Therefore differentiating with respect to τ both sides of (3.33):

$$D_j\left[\chi^{-1}(\tau,t)\omega(\varphi(\tau,x))\right] = \tilde{g}(\tau,x) + (\chi^1)^{-1}(\tau,x)\omega^1(\varphi(\tau,x)). \tag{3.40}$$

To prove the differentiability of g we need to check that $D_j\left[\chi^{-1}(\tau,t)\omega(\varphi(\tau,x))\right]$ is locally uniformly integrable with respect to x. In order to prove this fact and expression (3.31) for $\mathcal{S}(Dg(x))$ in Lemma 3.14, we prove the locally uniformly boundedness (with respect to x) by an integrable function of the right hand side of (3.40). Indeed, we have that $\omega^1 \in \mathcal{H}^{\nu-1+N}$ and that by Lemma 3.13, $B_{\mathcal{Q}^1} \geq B_{\mathcal{Q}} + A_{\mathbf{p}}$. We apply Lemma 3.12 with $\nu-1$, χ^1 and ω^1 instead of ν, χ and ω respectively and we obtain that the function

$$G^1(x) := \int_\infty^0 (\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))\,dt$$

belongs to $\mathcal{H}^\nu$ provided $\nu + \frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}} + \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}} > 0$. In fact, in the proof of Lemma 3.12 we checked that $(\chi^1)^{-1}(t,x)\omega^1(\varphi(t,x))$ is locally uniformly bounded with respect to x by an integrable function.

Now we deal with $\tilde{g}$. We first bound the first term in (3.39). Since $\mathcal{Q} \in \mathcal{H}^{N-1}$, there exists a constant $K > 0$ such that

$$\|D_j\big(\mathcal{Q}(\varphi(s,x))\big)\| \leq K\|\varphi(s,x)\|^{N-2}\|D_j\varphi(s,x)\|. \tag{3.41}$$

We recall that $D\varphi(\tau,x)$ is the fundamental solution of the linear system $\dot{\psi} = D\mathbf{p}(\varphi(\tau,x))\psi$ such that $D\varphi(0,x) = \mathrm{Id}$. Hence we apply Lemma 3.11 to $D\varphi$ to obtain:

$$\|D\varphi(\tau,x)\| \leq \frac{1}{\left(1 + d_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right)^{\alpha\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}}} \tag{3.42}$$

(compare definition of $A_{\mathbf{p}}$ and definition of $A_{\mathbf{Q}}$ in (3.3)). Using (3.42) and the bound of $\|\varphi(t,x)\|$ given by Lemma 3.10 in (3.41), we get

$$\|D_j\big(\mathcal{Q}(\varphi(s,x))\big)\| \le K\frac{\|x\|^{N-2}}{\big(1+a_{\mathbf{p}}(N-1)s\|x\|^{N-1}\big)^{\alpha((N-2)+\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}})}}. \tag{3.43}$$

By Lemma 3.12, $\|g(x)\| \le K\|x\|^{\nu+1}$ for some constant $K>0$. Using the bounds of $\|\chi^{-1}(t,x)\|$ and $\|\varphi(t,x)\|$ given by Lemmas 3.11 and 3.10 respectively, we obtain:

$$\|\chi^{-1}(\tau,x)D_j\big(\mathcal{Q}(\varphi(\tau,x))\big)g(\varphi(\tau,x))\| \le K\frac{\|x\|^{\nu+N-1}}{\big(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\big)^{\kappa_0}} \tag{3.44}$$

with $\kappa_0=\alpha\left(\nu+1+\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}+N-2+\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right)$ and $\kappa_0>1$ by hypothesis.

We deal with $\|D_j\chi^{-1}(\tau,x)\|$ for $\tau\ge 0$ and $x\in V_{\varrho_0}$. $D_j\chi^{-1}(\tau,x)$ is the solution of

$$\frac{d}{d\tau}D_j\chi^{-1}(\tau,x)=-D_j\chi^{-1}(\tau,x)\mathcal{Q}(\varphi(\tau,x))-\chi^{-1}(\tau,x)D_j\big(\mathcal{Q}(\varphi(\tau,x))\big)$$

satisfying the initial condition $D_j\chi^{-1}(0,x)=0$. We have then

$$\begin{aligned}D_j\chi^{-1}(\tau,x)&=-\left(\int_0^\tau \chi^{-1}(s,x)D_j\big(\mathcal{Q}(\varphi(s,x))\big)\chi(s,x)\,ds\right)\chi^{-1}(\tau,x)\\&=-\int_0^\tau \chi^{-1}(s,x)D_j\big(\mathcal{Q}(\varphi(s,x))\big)\chi^{-1}(\tau-s,\varphi(s,x))\,ds,\end{aligned} \tag{3.45}$$

where we have used (3.35) again.

For $\tau>s$, by Lemmas 3.10 and 3.11, a calculation (distinguishing the cases $B_{\mathcal{Q}}\ge 0$ and $B_{\mathcal{Q}}<0$) gives

$$\begin{aligned}\|\chi^{-1}(\tau-s,\varphi(s,x))\|\|\chi^{-1}(s,x)\| &\le \frac{\|\chi^{-1}(s,x)\|}{\big(1+c_{\mathbf{p}}(N-1)(\tau-s)\|\varphi(s,x)\|^{N-1}\big)^{\alpha\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}}}\\&\le \frac{1}{\big(1+c_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\big)^{\alpha\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}}}.\end{aligned} \tag{3.46}$$

Note that the bound is independent of s. If $d_{\mathbf{p}}\ne A_{\mathbf{p}}$, using bound (3.43) for $\|D_j\big(\mathcal{Q}(\varphi(s,x))\big)\|$:

$$\int_0^\tau D_j\big(\mathcal{Q}(\varphi(s,x))\big)\,ds \le K\|x\|^{-1}\big(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\big)^{\alpha\max\left\{0,1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}}.$$

Using previous computations for bounding the terms in formula (3.45), we obtain that

$$\|D_j\chi^{-1}(\tau,x)\| \le \frac{K\|x\|^{-1}}{\left(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right)^{\alpha\left(\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}-\max\left\{0,1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}\right)}}.$$

In addition, using that $\omega\in\mathcal{H}^{\nu+N}$ and the bound for $\|\varphi(t,x)\|$ in Lemma 3.10:

$$\|D_j\chi^{-1}(\tau,x)\omega(\varphi(\tau,x))\| \le K\|x\|^{\nu+N-1}\left(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right)^{-\kappa} \tag{3.47}$$

with $\kappa=\alpha\left(\nu+N+\frac{B_{\mathcal{Q}}}{c_{\mathbf{p}}}-\max\left\{0,1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}\right)$. By hypothesis $\kappa>1$. Also, $\kappa_0\ge\kappa$.

Now, to bound $\tilde{g}$ defined in (3.39), we use (3.44) and (3.47) and we get:

$$\|\tilde{g}(\tau,x)\| \le K\|x\|^{\nu+N-1}\left(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right)^{-\kappa}$$

which can be locally uniformly bounded with respect to x by an absolutely integrable function.

If $d_{\mathbf{p}}=A_{\mathbf{p}}$, an analogous argument leads to

$$\|\tilde{g}(\tau,x)\| \le K\|x\|^{\nu+N-1}\left(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right)^{-\kappa}\log\left(1+a_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\right).$$

Then g is differentiable and

$$D_jg(x)=\int_\infty^0 D_j\left(\chi^{-1}(t,x)\omega(\varphi(t,x))\right)dt.$$

Using (3.33), and the fact that $\lim_{\tau\to\infty}D_j\chi^{-1}(\tau,x)g(\varphi(\tau,x))=0$ we get (3.31).

Using again the homogeneity of g we extend the regularity properties of g from the domain V_ϱ to V. □

End of the proof of Theorem 3.2. Once Lemma 3.14 is proven, we can apply it to h with $\nu=\mathfrak{m}$, $\mathcal{Q}=\mathbf{Q}$ and $\omega=\mathbf{w}$ to deduce that h is $\mathcal{C}^1$. Then we are ready to prove that indeed h is a solution of (3.1). From the expression of h and the fact that V_ϱ is positively invariant by φ we can write

$$h(\varphi(s,x))=M(s,x)\int_\infty^s M^{-1}(t,x)\mathbf{w}(\varphi(t,x))\,dt, \qquad x\in V_\varrho,$$

where we have used (3.35) with $\chi=M$. Taking derivatives with respect to s we obtain

$$Dh(\varphi(s,x))\mathbf{p}(\varphi(s,x))=\mathbf{Q}(\varphi(s,x))h(\varphi(s,x))+\mathbf{w}(\varphi(s,x)) \tag{3.48}$$

and evaluating at $s=0$ we get (3.1).

It remains to check the higher regularity of h. Note that the analytic case follows directly from Lemma 3.12. For the differentiable case, we proceed by induction. Assume then that $\mathbf{p}$, $\mathbf{Q}$ and $\mathbf{w}$ are $\mathcal{C}^r$. Let $r_{\mathbf{p}}\le r$ be the degree of differentiability stated in Theorem 3.2 depending on the values of $B_{\mathbf{Q}}$, $A_{\mathbf{p}}$, $c_{\mathbf{p}}$ and $d_{\mathbf{p}}$.

We first introduce some notation. Let $\mathbf{Q}^0 = \mathbf{Q}$, $\mathbf{w}^0 = \mathbf{w}$, $H^0 = h$ and for $l \geq 1$

$$\mathbf{Q}^l(x) = \mathcal{B}_{\mathbf{Q}^{l-1}}(x) - \mathcal{I}_{D\mathbf{p}}^{n^{l-1}\cdot k}(x) = \mathrm{diag}(\mathbf{Q}^{l-1}(x), \dots, \mathbf{Q}^{l-1}(x)) - \mathcal{I}_{D\mathbf{p}}^{n^{l-1}\cdot k}(x),$$

where $\mathcal{B}_{\mathbf{Q}^{l-1}}$ and $\mathcal{I}_{D\mathbf{p}}^{n^{l-1}\cdot k}$ were defined in (3.25) and (3.26) respectively. We denote by $M^l(t,x)$ the fundamental matrix of

$$\frac{d}{dt}\psi = \mathbf{Q}^l(\varphi(t,x))\psi, \qquad \text{such that} \qquad M^l(0,x) = \mathrm{Id}\,.$$

In addition we set

$$\begin{aligned}
\mathbf{w}^l(x) &= \mathcal{S}(D\mathbf{w}^{l-1}(x)) + \Big((D_1\mathbf{Q}^{l-1}(x)H^{l-1}(x))^\top, \dots, (D_n\mathbf{Q}^{l-1}(x)H^{l-1}(x))^\top\Big)^\top, \\
H^l(x) &= \mathcal{S}(DH^{l-1}(x)),
\end{aligned}$$

provided the derivative exists, where the linear operator $\mathcal{S}$ is defined in (3.24). It is clear that

$$\mathbf{Q}^l(x) \in \mathcal{L}(\mathbb{R}^{n^l\cdot k}, \mathbb{R}^{n^l\cdot k}), \quad \mathbf{Q}^l \in \mathcal{H}^{N-1} \cap \mathcal{C}^{r-1}, \quad H^l(x) \in \mathbb{R}^{n^l\cdot k}, \quad \mathbf{w}^l(x) \in \mathbb{R}^{n^l\cdot k}.$$

We claim that for $0 \leq i \leq r_{\mathbf{p}}$ we have

$(a)_i$ $B_{\mathbf{Q}^i} \geq B_{\mathbf{Q}} + iA_{\mathbf{p}}$.
$(b)_i$ $\mathbf{w}^i \in \mathcal{H}^{\mathfrak{m}+N-i}$ and $\mathbf{w}^j \in \mathcal{C}^{i+1-j}$ for $0 \leq j \leq i$.
$(c)_i$ $H^i \in \mathcal{H}^{\mathfrak{m}+1-i}$, $H^j \in \mathcal{C}^{i-j}$ for $0 \leq j \leq i$ and

$$H^i(x) = \int_\infty^0 (M^i)^{-1}(t,x)\mathbf{w}^i(\varphi(t,x))\,dt. \tag{3.49}$$

We prove the claim by induction on i. The case $i = 0$ follows directly from the definitions and Lemma 3.12. Assume the claim holds for $i-1$, $1 \leq i \leq r_{\mathbf{p}} - 1$. Item $(a)_i$ follows from Lemma 3.13 applied to $\mathcal{Q} = \mathbf{Q}^i = \mathcal{B}_{\mathbf{Q}^{i-1}}(x) - \mathcal{I}_{D\mathbf{p}}^{n^{i-1}\cdot k}(x)$ which gives, together with the induction hypothesis $B_{\mathbf{Q}^i} \geq B_{\mathbf{Q}^{i-1}} + A_{\mathbf{p}} \geq B_{\mathbf{Q}} + iA_{\mathbf{p}}$.

Item $(b)_i$. Since, by the induction hypothesis, $\mathbf{w}^{i-1}$ is at least $\mathcal{C}^2$, from the definition of $\mathbf{w}^i$ we have that $\mathbf{w}^i \in \mathcal{H}^{\mathfrak{m}+N-i}$. From $j=0$, $\mathbf{w}^0 = \mathbf{w} \in \mathcal{C}^r \subset \mathcal{C}^{i+1}$. If $1 \leq j \leq i$, using $(b)_{i-1}$ and $(c)_{i-1}$,

$$\begin{aligned}
\mathbf{w}^j(x) &= \mathcal{S}(D\mathbf{w}^{j-1}(x)) + \Big((D_1\mathbf{Q}^{j-1}(x)H^{j-1}(x))^\top, \dots, (D_n\mathbf{Q}^{j-1}(x)H^{j-1}(x))^\top\Big)^\top \\
&\in \mathcal{C}^{i+1-j}.
\end{aligned}$$

Item $(c)_i$. We apply Lemma 3.14 with $\mathcal{Q} = \mathbf{Q}^{i-1}$, $\omega = \mathbf{w}^{i-1}$ and $\nu = \mathfrak{m} - i + 1$ so that $\mathcal{Q}^1 = \mathbf{Q}^i$, $\chi^1 = M^i$ and $\omega^1 = \mathbf{w}^i$. We have to check (3.30). For that we will use that $i \leq r_{\mathbf{p}}$ and (3.9). Let $c_{\mathbf{p}}^{i-1}$ be the constant $c_{\mathbf{p}}$ corresponding to $\mathbf{Q}^{i-1}$ (see definition (3.3)).

When $A_{\mathbf{p}} < d_{\mathbf{p}}$,

$$\nu + 1 + \frac{B_{\mathbf{Q}^{i-1}}}{c_{\mathbf{p}}^{i-1}} \geq \mathfrak{m} - i + 2 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} + (i-1)\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}} > 1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}} > 0.$$

When $A_{\mathbf{p}} \geq d_{\mathbf{p}}$,

$$\nu + 1 + \frac{B_{\mathbf{Q}^{i-1}}}{c_{\mathbf{p}}} \geq \mathfrak{m} - i + 2 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} + (i-1)\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}} > (i-1)\left(\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}} - 1\right) \geq 0.$$

Then $H^{i-1} \in \mathcal{C}^1$ and $H^i = \mathcal{S}\big(DH^{i-1}(x)\big)$ can be written as (3.49). Therefore, by the definition of H^j, $H^j \in \mathcal{C}^{i-j}$, $0 \leq j \leq i$, and the claim is proven.

As a consequence of the claim, we have that $h \in \mathcal{C}^{r_{\mathbf{p}}}$ in V_ϱ in all cases. By the homogeneity we extend the regularity from V_ϱ to V. When $A_{\mathbf{p}} \geq b_{\mathbf{p}}$, if $r = \infty$, we also obtain $h \in \mathcal{C}^\infty$. $\square$

Proof of Corollary 3.5. Assume that we have a homogeneous solution $h \in \mathcal{H}^\nu$ of equation (3.1). Then, it has to satisfy the ordinary differential equation (3.48) so that

$$M^{-1}(t,x)h(\varphi(t,x)) = h(x) + \int_0^t M^{-1}(s,x)\mathbf{w}(\varphi(s,x))\,ds.$$

Since $h \in \mathcal{H}^\nu$, by Lemmas 3.10 and 3.11,

$$\|M^{-1}(t,x)h(\varphi(t,x))\| \leq \big(1 + a_{\mathbf{p}}(N-1)t\|x\|^{N-1}\big)^{-\alpha\left(\frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}}+\nu\right)}$$

which is bounded as $t \to \infty$ provided $B_{\mathbf{Q}}/c_{\mathbf{p}} + \nu \geq 0$. Thus, the result is proven. $\square$

4. Proof of Theorems 2.2 and 2.9

As we will see in Section 4.5 below, Theorem 2.9 can be deduced following the same lines as Theorem 2.2. For that reason we first focus on the maps case.

We first notice that, for R such that $R(x) - (x + p(x,0)) \in \mathcal{H}^{\geq N+1}$ then, by Lemma 3.8, $R(V_\varrho) \subset V_\varrho$ (taking ϱ slightly smaller if necessary). Hence, if the domain of K is V_ϱ (as we will see), the composition $K \circ R$ is always well defined. Moreover, for K such that $K(x) - (x,0) \in \mathcal{H}^{\geq 2}$, if $x \in V_\varrho$ then $K(x) \in U$ and consequently $F \circ K$ is well defined as well.

For h such that its projections have different orders, we will write $h \in \mathcal{H}^{\geq l_1} \times \mathcal{H}^{\geq l_2}$ if $h_x \in \mathcal{H}^{\geq l_1}$ and $h_y \in \mathcal{H}^{\geq l_2}$. We will use the same notation for the spaces $\mathcal{H}^{>l}$ and $\mathcal{H}^l$.

4.1. Preliminaries of the induction procedure: the cohomological equations

Given $N \leq \ell \leq r$ and $j \in \mathbb{N}$ such that $1 \leq j \leq \ell - N + 1$ we proceed by induction over j to prove first that there exist $K^{\leq j}$ and $R^{\leq j+N-1}$ of the form

$$K^{\leq j}(x) = \sum_{l=1}^{j} K^l(x), \qquad R^{\leq j+N-1}(x) = x + \sum_{l=N}^{j+N-1} R^l(x), \tag{4.1}$$

with $K^1(x) = (x, 0)^\top$ and $R^N(x) = p(x, 0)$, satisfying

$$E^{>j} := F \circ K^{\leq j} - K^{\leq j} \circ R^{\leq j+N-1} = (E_x^{>j}, E_y^{>j}) \in \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}. \tag{4.2}$$

Concerning property (2.5) in Theorem 2.2, if $L = N$, it is a consequence of (4.2) taking $j = \ell - N + 1$. If $L = M < N$, we have to perform an extra induction procedure for values of j such that $\ell - N + 2 \leq j \leq \ell - L + 1$.

The case $j = 1$ follows immediately taking $K^{\leq 1}(x) = (x, 0)^\top$ and $R^{\leq N}(x) = x + p(x, 0)$. Indeed:

$$\begin{aligned} E_x^{>1}(x) &= x + p(x,0) + f(x,0) - R^{\leq N}(x) = f(x,0) \in \mathcal{H}^{\geq N+1} \subset \mathcal{H}^{>N}, \\ E_y^{>1}(x) &= g(x,0) \in \mathcal{H}^{\geq M+1} \subset \mathcal{H}^{>L}, \end{aligned}$$

where we have used that, by hypothesis H2, $q(x, 0) = 0$.

Suppose that (4.2) holds true for $j - 1 \geq 1$, $K^{\leq j-1}$ and $R^{\leq j+N-2}$. We will find the condition that $K^j \in \mathcal{H}^j$ and $R^{j+N-1} \in \mathcal{H}^{j+N-1}$ have to satisfy in order to ensure that (4.2) holds for j, $K^{\leq j} = K^{\leq j-1} + K^j$ and $R^{\leq j+N-1} = R^{\leq j+N-2} + R^{j+N-1}$.

We claim that, since $j - 1 + N \leq \ell \leq r$, there exists $E = (E_x^{j+N-1}, E_y^{j+L-1})$ with $E_x^{j+N-1} \in \mathcal{H}^{j+N-1}$ and $E_y^{j+L-1} \in \mathcal{H}^{j+L-1}$ such that

$$E_x^{>j-1} - E_x^{j+N-1} \in \mathcal{H}^{>j+N-1}, \qquad E_y^{>j-1} - E_y^{j+L-1} \in \mathcal{H}^{>j+L-1}. \tag{4.3}$$

Indeed, by Taylor's theorem

$$\begin{aligned} F_x(x,y) &= x + p(x,y) + F_x^{N+1}(x,y) + \cdots + F_x^r(x,y) + F_x^{>r}(x,y), \\ F_y(x,y) &= y + q(x,y) + F_y^{M+1}(x,y) + \cdots + F_y^r(x,y) + F_y^{>r}(x,y), \end{aligned} \tag{4.4}$$

with $F_x^l, F_y^l \in \mathcal{H}^l$ and $F_x^{>r}, F_y^{>r} \in \mathcal{H}^{>r}$. Moreover, $K^{\leq j-1}$ and $R^{\leq j+N-2}$ are sums of homogeneous functions. By the induction hypothesis it is easily checked that

$$\begin{aligned} E_x^{>j-1} &= F_x \circ K^{\leq j-1} - K_x^{\leq j-1} \circ R^{\leq j+N-2} = E_x^{j+N-1} + \widehat{E}_x^{>j}, \\ E_y^{>j-1} &= F_y \circ K^{\leq j-1} - K_y^{\leq j-1} \circ R^{\leq j+N-2} = E_y^{j+L-1} + \widehat{E}_y^{>j} \end{aligned}$$

with $E_{x,y}^l \in \mathcal{H}^l$ and $\widehat{E}_x^{>j} \in \mathcal{H}^{>j+N-1}$, $\widehat{E}_y^{>j} \in \mathcal{H}^{>j+L-1}$ and hence (4.3) is satisfied. We decompose $F \circ K^{\leq j} - K^{\leq j} \circ R^{\leq j+N-1}$ as

$$\begin{aligned} F \circ K^{\leq j} - K^{\leq j} \circ R^{\leq j+N-1} =& E^{>j-1} + \left[F \circ K^{\leq j} - F \circ K^{\leq j-1} - DF(K^{\leq j-1}) \cdot K^j\right] \\ &+ DF(K^{\leq j-1}) \cdot K^j - K^j \circ R^{\leq j+N-2} \\ &- \left[K^{\leq j} \circ R^{\leq j+N-1} - K^{\leq j} \circ R^{\leq j+N-2}\right]. \end{aligned}$$

Next we study each term of the above decomposition. In doing that we introduce several new remainders e_i. By Taylor's theorem, and using that $j - 1 \geq 1$,

$$e_1 := F \circ K^{\le j} - F \circ K^{\le j-1} - DF(K^{\le j-1}) \cdot K^j \in \mathcal{H}^{\ge N-2+2j} \times \mathcal{H}^{\ge M-2+2j}$$
$$\subset \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}.$$

We denote $\iota(x) = (x, 0)$. Taking into account that $K^{\le j-1} - \iota \in \mathcal{H}^{\ge 2}$ we can write

$$DF(K^{\le j-1}) \cdot K^j = DF \circ \iota \cdot K^j + e_2 = \begin{pmatrix} [\mathrm{Id} + D_x p \circ \iota] \cdot K_x^j + D_y p \circ \iota \cdot K_y^j \\ [\mathrm{Id} + D_y q \circ \iota] \cdot K_y^j \end{pmatrix} + e_2,$$

with $e_2 \in \mathcal{H}^{\ge j+N} \times \mathcal{H}^{\ge j+M} \subset \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}$. Since $R^{\le j+N-2}(x) - x - p(x,0) \in \mathcal{H}^{\ge N+1}$ and $N \ge 2$,

$$K^j \circ R^{\le j+N-2}(x) = K^j(x) + DK^j(x) \cdot p(x,0) + e_3(x)$$

with $e_3 \in \mathcal{H}^{\ge j-2+2N} \cup \mathcal{H}^{\ge j+N} \subset \mathcal{H}^{>j+N-1}$. Finally

$$\begin{aligned} K^{\le j} \circ R^{\le j+N-1} - K^{\le j} \circ R^{\le j+N-2} &= DK^{\le j}(R^{\le j+N-2}) \cdot R^{j+N-1} + e_4 \\ &= \begin{pmatrix} R^{j+N-1} \\ 0 \end{pmatrix} + e_5 + e_4, \end{aligned}$$

where $e_4 \in \mathcal{H}^{\ge 2(j+N-1)} \subset \mathcal{H}^{>j+N-1}$ and $e_5 \in \mathcal{H}^{\ge j+N} \subset \mathcal{H}^{>j+N-1}$.

In conclusion, $e_l \in \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}$ for $l = 1, \cdots, 5$. Using (4.3) and the previous computations, we have that

$$\begin{aligned} &F \circ K^{\le j} - K^{\le j} \circ R^{\le j+N-1} \\ &\quad = \begin{pmatrix} E_x^{j+N-1} \\ E_y^{j+L-1} \end{pmatrix} + \begin{pmatrix} D_x p \circ \iota \cdot K_x^j + D_y p \circ \iota \cdot K_y^j - R^{j+N-1} \\ D_y q \circ \iota \cdot K_y^j \end{pmatrix} - DK^j \cdot p \circ \iota + \tilde{E}^{>j}, \end{aligned}$$

where $\tilde{E}^{>j} = E^{>j-1} - (E_x^{j+N-1}, E_y^{j+L-1})^\top + e_1 + e_2 - e_3 - e_4 - e_5 \in \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}$.

In order to get property (4.2) for j, we have to choose $K^j \in \mathcal{H}^j$ and $R^{j+N-1} \in \mathcal{H}^{j+N-1}$ such that

$$DK_x^j(x) \cdot p(x,0) - D_x p(x,0) \cdot K_x^j(x) - D_y p(x,0) \cdot K_y^j(x) + R^{j+N-1}(x) = E_x^{j+N-1}(x) \quad (4.5)$$

and, taking into account that M and N may be different,

$$DK_y^j(x) \cdot p(x,0) - D_y q(x,0) \cdot K_y^j(x) - E_y^{j+L-1}(x) \in \mathcal{H}^{>j+L-1}. \quad (4.6)$$

As usual in the parametrization method we have a lot of freedom to choose solutions of the above equations. On the one hand, we expect that equation (4.6) for K_y^j has a unique homogeneous solution. On the other hand, it is clear that equation (4.5) for K_x^j and R^{j+N-1} admits several homogenous solutions. Despite the fact that we could solve first (4.6) for $K_y^j \in \mathcal{H}^j$ and then, take $K_x^j \equiv 0$ and

$$R^{j+N-1}(x) = E_x^{j+N-1}(x) + D_y p(x,0) \cdot K_y^j(x)$$

to solve (4.5), we are also interested in looking for the simplest representation of the dynamics on the stable manifold, that is, we ask $R^{\le j+N-1}$ to be as simple as possible, for instance taking $R^{j+N-1} = 0$ if we can solve the following equation

$$DK_x^j(x) \cdot p(x,0) - D_x p(x,0) \cdot K_x^j(x) = E_x^{j+N-1}(x) + D_y p(x,0) \cdot K_y^j(x). \tag{4.7}$$

We distinguish three cases to obtain an equation for K_y^j so that condition (4.6) holds:

- If $N < M$, then condition (4.6) is satisfied if

$$DK_y^j(x) \cdot p(x,0) = E_y^{j+L-1}(x). \tag{4.8}$$

- If $N = M$,

$$DK_y^j(x) \cdot p(x,0) - D_y q(x,0) \cdot K_y^j(x) = E_y^{j+L-1}(x). \tag{4.9}$$

- If $N > M$, then we get an algebraic equation:

$$-D_y q(x,0) \cdot K_y^j(x) = E_y^{j+L-1}(x) \tag{4.10}$$

  which can be solved by using that, by hypothesis H2, $D_y q(x,0)$ is invertible. We also have that $[D_y q(x,0)]^{-1} \in \mathcal{H}^{-M+1}$. This equation clearly illustrates the fact that the solutions K^j are not necessarily polynomials.

Assume that we are able to find appropriate solutions K_x^j of equation (4.7) and K_y^j of (4.8), (4.9) or (4.10). We recall that we were dealing with values of $j = 2, \cdots, \ell - N + 1$. When $L = N \le M$, (2.5) and (2.6) follows from (4.2) by taking $j = \ell - N + 1$ so in this case we are done. However, in the case $L = M < N$ we also have to deal with the equation for K^j when $j = \ell - N + 2, \cdots, \ell - L + 1$. That is, we need to add some extra homogeneous terms to K_y to obtain (2.5) and (2.6). Indeed, for any given ℓ, assume that $K^{\le \ell - N+1}$, $R^{\le \ell}$ are of the form (4.1) and they satisfy (4.2) for $j = \ell - N + 1$. We prove by induction on j that, for any $\ell - N + 2 \le j \le \ell - L + 1$, we can find

$$K^{\le j} = K^{\le \ell - N+1} + \sum_{l=\ell-N+2}^{j} K^l, \qquad K^l \in \mathcal{H}^l, \quad \text{with} \quad K_x^l \equiv 0$$

in such a way that $E^{>j} = F \circ K^{\le j} - K^{\le j} \circ R^{\le \ell} \in \mathcal{H}^{>\ell} \times \mathcal{H}^{>j+L-1}$.

Assume that the result holds for $j-1$. Then, since $j + L - 1 \le \ell \le r$, decomposition (4.3) of $E_y^{>j-1}$ is also true in this case. Taking K_x^j, $R^{j+N-1} \equiv 0$ in the above computations we also have that

$$F_y \circ K^{\le j} - K_y^{\le j} \circ R = -DK_y^j \cdot p \circ \iota + D_y q \circ \iota \cdot K_y^j + E_y^{j+L-1} + \tilde{E}_y^{>j}$$

with $E_y^{j+L-1} \in \mathcal{H}^{j+L-1}$, $\tilde{E}_y^{>j} \in \mathcal{H}^{>j+L-1}$ and

$$E_x^{>j} = F_x \circ K^{\leq j} - K_x^{\leq j} \circ R = D_y p \circ \iota \cdot K_y^j + \tilde{E}_x^{>j},$$

with $\tilde{E}_x^{>j} \in \mathcal{H}^{>j+N-1} \subset \mathcal{H}^{>\ell}$.

Since $M < N$, if $K_y^j \in \mathcal{H}^j$ and satisfies the equation

$$D_y q(x,0) \cdot K_y^j(x) = -E_y^{j+L-1}(x),$$

then $D_y p \circ \iota \cdot K_y^j \in \mathcal{H}^{\geq j+N-1} \subset \mathcal{H}^{>\ell}$ and $E_x^{>j} \in \mathcal{H}^{>\ell}$. Therefore, we can follow this procedure $N - L$ times until (4.3) holds true. After that, the order of the remainder $E^{>\ell-L+1}$ will be ℓ and K_x will have the form given in Theorem 2.2 and property (2.5) will be satisfied.

We remark that the equation for K_y^j, $j = \ell - N + 2, \cdots, \ell - L + 1$, is the same algebraic equation (4.10) as the one corresponding to $j = 2, \cdots, \ell - N + 1$.

4.2. *Resolution of the linear equations (4.8)-(4.10) for K_y^j*

We take $2 \leq j \leq \ell - L + 1$. In the case $M < N$, K_y^j is a solution of the algebraic equation (4.10). Since $D_y q(x,0)$ is invertible, the unique solution of this equation is

$$K_y^j(x) = -\big(D_y q(x,0)\big)^{-1} E_y^{j+L-1}(x).$$

Clearly, K_y^j is a homogeneous function of order j which is analytic in V. Nevertheless, it is only $j - 1$ times differentiable at the origin according to Definition 2.1.

Let $M \geq N$. In this case K_y^j has to satisfy either equation (4.8), if $N < M$, or (4.9), if $N = M$. We write them in a unified way as

$$DK_y^j(x) \cdot p(x,0) - \mathbf{Q}(x) \cdot K_y^j(x) = E_y^{j+L-1}(x),$$

where $\mathbf{Q}(x) = 0$ if $N < M$ and $\mathbf{Q}(x) = D_y q(x,0)$ if $M = N$. Hence this case follows from Theorem 3.2 taking $\mathbf{p}(x) = p(x,0)$ and $\mathbf{Q}$ as indicated. We claim that under the current hypotheses, $\mathbf{p}$ and $\mathbf{Q}$ satisfy the conditions of Theorem 3.2. Indeed, the constants $A_{\mathbf{p}}, a_{\mathbf{p}}, b_{\mathbf{p}}$ in Theorem 3.2 are

$$a_{\mathbf{p}} = a_p > 0 \quad \text{(by H1)}, \qquad b_{\mathbf{p}} = b_p > 0 \quad \text{(by definition)} \qquad A_{\mathbf{p}} = A_p.$$

As for $B_{\mathbf{Q}}$, by definition (2.3), if $M > N$, $B_{\mathbf{Q}} = 0$. If $M = N$, $B_{\mathbf{Q}} = B_q$ and by hypotheses H1 and H2 the condition $j + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\left\{1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}, 0\right\}$ is satisfied in both cases. Then Theorem 3.2 provides a solution $K_y^j \in \mathcal{H}^j$ for $2 \leq j \leq \ell - L + 1$.

4.3. *Resolution of the linear equation (4.5) for K_x^j*

Consider $2 \leq j \leq \ell - N + 1$. We have to find K_x^j satisfying equation (4.5) which we recall here:

$$DK_x^j(x) \cdot p(x,0) - D_x p(x,0) \cdot K_x^j(x) + R^{j+N-1}(x) = E_x^{j+N-1}(x) + D_y p(x,0) \cdot K_y^j(x)$$

being E_x^{j+N-1} a homogenous function of order $j+N-1$ and $K_y^j \in \mathcal{H}^j$ the solution of the linear equation considered in Section 4.2. Since $D_y p \circ \iota \cdot K_y^j \in \mathcal{H}^{j+N-1}$ we can add this term to E_x^{j+N-1} and denote the resulting term again by E_x^{j+N-1} and hence we end up with equation

$$DK_x^j(x)\cdot p(x,0) - D_x p(x,0)\cdot K_x^j(x) + R^{j+N-1}(x) = E_x^{j+N-1}(x). \tag{4.11}$$

As we mentioned in Section 4.1, to solve (4.11), one possibility is to take K_x^j as any function in $\mathcal{H}^j$ and R^{j+N-1} as the solution of the resulting equation. If we proceed in this form, we are always able to solve the equation, but we do not have a normal form result for R in the sense that R is not simple at all. In the other extreme, we can try to choose $R^{j+N-1}=0$ and use Theorem 3.2 with $\mathbf{p}(x)=p(x,0)$ and $\mathbf{Q}(x)=D_x p(x,0)$ to solve

$$DK_x^j(x)\cdot p(x,0) - D_x p(x,0)\cdot K_x^j(x) = E_x^{j+N-1}(x), \quad \text{for } K_x^j. \tag{4.12}$$

However, this equation may not have solutions if j is not large enough. Indeed, in this case, since $\mathbf{p}(x)=p(x,0)$ and $\mathbf{Q}(x)=D_x p(x,0)$, by hypothesis H1 and Lemma 3.6 the constant $B_{\mathbf{Q}} = -B_p \le -Na_p < 0$ and hence equation (4.12) cannot be solved unless j is large enough. Concretely, the sufficient condition to have solutions is $j - \frac{B_p}{a_p} > \max\left\{1-\frac{A_p}{d_p}, 0\right\}$. Therefore, if $j\in\mathbb{N}$ satisfies

$$j > \frac{B_p}{a_p} + \max\left\{1-\frac{A_p}{d_p}, 0\right\},$$

equation (4.12) has a unique homogeneous solution $K_x^j \in \mathcal{H}^j$.

In conclusion, if $j > \frac{B_p}{a_p} + \max\{1-\frac{A_p}{d_p}, 0\}$, we take $R^{j+N-1}\equiv 0$ and K_x^j a homogeneous solution of (4.12). Otherwise, K_x^j is free and we take as R^{j+N-1} the solution of (4.11).

4.4. Regularity of K^j and R^{j+N-1}

When $A_p > d_p$, since (E_x^{N+1}, E_y^{M+1}) is analytic, from Theorem 3.2, K^2 is an analytic function in V and consequently, by induction K^j is also analytic.

If $M<N$, we solve equation (4.11) for $j=2$ by taking $K_x^2\equiv 0$ and $R^{N+1}=E_x^{N+1}$. Hence K^2 is analytic, since (E_x^{N+1}, E_y^{M+1}) is analytic. Then by induction K^j is also analytic provided we solve equation (4.11) in some appropriate way, for instance, by taking $K_x^j\equiv 0$ and $R^{j+N-1} = E_x^{j+N-1}$.

In the case $A_p = d_p$ and $M\ge N$, even if (E_x^{N+1}, E_y^{M+1}) is analytic, Theorem 3.2 only provides $\mathcal{C}^\infty$ solutions in V. Consequently, K^2 is only $\mathcal{C}^\infty$ and inductively we obtain that K^j is $\mathcal{C}^\infty$.

Finally we consider the case $A_p < d_p$ and $M\ge N$, where we lose regularity. Concretely, $K_y^2 \in C^{r_*}$, on V where r_*, given in (2.4), is the maximum integer k such that

$$\left(1-\frac{A_p}{d_p}\right)k < 2 + \frac{B_q}{c_p}.$$

In order to deal with the cases $M = N$ and $M > N$ jointly, from now on we understand that $B_q = 0$ if $M > N$. Recall that $K_x^j \equiv 0$ for $j = 2, \dots, \ell_* - N + 1$ so that the differentiability of K_y^j for these values of j only depends on the smoothness of K_y^i, for $i = 2, \cdots, j-1$, which is r_* by induction.

When $j = \ell_* - N + 2$, from Theorem 3.2, we have that $K_x^{\ell_* - N+2} \in \mathcal{C}^{r_x}$, with r_x the maximum integer k satisfying

$$\left(1 - \frac{A_p}{d_p}\right) k < \ell_* - N + 2 - \frac{B_p}{c_p}.$$

The maximum differentiability, r_*, is obtained by choosing ℓ_* to be the smallest integer satisfying

$$\ell_* > \left(1 - \frac{A_p}{d_p}\right) r_* + N - 2 + \frac{B_p}{c_p}$$

which justifies the definition (2.7) of ℓ_* in the present case.

By induction one checks that $K^j = (K_x^j, K_y^j)$ is also a $\mathcal{C}^{r_*}$ function.

By construction, R^{j+N-1} has the same regularity in all cases.

4.5. *The flow case. Proof of Theorem 2.9 without parameters*

In the case of flows we have to find $K^{\le j}(x,t)$ and $Y^{\le j+N-1}(x)$ of the form

$$K^{\le j}(x,t) = \sum_{l=1}^{j} K^{(l)}(x,t), \qquad Y^{\le j+N-1}(x) = \sum_{l=N}^{j+N-1} Y^l(x)$$

being $K_1(x,t) = (x,0)^\top$, $Y^N(x) = p(x,0)$. For technical reasons, we look for $K^{(l)}$ as a sum of two homogeneous functions: one of degree l independent of t and the other belonging to $\mathcal{H}^{>l+N-1} \times \mathcal{H}^{>l+L-1}$. The homogeneous terms K^l in the statement of the theorem are obtained by rearranging the sum above. $K^{\le j}$ have to satisfy the invariance condition (2.13) up to some order j in the sense that the error term

$$E^{>j}(x,t) := X(K^{\le j}(x,t),t) - DK^{\le j}(x,t) Y^{\le j+N-1}(x) - \partial_t K^{\le j}(x,t)$$

satisfies

$$E^{>j} = (E_x^{>j}, E_y^{>j}) \in \mathcal{H}^{>j+N-1} \times \mathcal{H}^{>j+L-1}. \tag{4.13}$$

As we have noticed in Section 4.1, in the case $L = N$ condition (4.13) implies (2.14).

Following the same induction arguments as in Section 4.1 we obtain that Y^j and $K^{(j)} = (K_x^{(j)}, K_y^{(j)})$ have to satisfy the conditions:

$$\begin{aligned} DK_x^{(j)}(x,t)p(x,0) - D_x p(x,0) K_x^{(j)}(x,t) - D_y p(x,0) K_y^{(j)}(x,t) \\ + Y^{j+N-1}(x) + \partial_t K_x^{(j)}(x,t) - E_x^{j+N-1}(x,t) \in \mathcal{H}^{>j+N-1}, \end{aligned} \tag{4.14}$$

and

$$DK_y^{(j)}(x,t)p(x,0) - D_y q(x,0)K_y^{(j)}(x,t) + \partial_t K_y^{(j)}(x,t) - E_y^{j+L-1}(x,t) \in \mathcal{H}^{>j+L-1} \quad (4.15)$$

which are the analogous in the case of flows for (4.5) and (4.6) respectively. We will skip the computations which are pretty similar as the ones in the previous section. However we will not ask $K_x^{(j)}$, $K_y^{(j)}$ to satisfy their corresponding partial differential equation (vanishing the terms E_x^{j+N-1} and E_y^{j+L-1}) but we allow them to include new terms of higher order.

With this strategy in mind, we are going to explain how to solve these equations. For a given T-periodic function h, we denote by $\overline{h}$ its average and by $\widetilde{h} = h - \overline{h}$ its oscillatory part (with zero average). Clearly, since we look for $K^{(j)}$ periodic, one choice is to ask that the average $\overline{K^{(j)}}$ satisfies the equations

$$\begin{aligned} D\overline{K_x^{(j)}}(x)p(x,0) - D_x p(x,0)\overline{K_x^{(j)}}(x) - D_y p(x,0)\overline{K_y^{(j)}}(x) \\ +Y^{j+N-1}(x) - \overline{E_x^{j+N-1}}(x) = 0, \\ D\overline{K_y^{(j)}}(x)p(x,0) - D_y q(x,0)\overline{K_y^{(j)}}(x) - \overline{E_y^{j+L-1}}(x) \in \mathcal{H}^{>j+L-1}. \end{aligned} \quad (4.16)$$

We can solve equations (4.16) as in the map case, following the arguments in Sections 4.2 and 4.3 for solving equations (4.5) and (4.6). Concerning regularity, the arguments in Section 4.4 leads to the same regularity as in the map case for the average of $K^{(j)}$ and Y^j. As a conclusion, we have solutions of equations (4.16) $\overline{K^{(j)}}$ and Y^{j+N-1} belonging to $\mathcal{H}^j$ and $\mathcal{H}^{j+N-1}$ respectively.

We take the oscillatory part $\widetilde{K^{(j)}}$ with zero average and satisfying

$$\partial_t \widetilde{K^{(j)}}(x,t) = (\widetilde{E_x^{j+N-1}}(x,t), \widetilde{E_y^{j+L-1}}(x,t)). \quad (4.17)$$

Consequently, $\widetilde{K^{(j)}} \in \mathcal{H}^{j+N-1} \times \mathcal{H}^{j+L-1}$.

It only remains to see that $K^{(j)} = \overline{K^{(j)}} + \widetilde{K^{(j)}}$ and Y^{j+N-1} satisfy equations (4.14) and (4.15). Indeed, when we compute the left-hand side of, for instance, equation (4.14) we obtain

$$D\widetilde{K_x^{(j)}}(x,t)p(x,0) - D_x p(x,0)\widetilde{K_x^{(j)}}(x,t) - D_y p(x,0)\widetilde{K_y^{(j)}}(x,t)$$

which belongs to $\mathcal{H}^{j+L-1+N-1} \subset \mathcal{H}^{>j+N-1}$ since $L \geq 2$. Analogously for equation (4.15). Therefore, we conclude that $K^{(j)} = \overline{K^{(j)}} + \widetilde{K^{(j)}}$ and Y^{j+N-1} satisfy equations (4.14) and (4.15) and then (4.13) is satisfied.

The regularity of the oscillatory part follows from the fact that it satisfies equation (4.17).

As in Section 4.1, if $L = N$, we are done. The case $L = M$ needs an extra argument which is totally analogous to the one in Section 4.1.

Remark 4.1. The vector field Y can be chosen independent of t. This is due to the fact that we can perform the averaging procedure so that for any given ℓ we can move the dependence on t of the vector field X up to order $\|z\|^\ell$. If we take $\ell \geq \ell_* + 1$, then the formal procedure is independent of t and we obtain (for the averaged vector field $\overline{X}$) a parametrization $\overline{K}^{\leq}$ and a vector field $\overline{Y}$ satisfying the invariance condition (2.13) up to order $\|x\|^\ell$ which do not depend on t.

Nevertheless we can add t-depending terms to Y in order to have a more simple K_x.

5. Dependence on parameters. Proof of Theorems 2.8 and 2.9

In this section we prove Theorems 2.8 and 2.9 which give us the dependence on parameters of the functions K and R given in Theorem 2.2 as a sum of homogeneous functions.

We first emphasize that the methodology developed in Section 4 can also be applied in the parametric case so that the cohomological equations (4.5) and (4.6) for K^j are the same in this context but involving the dependence on λ. For a given value of the parameter λ, the discussion about how to solve the cohomological equations for K_y^j distinguishing the different cases ($N > M$, $N = M$ and $N < M$) and the different strategies to solve the cohomological equations for K_x^j are also valid. Therefore, even in the parametric case, the existence of $K^j(\cdot, \lambda)$ is already proven. Next we study the regularity with respect to λ both for maps and flows.

5.1. The cohomological equation in the parametric case

The case $N \geq M$, can be treated by using the auxiliary equation (3.1) studied in Section 3. See the strategy of how to proceed in Sections 4.2 and 4.3. As a consequence, we are lead to deal with the dependence on parameters of the homogeneous solution h

$$h(x, \lambda) = \int_{\infty}^{0} M^{-1}(t, x, \lambda)\mathbf{w}(\varphi(t, x, \lambda), \lambda)\, dt$$

of the auxiliary equation:

$$D_x h(x, \lambda)\mathbf{p}(x, \lambda) - \mathbf{Q}(x, \lambda)h(x, \lambda) = \mathbf{w}(x, \lambda), \tag{5.1}$$

given by Theorem 3.2 for any $\lambda \in \Lambda$, where $\mathbf{p}$, $\mathbf{Q}$ and $\mathbf{w}$ are homogeneous functions of degree N, $N-1$, $\mathfrak{m}+N$ respectively. We will write $\mathbf{p} \in \mathcal{H}^N$, $\mathbf{Q} \in \mathcal{H}^{N-1}$ and $\mathbf{w} \in \mathcal{H}^{\mathfrak{m}+N}$.

In this setting, the constants defined in (3.3), HP1 and HP2 depend on λ. We denote them by $A_{\mathbf{p}}^\lambda$, $A_{\mathbf{Q}}^\lambda$, $B_{\mathbf{Q}}^\lambda$, $a_V^{\mathbf{p},\lambda}$, $a_{\mathbf{p}}^\lambda$, $b_{\mathbf{p}}^\lambda$, $c_{\mathbf{p}}^\lambda$ and $d_{\mathbf{p}}^\lambda$. In order to obtain uniform bounds with respect to λ we redefine

$$\begin{aligned} a_{\mathbf{p}} &= \inf_{\lambda\in\Lambda} a_{\mathbf{p}}^\lambda, & b_{\mathbf{p}} &= \sup_{\lambda\in\Lambda} b_{\mathbf{p}}^\lambda, & A_{\mathbf{p}} &= \inf_{\lambda\in\Lambda} A_{\mathbf{p}}^\lambda, \\ B_{\mathbf{Q}} &= \inf_{\lambda\in\Lambda} B_{\mathbf{Q}}^\lambda, & A_{\mathbf{Q}} &= \sup_{\lambda\in\Lambda} A_{\mathbf{Q}}^\lambda, & a_V^{\mathbf{p}} &= \inf_{\lambda\in\Lambda} a_V^{\mathbf{p},\lambda} \end{aligned} \tag{5.2}$$

and $c_{\mathbf{p}}$, $d_{\mathbf{p}}$ as in (3.3). Notice that, with this definition of the constants, all the bounds in Section 3 will be also true uniformly for any $\lambda \in \Lambda$.

To study equation (5.1), we will assume the following:

HPλ Hypotheses HP1 and HP2 hold true for $a_{\mathbf{p}}$, $a_V^{\mathbf{p}}$ defined in (5.2).

To deal with the analytic case, for $\gamma > 0$, we define the complex extension of Λ

$$\Lambda(\gamma) = \{\lambda \,:\, \operatorname{Re}\lambda \in \Lambda,\ \|\operatorname{Im}\lambda\| < \gamma\}.$$

Lemma 5.1. *Let $\mathbf{p} \in \mathcal{H}^N$, $\mathbf{Q} \in \mathcal{H}^{N-1}$ and $\mathbf{w} \in \mathcal{H}^{\mathfrak{m}+N}$. Assume that $\mathbf{p}, \mathbf{Q}, \mathbf{w} \in \mathcal{C}^{\Sigma_{s,r}}$ and that $\mathbf{p}$ satisfies hypothesis HPλ for $\varrho_0 > 0$.*

Then, if

$$\mathfrak{m} + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\left\{1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}, 0\right\},$$

the solution $h : V \times \Lambda \to \mathbb{R}^k$ of (5.1) provided by Theorem 3.2 satisfies $h \in \mathcal{H}^{\mathfrak{m}+1}$ and we have the regularity results according to the cases:

(1) $A_{\mathbf{p}} \geq d_{\mathbf{p}}$. *If $1 \leq r \leq \infty$, then $h \in \mathcal{C}^{\Sigma_{s,r}}$ in $V \times \Lambda$.*
(2) $A_{\mathbf{p}} < d_{\mathbf{p}}$. *Let κ_0 be the maximum of $1 \leq i \leq r + s$ such that*

$$\mathfrak{m} + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} - i\left(1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right) > 0.$$

Then $h \in \mathcal{C}^{\Sigma_{s_0,r_0}}$ in $V \times \Lambda$ with $s_0 = \min\{s, \kappa_0\}$ and $r_0 = \kappa_0 - s_0$.
(3) $A_{\mathbf{p}} > d_{\mathbf{p}}$. *If $\mathbf{p}, \mathbf{Q}, \mathbf{w}$ are real analytic functions in $\Omega(\gamma_0) \times \Lambda(\gamma_0^2)$ for some γ_0 then h is analytic in $\Omega(\gamma) \times \Lambda(\gamma^2)$ for γ small enough. In particular it is real analytic in $V \times \Lambda$.*

To have an unified notation for all cases of the lemma, we introduce the differentiability degrees $r_{\mathbf{p}}, s_{\mathbf{p}}$ as:

$$r_{\mathbf{p}} = r,\ \ s_{\mathbf{p}} = s, \quad \text{if } A_{\mathbf{p}} \geq b_{\mathbf{p}}, \qquad r_{\mathbf{p}} = r_0,\ \ s_{\mathbf{p}} = s_0, \quad \text{otherwise}, \tag{5.3}$$

where r_0, s_0 are defined in Lemma 5.1. In this way, in all cases $h \in \mathcal{C}^{\Sigma_{r_{\mathbf{p}},s_{\mathbf{p}}}}$.

We proceed in a similar way as in Section 3.3. We introduce the function

$$g(x,\lambda) := \int_{\infty}^{0} M^{-1}(t,x,\lambda)\omega(\varphi(t,x,\lambda),\lambda)\,dt, \tag{5.4}$$

where $\varphi(t,x,\lambda)$ is the solution of $\dot{x} = \mathbf{p}(x,\lambda)$ such that $\varphi(0,x,\lambda) = x$, M is the fundamental matrix of $\dot{\psi} = \mathbf{Q}(\varphi(t,x,\lambda),\lambda)\psi$ such that $M(0,x,\lambda) = \mathrm{Id}$ and ω satisfies appropriate conditions to be specified later on. We first deal with the continuity of g and then with the differentiability with respect to the parameter λ. For that we check that the formal derivative $D_\lambda g$ is of the same form as g with a suitable different ω which implies the differentiability with respect to λ. This is done jointly in Lemma 5.3. Then using an induction argument we deal with the general differentiable case. Finally we deal with the analytic case, which needs an extra argument in this parametric setting.

For a given set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n'}$, it will be useful to consider the functional spaces:

$$\mathcal{B}^{\nu}_{\sigma,\kappa} = \{h : \mathcal{U} \to \mathbb{R}^k \,:\, h \in \mathcal{C}^{\Sigma_{\sigma,\kappa-\sigma}} \text{ and } h \in \mathcal{H}^{\nu}\}$$

if $\kappa, \sigma \in \mathbb{Z}^+$ and $\kappa \geq \sigma$.

Remark 5.2. Note that $\mathcal{B}^{\nu}_{s,r+s} = \mathcal{C}^{\Sigma_{s,r}} \cap \mathcal{H}^{\nu}$.

Lemma 5.3. *Let $\kappa \geq \sigma$ with $\sigma = 0, 1$. Assume that $\mathbf{p} \in \mathcal{B}^{N}_{\sigma,\kappa}$, $\mathbf{Q} \in \mathcal{B}^{N-1}_{\sigma,\kappa}$ and $\omega \in \mathcal{B}^{\nu+N}_{\sigma,\kappa}$. Let $\varrho_0 > 0$ be such that HPλ holds true.*

Then, if $\nu + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\left\{1 - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}, 0\right\}$, the function g defined by (5.4) belongs to $\mathcal{B}^{\nu+1}_{\sigma,\kappa_{\mathbf{p}}}$, where $\kappa_{\mathbf{p}} = r_{\mathbf{p}} + s_{\mathbf{p}}$ and $r_{\mathbf{p}}, s_{\mathbf{p}}$ are defined in (5.3).

In addition, when $\sigma = 1$, $D_\lambda g$ exists and

$$D_\lambda g(x,\lambda) = \int_{\infty}^{0} M^{-1}(t,x,\lambda)\omega^1(\varphi(t,x,\lambda),\lambda)\,dt \tag{5.5}$$

with $\omega^1 : V \times \Lambda \to \mathcal{L}(\mathbb{R}^{n'}, \mathbb{R}^k)$ (recall that $\Lambda \subset \mathbb{R}^{n'}$), given by:

$$\omega^1(x,\lambda) = D_\lambda \omega(x,\lambda) + D_\lambda \mathbf{Q}(x,\lambda) g(x,\lambda) - D_x g(x,\lambda) D_\lambda \mathbf{p}(x,\lambda). \tag{5.6}$$

Remark 5.4. We observe that $\kappa_{\mathbf{p}}$, the degree of differentiability stated for the case $\sigma = 0$, is the same as the one given in Theorem 3.2.

Remark 5.5. Note that $D_\lambda \mathbf{Q}(x,\lambda) g(x,\lambda) \in \mathcal{L}(\mathbb{R}^{n'}, \mathbb{R}^k)$ having the i-th column

$$D_{\lambda_i} \mathbf{Q}(x,\lambda) g(x,\lambda).$$

The same happens for $D_x \mathbf{Q}(x,\lambda) g(x,\lambda)$.

Proof. The case $\sigma = 0$ follows from Theorem 3.2, the dominated convergence theorem and the fact that the bounds are uniform in $\lambda \in \Lambda$.

The case $\sigma = 1$ is more involved. Its proof is analogous to the proof of Lemma 3.14. To shorten the notation we introduce $\varphi^x_\lambda(t) := \varphi(t,x,\lambda)$. First we check that

$$G^x_\lambda(\tau) := \int_{\tau}^{0} D_\lambda\left[M^{-1}(t,x,\lambda)\omega(\varphi^x_\lambda(t),\lambda)\right] dt$$

can be written as:

$$\begin{aligned} G^x_\lambda(\tau) :=& - D_\lambda M^{-1}(\tau,x,\lambda) g(\varphi^x_\lambda(\tau),\lambda) + \int_{\tau}^{0} M^{-1}(t,x,\lambda) D_\lambda\left[\omega(\varphi^x_\lambda(t),\lambda)\right] dt \\ &+ \int_{\tau}^{0} M^{-1}(t,x,\lambda) D_\lambda\left[\mathbf{Q}(\varphi^x_\lambda(t),\lambda)\right] g(\varphi^x_\lambda(t),\lambda)\,dt \end{aligned} \tag{5.7}$$

Indeed, since M^{-1} is the fundamental matrix of $\dot{\psi} = -\psi \mathbf{Q}(\varphi_\lambda^x(t), \lambda)$, we can use the variation of constants formula to the variational equation for $D_{\lambda_j} M^{-1}$ to obtain:

$$D_{\lambda_j} M^{-1}(t,x,\lambda) M(t,x,\lambda) = \int_t^0 M^{-1}(s,x,\lambda) D_{\lambda_j} \big[\mathbf{Q}(\varphi_\lambda^x(s),\lambda)\big] M(s,x,\lambda)\, ds$$

and, using expression (3.38) of $g(\varphi_\lambda^x(t), \lambda)$,

$$M^{-1}(t,x,\lambda)\omega(\varphi_\lambda^x(t),\lambda) = \frac{d}{dt}\big[M^{-1}(t,x,\lambda) g(\varphi_\lambda^x(t),\lambda)\big].$$

Then the result follows by integrating by parts

$$\int_\tau^0 \big[D_\lambda M^{-1}(t,x,\lambda) M(t,x,\lambda)\big]\big[M^{-1}(t,x,\lambda)\omega(\varphi_\lambda^x(t),\lambda)\big]\, dt.$$

Next we prove that

$$\widetilde{G}_\lambda^x(\tau) := \int_\tau^0 M^{-1}(t,x,\lambda) D_x g(\varphi_\lambda^x(t),\lambda) D_\lambda \mathbf{p}(\varphi_\lambda^x(t),\lambda)\, dt$$

can be written as

$$\begin{aligned} \widetilde{G}_\lambda^x(\tau) = &- M^{-1}(\tau,x,\lambda) D_x g(\varphi_\lambda^x(\tau),\lambda) D_\lambda \varphi_\lambda^x(t) \\ &- \int_\tau^0 M^{-1}(t,x,\lambda) D_x \mathbf{Q}(\varphi_\lambda^x(t),\lambda) g(\varphi_\lambda^x(\tau),\lambda) D_\lambda \varphi_\lambda^x(t) \\ &- \int_\tau^0 M^{-1}(t,x,\lambda) D_x \omega(\varphi_\lambda^x(t),\lambda) D_\lambda \varphi_\lambda^x(t). \end{aligned} \tag{5.8}$$

In order to prove (5.8), we will also integrate by parts. By using that $D_\lambda \varphi$ is the solution of

$$\frac{d}{dt}\psi = D_x \mathbf{p}(\varphi_\lambda^x(t),\lambda)\psi + D_\lambda \mathbf{p}(\varphi_\lambda^x(t),\lambda), \qquad \psi(0,x,\lambda) = 0$$

we deduce that:

$$D_\lambda \mathbf{p}(\varphi_\lambda^x(t),\lambda) = D_x \varphi_\lambda^x(t) \frac{d}{dt}\Big[(D_x \varphi_\lambda^x(t))^{-1} D_\lambda \varphi_\lambda^x(t)\Big].$$

Therefore, since $D_x\big[g(\varphi_\lambda^x(t),\lambda)\big] = D_x g(\varphi_\lambda^x(t),\lambda) D_x \varphi_\lambda^x(t)$,

$$\widetilde{G}^x_\lambda(\tau) = \int_\tau^0 M^{-1}(t,x,\lambda) D_x\big[g(\varphi^x_\lambda(t),\lambda)\big] \frac{d}{dt}\Big[(D_x\varphi^x_\lambda(t))^{-1} D_\lambda \varphi^x_\lambda(t)\Big]\, dt. \tag{5.9}$$

Applying (3.48) with $h = g$ we have

$$\frac{d}{dt}\big[g(\varphi^x_\lambda(t),\lambda)\big] = \mathbf{Q}(\varphi^x_\lambda(t),\lambda) g(\varphi^x_\lambda(t),\lambda) + \omega(\varphi^x_\lambda(t),\lambda)$$

which implies

$$\begin{aligned}\frac{d}{dt}\Big(M^{-1}(t,x,\lambda) D_x\big[g(\varphi^x_\lambda(t),\lambda)\big]\Big) = &- M^{-1}(t,x,\lambda)\mathbf{Q}(\varphi^x_\lambda(t),\lambda) D_x\big[g(\varphi^x_\lambda(t),\lambda)\big] \\ &+ M^{-1}(t,x,\lambda) D_x\big[\mathbf{Q}(\varphi^x_\lambda(t),\lambda) g(\varphi^x_\lambda(t),\lambda)\big] \\ &+ M^{-1}(t,x,\lambda) D_x\big[\omega(\varphi^x_\lambda(t),\lambda)\big].\end{aligned}$$

Finally, expression (5.8) follows from integrating by parts in (5.9). To do so we use that if $H(x,\lambda) := \mathbf{Q}(x,\lambda) g(x,\lambda)$ we have that $D_x\big[H(\varphi^x_\lambda(t),\lambda)\big] = D_x H(\varphi^x_\lambda(t),\lambda) D_x \varphi^x_\lambda(t)$ with

$$D_x H(\varphi^x_\lambda(t),\lambda) = D_x\mathbf{Q}(\varphi^x_\lambda(t),\lambda) g(\varphi^x_\lambda(t),\lambda) + \mathbf{Q}(\varphi^x_\lambda(t),\lambda) D_x g(\varphi^x_\lambda(t),\lambda).$$

Now we are going to relate expression (5.7) with (5.8). It is an straightforward computation (see Remark 5.4) to check that

$$\begin{aligned}D_\lambda[\mathbf{Q}(\varphi^x_\lambda(t),\lambda)] g(\varphi^x_\lambda(t),\lambda) = &D_x\mathbf{Q}(\varphi^x_\lambda(t),\lambda) g(\varphi^x_\lambda(t),\lambda) D_\lambda \varphi^x_\lambda(t) \\ &+ D_\lambda \mathbf{Q}(\varphi^x_\lambda(t),\lambda) g(\varphi^x_\lambda(t),\lambda).\end{aligned}$$

Substituting the above expression of $D_\lambda[\mathbf{Q}(\varphi^x_\lambda(t),\lambda)] g(\varphi^x_\lambda(t),\lambda)$ into (5.7), using (5.8) and the definition of $\widetilde{G}^x_\lambda$ we have

$$\begin{aligned}G^x_\lambda(\tau) = &- D_\lambda M^{-1}(\tau,x,\lambda) g(\varphi^x_\lambda(\tau),\lambda) - M^{-1}(\tau,x,\lambda) D_x g(\varphi^x_\lambda(\tau),\lambda) D_\lambda \varphi^x_\lambda(\tau) \\ &+ \int_\tau^0 M^{-1}(\tau,x,\lambda)\omega^1(\varphi^x_\lambda(t),\lambda)\, dt\end{aligned}$$

with ω^1 defined in (5.6).

To prove that $\lim_{\tau\to\infty} G^x_\lambda(\tau) = \int_\infty^0 M^{-1}(\tau,x,\lambda)\omega^1(\varphi^x_\lambda(t),\lambda)\, dt$ it remains to check that

$$\overline{h}^x_\lambda(\tau) := D_\lambda M^{-1}(\tau,x,\lambda) g(\varphi^x_\lambda(\tau),\lambda) + M^{-1}(\tau,x,\lambda) D_x g(\varphi^x_\lambda(\tau),\lambda) D_\lambda \varphi^x_\lambda(\tau)$$

goes to 0 as $\tau \to \infty$ uniformly in $(x,\lambda) \in V \times \Lambda$. Indeed, the result follows from

$$\|D_\lambda \varphi^x_\lambda(\tau)\| \le K\|x\|\big(1 + d_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\big)^{-\alpha\left(1-\max\left\{0,1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}\right)},$$

$$\|D_\lambda M^{-1}(\tau,x,\lambda)\| \le K\big(1 + c_{\mathbf{p}}(N-1)\tau\|x\|^{N-1}\big)^{-\alpha\left(\frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}}-\max\left\{0,1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}\right)}.$$

These bounds are obtained in a similar way as the ones of the corresponding derivatives with respect to x in Lemma 3.14. First we write adequately $D_\lambda\varphi_\lambda^x$ and $D_\lambda M^{-1}$ by taking into account the differential equations that they both satisfy and property (3.35):

$$D_\lambda\varphi_\lambda^x(\tau) = \int_0^\tau D_x\varphi(\tau - s, \varphi_\lambda^x(s), \lambda) D_\lambda \mathbf{p}(\varphi_\lambda^x(s), \lambda)\, ds,$$

$$D_\lambda M^{-1}(\tau, x, \lambda) = -\int_0^\tau M^{-1}(s, x, \lambda) D_\lambda\big[\mathbf{Q}(\varphi_\lambda^x(s), \lambda)\big] M^{-1}(\tau - s, \varphi_\lambda^x(s), \lambda)\, ds.$$

Bound (3.42) of $\|D_x\varphi_\lambda^x(s)\|$, bound of $\|\varphi_\lambda^x(s)\|$ in Lemma 3.10 and the fact that $D_\lambda \mathbf{p} \in \mathcal{H}^N$, lead to

$$\|D_\lambda\varphi_\lambda^x(\tau)\| \le \frac{K\|x\|^N}{\big(1 + (N-1)d_{\mathbf{p}}\tau\|x\|^{N-1}\big)^{\alpha\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}}} \int_0^\tau \frac{1}{\big(1 + (N-1)d_{\mathbf{p}}s\|x\|^{N-1}\big)^{\alpha\left(N - \frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right)}}\, ds$$

which gives the bound for $\|D_\lambda\varphi_\lambda^x(\tau)\|$. Since

$$D_\lambda\big[\mathbf{Q}(\varphi_\lambda^x(\tau), \lambda)\big] = D_x\mathbf{Q}(\varphi_\lambda^x(\tau), \lambda) D_\lambda\varphi_\lambda^x(\tau) + D_\lambda\mathbf{Q}(\varphi_\lambda^x(\tau), \lambda),$$

we have that

$$\|D_\lambda\big[\mathbf{Q}(\varphi_\lambda^x(\tau), \lambda)\big]\| \le K\big(1 + (N-1)d_{\mathbf{p}}\tau\|x\|^{N-1}\big)^{-\alpha\left(N-1-\max\left\{0, 1-\frac{A_{\mathbf{p}}}{d_{\mathbf{p}}}\right\}\right)}.$$

Then, using (3.46) with $\chi = M$, we obtain the bound for $\|D_\lambda M^{-1}(\tau, x, \lambda)\|$.

Finally we easily check that the three terms in $M^{-1}(t, x, \lambda)\omega^1(\varphi_\lambda^x(t), \lambda)$ have a uniform behavior of the form $t^{-a(\frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} + N - 1)}$ when t is big and $\alpha(\frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} + N - 1) > 1$. This proves that indeed, g is differentiable with respect to λ and formula (5.5) holds true.

Now assume that $\omega \in \mathcal{B}_{1,\kappa}^{\nu+N}$. Applying the result when $\sigma = 0$, we get that $g \in \mathcal{B}_{0,\kappa_{\mathbf{p}}}^{\nu+1}$ and in particular $D_x g \in \mathcal{B}_{0,\kappa_{\mathbf{p}}-1}^{\nu}$. Then we deduce that $\omega^1 \in \mathcal{B}_{0,\kappa_{\mathbf{p}}-1}^{\nu+N}$. Therefore, using again the present result for $\sigma = 0$, $D_\lambda g \in \mathcal{B}_{0,\kappa_{\mathbf{p}}-1}^{\nu+1}$, that is: $D_x^j D_\lambda g(x, \lambda)$ for $j \le \kappa_{\mathbf{p}} - 1$ are continuous and bounded and as a consequence $g \in \mathcal{B}_{1,\kappa_{\mathbf{p}}}^{\nu+1}$. $\square$

End of the proof of Lemma 5.1. We consider the differentiable and the analytic cases separately.

Assume that $\mathbf{p} \in \mathcal{B}_{s,\kappa}^N$, $\mathbf{Q} \in \mathcal{B}_{s,\kappa}^{N-1}$ and $\mathbf{w} \in \mathcal{B}_{s,\kappa}^{\mathfrak{m}+N}$ with $\kappa = r + s$. We apply Lemma 5.3 with $\omega = \mathbf{w}$ and $\nu = \mathfrak{m}$ and we obtain that the function h belongs to $\mathcal{B}_{1,\kappa_{\mathbf{p}}}^{\mathfrak{m}+1}$ with $\kappa_{\mathbf{p}} = r_{\mathbf{p}} + s_{\mathbf{p}}$ defined in (5.3). To finish the proof in the differentiable case we use induction. Assume that $h \in \mathcal{B}_{\sigma-1,\kappa_{\mathbf{p}}}^{\mathfrak{m}+1}$ with $\sigma \le s_{\mathbf{p}}$. By definition of $\mathcal{B}_{\sigma-1,\kappa_{\mathbf{p}}}^{\mathfrak{m}+1}$, we have that if $i + j \le \kappa_{\mathbf{p}}$ and $i \le \sigma - 1$, then $D_\lambda^i D_x^j h$ are continuous and bounded functions. We have to prove that indeed, $h \in \mathcal{B}_{\sigma,\kappa_{\mathbf{p}}}^{\mathfrak{m}+1}$.

We define $H^0 = h$, $\mathbf{w}^0 = \mathbf{w}$ and recurrently, for $1 \le i \le \sigma - 1$:

$$\begin{aligned} H^i(x,\lambda) &= D_\lambda H^{i-1}(x,\lambda), \\ \mathbf{w}^i(x,\lambda) &= D_\lambda \mathbf{w}^{i-1}(x,\lambda) + D_\lambda \mathbf{Q}(x,\lambda) H^{i-1}(x,\lambda) - D_x H^{i-1}(x,\lambda) D_\lambda \mathbf{p}(x,\lambda). \end{aligned}$$

Note that by expression (5.5) in Lemma 5.3, we have that

$$H^{\sigma-1}(x,\lambda) = \int_\infty^0 M^{-1}(t,x,\lambda)\mathbf{w}^{\sigma-1}(\varphi(t,x,\lambda),\lambda)\,dt.$$

Since by induction hypothesis $H^0 \in \mathcal{B}^{\mathfrak{m}+1}_{\sigma-1,\kappa_{\mathbf{p}}}$ then $H^i \in \mathcal{B}^{\mathfrak{m}+1}_{\sigma-1-i,\kappa_{\mathbf{p}}-i}$ and $D_x H^{i-1} \in \mathcal{B}^{\mathfrak{m}}_{\sigma-i,\kappa_{\mathbf{p}}-i}$. These facts imply that $\mathbf{w}^i \in \mathcal{B}^{\mathfrak{m}+N}_{\sigma-i,\kappa_{\mathbf{p}}-i}$. Applying the last formula for $i = \sigma - 1$, one has that $\mathbf{w}^{\sigma-1} \in \mathcal{B}^{\mathfrak{m}+N}_{1,\kappa_{\mathbf{p}}-\sigma+1}$. Therefore, applying Lemma 5.3 with $s = 1$, one concludes that $H^{\sigma-1} \in \mathcal{B}^{\mathfrak{m}+1}_{1,\kappa_{\mathbf{p}}-\sigma+1}$.

Now we are almost done because, on the one hand, if $1 \le i \le \sigma - 1$ and $1 \le i + j \le \kappa_{\mathbf{p}}$, all the derivatives $D^i_\lambda D^j_x h$ are bounded and continuous by induction hypothesis and on the other hand, since $H^{\sigma-1} = D^{\sigma-1}_\lambda h \in \mathcal{B}^{\mathfrak{m}+1}_{1,\kappa_{\mathbf{p}}-\sigma+1}$ the same happens for

$$D_\lambda D^j_x H^{\sigma-1} = D_\lambda D^j_x \left(D^{\sigma-1}_\lambda h\right)$$

if $1 + j \le \kappa_{\mathbf{p}} - \sigma + 1$, hence $D^\sigma_\lambda D^j_x h$ is continuous and bounded if $\sigma + j \le \kappa_{\mathbf{p}}$.

It remains to deal with the analytic case. We denote by $\varphi(t,x,\lambda)$ the flow of $\dot{x} = p(x,\lambda)$. We claim that, if ϱ, γ are small enough, the complex set $\Omega(\varrho,\gamma)$ is invariant by $\varphi(t,x,\lambda)$ for any $\lambda \in \Lambda(\gamma)$. Indeed, first we note that

$$\begin{aligned} \mathbf{p}(x,\lambda) =& \mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda) + \mathrm{i} D\mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda)[\operatorname{Im} x, \operatorname{Im}\lambda] \\ & - \int_0^1 (1-\mu) D^2\mathbf{p}(x(\mu),\lambda(\mu))[\operatorname{Im} x, \operatorname{Im}\lambda]^2\,d\mu, \end{aligned}$$

with $x(\mu) = \operatorname{Re} x + \mathrm{i}\mu \operatorname{Im} x$ and $\lambda(\mu) = \operatorname{Re}\lambda + \mathrm{i}\mu\operatorname{Im}\lambda$. We observe that, writing $z_\mu = (x(\mu),\lambda(\mu))$:

$$\begin{aligned} D\mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda)[\operatorname{Im} x, \operatorname{Im}\lambda] =& D_x\mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda)\operatorname{Im} x + D_\lambda \mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda)\operatorname{Im}\lambda, \\ D^2\mathbf{p}(z_\mu)[\operatorname{Im} x, \operatorname{Im}\lambda]^2 =& D^2_x\mathbf{p}(z_\mu)[\operatorname{Im} x, \operatorname{Im} x] + 2D_x D_\lambda \mathbf{p}(z_\mu)[\operatorname{Im} x, \operatorname{Im}\lambda] \\ & + D^2_\lambda \mathbf{p}(z_\mu)[\operatorname{Im}\lambda, \operatorname{Im}\lambda]. \end{aligned}$$

Then, since $\mathbf{p}$ is homogeneous and analytic, we have that $D_\lambda\mathbf{p}$, $D^2_\lambda\mathbf{p} \in \mathcal{H}^N$. Then, if $\lambda \in \Lambda(\gamma^2)$:

$$\mathbf{p}(x,\lambda) = \mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda) + \mathrm{i} D_x\mathbf{p}(\operatorname{Re} x, \operatorname{Re}\lambda)\operatorname{Im} x + \gamma^2 \mathcal{O}(\|x\|^N).$$

From the above equality we can proceed as in the proof of Lemma 3.9 to prove that $\Omega(\varrho,\gamma)$ is invariant if ϱ,γ are small enough. Then, the proof of the analytic case is completely analogous to the one of Theorem 3.2, using the dominated convergence theorem and the fact that the bounds are uniform for $\lambda\in\Lambda$. $\square$

5.2. *End of the proof of Theorems 2.8 and 2.9*

First we discuss the case of maps. No matter what strategy we choose for solving the cohomological equations for K^j we have to deal with the remainders E_x^{j+N-1} and E_y^{j+L-1} (Sections 4.2 and 4.3). Therefore, the first thing we need to do is to check what regularity with respect to (x,λ) they have. We deal with E_x^{j+N-1} being the case for E_y^{j+L-1} analogous. Recall that, as we prove in (4.3), E_x^{j+N-1} was the homogeneous part of the error term $E_x^{>j-1}=F_x\circ K^{\leq j-1}-K_x^{\leq j-1}\circ R^{\leq j+N-2}$. To prove this we used that by induction $K^{\leq j}$ and $R^{\leq j+N-1}$ are sums of homogeneous functions and Taylor's theorem by decomposing F_x as in (4.4):

$$F_x(x,y,\lambda)=x+p(x,y,\lambda)+F_x^{N+1}(x,y,\lambda)+\cdots+F_x^{r}(x,y,\lambda)+F_x^{>r}(x,y,\lambda).$$

Since p and F_x^l, $l=N+1,\cdots,r$, are homogeneous polynomials with respect to (x,y) and moreover $F\in\mathcal{C}^{\Sigma_{s,r}}$, we have that $p, F_x^l\in\mathcal{C}^{\Sigma_{s,\infty}}$ for $l=N+1,\cdots,r$. In fact they are analytic with respect to x and $\mathcal{C}^s$ with respect to λ. Analogously for E_y^{j+L-1}.

The cases $M<N$ or $A_p\geq b_p$ follows immediately from the strategy in Section 4.4 and Lemma 5.1.

When, $M\geq N$ and $A_p<b_p$ the first cohomological equation we solve is

$$D_xK_y^2(x,\lambda)p(x,0,\lambda)-\mathbf{Q}(x,\lambda)K_y^2(x,\lambda)=E_y^{M+1},$$

with $\mathbf{Q}\equiv 0$ if $M>N$ or $\mathbf{Q}(x,\lambda)=D_yq(x,0,\lambda)$ if $N=M$. Using Lemma 5.1 with $\mathbf{p}(x,\lambda)=p(x,0,\lambda)$, $\mathfrak{m}=1$ and $\mathbf{w}=E_y^{M+1}$, we have that $K_y^2\in\mathcal{C}^{\Sigma_{s_*,r_*-s_*}}$ where s_*,r_* are the given in Theorem 2.8. Proceeding by induction as in Section 4.4, we prove Theorem 2.8.

The proof of Theorem 2.9 is straightforward. Indeed, following the strategy in Section 4.5, we decompose $K^{(j)}=\overline{K^{(j)}}+\widetilde{K^{(j)}}$ where $\overline{K^{(j)}}$ is the time average of $K^{(j)}$ which satisfies equation (4.16). The same argument as in the case of maps leads to conclude that $\overline{K^{(j)}}\in\mathcal{C}^{\Sigma_{s_*,r_*-s_*}}$. Finally, $\widetilde{K^{(j)}}$ satisfies the equation

$$\partial_t\widetilde{K^{(j)}}=(\widetilde{E_x^{j+N-1}},\widetilde{E_y^{j+L-1}})$$

and therefore it is $\mathcal{C}^s$ with respect to (t,λ) and analytic with respect to x.

6. Examples

In this section we are going to see that our hypotheses are all of them necessary in order to be able to solve the cohomological equations for K_y^j.

In Section 6.1, we present an alternative (and easy) way for solving the cohomological equations in a particular setting. We also provide two examples of analytic maps (or even analytic vector fields) satisfying all the hypotheses, where the solution of the corresponding cohomological equations are only $\mathcal{C}^r$ in $\mathrm{int}(V)$. One of these examples, satisfies that $A_p = 0$ and the other one is such that $A_p > 0$. We will also check that the condition $A_p > d_p$ is essential to obtain analyticity. Moreover, we will also check that, when $A_p < d_p$, r_* is the maximum degree of differentiability.

Recall that the cohomological equations for K_x^j can be always solved by choosing R^j properly. However, it is interesting to obtain the simplest normal form, to be able to solve the cohomological equations for K_x^j with $R^j \equiv 0$. We present an example where the cohomological equation for K_x^j cannot be solved with $R^j \equiv 0$ if the degree $j \le \ell_*$ with ℓ_* the degrees of freedom to chose K_x^j defined in (2.7). In consequence, the normal form R^j stated in the main result, is the simplest one, generically.

6.1. Example 1. A particular form of p

Let F be a map of the form (2.1), satisfying hypotheses H1, H2 and H3.

Claim 6.1. *Let $\mathbf{p}(x) = p(x, 0)$. Assume that $\mathbf{p}(x) = \mathbf{p}_0(x)x$, with $\mathbf{p}_0 : V \to \mathbb{R}$ and $\mathbf{p}$ and V satisfy hypotheses HP1, HP2. Then the approximate parametrization $K^{\le}$ and the reparametrization R are rational functions (which in general are not polynomials). Moreover R can be chosen to be of the form $R(x) = x + \mathbf{p}_0(x)x + R^{2N-1}(x)$, as in the one dimensional case.*

Proof. HP1 implies $-2 < \mathbf{p}_0(x) < 0$, $x \in V$. Then, the auxiliary equation (3.1) reads

$$Dh(x)\mathbf{p}_0(x)x - \mathbf{Q}(x)h(x) = \mathbf{w}(x).$$

Since we look for homogeneous solutions of degree $\mathfrak{m} + 1$, using Euler's identity, namely $Dh(x)x = (\mathfrak{m} + 1)h(x)$, if h is homogeneous of degree $\mathfrak{m} + 1$, equation (3.1) can be written as:

$$\big[(\mathfrak{m} + 1)\mathbf{p}_0(x)\mathrm{Id} - \mathbf{Q}(x)\big]h(x) = \mathbf{w}(x).$$

Consequently, we can solve this equation for any homogeneous function $\mathbf{w} \in \mathcal{H}^{\mathfrak{m}+N}$ if and only if the matrix $(\mathfrak{m}+1)\mathbf{p}_0(x)\mathrm{Id} - \mathbf{Q}(x)$ is invertible for all $x \in V$. Assume the contrary, that is, there exists $x \in V$ and a eigenvector v, with $\|v\| = 1$, of the eigenvalue 0. For the next computations we assume that $\mathfrak{m} > 0$ and V is small enough so that $-1 < \mathbf{p}_0(x) < (\mathfrak{m} + 1)^{-1}$ if $x \in V$. Then $\mathbf{Q}(x)v = (\mathfrak{m} + 1)\mathbf{p}_0(x)v$ and

$$\|\big(\mathrm{Id} - \mathbf{Q}(x)\big)v\| = 1 - (\mathfrak{m} + 1)\mathbf{p}_0(x).$$

By definition (3.3) of $B_{\mathbf{Q}}$:

$$\|\big(\mathrm{Id} - \mathbf{Q}(x)\big)v\| \le \|\mathrm{Id} - \mathbf{Q}(x)\| \le 1 - B_{\mathbf{Q}}\|x\|^{N-1}$$

and by definition (3.2) of $a_{\mathbf{p}}$, $\mathbf{p}_0(x) \le -a_{\mathbf{p}}\|x\|^{N-1}$, then, we deduce that, if the matrix $(\mathfrak{m} + 1)p_0(x)\mathrm{Id} - \mathbf{Q}(x)$ is not invertible,

$$\mathfrak{m}+1+\frac{B_{\mathbf{Q}}}{a_{\mathbf{p}}}\leq 0.$$

Consequently, if $\mathfrak{m}+1+\frac{B_{\mathbf{Q}}}{a_{\mathbf{p}}}>0$, for any $x\in V$, the matrix $(\mathfrak{m}+1)\mathbf{p}_0(x)\mathrm{Id}-\mathbf{Q}(x)$ is invertible and moreover, the solution of the auxiliary equation is

$$h(x)=\left[(\mathfrak{m}+1)\mathbf{p}_0(x)\mathrm{Id}-\mathbf{Q}(x)\right]^{-1}\mathbf{w}(x).$$

Depending on the values of M,N, K_y^j has to satisfy the cohomological equations (4.8) if $N<M$, equation (4.9) if $N=M$ and (4.10) when $N>M$. Then, taking in the auxiliary equation $\mathbf{w}(x)=E_y^{j+L-1}$, and either $\mathbf{Q}(x)=0$ if $N<M$ or $\mathbf{Q}(x)=D_yq(x,0)$ if $N\geq M$, we have that

$$K_y^j(x)=\begin{cases}\dfrac{1}{j\mathbf{p}_0(x)}E_y^{j+N-1}(x), & N<M,\\[2ex] \left[j\mathbf{p}_0(x)\mathrm{Id}-D_yq(x,0)\right]^{-1}E_y^{j+N-1}(x), & N=M,\\[2ex] -D_yq(x,0)^{-1}E_y^{j+M-1}(x), & N>M.\end{cases}$$

To obtain K_x^j and R^{j+N-1} we have to deal with (4.5) which in abstract form reads

$$\begin{aligned}Dh(x)\mathbf{p}_0(x)x-D(\mathbf{p}_0(x)x)h(x)+\eta(x)&=\big(j\mathbf{p}_0(x)\mathrm{Id}-D(\mathbf{p}_0(x)x)\big)h(x)+\eta(x)\\&=\mathbf{w}(x),\end{aligned}$$

where $h=K_x^j$, $\eta=R^{j+N-1}$ and $\mathbf{w}(x)=E_x^{j+N-1}(x)+D_yp(x,0)K_y^j(x)$. Assume that the matrix in the above equation is not invertible for some $x\in V$. Then there exists $v\in\mathbb{R}^n$ with $\|v\|=1$ such that

$$(j-1)\mathbf{p}_0(x)v=(D\mathbf{p}_0(x)v)x.$$

This implies that x and v are linearly dependent: $v=\lambda x$ for some $\lambda\in\mathbb{R}\backslash\{0\}$. Then

$$(j-1)\mathbf{p}_0(x)\lambda x=\lambda(D\mathbf{p}_0(x)x)x=\lambda(N-1)\mathbf{p}_0(x)x$$

and hence $j=N$. As a consequence, for $j\geq 2$, $j\neq N$, the previous matrix is invertible, we can take $R^{j+N-1}\equiv 0$ and

$$K_x^j(x)=\left[j\mathbf{p}_0(x)\mathrm{Id}-D_xp(x,0)\right]^{-1}(E_x^{j+N-1}(x)+D_yp(x,0)K_y^j(x)).$$

When $j=N$, we can take K_x^N as any function in $\mathcal{H}^N$ and then

$$R^{2N-1}(x)=E_x^{2N-1}(x)-DK_x^N(x)\mathbf{p}_0(x)x+D_xp(x,0)K_x^N(x)+D_yp(x,0)K_y^N(x).\quad\square$$

6.2. Example 2. On the necessity of hypothesis H3

Consider the system of ordinary differential equation in $\mathbb{R}^2 \times \mathbb{R}$

$$\dot{x}_1 = -x_1^2, \qquad \dot{x}_2 = -ax_1x_2, \qquad \dot{y} = bx_1y + x_2^3$$

with $a, b > 0$ and $b + 3a \leq 1$. This system was also considered in Section 5.1 of [4]. There it was shown that the time 1 map F of the flow defined by the above system satisfies hypotheses H1 and H2 in a suitable domain V but that it has no invariant manifold over V.

Claim 6.2. *There exist $V \subset \mathbb{R}^2$, star-shaped with respect to the origin, where F satisfies hypotheses H1 and H2 but in which the cohomological equations (4.9) have no homogeneous solution in V. That is, H3 is needed both at a formal and at an analytical level.*

It is clear that F is a map of the form (2.1) with $N = M = 2$, $p(x, y) = (-x_1^2, -ax_1x_2)$ and $q(x, y) = bx_1y$.

We denote $x = (x_1, x_2)$. Let φ be the flow of $\dot{x} = p(x, 0)$, which can be explicitly computed:

$$\varphi(t, x) = \big(\varphi_{x_1}(t, x, y), \varphi_{x_2}(t, x)\big) = \left(\frac{x_1}{1 + tx_1}, \frac{x_2}{(1 + tx_1)^a}\right).$$

Proof. Hypotheses H1 and H2 are satisfied for F in the convex domain

$$W = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < (1 - a)x_1 < \frac{2}{a + 1}\right\}$$

with the supremum norm. Actually, $A_p = a^2$, $a_p = 1$ and $B_q = b$. However there is no open invariant set for F_x contained in W and, as a consequence, hypothesis H3 is not satisfied. Indeed, assume there is such open set and that $x^0 = (x_1^0, x_2^0) \in W$, $x_2^0 \neq 0$, and let

$$x^n = F_x(x^{n-1}, 0) = (F^n)_x(x^0, 0) = (F_x)^n(x^0, 0) = \left(\frac{x_1^0}{1 + nx_1^0}, \frac{x_2^0}{\left(1 + nx_1^0\right)^a}\right).$$

If the sequence $x^n \in W$, $\forall n \geq 0$, then $(1 - a)x_1^0 \geq |x_2^0|\big(1 + nx_1^0\big)^{1-a}$, $\forall n \geq 0$, which is false since $a < 1$.

Following the algorithm described in Section 4, we compute

$$E^{>1}(x) = F \circ K^{\leq 1}(x) - K^{\leq 1} \circ R^{\leq N}(x) = F(x, 0) - \big(x + p(x, 0), 0\big) = (0, 0, x_2^3) + \mathcal{O}(\|x\|^4\|).$$

Therefore, the first cohomological equation that we need to solve is

$$DK_y^2(x)p(x, 0) - D_yq(x, 0)K_y^2(x) = x_2^3. \tag{6.1}$$

Let $M_q(t, x) = (1 + tx_1)^b$ be the fundamental matrix of $\dot{z} = D_yq(\varphi(t, x), 0)z = b\varphi_{x_1}(t, x)z$. Formula (3.8) applied to $\mathbf{p}(x) = p(x, 0)$, $\mathbf{Q}(x) = D_yq(x, 0)$ and $\mathbf{w}(x) = x_2^3$ states that

$$K_y^2(x) = \int_\infty^0 M_q^{-1}(t,x)\mathbf{w}(\varphi(t,x))\,dt = x_2^3 \int_\infty^0 \frac{1}{(1+tx_1)^{b+3a}}\,dt$$

which, obviously, is not convergent if $b+3a \le 1$. In conclusion, our algorithm cannot be applied if H3 does not hold. Finally, we remark that, by Corollary 3.5, equation (6.1) has no homogeneous solution. $\square$

6.3. Example 3. The loss of differentiability

We consider the map $(x,y) \in \mathbb{R}^2 \times \mathbb{R} \mapsto F(x,y) \in \mathbb{R}^3$ given by

$$F(x,y) = \begin{pmatrix} x + p(x) \\ y + q_1(x)y + g(x) \end{pmatrix}, \qquad x = (x_1, x_2) \in \mathbb{R}^2,\ y \in \mathbb{R},$$

where

$$p(x) = \begin{pmatrix} -x_1^3 \\ -cx_2^3 \end{pmatrix}, \qquad q_1(x) = d(x_1^2 + x_2^2), \qquad g(x) = x_1^i x_2^j,$$

with $i + j \ge 4$ and $c, d > 0$.

Claim 6.3. *There exists $V \subset \mathbb{R}^2$, star-shaped with respect to the origin, where F satisfies hypotheses H1, H2 and H3.*

Let K be any approximate solution of (2.5) provided by Theorem 2.2. If the choice of i, j, c, d is such that $i + d = j + d/c = 4$, then K is only $j + 1$ times differentiable. This is the optimal regularity claimed by Theorem 2.2.

Possible choices are $i = j = d = 2$, $c = 1$ and $i = 3$, $j = d = 1$, $c = 1/3$.

Proof. We will compute the term K_y^2 explicitly and check that if has precisely the claimed regularity.

Let $V = B_{\varrho_0} \setminus \{0\} \subset \mathbb{R}^2$ with ϱ_0 small. We claim that, hypotheses H1, H2, H3 are satisfied in V for the Euclidean norm $\|\cdot\|_2$ (in fact, they are satisfied with any norm). Indeed, we have that V is invariant by $x \mapsto x + p(x)$ if ϱ_0 is small and

$$a_p = \frac{c}{1+c} + \mathcal{O}(\varrho_0^2) > 0, \qquad A_p = 0, \qquad b_p = \max\{1, c\}, \qquad B_q = d > 0.$$

We have that $E^{>1}(x) = (E_x^4, E_y^4)(x) = F(x,0) - (x + p(x), 0) = (0, g(x))$. Then, the first cohomological equation we have to solve is

$$DK_y^2(x)p(x) - q_1(x)K_y^2(x) = g(x) = x_1^i x_2^j,$$

which, according to (3.8), gives

$$K_y^2(x) = x_1^i x_2^j \int_\infty^0 \frac{1}{\left(1 + 2tx_1^2\right)^{\frac{i+d}{2}} \left(1 + 2tcx_2^2\right)^{\frac{j}{2}+\frac{d}{2c}}}\,dt.$$

According to Theorem 2.2, the degree of differentiability of K, given in (2.4), is the maximum integer satisfying

$$r_* < 2 + \frac{B_q}{b_p} = 2 + \frac{d}{\max\{1, c\}}.$$

Now we take values of i, j, c, d such that $i + d = j + d/c = 4$. It is a calculation to check that

$$K_y^2(x) = x_1^i x_2^j \left[\frac{cx_2^2 + x_1^2}{2(cx_2^2 - x_1^2)^2} - c\frac{x_1^2 x_2^2}{(cx_2^2 - x_1^2)^3} \log\left(\frac{cx_2^2}{x_1^2}\right)\right].$$

We study K_y^2 in the subdomain $W = \{|\sqrt{c}x_2| < |x_1|\}$ of V. On W, K_y^2 is

$$\begin{aligned} K_y^2(x) =& x_1^{i-j-2}\frac{x_2^j}{x_1^j}\left(1+\frac{cx_2^2}{x_1^2}\right)\left(1-\frac{cx_2^2}{x_1^2}\right)^{-2} \\ &+ 2x_1^{i-j-6}\frac{x_2^{j+2}}{x_1^{j+2}}\left(1-\frac{cx_2^2}{x_1^2}\right)^{-3} \log\left(\frac{\sqrt{c}|x_2|}{|x_1|}\right). \end{aligned}$$

To study the differentiability of K_y^2 on W is equivalent to study the derivability of $\chi(z) = z^{j+2}\log(|z|)$, which is only $\mathcal{C}^{j+1}$ at $z = 0$ but it is not $\mathcal{C}^{j+2}$. Consequently, K_y^2 is only $\mathcal{C}^{j+1}$ at the points $(x_1, 0) \in W \subset V$. Note that, with the two choices of the parameters i, j, d, c, we have that, $d = j$ and $c \leq 1$. Then, $r_* < 2 + j$, that is, $r_* = 1 + j$ which coincides with regularity of K_y^2 at $x_2 = 0$. □

6.4. The reparametrization R

We consider the map given by

$$F(x, y) = \begin{pmatrix} x + p(x) + f(x) \\ y + q_1(x)y + g(x) \end{pmatrix}, \qquad (x, y) \in \mathbb{R}^2 \times \mathbb{R},$$

with $p(x) = (-x_1^N, -cx_1^{N-1}x_2)$, $N \geq 2$, $q_1(x) = (x_1^2 + x_2^2)^{(M-1)/2}$, M odd and $M \geq 3$, $g \in \mathcal{H}^{\geq M+1}$ and $f \in \mathcal{H}^{\geq N+1}$.

Claim 6.4. *Assume $c > 1$. F satisfies hypotheses H1, H2 and H3 with the supremum norm in the set*

$$V = \{x \in \mathbb{R}^2 : |x_2| < x_1\}. \tag{6.2}$$

For any approximate solutions K and R given by Theorem 2.2, R has the form

$$R(x) = x + p(x) + \sum_{j=2}^{N} R^{j+N-1}(x), \qquad R^{j+N-1} \neq 0, \quad j = 2, \cdots, N.$$

In the case of one dimensional manifolds, it was proven in [3]) that one can always take $R^{j+N-1}=0$ if $j=2,\dots,N-1$.

Proof. It is easy to see that Hypotheses H1, H2 and H3 hold in V, as well as to compute the value of the constants $B_p=Nc$, $a_p=1$, $A_p=-c(N-2)$ and $b_p=c$. Consequently we have that

$$\ell_* > N-1+[Nc] \geq 2N-1.$$

What we are going to check is that, necessarily, for solving the cohomological equations (4.5) for K_x^j in Section 4 for values of $2 \leq j \leq \ell_*-N+1$, we have to take $R^{j+N-1} \not\equiv 0$. Indeed, if not, the cohomological equations (4.5) for $2 \leq j \leq N$ are

$$\begin{aligned} DK_x^j(x)p(x) - Dp(x)K_x^j(x) &= DK_x^j(x)\begin{pmatrix} -x_1^N \\ -cx_1^{N-1}x_2 \end{pmatrix} + \begin{pmatrix} Nx_1^{N-1} & 0 \\ c(N-1)x_1^{N-2}x_2 & cx_1^{N-1} \end{pmatrix} K_x^j(x) \\ &= E_x^{j+N-1}(x), \end{aligned}$$

where E_x^{j+N-1} is a homogeneous function of degree $j+N-1$.

We focus our attention to the equation for the first component of K_x^j,

$$x_1 D_1 K_{x_1}^j(x) + cx_2 D_2 K_{x_1}^j(x) - NK_{x_1}^j(x) = -x_1^{1-N} E_{x_1}^{j+N-1}(x). \tag{6.3}$$

We introduce the auxiliary functions $h(z)=K_{x_1}^j(1,z)$ and $T(z)=E_{x_1}^{j+N-1}(1,z)$. Notice that we can recover $K_{x_1}^j(x)$ from the identity:

$$K_{x_1}^j(x_1,x_2) = x_1^j h\big(x_2/x_1\big). \tag{6.4}$$

Using Euler's identity $jK_x^j(x) = DK_x^j(x)x$ and rearranging terms in (6.3), we obtain that h is a solution of the differential equation:

$$(c-1)\frac{d}{dz}h(z) = \frac{N-j}{z}h(z) - \frac{T(z)}{z}. \tag{6.5}$$

We study the solutions of (6.5). Assume the easiest case, that is $E_{x_1}^{j+N-1}$ is a homogeneous polynomial. Then $T(z)$ is a polynomial of degree $j+N-1$ which we write as: $T(z)=\sum_{l=0}^{j+N-1} a_l z^l$. From the form of (6.5) the solutions are defined for $z \in (0,\infty)$ and for $z \in (-\infty,0)$. When $j=N$, equation (6.5) yields:

$$(c-1)h(z) = C - a_0 \log|z| - \sum_{l=1}^{2N-1} \frac{a_l}{l} z^l$$

for some constant C. Then, by (6.4)

$$K^N_{x_1}(x) = \frac{x_1^N}{c-1}\left(C - a_0 \log\left|\frac{x_2}{x_1}\right| - \sum_{l=1}^{2N-1} a_l \frac{x_2^l}{l x_1^l}\right)$$

which is not defined for $x_2 = 0$ contained in the set V in (6.2). So that equation (6.3) cannot be solved in V for $j = N$. Even more, when $j \neq N$, denoting $\beta = (N - j)/(c - 1)$

$$h(z) = |z|^\beta C - |z|^\beta \int_1^z w^{-\beta-1} T(w)\, dw = |z|^\beta C - |z|^\beta \sum_{l=0}^{j+N-1} \int_1^z a_l w^{-\beta-1+l}\, dw.$$

When $\beta = l \in \{0, \cdots, j + N - 1\}$, h will have the term $a_l \log|z|$ and, as in the case $j = N$, $K^j_{x_1}$ will have the term $\log(|x_2|/|x_1|)$ which, again, is not defined in the set V. We realize this case for $j < N$ taking, for instance, $c = 2$ and $l = N - j$. On the contrary, $K^j_{x_1}$ is well defined if $j > N$. $\square$

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