

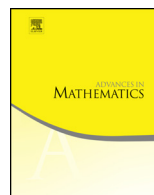


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Breakdown of homoclinic orbits to L_3 in the RPC3BP (I). Complex singularities and the inner equation



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ABSTRACT

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries. If the primaries perform circular motions and the massless body is coplanar with them, one has the Restricted Planar Circular 3-Body Problem (RPC3BP). In synodic coordinates, it is a two degrees of freedom Hamiltonian system with five critical points, L_1, \dots, L_5 , called the Lagrange points.

The Lagrange point L_3 is a saddle-center critical point which is collinear with the primaries and is located beyond the largest of the two. In this paper and its sequel [10], we provide an asymptotic formula for the distance between the one dimensional stable and unstable invariant manifolds of L_3 when the ratio between the masses of the primaries μ is small. It implies that L_3 cannot have one-round homoclinic orbits.

If the mass ratio μ is small, the hyperbolic eigenvalues are weaker than the elliptic ones by factor of order $\sqrt{\mu}$. This implies that the distance between the invariant manifolds is exponentially small with respect to μ and, therefore, the classical Poincaré–Melnikov method cannot be applied.

In this first paper, we approximate the RPC3BP by an averaged integrable Hamiltonian system which possesses a

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saddle center with a homoclinic orbit and we analyze the complex singularities of its time parameterization. We also derive and study the inner equation associated to the original perturbed problem. The difference between certain solutions of the inner equation gives the leading term of the distance between the stable and unstable manifolds of L_3 .

In the sequel [10] we complete the proof of the asymptotic formula for the distance between the invariant manifolds.

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1. Introduction

The understanding of the motions of the 3-Body Problem has been of deep interest in the last centuries. Since Poincaré, see [46], one of the fundamental problems is to

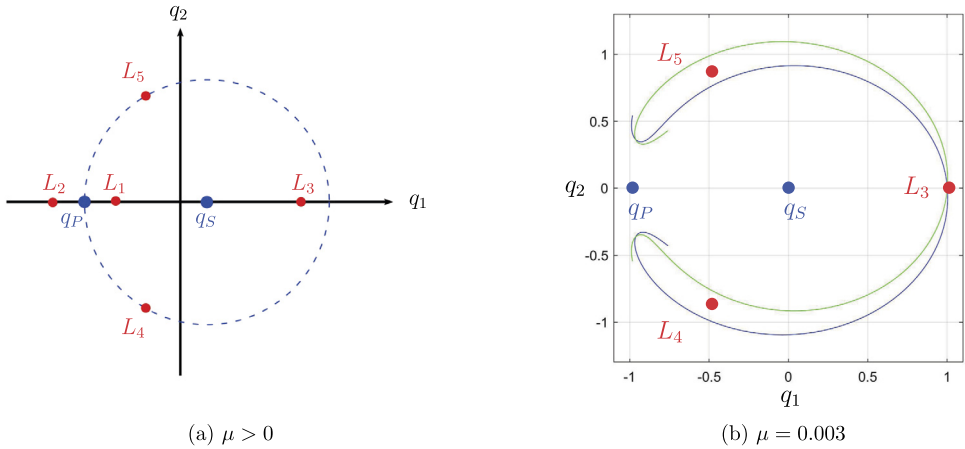


Fig. 1. (a) Projection onto the q -plane of the equilibrium points for the RPC3BP on rotating coordinates. (b) Plot of the stable (green) and unstable (blue) manifolds of L_3 projected onto the q -plane for $\mu = 0.003$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

understand how the invariant manifolds of its different invariant objects (periodic orbits, invariant tori) structure its global dynamics. Assume that one of the bodies (say the third) has mass zero. Then, one has the Restricted 3-Body Problem. In this model, the two first bodies, called the primaries, are not influenced by the massless one. As a result, their motions are governed by the classical Kepler laws. If one further assumes that the primaries perform circular motion and that the third body is coplanar with them, one has the Restricted Planar Circular 3-Body Problem (RPC3BP).

Let us name the two primaries S (star) and P (planet). Normalizing their masses, we can assume that $m_S = 1 - \mu$ and $m_P = \mu$, with $\mu \in (0, \frac{1}{2}]$. Since the primaries follow circular orbits, in rotating (usually also called synodic) coordinates, their positions can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$. Then, denoting by $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$ the position and momenta of the third body and taking appropriate units, the RPC3BP is a 2-degrees of freedom Hamiltonian system with respect to

$$h(q, p; \mu) = \frac{\|p\|^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1 - \mu)}{\|q - (\mu, 0)\|} - \frac{\mu}{\|q - (\mu - 1, 0)\|}. \tag{1}$$

For $\mu > 0$, it is a well known fact that h has five equilibrium points: L_1, L_2, L_3, L_4 and L_5 , called Lagrange points¹ (see Fig. 1(a)). On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic orbits with the same period as the two primaries, i.e. they lie on a 1:1 mean motion resonance. The three collinear points with the primaries, L_1, L_2 and L_3 , are of center-saddle type and, for small μ , the triangular ones, L_4 and L_5 , are of center-center type (see for instance [51]).

¹ For $\mu = 0$, the system has a circle of critical points (q, p) with $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$.

Since the points L_1 and L_2 are rather close to the small primary, their invariant manifolds have been widely studied for their interest in astrodynamics applications, (see [12,29,37]). The dynamics around the points L_4 and L_5 have also been considerably studied since, due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [13,28,47]).

On the contrary, the invariant manifolds of the Lagrange point L_3 have received somewhat less attention. Still, they structure the dynamics in regions of the phase space of the RPC3BP. In particular, the horseshoe-shaped orbits that explain the orbits of Saturn satellites Janus and Epimetheus lie “close” to the invariant manifolds of L_3 (see [44]). Moreover, the invariant manifolds of L_3 (more precisely its center-stable and center-unstable manifolds) act as effective boundaries of the stability domains around $L_{4,5}$ (see in [49]). See the companion paper [10] for more references about the dynamics of the RPC3BP in a neighborhood of L_3 and its invariant manifolds.

The purpose of this paper and its sequel [10] is to study the invariant manifolds of L_3 and, particularly, show that they do not intersect for $0 < \mu \ll 1$ (at their first round).

1.1. The unstable and stable invariant manifolds of L_3

The eigenvalues of the Lagrange point L_3 satisfy that

$$\text{Spec} = \{\pm\sqrt{\mu}\rho(\mu), \pm i\omega(\mu)\}, \quad \text{with} \quad \begin{cases} \rho(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2), \end{cases} \quad (2)$$

as $\mu \rightarrow 0$ (see [51]). Notice that, due to the different size in the eigenvalues, the system possesses two time scales, which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3 .

The one dimensional unstable and stable invariant manifolds have two branches each (see Fig. 1(b)). One pair, which we denote by $W^{u,+}(\mu)$ and $W^{s,+}(\mu)$ circumvents L_5 whereas the other, denoted as $W^{u,-}(\mu)$ and $W^{s,-}(\mu)$, circumvents L_4 . Since the Hamiltonian system associated to h in (1) is reversible with respect to the involution

$$\Phi(q, p) = (q_1, -q_2, -p_1, p_2), \quad (3)$$

the $+$ branches are symmetric to the $-$ ones. Thus, one can restrict the study to the first ones.

As already mentioned, the aim of this paper (and its sequel [10]) is to give an asymptotic formula for the distance between $W^{u,+}(\mu)$ and $W^{s,+}(\mu)$, for $0 < \mu \ll 1$ (in an appropriate transverse section). However, due to the rapidly rotating dynamics of the system (see (2)), the stable and unstable manifolds of L_3 are exponentially close to each other with respect to $\sqrt{\mu}$. This implies that the classical Melnikov Theory [42] cannot be applied and that obtaining the asymptotic formula is a rather involved problem.

The precise statement for the asymptotic formula for the distance is properly stated in the sequel of this paper [10]. Let us give here a more informal statement. Consider the classical symplectic polar coordinates (r, θ, R, G) , where R is the radial momentum and G the angular momentum, and the section $\Sigma = \{\theta = \pi/2, r > 1\}$. Then, if one denotes by P^u and P^s the first intersections of the invariant manifolds $W^{u,+}(\mu)$, $W^{s,+}(\mu)$ with Σ , the distance between these points, is given by

$$\text{dist}_{\Sigma}(P^u, P^s) = \sqrt[3]{4} \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|}\right) \right], \quad (4)$$

for $0 < \mu \ll 1$ and certain constants $A > 0$ and $\Theta \in \mathbb{C}$. The proof of this asymptotic formula spans both the present paper and the sequel [10]. In this first paper, we perform the first steps towards the proof (see Section 1.3 below). In particular, we obtain and describe the constants A and Θ appearing in (4).

A fundamental problem in dynamical systems is to prove that a model has chaotic dynamics (for instance a Smale Horseshoe). For many physically relevant problems, like those in Celestial Mechanics, this is usually a remarkably difficult problem. Certainly, the fact that the invariant manifolds of L_3 do not coincide does not lead to chaotic dynamics. However, one should expect the existence of Lyapunov periodic orbits which are exponentially close (with respect to $\sqrt{\mu}$) to L_3 and whose stable and unstable invariant manifolds intersect transversally. If so, the Smale-Birkhoff Theorem would imply the existence of a hyperbolic set whose dynamics is conjugated to that of the Bernoulli shift (in particular, with positive topological entropy) exponentially close to the invariant manifolds of L_3 .

1.2. Exponentially small splitting of separatrices

Even though there is a standard theory to analyze the breakdown of homoclinic and heteroclinic connections, the so called Poincaré-Melnikov Theory (see [42] and [35] for a more modern exposition), it cannot be applied to obtain (4) due to its exponential smallness. Indeed, the breakdown of homoclinic connections to L_3 fits into what is usually referred to as exponentially small splitting of separatrices problems. This *beyond all orders* phenomenon was first detected by Poincaré in [46] when he studied the non integrability of the 3-Body Problem.

The first obtention of an asymptotic formula for an exponentially small splitting of separatrices did not appear until the 1980's with the pioneering work by Lazutkin for the *standard map* [38,39]. Even if Lazutkin's work was not complete (the complete proof was achieved by Gelfreich in [24]), the ideas he developed have been very influential and have been the basis of many of the works in the field (and in particular of this paper and the sequel [10]). Other methods to deal with exponentially small splitting of separatrices are Treschev's *continuous averaging* (see [52]) or "direct" series methods (see [21]).

Usually, the exponentially small splitting of separatrices problems are classified as *regular* or *singular*.

In the regular cases, even if Melnikov theory cannot be straightforwardly applied, the Melnikov function gives the leading term for the distance between the perturbed invariant manifolds. That is, Melnikov theory provides the first order for the distance but leads to too crude estimates for the higher order terms. This phenomenon has been studied in rapidly forced periodic or quasi-periodic perturbations of 1-degree of freedom Hamiltonian systems, see [2,3,15,16,22,25,30,36], in close to the identity area preserving maps, see [14], and in Hamiltonian systems with two or more degrees of freedom which have hyperbolic tori with fast quasiperiodic dynamics [48]. In particular, [32] provides the first prove of exponentially small splitting of separatrices in a Celestial Mechanics problem (see also [33,34]).

In the singular cases, the exponentially small first order for the distance between the invariant manifolds is no longer given by the Melnikov function. Instead, one has to consider an auxiliary equation, usually called *inner equation*, which does not depend on the perturbative parameter and provides the first order for the distance. Some results on inner equations can be found on [1,4,5,23,26,45] and the application of the inner equation analysis to the original problem can be found in [6–9,20,31,40,41].

Due to the extreme sensitivity of the exponentially small splitting of separatrices phenomenon on the features of each particular model, most of the available results apply under quite restrictive hypothesis and, therefore, cannot be applied to analyze the invariant manifolds of L_3 .

1.3. Strategy to obtain an asymptotic formula for the breakdown of the invariant manifolds of L_3

For the limit problem h in (1) with $\mu = 0$, the five Lagrange point “disappear” into the circle of (degenerate) critical points $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$. As a consequence, the one-dimensional invariant manifolds of L_3 disappear when $\mu = 0$ too. For this reason, to analyze perturbatively these invariant manifolds, the first step is to perform a singular change of coordinates to obtain a “new first order” Hamiltonian which has a center saddle equilibrium point (close to L_3) with stable and unstable manifolds that coincide along a separatrix. To perform the change of coordinates we use the Poincaré planar elements (see [43]) plus a singular (with respect to μ) scaling.

In the following list, we present the main steps of our strategy to prove formula (4). We split the list in two. First we explain the steps performed in this paper and later those carried out in its sequel [10].

In this paper, we complete the following steps:

- A. We perform a change of coordinates which captures the slow-fast dynamics of the system. The new Hamiltonian becomes a (fast) oscillator weakly coupled to a 1-degree of freedom Hamiltonian with a saddle point and a separatrix associated to it.

- B. We analyze the analytical continuation of a time-parametrization of the separatrix. In particular, we obtain its maximal strip of analyticity (centered at the real line). We also describe the character and location of the singularities at the boundaries of this region.
- C. We derive the inner equation, which gives the first order of the original system close to the singularities of the separatrix described in Step B. This equation is independent of the perturbative parameter μ .
- D. We study two special solutions of the inner equation which are approximations of the perturbed invariant manifolds near the singularities. Moreover, we provide an asymptotic formula for the difference between these two solutions of the inner equation. We follow the approach presented in [5].

In [10] we complete the following steps:

- E. We prove the existence of the analytic continuation of suitable parametrizations of $W^{u,+}(\delta)$ and $W^{s,+}(\delta)$ in appropriate complex domains (and as graphs). These domains contain a segment of the real line and intersect a neighborhood sufficiently close to the singularities of the separatrix.
- F. By using complex matching techniques, we compare the solutions of the inner equation with the graph parametrizations of the perturbed invariant manifolds.
- G. Finally, we prove that the dominant term of the difference between manifolds is given by the term obtained from the difference of the solutions of the inner equation.

The structure of this paper goes as follows. In Section 2, we present the main results for the Steps A to D and introduce some heuristics to contextualize them. Sections 3-5 are devoted to the proof of the results in Section 2.

The constants in the asymptotic formula for the distance. The constant A in (4) is given by the height of the maximal strip of analyticity of the unperturbed separatrix (see Step B). Therefore, to obtain its value, one has to compute the imaginary part of the singularities of the separatrix which are closer to the real line.

On all the previous mentioned works on splitting of separatrices, either the separatrix of the unperturbed model has an analytic expression (see for example [7,15,20,31,32]) or otherwise certain properties of its analytic continuation are given as assumptions (see [2, 6,16,23]). In this case, we do not have an explicit expression for the time-parameterization of the separatrix and, to obtain its complex singularities, we need to rely on techniques of analytical continuation to analyze them (see Section 2.2). In particular, we describe the parametrization of the separatrix in terms of a multivalued function involving a complex integral and (see Theorem 2.2 below) we obtain

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx \approx 0.177744.$$

This value agrees with the numerical computations of the distance between the invariant manifolds given in [18,49].

Since we are in a singular case, the constant Θ in (4) is not correctly given by the Melnikov function but by the analysis of the inner equation of the system (see Step C above and also Sections 2.3 and 2.4). In particular, Θ corresponds to a Stokes constant and does not have a closed formula. By a numerical computation, we see that $|\Theta| \approx 1.63$ (see Remark 2.8). We expect that, by means of a computer assisted proof, it would be possible to obtain rigorous estimates and verify that $|\Theta| \neq 0$, see [11].

2. Main results

We devote this section to state the main results concerning the Steps A, B, C and D explained in Section 1.3. First, in Section 2.1, we present the changes of coordinates involved to rewrite the Hamiltonian h in (1) as a singular perturbation problem given by a fast oscillator weakly coupled with a one degree of freedom Hamiltonian with a saddle point and a separatrix (Step A). In Section 2.2, we consider the time-parametrization of the separatrix and analyze the properties of its analytical continuation (Step B). In Section 2.3 we give some heuristic ideas regarding the parametrization of the perturbed manifolds on certain complex domains (Step E) and deduce the singular change of variables which leads to the inner equation (Step C). Finally, in Section 2.4, we present the study of certain solutions of the inner equation and give an asymptotic formula for their difference (Step D).

2.1. A singular perturbation formulation of the problem

When studying a close to integrable Hamiltonian system at a resonance, it is usual to “blow-up” the “resonant zone” to capture the slow-fast time scales. In this section we present the singular change of coordinates which transforms the Hamiltonian h in (1) into a pendulum-like Hamiltonian plus a fast oscillator with a small coupling, namely

$$H(\lambda, \Lambda, x, y) = -\frac{3}{2}\Lambda^2 + V(\lambda) + \frac{xy}{\sqrt{\mu}} + o(1),$$

with respect to the symplectic form $d\lambda \wedge d\Lambda + idx \wedge dy$. In these coordinates, the first order of the Hamiltonian has a saddle in the (λ, Λ) -plane and a center in the (x, y) -plane. Notice that the system possesses two time scales (~ 1 and $\sim 1/\sqrt{\mu}$). Recall that this two time scales are also present in the eigenvalues of L_3 in (2).

We consider Poincaré coordinates for the RPC3BP (see (1)) in order to write the system as a close to integrable Hamiltonian system and decouple (at first order) the

saddle and the center behavior. To this end, we first consider the symplectic polar and Delaunay coordinates.

Polar coordinates. Let us consider the change of coordinates:

$$\phi_{\text{pol}} : (r, \theta, R, G) \mapsto (q, p),$$

where r is the radius, θ the argument of q , R the linear momentum in the r direction and G is the angular momentum. Then, the Hamiltonian (1), becomes

$$H^{\text{pol}} = H_0^{\text{pol}} + \mu H_1^{\text{pol}}, \tag{5}$$

where

$$\begin{aligned} H_0^{\text{pol}}(r, R, G) &= \frac{1}{2} \left(R^2 + \frac{G^2}{r^2} \right) - \frac{1}{r} - G, \\ \mu H_1^{\text{pol}}(r, \theta; \mu) &= \frac{1}{r} - \frac{1 - \mu}{\sqrt{r^2 - 2\mu r \cos \theta + \mu^2}} \\ &\quad - \frac{\mu}{\sqrt{r^2 + 2(1 - \mu)r \cos \theta + (1 - \mu)^2}}. \end{aligned} \tag{6}$$

The critical point L_3 (see [51] for the details) satisfies that, as $\mu \rightarrow 0$,

$$(r, \theta, R, G) = (d_\mu, 0, 0, d_\mu^2), \quad \text{with } d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3). \tag{7}$$

Delaunay coordinates. The Delaunay elements, denoted (ℓ, L, \hat{g}, G) , are action–angle variables for the 2-Body Problem (for negative energy) in non-rotating coordinates. The variable ℓ is the mean anomaly, \hat{g} is the argument of the pericenter, L is the square root of the semi major axis and G is the angular momentum, (see [43]).

Let us introduce some formulae to describe these elements from the non-rotating polar coordinates $(r, \hat{\theta}, R, G)$, namely $\hat{\theta} = \theta + t$. The action L is defined by

$$-\frac{1}{2L^2} = \frac{1}{2} \left(R^2 + \frac{G^2}{r^2} \right) - \frac{1}{r},$$

and the (osculating) eccentricity of the body is expressed as

$$e = \sqrt{1 - \frac{G^2}{L^2}} = \frac{\sqrt{(L - G)(L + G)}}{L}. \tag{8}$$

Let us recall now the ‘‘anomalies’’: the three angular parameters that define a position at the (osculating) ellipse. These are the mean anomaly ℓ , the eccentric anomaly u , and the true anomaly f , which satisfy

$$r = L^2(1 - e \cos u) \quad \text{and} \quad \hat{\theta} = f + \hat{g}. \quad (9)$$

To use these elements in a rotating frame, we consider rotating Delaunay coordinates (ℓ, L, g, G) , where the new angle is defined as $g = \hat{g} - t$ (the argument of the pericenter with respect to the line defined by the primaries S and J). As a result,

$$\theta = f + g, \quad (10)$$

and the unperturbed Hamiltonian H_0^{pol} becomes

$$H_0^{\text{pol}} = -\frac{1}{2L^2} - G.$$

The eccentric and true anomalies are related by

$$\cos f = \frac{\cos u - e}{1 - e \cos u}, \quad \sin f = \frac{\sqrt{1 - e^2} \sin u}{1 - e \cos u}, \quad (11)$$

and the mean anomaly u is given by Kepler's equation

$$u - e \sin u = \ell. \quad (12)$$

The critical point L_3 (see (7)) satisfies $\theta = \ell + g = 0$ and

$$L = \sqrt{\frac{d_\mu}{2 - d_\mu^3}} = 1 + \mathcal{O}(\mu), \quad G = d_\mu^2 = 1 + \mathcal{O}(\mu), \quad L - G = \mathcal{O}(\mu^2). \quad (13)$$

Note that the Delaunay coordinates are not well defined for circular orbits ($e = 0$), since the pericenter, and as a consequence the angle g , are not well defined.

Poincaré coordinates. To “blow-down” the singularity of the Delaunay coordinates at circular motions, we use the classical Poincaré coordinates, which can be expressed by means of (rotating) Delaunay variables. Let us define

$$\phi_{\text{Poi}} : (\lambda, L, \eta, \xi) \mapsto (r, \theta, R, G), \quad (14)$$

given by

$$\lambda = \ell + g, \quad \eta = \sqrt{L - Ge}^{ig}, \quad \xi = \sqrt{L - Ge}^{-ig}. \quad (15)$$

These coordinates are symplectic and analytic. Moreover, even though they are defined through the Delaunay variables, they are also analytic when the eccentricity tends to zero (i.e. at $L = G$), see [17,43].

The Hamiltonian equation associated to (5), expressed in Poincaré coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i d\eta \wedge d\xi$ and

$$H^{\text{Poi}} = H_0^{\text{Poi}} + \mu H_1^{\text{Poi}},$$

where

$$H_0^{\text{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \quad \text{and} \quad H_1^{\text{Poi}} = H_1^{\text{pol}} \circ \phi_{\text{Poi}}. \tag{16}$$

In Poincaré coordinates, the critical point L_3 , as given in (13), satisfies

$$\lambda = 0, \quad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu).$$

The linearized part of the vector field associated to this point has, at first order, an uncoupled nilpotent and center blocks,

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu).$$

The center is found on the projection to coordinates (η, ξ) and the degenerate behavior on the projection to (λ, L) .

The perturbative term μH_1^{Poi} is not explicit. We overcome this problem by computing the first terms of the series of μH_1^{Poi} in powers of (η, ξ) , (see Lemma 4.1).

Notice that, on the original coordinates, Hamiltonian h (see (1)) is analytic at points away from collision with the primaries. However, the collisions are not as easily defined in Poincaré coordinates.

A singular scaling. We consider the parameter

$$\delta = \mu^{\frac{1}{4}}$$

and we define the symplectic scaling

$$\phi_{\text{sc}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \quad L = 1 + \delta^2 \Lambda, \quad \eta = \delta x, \quad \xi = \delta y, \tag{17}$$

and the time reparameterization $t = \delta^{-2} \tau$. The transformed equations are Hamiltonian with respect to the Hamiltonian

$$\delta^{-4} (H_0^{\text{Poi}} \circ \phi_{\text{sc}}) + (H_1^{\text{Poi}} \circ \phi_{\text{sc}})$$

and the symplectic form $d\lambda \wedge d\Lambda + idx \wedge dy$. The Hamiltonian (up to a constant) satisfies

$$\begin{aligned} \delta^{-4} (H_0^{\text{Poi}} \circ \phi_{\text{sc}}) &= -\frac{3}{2} \Lambda^2 + \frac{1}{\delta^4} F_{\text{P}}(\delta^2 \Lambda) + \frac{xy}{\delta^2}, \\ (H_1^{\text{Poi}} \circ \phi_{\text{sc}}) &= V(\lambda) + \mathcal{O}(\delta), \end{aligned}$$

with

$$\begin{aligned}
 V(\lambda) &= H_1^{\text{Poi}}(\lambda, 1, 0, 0; 0), \\
 F_p(z) &= \left(-\frac{1}{2(1+z)^2} - (1+z) \right) + \frac{3}{2} + \frac{3}{2}z^2 = \mathcal{O}(z^3).
 \end{aligned}
 \tag{18}$$

The function $V(\lambda)$, which we call the potential, has an explicit formula:

$$V(\lambda) = H_1^{\text{pol}}(1, \lambda; 0) = 1 - \cos \lambda - \frac{1}{\sqrt{2 + 2 \cos \lambda}},
 \tag{19}$$

where H_1^{pol} is defined in (6). Indeed, taking $\delta = 0$ on the change of coordinates (17), we have that $(\lambda, L, \eta, \xi) = (\lambda, 1, 0, 0)$. These coordinates, correspond with a circular orbit, $e = 0$, and applying (9) and (10), we obtain that $(r, \theta) = (1, \lambda)$.

We summarize the previous results in the following theorem.

Theorem 2.1. *The Hamiltonian system given by h in (1) expressed in coordinates (λ, Λ, x, y) defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + idx \wedge dy$ and the Hamiltonian*

$$H = H_p + H_{\text{osc}} + H_1,
 \tag{20}$$

with

$$H_p(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \quad H_{\text{osc}}(x, y; \delta) = \frac{xy}{\delta^2},
 \tag{21}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = (H_1^{\text{Poi}} \circ \phi_{\text{sc}}) - V(\lambda) + \frac{1}{\delta^4} F_p(\delta^2 \Lambda),
 \tag{22}$$

and H_1^{Poi} , F_p and V defined in (16), (18) and (19), respectively. Moreover, the Hamiltonian H is real-analytic² away from collision with the primaries.

Moreover, for $\delta > 0$ small enough:

- The critical point L_3 (see (7)) expressed in coordinates (λ, Λ, x, y) is given by

$$\mathfrak{L}(\delta) = (0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)),
 \tag{23}$$

with $|\mathfrak{L}_\Lambda(\delta)|, |\mathfrak{L}_x(\delta)|, |\mathfrak{L}_y(\delta)| \leq C$, for some constant $C > 0$ independent of δ .

- The point $\mathfrak{L}(\delta)$ is a saddle-center equilibrium point and its linear part is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

² Real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, \overline{y}, \overline{x}; \overline{\delta})$.

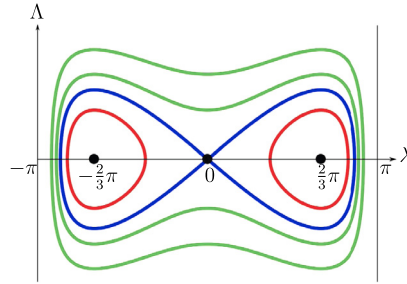


Fig. 2. Phase portrait of equation (26). On blue the two separatrices.

Therefore, it possesses one-dimensional stable and unstable manifolds and a two-dimensional center manifold.

The proof of Theorem 2.1 follows from the results obtained through Section 2.1.

Since the original Hamiltonian is symmetric with respect to the involution Φ in (3), the Hamiltonian H is reversible with respect to the involution

$$\tilde{\Phi}(\lambda, \Lambda, x, y) = (-\lambda, \Lambda, y, x). \tag{24}$$

From now on, we consider as “new” unperturbed Hamiltonian

$$H_0(\lambda, \Lambda, x, y; \delta) = H_p(\lambda, \Lambda) + H_{\text{osc}}(x, y; \delta), \tag{25}$$

which corresponds to an uncoupled pendulum-like Hamiltonian H_p and an oscillator H_{osc} , and we refer to H_1 as the perturbation.

2.2. The Hamiltonian H_p and its separatrices

In this section we analyze the 1-degree of freedom Hamiltonian $H_p(\lambda, \Lambda)$ introduced in (21),

$$H_p(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \quad V(\lambda) = 1 - \cos \lambda - \frac{1}{\sqrt{2 + 2 \cos \lambda}},$$

and the associated Hamiltonian system

$$\dot{\lambda} = -3\Lambda, \quad \dot{\Lambda} = -\sin \lambda \left(1 - \frac{1}{(2 + 2 \cos \lambda)^{\frac{3}{2}}} \right). \tag{26}$$

This Hamiltonian system has a singularity at $\lambda = \pi$, which corresponds to the collision with the small primary P , and a saddle at $(\lambda, \Lambda) = (0, 0)$ with two homoclinic connections or separatrices, see Fig. 2. From now on, we only consider the separatrix on the right; by symmetry (see (24)), the results obtained below are analogous for the separatrix on the left.

We consider the real-analytic time parametrization of the separatrix,

$$\begin{aligned} \sigma : \mathbb{R} &\rightarrow \mathbb{T} \times \mathbb{R} \\ t &\mapsto \sigma(t) = (\lambda_h(t), \Lambda_h(t)), \end{aligned} \tag{27}$$

with initial condition

$$\sigma(0) = (\lambda_0, 0) \quad \text{where} \quad \lambda_0 \in \left(\frac{2}{3}\pi, \pi\right).$$

Theorem 2.2. *The real-analytic time parametrization σ in (27) satisfies:*

- *It extends analytically to the strip*

$$\Pi_A = \{t \in \mathbb{C} : |\text{Im } t| < A\}, \tag{28}$$

where

$$A = \int_0^{a_+} \frac{1}{1-x} \sqrt{\frac{x}{3(x+1)(a_+ - x)(x - a_-)}} dx \approx 0.177744, \tag{29}$$

with $a_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$.

- *It has only two singularities in $\partial\Pi_A$ at $t = \pm iA$.*
- *There exists $\nu > 0$ such that, for $t \in \mathbb{C}$ with $|t - iA| < \nu$ and $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$, $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ can be expressed as*

$$\begin{aligned} \lambda_h(t) &= \pi + 3\alpha_+(t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}}, \\ \Lambda_h(t) &= -\frac{2\alpha_+}{3} \frac{1}{(t - iA)^{\frac{1}{3}}} + \mathcal{O}(t - iA)^{\frac{1}{3}}, \end{aligned} \tag{30}$$

with $\alpha_+ \in \mathbb{C}$ such that $\alpha_+^3 = \frac{1}{2}$.

An analogous result holds for $|t + iA| < \nu$, $\arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ and $\alpha_- = \overline{\alpha_+}$.

In a perturbed 2-body problem, the motion of a collision singularity corresponds to a branching point of order $\frac{2}{3}$, see [50]. This fact agrees with the results in Theorem 2.2, since the singularity $t = \pm iA$ corresponds to a complex collision with the small primary P .

We can also describe the zeroes of $\Lambda_h(t)$ in $\overline{\Pi_A}$.

Proposition 2.3. *Consider the real-analytic time parametrization $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ and the domain Π_A defined in (27) and (28) respectively. Then, $\Lambda_h(t)$ has only one zero in $\overline{\Pi_A}$ at $t = 0$.*

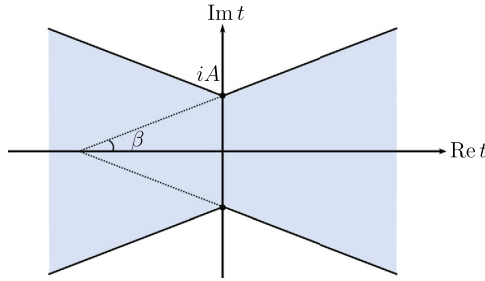


Fig. 3. Representation of the domain $\Pi_{A,\beta}^{\text{ext}}$ in (31).

We can expand the region of analyticity of the time parametrization σ .

Corollary 2.4. *There exists $0 < \beta < \frac{\pi}{2}$ such that the real-analytic time parametrization $\sigma(t)$ extends analytically to*

$$\begin{aligned} \Pi_{A,\beta}^{\text{ext}} = \{t \in \mathbb{C} : |\text{Im } t| < \tan \beta \text{Re } t + A\} \cup \\ \{t \in \mathbb{C} : |\text{Im } t| < -\tan \beta \text{Re } t + A\}. \end{aligned} \tag{31}$$

(See Fig. 3.) Moreover,

1. σ has only two singularities on $\partial\Pi_{A,\beta}^{\text{ext}}$ at $t = \pm iA$.
2. Λ_h has only one zero in the closure of $\Pi_{A,\beta}^{\text{ext}}$ at $t = 0$.

Proof. By [19], there exists $T > 0$ such that $\sigma(t)$ is analytic in $\{|\text{Re } t| > T\}$. Moreover, applying Theorem 2.2, $\sigma(t)$ has two branching points at $t = \pm iA$ and can be expressed as in (30) in the domains

$$\begin{aligned} D_1 = \{|t - iA| < \nu, \arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})\} \cup \\ \{|t + iA| < \nu, \arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})\}, \end{aligned}$$

for some $\nu > 0$. This implies that the only singularities in D_1 are at $t = \pm iA$.

Thus, we only need to check Item 1 in

$$(\Pi_{A,\beta}^{\text{ext}} \cap \{|\text{Re } t| \leq T\}) \setminus D_1.$$

To this end, note that, by Theorem 2.2, $\sigma(t)$ is analytic in the compact set $D_2 = (\overline{\Pi_A} \cap \{|\text{Re } t| \leq T\}) \setminus D_1$. Therefore, there exists a cover of ∂D_2 by open balls centered in ∂D_2 where $\sigma(t)$ is analytic. Moreover, since ∂D_2 is compact, it has a finite subcover. This implies that there exists $\eta > 0$ such that we can extend the analyticity domain of $\sigma(t)$ to

$$(\Pi_{A+\eta} \cap \{|\text{Re } t| \leq T\}) \setminus D_1.$$

Then, taking $\beta = \arctan(\eta/T)$, $\sigma(t)$ is analytic in $(\Pi_{A,\beta}^{\text{ext}} \cap \{| \text{Re } t | \leq T\}) \setminus D_1$. The prove of Item 2 follows the same lines. \square

2.3. Derivation of the inner equation

The inner equation associated to the Hamiltonian H in (20) describes the dominant behavior of suitable complex parametrizations of the invariant manifolds close to (one of) the singularities $\pm iA$ of the unperturbed separatrix. Let us explain how this equation arises from the Hamiltonian H .

First, we consider the translation of the equilibrium point $\mathfrak{L}(\delta)$ to the origin,

$$\phi_{\text{eq}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, \Lambda, x, y) + \mathfrak{L}(\delta). \tag{32}$$

Second, to measure the distance of the stable and unstable manifolds, we parameterize them as graphs. In the unperturbed case, we know that the invariant manifolds coincide along the separatrix $(\lambda_h(t), \Lambda_h(t), 0, 0)$. Since we need to involve, in some sense, the time; we consider as a new independent variable u such that $\lambda_h(u) = \lambda$. Notice that $\dot{u} = 1$ for the unperturbed system. To this end, we consider the symplectic change of coordinates

$$\phi_{\text{sep}} : (u, w, x, y) \rightarrow (\lambda, \Lambda, x, y), \quad \lambda = \lambda_h(u), \quad \Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}, \tag{33}$$

where $\sigma = (\lambda_h, \Lambda_h)$ is the parametrization of the separatrix studied in Theorem 2.2. Notice that, except for $u = 0$ (see Proposition 2.3), the perturbed manifolds can be expressed as a graph and the change (33) is well defined.

The Hamiltonian H , written in these coordinates and after the translation ϕ_{eq} in (32), becomes

$$H^{\text{sep}} = H_0^{\text{sep}} + H_1^{\text{sep}}, \tag{34}$$

with

$$H_0^{\text{sep}}(w, x, y) = w + \frac{xy}{\delta^2}, \quad H_1^{\text{sep}} = H \circ (\phi_{\text{eq}} \circ \phi_{\text{sep}}) - H_0^{\text{sep}}.$$

Since we look for the perturbed manifolds as graphs with respect to u , we consider parametrizations

$$z^\diamond(u) = (w^\diamond(u), x^\diamond(u), y^\diamond(u))^T, \quad \text{for } \diamond = \text{u, s,}$$

such that the unstable and stable invariant manifolds of H associated to $\mathfrak{L}(\delta)$ can be expressed as

$$W^\diamond = \left\{ \left(\lambda_h(u), \Lambda_h(u) - \frac{w^\diamond(u)}{3\Lambda_h(u)}, x^\diamond(u), y^\diamond(u) \right) + \mathfrak{L}(\delta) \right\}, \quad \text{for } \diamond = \text{u, s,}$$

with u belonging to appropriate domains. The proof of existence of z^u and z^s defined in appropriate (complex) domains requires a significant amount of technicalities. We present this result in the companion paper, see [10].

Due to the slow-fast character of the system, to capture the asymptotic first order of the difference $\Delta z = z^u - z^s$, we need to give the main terms of this difference close to the singularities, concretely, up to distance of order δ^2 . To this end, we derive the inner equation, see [1,5], which contains the first order of the Hamiltonian H^{sep} (see (34)) close to (one of) the singularities and is independent of the small parameter δ . That is, we look, for instance, for a Hamiltonian which is good approximations of H^{sep} in a neighborhood of $u = iA$. Here, we focus on a domain near the singularity $u = iA$, but a similar analysis can be done near $u = -iA$.

Since we need to control the difference up to distance of order δ^2 of the singularity $u = iA$, we consider U such that

$$u - iA = \delta^2 U.$$

Notice that we can take $|U| \gg 1$ independent of δ . Close to the singularity $u = iA$, the homoclinic connection is not the dominant term of the perturbed invariant manifolds anymore. Let us be more precise, take $\Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}$, and recall that, by Theorem 2.2, we have

$$\Lambda_h(u) \sim -\frac{2\alpha_+}{3}(u - iA)^{-\frac{1}{3}}, \quad \text{for } |u - iA| \leq \nu,$$

or equivalently,

$$\Lambda_h(iA + \delta^2 U) \sim -\frac{2\alpha_+}{3\delta^{\frac{2}{3}}U^{\frac{1}{3}}} \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{2}{3}}U^{\frac{1}{3}}}\right).$$

Then,

$$w^\diamond(iA + \delta^2 U) \sim 3\Lambda_h^2(iA + \delta^2 U) \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{4}{3}}U^{\frac{2}{3}}}\right). \tag{35}$$

In addition, the unperturbed Hamiltonian must have all of its terms of the same order. Therefore,

$$\frac{x^\diamond(iA + \delta^2 U)y^\diamond(iA + \delta^2 U)}{\delta^2} \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{4}{3}}U^{\frac{2}{3}}}\right). \tag{36}$$

By symmetry, $x^\diamond(iA + \delta^2 U), y^\diamond(iA + \delta^2 U) \sim \mathcal{O}(\delta^{\frac{1}{3}}U^{-\frac{1}{3}})$.

To avoid the dependence on the inner equation with respect to α_+ (see Theorem 2.2) and to keep the symplectic character, we perform the scaling

$$\phi_{\text{in}} : (U, W, X, Y) \rightarrow (u, w, x, y), \tag{37}$$

given by

$$U = \frac{u - iA}{\delta^2}, \quad W = \delta^{\frac{4}{3}} \frac{w}{2\alpha_+^2}, \quad X = \frac{x}{\delta^{\frac{1}{3}} \sqrt{2}\alpha_+}, \quad Y = \frac{y}{\delta^{\frac{1}{3}} \sqrt{2}\alpha_+},$$

and the time scaling $\tau = \delta^2 t$. The heuristics above lead us to assume that $(U, W, X, Y) = \mathcal{O}(1)$ when $u - iA = \mathcal{O}(\delta^2)$. In the following proposition, by applying the change of coordinates ϕ_{in} , we obtain the inner equation of the Hamiltonian H^{sep} .

Proposition 2.5. *The Hamiltonian equations associated to (34) expressed in inner coordinates (see (37)) are Hamiltonian with respect to*

$$H^{\text{in}} = \mathcal{H} + H_1^{\text{in}},$$

where

$$\mathcal{H}(U, W, X, Y) = H^{\text{in}}(U, W, X, Y; \delta)|_{\delta=0} = W + XY + \mathcal{K}(U, W, X, Y), \quad (38)$$

with

$$\mathcal{K}(U, W, X, Y) = -\frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}} \left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1 \right) \quad (39)$$

and

$$\begin{aligned} \mathcal{J}(U, W, X, Y) = & \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X + Y)}{9U} \left(W - \frac{2}{3U^{\frac{2}{3}}} \right) \\ & - \frac{4i(X - Y)}{3U^{\frac{2}{3}}} - \frac{X^2 + Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}. \end{aligned} \quad (40)$$

Moreover, if $c_1^{-1} \leq |U| \leq c_1$ and $|(W, X, Y)| \leq c_2$ for some $c_1 > 1$ and $0 < c_2 < 1$, we have that there exist $b_0, \gamma_1, \gamma_2 > 0$ independent of δ, c_1, c_2 such that

$$|H_1^{\text{in}}(U, W, X, Y; \delta)| \leq b_0 c_1^{\gamma_1} c_2^{\gamma_2} \delta^{\frac{4}{3}}.$$

Remark 2.6. The change of coordinates (37) allows us to study an approximation of the invariant manifolds $z^{\text{u,s}}(u)$ near the singularity $u = iA$. To obtain an approximation near $u = -iA$, one can proceed analogously by

$$U = \frac{u + iA}{\delta^2}, \quad W = \delta^{\frac{4}{3}} \frac{w}{2\alpha_-^2}, \quad X = \frac{x}{\delta^{\frac{1}{3}} \sqrt{2}\alpha_-}, \quad Y = \frac{y}{\delta^{\frac{1}{3}} \sqrt{2}\alpha_-},$$

where $\alpha_- = \overline{\alpha_+}$ (see Theorem 2.2).

2.4. *The solutions of the inner equation and their difference*

We devote this section to study two special solutions of the inner equation given by the Hamiltonian \mathcal{H} in (38). We introduce $Z = (W, X, Y)$ and the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{41}$$

Then, the equation associated to the Hamiltonian \mathcal{H} can be written as

$$\begin{cases} \dot{U} = 1 + g(U, Z), \\ \dot{Z} = \mathcal{A}Z + f(U, Z), \end{cases} \tag{42}$$

where $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g = \partial_W \mathcal{K}$.

We look for solutions of this equation parametrized as graphs with respect to U , namely we look for functions

$$Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T, \quad \text{for } \diamond = \text{u, s,}$$

satisfying the invariance condition given by (42), that is

$$\partial_U Z_0^\diamond = \mathcal{A}Z_0^\diamond + \mathcal{R}[Z_0^\diamond], \quad \text{for } \diamond = \text{u, s,} \tag{43}$$

where

$$\mathcal{R}[\varphi](U) = \frac{f(U, \varphi) - g(U, \varphi)\mathcal{A}\varphi}{1 + g(U, \varphi)}. \tag{44}$$

In order to “select” the solutions we are interested in, we point out that, since we need some uniformity with respect to δ and $U = \delta^{-2}(u - iA)$, then $\text{Re } U \rightarrow \pm\infty$ as $\delta \rightarrow 0$, depending on the sign of $\text{Re } U$. Then, according to (35) and (36), we deduce that $(W, X, Y) \rightarrow 0$ as $\text{Re } U \rightarrow \pm\infty$. For that reason, we look for Z_0^\diamond satisfying the asymptotic conditions

$$\lim_{\text{Re } U \rightarrow -\infty} Z_0^{\text{u}}(U) = 0, \quad \lim_{\text{Re } U \rightarrow +\infty} Z_0^{\text{s}}(U) = 0. \tag{45}$$

In fact, for a fixed $\beta_0 \in (0, \frac{\pi}{2})$, we look for functions Z_0^{u} and Z_0^{s} satisfying (43), (45) defined in the domains

$$\mathcal{D}_\kappa^{\text{u}} = \{U \in \mathbb{C} : |\text{Im } U| \geq \tan \beta_0 \text{Re } U + \kappa\}, \quad \mathcal{D}_\kappa^{\text{s}} = -\mathcal{D}_\kappa^{\text{u}}, \tag{46}$$

respectively, for some $\kappa > 0$ big enough (see Fig. 4). We analyze the difference $\Delta Z_0 = Z_0^{\text{u}} - Z_0^{\text{s}}$ in the overlapping domain

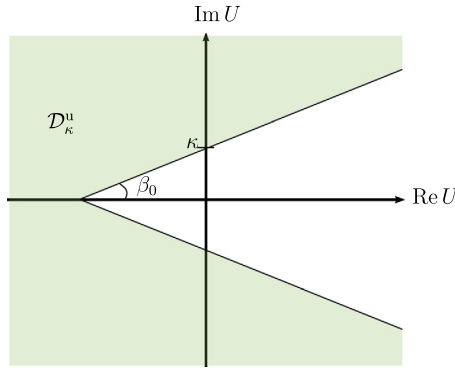


Fig. 4. The inner domain, \mathcal{D}_κ^u , for the unstable case.

$$\mathcal{E}_\kappa = \mathcal{D}_\kappa^u \cap \mathcal{D}_\kappa^s \cap \{U \in \mathbb{C} : \text{Im } U < 0\}. \tag{47}$$

Theorem 2.7. *There exist $\kappa_0, b_1 > 0$ such that for any $\kappa \geq \kappa_0$, the equation (43) has analytic solutions $Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T$, for $U \in \mathcal{D}_\kappa^\diamond$ and $\diamond = u, s$, satisfying*

$$|U^{\frac{8}{3}}W_0^\diamond(U)| \leq b_1, \quad |U^{\frac{4}{3}}X_0^\diamond(U)| \leq b_1, \quad |U^{\frac{4}{3}}Y_0^\diamond(U)| \leq b_1. \tag{48}$$

In addition, there exist $\Theta \in \mathbb{C}$ and $b_2 > 0$ independent of κ , and a function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that

$$\Delta Z_0(U) = Z_0^u(U) - Z_0^s(U) = \Theta e^{-iU} \left((0, 0, 1)^T + \chi(U) \right), \tag{49}$$

and, for $U \in \mathcal{E}_\kappa$,

$$|U^{\frac{7}{3}}\chi_1(U)| \leq b_2, \quad |U^2\chi_2(U)| \leq b_2, \quad |U\chi_3(U)| \leq b_2.$$

Remark 2.8. This theorem implies that $\Theta = \lim_{\text{Im } U \rightarrow -\infty} \Delta Y_0(U)e^{iU}$. Thus, we can obtain a numerical approximation of the constant Θ . Indeed, for $\rho > \kappa_0$, we can define

$$\Theta_\rho = |\Delta Y_0(-i\rho)| e^\rho, \tag{50}$$

which, for ρ big enough, satisfies $\Theta_\rho \approx |\Theta|$.

To compute $\Delta Y_0(-i\rho) = Y_0^u(-i\rho) - Y_0^s(-i\rho)$, we first look for good approximations of $Z_0^u(U)$ for $\text{Re } U \ll -1$ and of $Z_0^s(U)$ for $\text{Re } U \gg 1$, as power series in $U^{-\frac{1}{3}}$. One can easily check that $Z_0^u(U)$ as $\text{Re } U \rightarrow -\infty$ and $Z_0^s(U)$ as $\text{Re } U \rightarrow +\infty$ have the same asymptotics expansion:

$$\begin{aligned} W_0^\diamond(U) &= \frac{4}{243 U^{\frac{8}{3}}} - \frac{172}{2187 U^{\frac{14}{3}}} + \mathcal{O}\left(U^{-\frac{20}{3}}\right), \\ X_0^\diamond(U) &= -\frac{2i}{9 U^{\frac{4}{3}}} + \frac{28}{81 U^{\frac{7}{3}}} + \frac{20i}{27 U^{\frac{10}{3}}} - \frac{16424}{6561 U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right), \end{aligned}$$

Table 1
 Computation of Θ_ρ , as defined in (50), for different values of $\rho \leq 23$.

ρ	$ \Delta Y_0(-i\rho) $	e^ρ	Θ_ρ
13	$3.7 \cdot 10^{-6}$	$4.4 \cdot 10^5$	1.6373
14	$1.4 \cdot 10^{-6}$	$1.2 \cdot 10^6$	1.6361
15	$5.0 \cdot 10^{-7}$	$3.3 \cdot 10^6$	1.6351
16	$1.8 \cdot 10^{-7}$	$8.9 \cdot 10^6$	1.6341
17	$3.7 \cdot 10^{-8}$	$2.4 \cdot 10^7$	1.6333
18	$6.8 \cdot 10^{-8}$	$6.6 \cdot 10^7$	1.6326
19	$9.1 \cdot 10^{-9}$	$1.8 \cdot 10^8$	1.6320
20	$3.4 \cdot 10^{-9}$	$4.9 \cdot 10^8$	1.6315
21	$1.2 \cdot 10^{-9}$	$1.3 \cdot 10^9$	1.6312
22	$4.6 \cdot 10^{-10}$	$3.6 \cdot 10^9$	1.6313
23	$1.7 \cdot 10^{-10}$	$9.7 \cdot 10^9$	1.6323

$$Y_0^\diamond(U) = \frac{2i}{9U^{\frac{4}{3}}} + \frac{28}{81U^{\frac{7}{3}}} - \frac{20i}{27U^{\frac{10}{3}}} - \frac{16424}{6561U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right).$$

We use these expressions to set up the initial conditions for the numerical integration for computing $\Delta Y_0(-i\rho)$. We take as initial points the value of the truncated power series at order $U^{-\frac{13}{3}}$ at $U = 1000 - i\rho$ (for $\diamond = s$) and $U = -1000 - i\rho$ (for $\diamond = u$). (See Table 1.) We perform the numerical integration for different values of $\rho \leq 23$ and an integration solver with tolerance 10^{-12} .

Table 1 shows that the constant Θ is approximately 1.63 which indicates that it is not zero. We expect that this computation method can be implemented rigorously [11].

3. Analytic continuation of the separatrix

In this section we prove Theorem 2.2 and Proposition 2.3, which deal with the study of the complex singularities and zeroes of the analytic extension of the time-parametrization $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ of the homoclinic connection given in (27).

Let us recall that $\sigma(t)$ is a solution of the Hamiltonian system H_p in (21) and it is found at the energy level $H_p = -\frac{1}{2}$. Therefore,

$$(\dot{\lambda})^2 = 15 - 12 \cos^2\left(\frac{\lambda}{2}\right) - \frac{3}{\cos\left(\frac{\lambda}{2}\right)}. \tag{51}$$

Equation (51) can be solved as $t = F(\lambda)$, where F is a function defined by means of an integral. Prove Theorem 2.2 and Proposition 2.3 boils down to studying the analytic continuation of F^{-1} .

We divide the proof of Theorem 2.2 into three main steps. First, in Section 3.1, we perform the change of variables $q = \cos(\frac{\lambda}{2})$ and rephrase Theorem 2.2 in terms of $q(t)$ (Theorem 3.1). Then, in Section 3.2, we analyze all the possible types of singularities that $q(t)$ may have (Proposition 3.2), which turn out to be poles or branching points. In

addition we prove that all the singularities have to be given by integrals along suitable complex paths. Finally, in Section 3.3, taking into account all complex paths leading to singularities, we prove that the singularities of $q(t)$ with smaller imaginary part (in the first Riemann sheet of $q(t)$) are $t = \pm iA$.

Finally, in Section 3.4, we use the results obtained in the previous sections, to analyze the zeroes of $\Lambda_h(t)$ in the strip of analyticity Π_A (see (28)), thus proving Proposition 2.3.

In order to simplify the notation, through the rest of the section we denote by C any positive constant independent of t .

3.1. Reformulation of Theorem 2.2

To prove Theorem 2.2, it is more convenient to work with the variable $q = \cos(\frac{\lambda}{2})$ instead of λ . Notice that this change of coordinates, when restricted to $\lambda \in (0, \pi)$, is a diffeomorphism.

Theorem 3.1. Consider $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ the real-analytic time parametrization introduced in (27) and denote $a_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. Then, $q(t) = \cos\left(\frac{\lambda_h(t)}{2}\right)$ satisfies

$$q(t) \in [a_+, 1) \quad \text{for } t \in \mathbb{R}, \quad q(0) = a_+, \tag{52}$$

and the differential equation

$$\dot{q}^2 = \frac{3}{q}(q-1)^2(q+1)(q-a_-)(q-a_+). \tag{53}$$

Moreover, we have that:

- The function $q(t)$ extends analytically to the strip Π_A defined in (28).
- The function $q(t)$ has only two singularities on $\partial\Pi_A$ at $t = \pm iA$.
- There exists $\nu > 0$ such that, for $t \in \mathbb{C}$ with $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ and $|t - iA| < \nu$, we have

$$q(t) = -\frac{3\alpha_+}{2}(t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}}, \tag{54}$$

with $\alpha_+ \in \mathbb{C}$ such that $\alpha_+^3 = \frac{1}{2}$.

An analogous result holds for $|t + iA| < \nu$ and $\arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ with $\alpha_- = \overline{\alpha_+}$.

Theorem 2.2 is a corollary of Theorem 3.1.

Proof of Theorem 2.2. To obtain Theorem 2.2 from Theorem 3.1 it is enough to prove that $\Lambda_h(t)$ has no singularities in $\overline{\Pi_A} \setminus \{\pm iA\}$ and that $(\lambda_h(t), \Lambda_h(t))$ can be expressed as in (30) close to $t = \pm iA$.

Since $\dot{\lambda}_h = -3\Lambda_h$ and using the change of coordinates $q = \cos(\frac{\lambda}{2})$, we have that

$$\Lambda_h^2(t) = \frac{4}{9} \frac{\dot{q}^2(t)}{(1 - q^2(t))}. \tag{55}$$

We claim that, if $\Lambda_h(t)$ has a singularity at $t = t^*$, then $\Lambda_h^2(t)$ has a singularity at $t = t^*$ as well. Indeed, the only case when the previous affirmation could be false is if t^* is a branching point of order $\frac{k}{2}$ with $k \geq 1$ an odd natural number. In this case,

$$\lambda_h(t) = \lambda_h(t^*) + C(t - t^*)^{\frac{k}{2}-1} (1 + \mathcal{O}(t - t^*)^\beta), \quad \text{when } 0 < |t - t^*| \ll 1,$$

for some $\beta > 0$. Replacing this expression in (51) and comparing orders we see that this case is not possible.

Thus, we proceed to prove that $\Lambda_h(t)$ has no singularities in $\overline{\Pi_A} \setminus \{\pm iA\}$. Let us assume it has. That is, there exists $t^* \in \overline{\Pi_A} \setminus \{\pm iA\}$ such that $\Lambda_h(t)$ is singular at $t = t^*$. Note that Theorem 3.1 implies that $q(t)$ and $\dot{q}(t)$ are analytic in a neighborhood of $t^* \in \overline{\Pi_A} \setminus \{\pm iA\}$.

1. If $q^2(t^*) \neq 1$, $1/(1 - q^2(t))$ is analytic for $0 < |t - t^*| \ll 1$. Since $\dot{q}(t)$ is also analytic in this neighborhood, (55) implies that $\Lambda_h^2(t)$ has no singularity at $t = t^*$ and we reach a contradiction.
2. If $q^2(t^*) = 1$, by (53) and (55), we deduce that

$$\Lambda_h^2 = \frac{4}{3q} (1 - q)(q - a_+)(q - a_-). \tag{56}$$

Since by Theorem 3.1 q is analytic in $\overline{\Pi_A} \setminus \{\pm iA\}$, then Λ_h^2 must be as well.

Finally, we notice that, by equations (54) and (55), we have

$$\Lambda_h^2(t) = \frac{4}{9} \alpha_\pm^2 (t \mp iA)^{-\frac{2}{3}} + \mathcal{O}(1), \quad \text{when } 0 < |t \mp iA| \ll 1.$$

Therefore, $\Lambda_h(t)$ has branching points of order $-\frac{1}{3}$ in $t = \pm iA$. Moreover, integrating the expression for $\Lambda_h(t)$ and applying that $q(t) = \cos(\frac{\lambda_h(t)}{2})$ (and (54)), it is immediate to see that $\lambda_h(t)$ has branching points of order $\frac{2}{3}$ at $t = \pm iA$ and can be expressed as in (30) close to $t = \pm iA$. \square

We devote Sections 3.2 and 3.3 to prove Theorem 3.1. The statements (52) and (53) are straightforward by applying the change of coordinates $q = \cos(\frac{\lambda}{2})$ to equation (51).

We divide the rest of the proof of Theorem 3.1 into two parts. In Section 3.2 we classify the singularities of $q(t)$ and introduce a way to compute them using integration in complex paths. Finally, in Section 3.3 we prove that the singularities of $q(t)$ with smallest imaginary part are $t = \pm iA$ and are branching points of order $\frac{2}{3}$.

3.2. Classification of the singularities of $q(t)$

Equation (53) with initial condition $q(0) = a_+ = -\frac{1}{2} + \frac{\sqrt{2}}{2}$ is equivalent to

$$t = \int_{a_+}^{q(t)} f(s)ds, \quad \text{for } t \in \mathbb{R},$$

where

$$f(q) = \frac{1}{q-1} \sqrt{\frac{q}{3(q+1)(q-a_+)(q-a_-)}},$$

is defined in $\mathbb{R} \setminus \{[a_-, -1] \cup (0, a_+] \cup \{1\}\}$ with $a_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. From [19], we know that there exist $v > 0$ such that $q(t)$ can be extended to the open complex strip

$$\Pi_v = \{t \in \mathbb{C} : |\text{Im } t| < v\},$$

and $q(t)$ has singularities in $\partial\Pi_v$. Namely,

$$t = \int_{a_+}^{q(t)} f(q)dq, \quad \text{for } t \in \Pi_v. \tag{57}$$

Since f is a multi-valued function in the complex plane, in order to analyze the possible values of $\int_{a_+}^{q(t)} f(s)ds$, we consider its complete analytic continuation. That is,

$$\begin{aligned} \hat{f} : \mathcal{R}_f &\rightarrow \mathbb{C} \\ (q; \arg \mathbf{g}(q)) &\mapsto \frac{\sqrt{\mathbf{g}(q)}}{q-1}, \quad \text{where } \mathbf{g}(q) = \frac{q}{3(q+1)(q-a_+)(q-a_-)}, \end{aligned} \tag{58}$$

and \mathcal{R}_f is the Riemann surface associated to f . We define $\mathbf{p} : \mathcal{R}_f \rightarrow \mathbb{C}$ as the projection to the complex plane. We choose the first Riemann sheet to correspond to $\arg \mathbf{g}(q) \in (-\pi, \pi]$. Accordingly the second Riemann sheet corresponds to $(\pi, 3\pi]$.

To integrate \hat{f} along a path $\gamma \subset \mathcal{R}_f$, we introduce the notation

$$\int_{\gamma} \hat{f}(q)dq = \int_{q_0}^{q_{\text{end}}} \hat{f}d\gamma = \int_{s_0}^{s_{\text{end}}} \hat{f}(\gamma(s))\gamma'(s)ds,$$

such that $\gamma : (s_0, s_{\text{end}}) \rightarrow \mathcal{R}_f$ where $\lim_{s \rightarrow s_0} \mathbf{p}\gamma(s) = q_0$ and $\lim_{s \rightarrow s_{\text{end}}} \mathbf{p}\gamma(s) = q_{\text{end}}$. Moreover, we assume that the paths $\gamma \subset \mathcal{R}_f$ are \mathcal{C}^0 and \mathcal{C}^1 -piecewise. Therefore, by (57), we have that

$$t = \int_{a_+}^{q(t)} \hat{f} d\gamma, \quad \text{for } t \in \Pi_v. \tag{59}$$

Now, for $q \in \mathcal{R}_f$ and an integration path $\gamma \subset \mathcal{R}_f$, we define the function G as the right hand side of (59),

$$G(q) = \int_{a_+}^q \hat{f} d\gamma.$$

Notice that for a given $q \in \mathcal{R}_f$, $G(q)$ may depend on the integration path, and therefore, G may be multi-valued on \mathcal{R}_f . However, by (59), G is single-valued when

$$t = G(q(t)), \quad \text{for } t \in \Pi_v.$$

We use G to characterize and locate the singularities of $q(t)$. Indeed, if function $G(q)$ is biholomorphic at $q = q^*$, then $q(t)$ is analytic at a neighborhood of all values of t such that $q(t) = q^*$. Therefore, $q(t)$ may have singularities at $t = t^*$ when the hypothesis of the Inverse Function Theorem is not satisfied for G . That is, for $q(t^*) = q^*$ such that either

$$G'(q^*) = 0, \quad G \notin \mathcal{C}^1 \text{ at } q = q^*, \quad \text{or} \quad |q^*| \rightarrow \infty. \tag{60}$$

Namely, when there exist q^* and $\gamma \subset \mathcal{R}_f$ satisfying (60), such that

$$t^* = G(q^*) = \int_{a_+}^{q^*} \hat{f} d\gamma. \tag{61}$$

Since G is a multivalued function, the values of t^* can, and in fact will, depend on the integration path on $\gamma \subset \mathcal{R}_f$. From (58) and (60), one deduces that the singularities may take place only if $q(t^*) = q^*$ with

$$q^* = 0, 1, -1, a_+, a_- \quad \text{and} \quad |q^*| \rightarrow \infty. \tag{62}$$

The following proposition proves that we only need to consider $|q^*| \rightarrow \infty$ and $q^* = 0$.

Proposition 3.2. *Let $q(t)$ be a solution of equation (53) with initial condition $q(0) = a_+$. Then, the singularities $t^* \in \mathbb{C}$ of the analytic extension of $q(t)$ are characterized by either*

$$t^* = \int_{a_+}^0 \hat{f} d\gamma, \quad \text{or} \quad t^* = \int_{a_+}^{\infty} \hat{f} d\gamma,$$

for some path $\gamma \subset \mathcal{R}_f$.

Moreover:

- If $t^* = \int_{a_+}^0 \hat{f} d\gamma$ and $\text{Im } t^* > 0$ with $\arg(t - t^*) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$, then

$$q(t) = -\frac{3\alpha}{2}(t - t^*)^{\frac{2}{3}} + \mathcal{O}(t - t^*)^{\frac{4}{3}}, \quad \text{for } 0 < |t - t^*| \ll 1, \tag{63}$$

where $\alpha \in \mathbb{C}$ satisfies $\alpha^3 = \frac{1}{2}$. If $\text{Im } t^* < 0$ and $\arg(t - t^*) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, the same holds true.

- If $t^* = \int_{a_+}^\infty \hat{f} d\gamma$, then

$$q(t) = -\frac{1}{\sqrt{3}(t - t^*)} (1 + \mathcal{O}(t - t^*)), \quad \text{for } 0 < |t - t^*| \ll 1. \tag{64}$$

Proof. To prove this result first we need to analyze all the possible values of q^* that may lead to singularities, (see (62)). We will use the expressions of t and t^* given in (59) and (61), respectively.

1. If $|q^*| \rightarrow \infty$, we have

$$t - t^* = \int_{\infty}^{q(t)} \hat{f} d\gamma.$$

Then, since

$$\hat{f}(q) = \frac{1}{\sqrt{3}q^2} + \mathcal{O}\left(\frac{1}{q^3}\right), \quad \text{for } |q| \gg 1,$$

we obtain

$$t - t^* = -\frac{1}{\sqrt{3}q} + \mathcal{O}\left(\frac{1}{q^2}\right),$$

which implies (64).

2. If $q^* = a_+$, we have that

$$t - t^* = \int_{a_+}^q \hat{f} d\gamma. \tag{65}$$

The function \hat{f} can be written as

$$\hat{f}(q) = \frac{h_{a_+}(q)}{\sqrt{q - a_+}},$$

for some function h_{a_+} which is analytic and non-zero in a neighborhood of a_+ . Then, h_{a_+} can be written as $h_{a_+}(q) = \sum_{k=0}^{\infty} c_k(q - a_+)^k$, with $c_0 \neq 0$ and, for $0 < |q - a_+| \ll 1$, we obtain from (65)

$$t - t^* = \sqrt{q - a_+} \sum_{k=0}^{\infty} \frac{c_k}{k + \frac{1}{2}} (q - a_+)^k,$$

which implies $(t - t^*)^2 = g_{a_+}(q)$ with

$$g_{a_+}(q) = (q - a_+) \left(\sum_{k=0}^{\infty} \frac{c_k}{k + \frac{1}{2}} (q - a_+)^k \right)^2.$$

The function $g_{a_+}(q)$ is analytic on a neighborhood of a_+ and satisfies $g_{a_+}(a_+) = 0$, $g'_{a_+}(a_+) = 4c_0^2 \neq 0$. Thus, applying the Inverse Function Theorem,

$$q(t) = g_{a_+}^{-1}((t - t^*)^2), \quad \text{for } 0 < |t - t^*| \ll 1. \tag{66}$$

Therefore, $q(t)$ is analytic for $|t - t^*| \ll 1$. One can analogously prove that the same happens at $q^* = a_-$ and $q^* = -1$.

- 3. The value $q^* = 1$ corresponds to the saddle point $(\lambda, \Lambda) = (0, 0)$ of H_p (see (21)). Indeed,

$$\int_1^q \hat{f} d\gamma \text{ is divergent.} \tag{67}$$

This implies that $q(t) \neq 1$ for any complex t .

- 4. If $q^* = 0$,

$$t - t^* = \int_0^q \hat{f} d\gamma.$$

We can introduce $h_0(q) = \frac{1}{\sqrt{q}} \hat{f}(q)$ which is analytic and of the form $h_0(q) = \sum_{k=0}^{\infty} c_k q^k$ with $c_0 \neq 0$. Then, for $0 < |q| \ll 1$, we obtain

$$(t - t^*)^{\frac{2}{3}} = g_0(q) = q \left(\sum_{k=0}^{\infty} \frac{c_k}{k + \frac{3}{2}} q^k \right)^{\frac{2}{3}} = q \left(\frac{2c_0}{3} + \mathcal{O}(q) \right)^{\frac{2}{3}},$$

where $g_0(q)$ is analytic in a neighborhood of $q = 0$ and satisfies $g_0(0) = 0$, $g'_0(0) = (\frac{2}{3}c_0)^{\frac{2}{3}} \neq 0$. Thus, applying the Inverse Function Theorem,

$$q(t) = g_0^{-1} \left((t - t^*)^{\frac{2}{3}} \right) = \sum_{k=1}^{\infty} C_k (t - t^*)^{\frac{2k}{3}}, \text{ for } 0 < |t - t^*| \ll 1, \tag{68}$$

for some $C_k \in \mathbb{C}$ and choosing the Riemann sheet $\arg(t - t^*) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ for $\text{Im } t^* > 0$ and $\arg(t - t^*) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ for $\text{Im } t^* < 0$. Replacing (68) in equation (53) we obtain $C_1 = -\frac{3\alpha}{2}$, where $\alpha \in \mathbb{C}$ satisfies $\alpha^3 = \frac{1}{2}$, which implies (63). \square

3.3. Singularities closest to the real axis

Proposition 3.2 provides the type of singularities that $q(t)$ may posses in its first Riemann sheet. Then, to prove Theorem 3.1, we look for those singularities which are closest to the real axis. To do so, we analyze

$$\int_{a_+}^0 \hat{f}d\gamma, \quad \int_{a_+}^{\infty} \hat{f}d\gamma, \tag{69}$$

along all paths $\gamma \subset \mathcal{R}_f$ with such endpoints and prove that, for all possible paths, the only singularities in the complex strip $\overline{\Pi}_A$ (see (28)) are $t = \pm iA$.

We introduce the following paths

$$\mathcal{P}_0 = \left\{ \gamma : (s_0, s_{\text{end}}) \rightarrow \mathcal{R}_f : \lim_{s \rightarrow s_0} (\mathbf{p}\gamma(s), \arg \mathbf{g}(\gamma(s))) = (a_+, 0), \lim_{s \rightarrow s_{\text{end}}} \mathbf{p}\gamma(s) = 0 \right\},$$

$$\mathcal{P}_{\infty} = \left\{ \gamma : (s_0, s_{\text{end}}) \rightarrow \mathcal{R}_f : \lim_{s \rightarrow s_0} (\mathbf{p}\gamma(s), \arg \mathbf{g}(\gamma(s))) = (a_+, 0), \lim_{s \rightarrow s_{\text{end}}} |\mathbf{p}\gamma(s)| = \infty \right\},$$

with the natural projection $\mathbf{p} : \mathcal{R}_f \rightarrow \mathbb{C}$.

We have chosen $\theta_0 := \lim_{s \rightarrow s_0} \arg \mathbf{g}(\gamma(s)) = 0$ without loss of generality. Indeed, since $q(t) \in [a_+, 1)$ for $t \in \mathbb{R}$ (see (52)), it could be either 0 or 2π . The paths with asymptotic argument 2π can be analyzed analogously and lead to singularities with opposed signed with respect to those given by paths in $\mathcal{P}_0, \mathcal{P}_{\infty}$.

Furthermore, the asymptotic argument of the paths at its endpoints is not specified since it is given by the path itself.

For a given path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_{\infty}$, we can define a path $T[\gamma] : [s_0, s_{\text{end}}) \rightarrow \mathbb{C}$ in the t -plane (or, more precisely, on the Riemann surface of $q(t)$) as

$$T[\gamma](s) = \int_{s_0}^s \hat{f}(\gamma(\tau))\gamma'(\tau)d\tau, \quad \text{for } s \in [s_0, s_{\text{end}}). \tag{70}$$

Note that then the value of the integrals in (69) is just

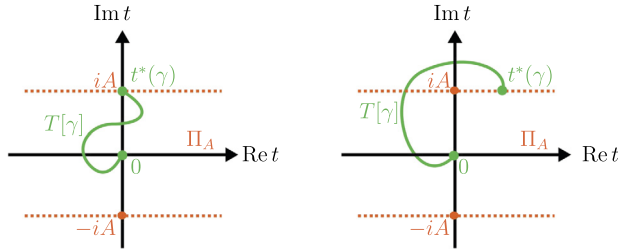


Fig. 5. Example of paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$. Left: $T[\gamma] \subset \Pi_A$ and $t^*(\gamma) = iA$. Right: $T[\gamma] \not\subset \Pi_A$.

$$t^*(\gamma) = \lim_{s \rightarrow s_{\text{end}}} T[\gamma](s) = \int_{\gamma} \hat{f}(q) dq. \tag{71}$$

Since we are interested in the singularities of $q(t)$ on its first Riemann sheet, we only consider the paths $T[\gamma]$ which belong to the complex strip Π_A (except, of course, the endpoint of the path $t^*(\gamma) \in \partial\Pi_A$). See Fig. 5.

The following definition characterizes the paths that we consider.

Definition 3.3. We say that a singularity t^* of $q(t)$ is *visible* if there exists a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that

- $t^* = t^*(\gamma)$,
- $T[\gamma](s) \in \Pi_A$, for $s \in [s_0, s_{\text{end}})$.

Remark 3.4. In [27], the authors use a different definition of visible singularity: $t^* \in \mathbb{C}$ is considered a visible singularity if $q(t)$ can be continued from the real axis and then along the vertical line with a path of the form

$$\zeta(t) = \text{Re } t^* + it, \quad \text{for } t \in [0, \text{Im } t^*).$$

This condition on the paths is more restrictive than merely imposing that $T[\gamma] \subset \Pi_v$ for $v = \text{Im } t^*$. However, to compute $t^*(\gamma)$, they are equivalent since both paths belong to Π_v .

Theorem 3.1 is a consequence of Proposition 3.2 and the following result.

Proposition 3.5. *There exist two paths $\gamma_{\pm} \in \mathcal{P}_0$ yielding the visible singularities $t^*(\gamma_{\pm}) = \pm iA$. Moreover, these are the only two visible singularities of $q(t)$.*

We devote the rest of this section to prove Proposition 3.5. Let us introduce some tools and considerations to simplify the analysis of the integrals in (69).

- If $\gamma \subset \mathcal{R}_f$ is an integration path then, defining $\eta = \overline{\gamma}$,³ we have that

$$\int_{\eta} \hat{f}(q) dq = \overline{\int_{\gamma} \hat{f}(q) dq}. \tag{72}$$

- Notice that the paths considered cannot contain the singularities of \hat{f} (except in their endpoints) since $0, a_{\pm}, \pm 1 \notin \mathcal{R}_f$.
- When saying that a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_{\infty}$ crosses \mathbb{R} , we refer to the two lines whose complex projection onto \mathcal{R}_f coincides with \mathbb{R} . Analogously for any other interval.
- Instead of detailing the paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_{\infty}$, we only describe their projections $p\gamma \subset \mathbb{C}$. This omission makes sense since paths γ are continuous on \mathcal{R}_f and, as a result, $\arg g(\gamma)$ must be continuous as well (see (58)). Therefore, we can let the natural arguments of $p\gamma$ and the initial point $(a_+, 0)$ of the path define $\arg g(\gamma)$.

To prove Proposition 3.5, we classify the paths as follows.

- A. Paths not crossing \mathbb{R} :
 - A.1. Paths in \mathcal{P}_0 not crossing \mathbb{R} : Lemma 3.6.
 - A.2. Paths in \mathcal{P}_{∞} not crossing \mathbb{R} : Lemma 3.7.
- B. Paths first crossing the real axis at $\mathbb{R} \setminus [0, 1]$:
 - B.1. First crossing of \mathbb{R} at $(1, +\infty)$: Lemma 3.10.
 - B.2. First crossing of \mathbb{R} at $(-\infty, a_-)$: Lemma 3.11.
 - B.3. First crossing of \mathbb{R} at $(-1, 0)$: Lemma 3.12.
 - B.4. First crossing of \mathbb{R} at $(a_-, -1)$: Lemma 3.13.
- C. Paths first crossing the real axis at $(0, 1)$:
 - C.1. Paths in \mathcal{P}_0 only crossing \mathbb{R} at $(0, 1)$: Lemma 3.14.
 - C.2. Paths in \mathcal{P}_{∞} only crossing \mathbb{R} at $(0, 1)$: Lemma 3.15.
 - C.3. Paths also crossing $\mathbb{R} \setminus [0, 1]$: Lemma 3.16.

3.3.1. Paths not crossing the real axis

In this section, we check the singularities resulting from the paths A.1 and A.2.

Lemma 3.6. *There exist only two singularities, $t_{1,\pm}^*$, given by the paths $\gamma \in \mathcal{P}_0$ not crossing the real axis. These singularities are visible and*

$$t_{1,\pm}^* = \mp iA,$$

with A defined in (29) and satisfying $A \in [\frac{3}{50}, \frac{3}{10}]$.

³ We define the conjugation on a Riemann surface as the natural continuation of the conjugation in the complex plane. That is, for $z = (x, \theta) \in \mathcal{R}_f$, its conjugated is $\bar{z} = (\bar{x}, -\theta) \in \mathcal{R}_f$.

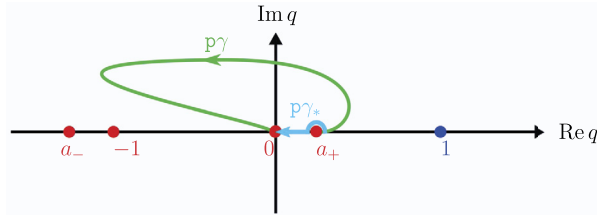


Fig. 6. Example of a path $\gamma \in \mathcal{P}_0$ such that $p\gamma \subset \mathbb{C}^+$. Path γ_* as defined in (73).

Proof. Let us consider paths $\gamma \in \mathcal{P}_0$ such that $p\gamma \subset \mathbb{C}^+ = \{\text{Im } z > 0\}$. Notice that, since γ does not cross the real axis, it does not encircle any singularity of \hat{f} and, by Cauchy’s Integral Theorem, all the paths considered generate the same singularity $t_{1,+}^*$. The singularity $t_{1,-}^*$ is given by the conjugated paths (see (72)).

Let us consider the path $\gamma_* = \gamma_*^1 \vee \gamma_*^2 \vee \gamma_*^3$ with

$$\begin{cases} \gamma_*^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ p\gamma_*^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ p\gamma_*^3(q) = q & \text{with } q \in [a_+ - \varepsilon, 0), \end{cases} \tag{73}$$

for $\varepsilon > 0$ small enough, (see Fig. 6). Then, the resulting singularity is

$$t_{1,+}^* = t^*(\gamma_*) = \int_{\gamma_*} \hat{f}(q) dq = \sum_{j=1}^3 \int_{\gamma_*^j} \hat{f}(q) dq.$$

Since $\int_{\gamma_*^j} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2$, taking the limit $\varepsilon \rightarrow 0$, we have

$$t_{1,+}^* = \lim_{\varepsilon \rightarrow 0} \int_{\gamma_*^3} \hat{f}(q) dq.$$

Then, by following the natural arguments of the path γ_* , we obtain

$$\arg(\gamma_*^3) = 0, \quad \arg(\gamma_*^3 - a_+) = \pi, \quad \arg(\gamma_*^3 + 1) = 0, \quad \arg(\gamma_*^3 - a_-) = 0,$$

and, as a consequence, by the definition of A in (29), we have

$$\begin{aligned} t_{1,+}^* &= \lim_{\varepsilon \rightarrow 0} \int_{\gamma_*^3} \hat{f}(q) dq \\ &= \int_{a_+}^0 \frac{1}{x-1} \sqrt{\frac{x}{3(x+1)|x-a_+|e^{i\pi}(x-a_-)}} dx = -iA. \end{aligned} \tag{74}$$

Moreover,

$$A \leq \frac{\sqrt{a_+}}{(1 - a_+)\sqrt{3|a_-|}} \int_0^{a_+} \frac{dx}{\sqrt{a_+ - x}} = \frac{4\sqrt{3}}{3} \frac{a_+^{3/2}}{1 - a_+} \leq \frac{3}{10},$$

$$A \geq \frac{1}{\sqrt{3a_+(a_+ + 1)(a_+ - a_-)}} \int_0^{a_+} \sqrt{x} dx = \frac{2\sqrt[4]{74}}{9} a_+^{3/2} \geq \frac{3}{50}.$$

Now, it just remains to see that $t_{1,+}^*$ is visible, namely, we check that $T[\gamma_*] \subset \Pi_A$. Indeed, for $s \in (0, a_+]$, we have that

$$|\operatorname{Im} T[\gamma_*](s)| = \int_s^{a_+} \frac{1}{1-x} \sqrt{\frac{x}{3(x+1)(a_+ - x)(x - a_-)}} dx < A. \quad \square$$

Lemma 3.7. *There exist only two singularities, $t_{2,\pm}^*$, resulting from the paths $\gamma \in \mathcal{P}_\infty$ not crossing the real axis. These singularities are not visible and have imaginary part*

$$\operatorname{Im}(t_{2,\pm}^*) = \mp \pi \sqrt{\frac{2}{21}}.$$

Proof. Let us consider paths $\gamma \in \mathcal{P}_\infty$ such that $\mathfrak{p}\gamma \subset \mathbb{C}^+$. Then, since $\hat{f}(q)$ decays with a rate of $|q|^{-2}$ as $|q| \rightarrow \infty$, all paths considered generate the same singularity, $t_{2,+}^*$. The singularity $t_{2,-}^*$ is given by the conjugated paths.

Let us consider the path $\gamma_* = \gamma_*^1 \vee \gamma_*^2 \vee \gamma_*^3$ where

$$\begin{cases} \gamma_*^1(q) = (q, 0) & \text{with } q \in (a_+, 1 - \varepsilon], \\ \mathfrak{p}\gamma_*^2(\phi) = 1 + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ \mathfrak{p}\gamma_*^3(q) = q & \text{with } q \in [1 + \varepsilon, +\infty), \end{cases} \quad (75)$$

for any small enough $\varepsilon > 0$. (See Fig. 7.) Then, the resulting singularity is

$$t_{2,+}^* = t^*(\gamma_*) = \sum_{j=1}^3 \int_{\gamma_*^j} \hat{f}(q) dq.$$

Since $\hat{f}(q) \in \mathbb{R}$ when $\mathfrak{p}(q) \in (a_+, 1) \cup (1, +\infty) \subset \mathbb{R}$, the integrals on γ_*^1 and γ_*^3 take real values. Therefore, γ_*^2 is the only path that contributes to the imaginary part to the singularity. Notice that the path γ_*^2 partially encircles the pole $q = (1, 0)$ of $\hat{f}(q)$. Then, one has

$$\operatorname{Im}(t_{2,+}^*) = \operatorname{Im} \int_{\gamma_*^2} \hat{f}(q) dq = -\pi \operatorname{Res}(\hat{f}, (1, 0)) = -\pi \sqrt{\frac{2}{21}}.$$

Since, by Lemma 3.6, $|\operatorname{Im}(t_{2,+}^*)| > A$, the singularity $t_{2,+}^*$ is not visible. \square

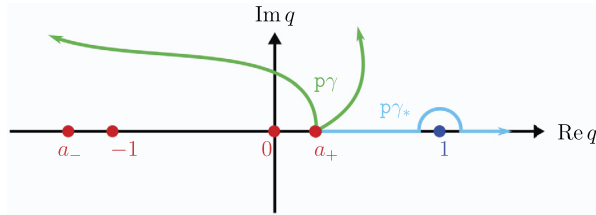


Fig. 7. Example of a path $\gamma \in \mathcal{P}_\infty$ such that $\text{p}\gamma \subset \mathbb{C}^+$. Path γ_* as defined in (75).

Remark 3.8. Using mathematical software, one can see that the singularities $t_{2,\pm}^*$ in Lemma 3.7 satisfy $t_{2,\pm}^* \approx -0.086697 \mp 0.969516i$.

3.3.2. Paths first crossing the real axis at $\mathbb{R} \setminus [0, 1]$

In this section, we continue the proof of Proposition 3.5 by checking that the singularities generated by paths B.1 to B.4 (see the list in Section 3.3) are not visible.

First, we introduce some concepts. Let us consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$. We define the *parameter of the first crossing* of the real line as

$$s_1(\gamma) = \inf \{s \in (s_0, s_{\text{end}}) : \text{Im } \text{p}\gamma(s) = 0\},$$

the *location of the first crossing* as

$$q_1(\gamma) = \text{p}\gamma(s_1(\gamma)) \in \mathbb{R} \setminus \{a_-, -1, 0, a_+, 1\},$$

the *piece of the path* before the first crossing of the real line as

$$\gamma_1(\gamma) = \{\gamma(s) : s \in (s_0, s_1(\gamma))\},$$

and the *time of the first crossing* as

$$t_1(\gamma) = \int_{\gamma_1(\gamma)} \hat{f}(q) dq.$$

In the following lemmas, we focus on the paths that stay in \mathbb{C}^+ until the first crossing of the real line, that is $\text{p}\gamma_1(\gamma) \subset \mathbb{C}^+$ (see (72) for the conjugate paths, i.e. $\text{p}\gamma_1(\gamma) \subset \mathbb{C}^-$).

Remark 3.9. To prove that a singularity $t^*(\gamma)$ is not visible (see Definition 3.3) it is sufficient to check that $|\text{Im } t_1(\gamma)| \geq A$.

Lemma 3.10. The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (1, +\infty)$ are not visible.

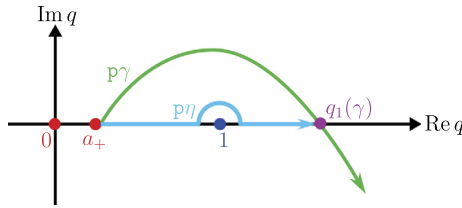


Fig. 8. Example of a path $\gamma \in \mathcal{H}_{q,\infty}$ such that $q_1(\gamma) \in (1, +\infty)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. The path η has been defined in (76).

Proof. Consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (1, +\infty)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. Integrating the function \hat{f} along the path $\gamma_1(\gamma)$ is equivalent to integrate \hat{f} along the path $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ where

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, 1 - \varepsilon], \\ p\eta^2(\phi) = 1 + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ p\eta^3(q) = q & \text{with } q \in [1 + \varepsilon, q_1), \end{cases} \tag{76}$$

for $\varepsilon > 0$ small enough, (see Fig. 8). Then,

$$t_1(\gamma) = \int_\eta \hat{f}(q) dq = \sum_{j=1}^3 \int_{\eta^j} \hat{f}(q) dq,$$

and the integrals on η^1 and η^3 take real values since $\hat{f}(q) \in \mathbb{R}$ when $p(q) \in (a_+, 1) \cup (1, +\infty) \subset \mathbb{R}$. Analogously to the proof of Lemma 3.7, we have that

$$|\text{Im } t_1(\gamma)| = \left| \text{Im} \int_{\eta^2} \hat{f}(q) dq \right| = \pi \left| \text{Res} \left(\hat{f}, (1, 0) \right) \right| = \pi \sqrt{\frac{2}{21}} > A.$$

Therefore, $t^*(\gamma)$ is not visible (see Remark 3.9). \square

Lemma 3.11. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-\infty, a_-)$ are not visible.*

Proof. Consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (-\infty, a_-)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. To compute $t_1(\gamma)$ we introduce the auxiliary path $\eta_\infty = \gamma_1(\gamma) \vee \tilde{\eta}$, where

$$p\tilde{\eta}(q) = q \quad \text{with } q \in [q_1, -\infty), \tag{77}$$

(see Fig. 9). Then, taking into account that $\hat{f}|_{\tilde{\eta}} \subset \mathbb{R}$,

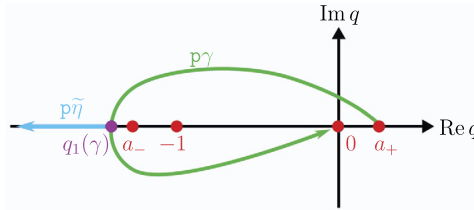


Fig. 9. Example of a path $\gamma \in \mathcal{P}_0$ such that $q_1(\gamma) \in (-\infty, a_-)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. The path $\tilde{\eta}$ has been defined in (77).

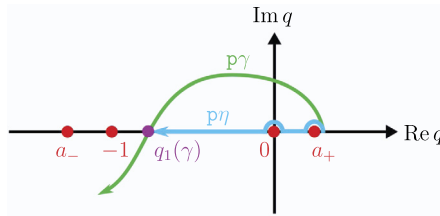


Fig. 10. Example of a path $\gamma \in \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-1, 0)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. The path η has been defined in (78).

$$|\operatorname{Im} t_1(\gamma)| = \left| \operatorname{Im} \int_{\eta_\infty} \hat{f}(q) dq \right| = \left| \operatorname{Im} \int_{\gamma^*} \hat{f}(q) dq \right| = \pi \sqrt{\frac{2}{21}} > A,$$

where γ^* is the path defined in (75). Therefore $t^*(\gamma)$ is not visible. \square

Lemma 3.12. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-1, 0)$ are not visible.*

Proof. Let $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ be a path such that $q_1 = q_1(\gamma) \in (-1, 0)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. Integrating the function \hat{f} along the path $\gamma_1(\gamma)$ is equivalent to integrating \hat{f} along $\eta = \eta^1 \vee \eta^2 \vee \eta^3 \vee \eta^4 \vee \eta^5$ where

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ p\eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ p\eta^3(q) = q & \text{with } q \in [a_+ - \varepsilon, \varepsilon], \\ p\eta^4(\phi) = \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ p\eta^5(q) = q & \text{with } q \in [-\varepsilon, q_1], \end{cases} \quad (78)$$

for $\varepsilon > 0$ small enough, (see Fig. 10). Using that $\int_{\eta^j} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2, 4$, that $\hat{f}|_{\eta^5} \subset \mathbb{R}$, and (74), one has that $t^*(\gamma)$ is not visible since

$$|\operatorname{Im} t_1(\gamma)| = \lim_{\varepsilon \rightarrow 0} \left| \int_{\eta^3} \hat{f}(q) dq \right| = A. \quad \square$$

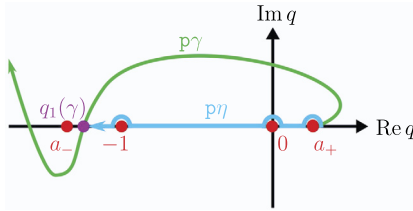


Fig. 11. Example of a path $\gamma \in \mathcal{P}_\infty$ such that $q_1(\gamma) \in (a_-, -1)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. Path η as defined in (79).

Lemma 3.13. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (a_-, -1)$ are not visible.*

Proof. Take a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (a_-, -1)$ and $p\gamma_1(\gamma) \subset \mathbb{C}^+$. The integral of the function \hat{f} along the path $\gamma_1(\gamma)$ coincides with the integral along the path

$$\eta = \bigvee_{j=1}^7 \eta^j, \tag{79}$$

where the paths η^j , $j = 1, 2, 3, 4$, are defined in (78) and

$$\begin{cases} p\eta^5(q) = q & \text{with } q \in [-\varepsilon, -1 + \varepsilon], \\ p\eta^6(\phi) = -1 + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ p\eta^7(q) = q & \text{with } q \in [-1 - \varepsilon, q_1], \end{cases}$$

for small enough $\varepsilon > 0$, (see Fig. 11). Then, proceeding analogously to the proof of Lemma 3.12,

$$\text{Im } t_1(\gamma) = \lim_{\varepsilon \rightarrow 0} \text{Im} \int_{\eta} \hat{f}(q) dq = -A + \sum_{j=5}^7 \lim_{\varepsilon \rightarrow 0} \text{Im} \int_{\eta^j} \hat{f}(q) dq. \tag{80}$$

Since $\hat{f}|_{\eta^5} \subset \mathbb{R}$ and $\int_{\eta^6} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$, it only remains to compute the integral on η^7 . Following the natural arguments of the path η , we obtain

$$\arg(\eta^7) = \pi, \quad \arg(\eta^7 - a_+) = \pi, \quad \arg(\eta^7 + 1) = \pi, \quad \arg(\eta^7 - a_-) = 0$$

and, as a consequence,

$$\lim_{\varepsilon \rightarrow 0} \int_{\eta^7} \hat{f}(q) dq = \int_{-1}^{q_1} \frac{1}{q-1} \sqrt{\frac{|q|e^{i\pi}}{3|q+1|e^{i\pi}|q-a_+|e^{i\pi}(q-a_-)}} dq = -iB(q_1),$$

where $B(q_1)$ is a real-valued, positive and strictly decreasing function for $q_1 \in (a_-, -1)$. Therefore, $t^*(\gamma)$ is not visible since, by (80), one has that $|\text{Im } t_1(\gamma)| = A + B(q_1) > A$. \square

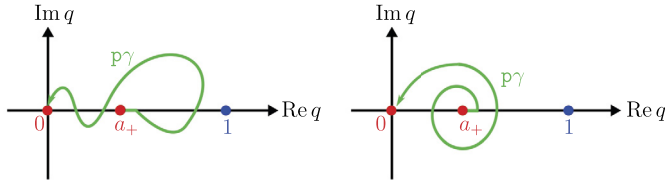


Fig. 12. Example of paths $\gamma \in \mathcal{P}_0$ only crossing \mathbb{R} at $(0, 1)$. Left: $\theta_{\text{end}}(\gamma) = \pi$. Right: $\theta_{\text{end}}(\gamma) = 3\pi$.

3.3.3. Paths first crossing the real axis at $[0, 1]$

In this section, we check that the singularities generated by paths C.1 to C.3 (see the list in Section 3.3) are either not visible or $\pm iA$.

Lemma 3.14. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0$ only crossing \mathbb{R} at $(0, 1)$ are either $t^*(\gamma) = iA$ or $t^*(\gamma) = -iA$.*

Proof. The paths considered in this lemma can turn around the branching point $q = a_+$, but not around the other branching points nor the pole. Therefore, we classify these paths depending on how many turns they perform around $q = a_+$. In order to do so, we define

$$\theta_{\text{end}}(\gamma) = \lim_{s \rightarrow s_{\text{end}}} \arg(\gamma(s) - a_+). \tag{81}$$

The considered paths satisfy $\theta_{\text{end}}(\gamma) = (2k+1)\pi$ for some $k \in \mathbb{Z}$ (see Fig. 12). Integrating the function \hat{f} along the path γ is equivalent to integrating along $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ with

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ - \varepsilon], \\ \eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, (2k+1)\pi], \\ \eta^3(q) = q & \text{with } q \in [a_+ - \varepsilon, 0), \end{cases} \tag{82}$$

for small enough $\varepsilon > 0$. Since $\int_{\eta^j} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2$,

$$\begin{aligned} t^*(\gamma) &= \lim_{\varepsilon \rightarrow 0} \int_{\eta^3} \hat{f}(q) dq \\ &= \int_{a_+}^0 \frac{1}{q-1} \sqrt{\frac{q}{3(q+1)|q-a_+| e^{i(2k+1)\pi}(q-a_-)}} dq = (-1)^{k+1} iA. \quad \square \end{aligned}$$

Lemma 3.15. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_\infty$ only crossing \mathbb{R} at $(0, 1)$ are not visible.*

Proof. We analyze the paths $\gamma \in \mathcal{P}_\infty$ such that $p\gamma$ goes to infinity on \mathbb{C}^+ (by (72), the paths on \mathbb{C}^- give conjugated results). Following the proof of Lemma 3.14, we classify

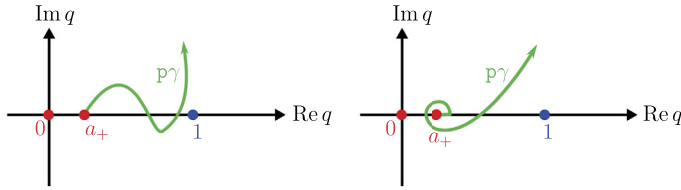


Fig. 13. Example of paths $\gamma \in \mathcal{P}_\infty$ only crossing \mathbb{R} at $(0, 1)$ such that $\mathbf{p}\gamma$ ends on the positive complex plane. Left: $\theta_{\text{end}}(\gamma) \in (0, \pi)$. Right: $\theta_{\text{end}}(\gamma) \in (2\pi, 3\pi)$.

the paths γ depending on $\theta_{\text{end}}(\gamma)$, the final argument with respect to a_+ (see (81) and Fig. 13). The paths considered satisfy

$$\theta_{\text{end}}(\gamma) \in (2\pi k, (2k + 1)\pi), \quad \text{for some } k \in \mathbb{Z}.$$

We compute $t^*(\gamma)$ using the path $\eta = \eta^1 \vee \eta^2 \vee \eta^3 \vee \eta^4 \vee \eta^5$ where the paths η^1, η^2 are defined in (82) and

$$\begin{cases} \mathbf{p}\eta^3(q) = q & \text{with } q \in [a_+ + \varepsilon, 1 - \varepsilon], \\ \mathbf{p}\eta^4(\phi) = 1 + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ \mathbf{p}\eta^5(q) = q & \text{with } q \in [1 + \varepsilon, +\infty), \end{cases}$$

for small enough $\varepsilon > 0$. Since the integrals on η^3 and η^5 take real values and applying the results in Lemma 3.14 for η^1 and η^2 , we obtain

$$\text{Im } t^*(\gamma) = \text{Im} \int_{\eta^4} \hat{f}(q) dq.$$

Proceeding as in the proof of Lemma 3.7 and following the natural arguments of the path η , one deduces that

$$\text{Im } t^*(\gamma) = -\pi \text{Res} \left(\hat{f}, (1, 2\pi k) \right) = (-1)^{k+1} \pi \sqrt{\frac{2}{21}}.$$

Therefore, since $|\text{Im } t^*(\gamma)| > A$, the singularity is not visible. \square

Lemma 3.16. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ both crossing $(0, 1)$ and $\mathbb{R} \setminus [0, 1]$ are not visible.*

Proof. Let us define the parameter of the first crossing at $\mathbb{R} \setminus [0, 1]$ as

$$s_2(\gamma) = \inf \{s \in (s_0, s_{\text{end}}) : \text{Im } \mathbf{p}\gamma(s) = 0, \text{Re } \mathbf{p}\gamma(s) \notin [0, 1]\}$$

and the corresponding point

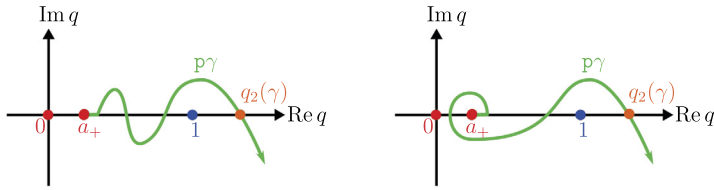


Fig. 14. Example of paths $\gamma \in \mathcal{P}_\infty$ crossing both $(0, 1)$ and $\mathbb{R} \setminus [0, 1]$ with $q_2(\gamma) \in (1, +\infty)$ and such that $p\gamma$ approaches $q_2(\gamma)$ from \mathbb{C}^+ . Left: $\theta_2(\gamma) = 0$. Right: $\theta_2(\gamma) = 2\pi$.

$$q_2(\gamma) = p\gamma(s_2(\gamma)) \in \mathbb{R} \setminus \{[0, 1], a_-, -1\}.$$

We consider paths γ with $q_2 = q_2(\gamma) \in (1, +\infty)$ and such that $p\gamma$ approaches $q_2(\gamma)$ from \mathbb{C}^+ (see (72)). The cases $q_2 \in (-\infty, a_-)$, $q_2 \in (a_-, -1)$ and $q_2 \in (-1, 0)$ are proved analogously.

The strategy is to classify the paths γ depending on how many turns they perform around $q = a_+$ before crossing $\mathbb{R} \setminus [0, 1]$. To this end, we define $\theta_2(\gamma) = \arg(q_2(\gamma) - a_+)$ (see Fig. 14). The paths we are considering satisfy $\theta_2(\gamma) = 2\pi k$ for some $k \in \mathbb{Z}$. We also define the piece of path before the crossing as $\gamma_2(\gamma) = \{\gamma(s) : s \in (s_0, s_2(\gamma))\}$ and the corresponding time

$$t_2(\gamma) = \int_{\gamma_2(\gamma)} \hat{f}(q) dq.$$

To prove that a singularity $t^*(\gamma)$ is not visible, it is sufficient to check that $|\text{Im } t_2(\gamma)| \geq A$.

1. Consider $\theta_2(\gamma) = 2\pi k$ with k an even number. Let us consider the path η as defined in (76) replacing q_1 by q_2 in its definition. This path η lies entirely on the first Riemann sheet, that is $\arg g(\eta) \in (-\pi, \pi]$. Integrating the function \hat{f} along $\gamma_2(\gamma)$ is equivalent to integrating it along

$$\xi = \tilde{\eta}^1 \vee \tilde{\eta}^2 \vee \tilde{\eta}^3 \vee \eta,$$

where

$$\begin{cases} \tilde{\eta}^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ p\tilde{\eta}^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, 2\pi k], \\ p\tilde{\eta}^3(q) = q & \text{with } q \in [a_+ + \varepsilon, a_+), \end{cases} \tag{83}$$

for $\varepsilon > 0$ small enough. Notice that this construction makes sense since the path $\tilde{\eta}^3$ has argument $\arg g(\tilde{\eta}^3) = 2\pi k$ (which belongs to the first Riemann sheet).

Then, since $\int_{\tilde{\eta}^j} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2, 3$ and applying Lemma 3.10, we have

$$|\operatorname{Im} t_2(\gamma)| = \lim_{\varepsilon \rightarrow 0} \left| \operatorname{Im} \int_{\xi} \hat{f}(q) dq \right| = \left| \operatorname{Im} \int_{\eta} \hat{f}(q) dq \right| > A.$$

2. Consider $\theta_2(k) = 2\pi k$ with k an odd integer. We define the path η^* , lying on the second Riemann sheet, as

$$\eta^* = (\mathfrak{p}\eta, \arg \mathfrak{g}(\eta) + 2\pi) \in \mathbb{C} \times (\pi, 3\pi],$$

where η is the path introduced in (76) (replacing q_1 by q_2 in its definition). Note that the path η lies on the first Riemann sheet ($\arg \mathfrak{g}(\eta) \in (-\pi, \pi]$).

It can be easily checked that switching the Riemann sheet implies a change in sign. That is,

$$\int_{\eta^*} \hat{f}(q) dq = - \int_{\eta} \hat{f}(q) dq. \tag{84}$$

Then, integrating the function \hat{f} along the path $\gamma_2(\gamma)$ is equivalent to integrating it over

$$\xi = \tilde{\eta}^1 \vee \tilde{\eta}^2 \vee \tilde{\eta}^3 \vee \eta^*,$$

where paths $\tilde{\eta}^j$ for $j = 1, 2, 3$, are defined on (83). This construction makes sense since the path $\tilde{\eta}^3$ has argument $\arg \mathfrak{g}(\tilde{\eta}^3) = 2\pi k$ (which belongs to the second Riemann sheet).

Then, since $\int_{\tilde{\eta}^j} \hat{f}(q) dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2, 3$ and applying Lemma 3.10 and formula (84), we have

$$|\operatorname{Im} t_2(\gamma)| = \lim_{\varepsilon \rightarrow 0} \left| \operatorname{Im} \int_{\xi} \hat{f}(q) dq \right| = \left| \operatorname{Im} \int_{\eta} \hat{f}(q) dq \right| > A. \quad \square$$

3.4. Proof of Proposition 2.3

For $t \in \mathbb{R}$, $\Lambda_h(t)$ satisfies $\Lambda_h(t) = 0$ if and only if $t = 0$ (see Fig. 2). To prove Proposition 2.3, we follow the same techniques used in the proof of Theorem 2.2.

Let us consider $q(t) = \cos(\frac{\lambda_h(t)}{2})$ as introduced in Theorem 3.1. Then, by (56),

$$\Lambda_h^2(t) = \frac{4}{3q(t)} (1 - q(t))(q(t) - a_+)(q(t) - a_-).$$

Let $\Lambda_h(t^*) = 0$ for a given t^* . Then, defining $q^* = q(t^*)$, we have three options:

$$q^* = 1, a_+, a_-.$$

We have seen that $q^* = 1$ corresponds to the saddle equilibrium point, namely $|t^*| \rightarrow \infty$, (see (67)). Therefore, it cannot lead to zeroes of $\Lambda_h(t)$. On the contrary, $q^* = a_{\pm}$ leads to zeroes of $\Lambda_h(t)$, since we have seen that $q(t)$ is well defined and analytic in a neighborhood of such t^* (see (66)).

To prove Proposition 2.3 it only remains to compute all possible values of $t^* \in \overline{\Pi_A}$ such that $q^* = q(t^*)$ with $q^* = a_+, a_-$. To do so, we use the techniques and results presented in Section 3.3.

From now on, we consider integration paths $\gamma : (s_0, s_{\text{end}}) \rightarrow \mathcal{R}_f$ with initial point $\lim_{s \rightarrow s_0} \gamma(s) = (a_+, 0)$ and endpoint $\lim_{s \rightarrow s_0} p\gamma(s) = q^* = a_{\pm}$. Moreover, we say that a zero t^* of Λ_h is *visible* if there exist a path γ such that $t^* = t^*(\gamma) \in \overline{\Pi_A}$ and $T[\gamma](s) \in \Pi_A$ for $s \in [s_0, s_{\text{end}})$, (see (70) and (71)).

First, we recall some of the results obtained in Sections 3.3.2 and 3.3.3.

- Consider $q_1 \in (-\infty, a_-) \cup (a_-, -1) \cup (1, +\infty)$. In the proofs of Lemmas 3.10, 3.11, 3.13 and 3.16 we have seen that

$$\left| \text{Im} \int_{a_+}^{q_1} \hat{f} d\gamma \right| > A. \tag{85}$$

- Consider $q_1 \in (-1, 0)$. In the proofs of Lemmas 3.12 and 3.16, we have seen that

$$\left| \text{Im} \int_{a_+}^{q_1} \hat{f} d\gamma \right| = A. \tag{86}$$

Now, we classify the paths depending on its endpoint q^* .

1. Consider $q^* = a_-$. Analogously to the proof of (85), it can be seen that

$$|\text{Im } t^*(\gamma)| = \left| \text{Im} \int_{a_+}^{a_-} \hat{f} d\gamma \right| > A.$$

Therefore, $q^* = a_-$ does not lead to any visible zero.

2. Consider $q^* = a_+$. Notice that, by (85) and (86), any path crossing $\mathbb{R} \setminus [0, 1]$ leads to non-visible zeroes. Therefore, we only consider paths γ either crossing $(0, 1)$ or not crossing \mathbb{R} .

Since in $(0, 1)$ the only singularity of $\hat{f}(q)$ is the branching point $q = a_+$, there exists a homotopic path $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ defined by

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ \mathfrak{p}\eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, 2\pi k], \\ \mathfrak{p}\eta^3(q) = q & \text{with } q \in [a_+ + \varepsilon, a_+), \end{cases}$$

for some $k \in \mathbb{Z}$ and $\varepsilon > 0$ small enough. Then,

$$t^*(\gamma) = \int_{a_+}^{a_+} \hat{f} d\gamma = \lim_{\varepsilon \rightarrow 0} \int_{\eta} \hat{f}(q) dq = 0.$$

Therefore, these paths lead to the only visible zero $t^* = 0$. \square

4. The inner system of coordinates

This section is devoted to prove Proposition 2.5. That is we perform the suitable changes of coordinates, described in Section 2.3, to Hamiltonian H obtained in Theorem 2.1 (see (20)) to obtain the inner Hamiltonian \mathcal{H} . However, recall that the Hamiltonian H is defined by means of H_1^{Poi} (see (16)) which does not have a closed form. For this reason, a preliminary step to prove Proposition 2.5 is to provide suitable expansions for H_1^{Poi} in an appropriate domain. This is done in Section 4.1. Then, in Section 4.2, we apply the changes of coordinates introduced in Section 2.3 to conclude the proof of the proposition.

4.1. The Hamiltonian in Poincaré variables

First, we give some formulae to translate the Delaunay variables and other orbital elements into Poincaré coordinates (see (14)).

- **Eccentricity** e (see (8)): It can be written as

$$e = 2 \tilde{e}(L, \eta, \xi) \sqrt{\eta \xi}, \quad \tilde{e}(L, \eta, \xi) = \frac{\sqrt{2L - \eta \xi}}{2L} = \frac{1}{\sqrt{2L}} + \mathcal{O}(\eta \xi). \tag{87}$$

Notice that \tilde{e} is analytic for $(L, \eta, \xi) \sim (1, 0, 0)$.⁴

- **Argument of the perihelion** g : By the definition of η and ξ in (15),

$$\cos g = \frac{\eta + \xi}{2\sqrt{\eta \xi}}, \quad \sin g = -i \frac{\eta - \xi}{2\sqrt{\eta \xi}}. \tag{88}$$

⁴ This expansion is valid as long as $L \neq 0$. However, since our analysis focuses on $L \sim 1$, to simplify notation we use this more restrictive domain.

– **Mean anomaly** ℓ : Since $\lambda = \ell + g$, we have that

$$\cos \ell = \frac{1}{2\sqrt{\eta\xi}} (e^{-i\lambda}\eta + e^{i\lambda}\xi), \quad \sin \ell = \frac{i}{2\sqrt{\eta\xi}} (e^{-i\lambda}\eta - e^{i\lambda}\xi).$$

These expressions are not analytic at $(\eta, \xi) = (0, 0)$. However, by (87),

$$\begin{aligned} e \cos \ell &= \tilde{e}(L, \eta, \xi) (e^{-i\lambda}\eta + e^{i\lambda}\xi), \\ e \sin \ell &= i\tilde{e}(L, \eta, \xi) (e^{-i\lambda}\eta - e^{i\lambda}\xi), \end{aligned} \tag{89}$$

are analytic for $(L, \eta, \xi) \sim (1, 0, 0)$.

– **Eccentric anomaly** u : It can be implicitly defined by $u = \ell + e \sin u$ (see (12)), which implies

$$u = \ell + e \sin \ell + e^2 \cos \ell \sin \ell + \mathcal{O}(e \sin \ell, e \cos \ell)^3.$$

Then, by (87) and (89),

$$\begin{aligned} e \cos u &= \frac{1}{\sqrt{2L}} (e^{-i\lambda}\eta + e^{i\lambda}\xi) + \frac{1}{2L} (e^{-i\lambda}\eta - e^{i\lambda}\xi)^2 \\ &\quad + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^3, \\ e \sin u &= \frac{i}{\sqrt{2L}} (e^{-i\lambda}\eta - e^{i\lambda}\xi) + \frac{i}{2L} (e^{-2i\lambda}\eta^2 - e^{2i\lambda}\xi^2) \\ &\quad + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^3, \end{aligned} \tag{90}$$

which are also analytic for $(L, \eta, \xi) \sim (1, 0, 0)$.

For any $\zeta \in [-1, 1]$, we define the function

$$D[\zeta] = (r^2 - 2\zeta r \cos \theta + \zeta^2) \circ \phi_{\text{Poi}}. \tag{91}$$

By the definition of μH_1^{Poi} in (16), we have that

$$\mu H_1^{\text{Poi}} = \frac{1}{\sqrt{D[0]}} - \frac{1 - \mu}{\sqrt{D[\mu]}} - \frac{\mu}{\sqrt{D[\mu - 1]}}. \tag{92}$$

Lemma 4.1. For $|(L - 1, \eta, \xi)| \ll 1$ and any $\zeta \in [-1, 1]$, one can split $D[\zeta]$ as

$$D[\zeta] = D_0[\zeta] + D_1[\zeta] + D_2[\zeta] + D_{\geq 3}[\zeta],$$

where

$$\begin{aligned}
 D_0[\zeta](\lambda, L) &= L^4 - 2\zeta L^2 \cos \lambda + \zeta^2, \\
 D_1[\zeta](\lambda, L, \eta, \xi) &= \eta \frac{\sqrt{2L^3}}{2} (3\zeta - 2L^2 e^{-i\lambda} - \zeta e^{-2i\lambda}) \\
 &\quad + \xi \frac{\sqrt{2L^3}}{2} (3\zeta - 2L^2 e^{i\lambda} - \zeta e^{2i\lambda}), \\
 D_2[\zeta](\lambda, L, \xi, \eta) &= -\eta^2 \frac{L e^{-i\lambda}}{4} (\zeta + 2L^2 e^{-i\lambda} + 3\zeta e^{-2i\lambda}) \\
 &\quad - \xi^2 \frac{L e^{i\lambda}}{4} (\zeta + 2L^2 e^{i\lambda} + 3\zeta e^{2i\lambda}) + \eta \xi L (3L^2 + 2\zeta \cos \lambda).
 \end{aligned}$$

Fix $\varrho \geq 0$. Then, for $|\operatorname{Im} \lambda| \leq \varrho$, the function $D_{\geq 3}[\zeta]$ is analytic and satisfies

$$|D_{\geq 3}[\zeta](\lambda, L, \eta, \xi)| \leq C |(\eta, \xi)|^3, \tag{93}$$

with $C = C(\varrho)$ a positive constant independent of $\zeta \in [-1, 1]$.

Proof of Lemma 4.1. In view of the definition of $D[\zeta]$ in (91), we look for expansions for r^2 and $r \cos \theta$ (expressed in Poincaré coordinates) in powers of (η, ξ) .

Let us consider first r^2 . Taking into account that $r = L^2(1 - e \cos u)$ (see (9)) and the expansions in (90) we obtain

$$\begin{aligned}
 r^2 &= L^4 - L^3 \sqrt{2L} e^{-i\lambda} \eta - L^3 \sqrt{2L} e^{i\lambda} \xi + 3L^3 \eta \xi \\
 &\quad - \frac{L^3}{2} e^{-2i\lambda} \eta^2 - \frac{L^3}{2} e^{2i\lambda} \xi^2 + \mathcal{O}(e^{-i\lambda} \eta, e^{i\lambda} \xi)^3.
 \end{aligned} \tag{94}$$

Now, we compute an expansion for $r \cos \theta$. Taking into account (10) and (11),

$$r \cos \theta = L^2 \left(\cos(g + u) - e \cos g - \left(\sqrt{1 - e^2} - 1 \right) \sin u \sin g \right).$$

Notice that, since $\lambda = \ell + g$ and $u = \ell + e \sin u$, we have that $\cos(g + u) = \cos(\lambda + e \sin u)$ is analytic at $(\eta, \xi) = (0, 0)$. Then, using (87), (88) and (90), we deduce

$$\begin{aligned}
 r \cos \theta &= L^2 \cos \lambda - \eta \frac{\sqrt{2L^3}}{2} (1 + i e^{-i\lambda} \sin \lambda) - \xi \frac{\sqrt{2L^3}}{2} (1 - i e^{i\lambda} \sin \lambda) \\
 &\quad - \eta \xi L \cos \lambda + \frac{L}{4} \eta^2 (e^{-i\lambda} + e^{-2i\lambda} \cos \lambda - 2i e^{-2i\lambda} \sin \lambda) \\
 &\quad + \xi^2 \frac{L}{4} (e^{i\lambda} + e^{2i\lambda} \cos \lambda + 2i e^{2i\lambda} \sin \lambda) + \mathcal{O}(e^{i\lambda} \eta, e^{-i\lambda} \xi)^3.
 \end{aligned} \tag{95}$$

Then, joining the results in (94) and (95) with the definition of $D[\zeta]$ in (91), we obtain its expansion in (η, ξ) . Moreover, since $D[\zeta]$ is analytic for $(L, \eta, \xi) \sim (1, 0, 0)$ and $|\operatorname{Im} \lambda| \leq \varrho$, the terms of order 3 satisfy the estimate in (93). \square

Remark 4.2. Observe that the Hamiltonian $H^{\text{Poi}} = H_0^{\text{Poi}} + \mu H_1^{\text{Poi}}$ in (16) is analytic away from collision with the primaries. By the decomposition of μH_1^{Poi} in (92), collisions with the primary S are given by the zeroes of the function $D[\mu]$ and collisions with P are given by the zeroes of $D[\mu - 1]$.

Since our analysis is performed for $|(L - 1, \eta, \xi)| \leq \varepsilon \ll 1$ and $0 < \mu \ll 1$, by Lemma 4.1, one has

$$D[\mu] = 1 + \mathcal{O}(\mu, \varepsilon), \quad D[\mu - 1] = 2 + 2 \cos \lambda + \mathcal{O}(\mu, \varepsilon).$$

That is, collisions with S are not possible whereas collisions with P may take place when $\lambda \sim \pi$.

4.2. Proof of Proposition 2.5

To prove Proposition 2.5, we analyze the Hamiltonian H^{in} which is given (up to a constant) by

$$\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} (H \circ \phi_{\text{eq}} \circ \phi_{\text{sep}} \circ \phi_{\text{in}}),$$

where the changes ϕ_{eq} , ϕ_{sep} and ϕ_{in} are defined in (32), (33) and (37), and $H = H_0 + H_1$ (see (25) and H_1 in (22)).

In the rest of the section, when performing changes of coordinates, to simplify notation, we omit the constant terms in the Hamiltonians.

Using the formulas for H_1^{Poi} in (92) and Lemma 4.1, we split H_1^{Poi} into two terms: one for the perturbation coming from the massive primary (S) and the other coming from the small primary (P),

$$H_1^{\text{Poi}} = H_1^{\text{Poi},S} + H_1^{\text{Poi},P},$$

which, recalling that $\mu = \delta^4$, are defined as

$$H_1^{\text{Poi},S} = \frac{1}{\delta^4} \left(\frac{1}{\sqrt{D[0]}} - \frac{1 - \delta^4}{\sqrt{D[\delta^4]}} \right) \quad \text{and} \quad H_1^{\text{Poi},P} = -\frac{1}{\sqrt{D[\delta^4 - 1]}}. \tag{96}$$

We also define the Hamiltonian $H^{\text{eq}} = H \circ \phi_{\text{eq}}$, which can be split as

$$H^{\text{eq}} = H_0 + R^{\text{eq}} + H_1^{\text{eq},P} + H_1^{\text{eq},S},$$

with

$$\begin{aligned}
 H_1^{\text{eq},*}(\lambda, \Lambda, x, y; \delta) &= H_1^{\text{Poi},*} \circ \phi_{\text{sc}} \circ \phi_{\text{eq}}, \quad \text{for } * = S, P, \\
 R^{\text{eq}}(\lambda, \Lambda, x, y; \delta) &= -V(\lambda) + \frac{1}{\delta^4} F_P(\delta^2 \Lambda + \delta^4 \mathfrak{L}_\Lambda(\delta)) \\
 &\quad - 3\delta^2 \Lambda \mathfrak{L}_\Lambda(\delta) + \delta(x \mathfrak{L}_y(\delta) + y \mathfrak{L}_x(\delta)).
 \end{aligned}
 \tag{97}$$

We recall that ϕ_{sc} is the scaling given in (17), V is the potential in (19), F_P is the function (18) and $(\mathfrak{L}_\Lambda, \mathfrak{L}_x, \mathfrak{L}_y)$ are introduced in (23).

Then, the Hamiltonian H^{in} can be written as

$$H^{\text{in}} = \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \left(H_0 \circ \Psi + R^{\text{eq}} \circ \Psi + H_1^{\text{Poi},P} \circ \Phi + H_1^{\text{Poi},S} \circ \Phi \right), \tag{98}$$

where

$$\Psi = \phi_{\text{sep}} \circ \phi_{\text{in}} \quad \text{and} \quad \Phi = \phi_{\text{sc}} \circ \phi_{\text{eq}} \circ \phi_{\text{sep}} \circ \phi_{\text{in}}.$$

In the following lemmas, we introduce expressions for the changes Ψ and Φ .

Lemma 4.3. *The change of coordinates $\Psi = (\Psi_\lambda, \Psi_\Lambda, \Psi_x, \Psi_y)$ satisfies*

$$\begin{aligned}
 \Psi_\lambda(U) &= \pi + 3\alpha_+ \delta^{\frac{4}{3}} U^{\frac{2}{3}} (1 + g_\lambda(\delta^2 U)), \\
 \Psi_\Lambda(U, W) &= -\frac{2\alpha_+}{3\delta^{\frac{2}{3}} U^{\frac{1}{3}}} (1 + g_\Lambda(\delta^2 U)) + \frac{\alpha_+ U^{\frac{1}{3}} W}{\delta^{\frac{2}{3}}} (1 + \tilde{g}_\Lambda(\delta^2 U)), \\
 \Psi_x(X) &= \delta^{\frac{1}{3}} \sqrt{2} \alpha_+ X, \\
 \Psi_y(Y) &= \delta^{\frac{1}{3}} \sqrt{2} \alpha_+ Y,
 \end{aligned}$$

where $g_\lambda(z), g_\Lambda(z), \tilde{g}_\Lambda(z) \sim \mathcal{O}(z^{\frac{2}{3}})$. Moreover, taking into account the time-parametrization of the separatrix (λ_h, Λ_h) given in (27), we have that

$$\Lambda_h \circ \phi_{\text{in}} = -\frac{2\alpha_+}{3\delta^{\frac{2}{3}} U^{\frac{1}{3}}} (1 + g_\Lambda(\delta^2 U)). \tag{99}$$

Lemma 4.4. *The change of coordinates $\Phi = (\Phi_\lambda, \Phi_L, \Phi_\eta, \Phi_\xi)$ satisfies*

$$\begin{aligned}
 \Phi_\lambda(U) &= \Psi_\lambda(U), & \Phi_L(U, W) &= 1 + \delta^2 \Psi_\Lambda(U, W) + \delta^4 \mathfrak{L}_\Lambda(\delta), \\
 \Phi_\eta(X) &= \delta \Psi_x(X) + \delta^4 \mathfrak{L}_x(\delta), & \Phi_\xi(Y) &= \delta \Psi_y(Y) + \delta^4 \mathfrak{L}_y(\delta),
 \end{aligned}$$

where $\Psi = (\Psi_\lambda, \Psi_\Lambda, \Psi_x, \Psi_y)$ is the change of coordinates given in Lemma 4.3.

We omit the proofs of these lemmas since they are a straightforward consequence of Theorem 2.2 and the definitions of the changes of coordinates (see (17), (32), (33) and (37)).

End of the proof of Proposition 2.5. We analyze each component of (98).

We denote by $C(c_1, c_2) > 0$ any constant satisfying that there exist $b_0, \gamma_1, \gamma_2 > 0$ independent of c_1, c_2, δ such that $C(c_1, c_2) \leq b_0 c_1^{\gamma_1} c_2^{\gamma_2}$.

1. We compute the first term of the Hamiltonian H^{in} in (98). Since $H_p(\lambda_h, \Lambda_h) = H_p(0, 0) = -\frac{1}{2}$ and taking into account (99), we have

$$\begin{aligned} \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_0 \circ \Psi &= \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \left(w - \frac{w^2}{6\Lambda_h^2(u)} + \frac{xy}{\delta^2} \right) \circ \phi_{\text{in}} \\ &= W + XY - \frac{3}{4} U^{\frac{2}{3}} W^2 \left(\frac{1}{1 + g_\Lambda(\delta^2 U)} \right)^2 \\ &= W + XY - \frac{3}{4} U^{\frac{2}{3}} W^2 + \mathcal{O} \left(\delta^{\frac{4}{3}} U^{\frac{2}{3}} W^2 \right). \end{aligned}$$

Since $|U| \leq c_1$ and $|W| \leq c_2$, the error term $\mathcal{O}(\delta^{\frac{4}{3}} U^{\frac{2}{3}} W^2)$ can be bounded by $C(c_1, c_2) \delta^{\frac{4}{3}}$.

For the other terms in (98), to simplify the notation, we are not specifying the dependence of the error terms on the variables (U, W, X, Y) . Moreover, when referring to error terms of order $\mathcal{O}(\delta^a)$, we mean that they can be bounded by $C(c_1, c_2) \delta^a$.

2. For the second term of the Hamiltonian H^{in} in (98) (see by (97)) we have

$$\begin{aligned} \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} R^{\text{eq}} \circ \Psi &= -\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} V(\Psi_\lambda) + \frac{1}{2\alpha_+^2 \delta^{\frac{8}{3}}} F_p(\delta^2 \Psi_\Lambda + \delta^4 \mathfrak{L}_\Lambda(\delta)) \\ &\quad - \frac{3\delta^{\frac{10}{3}} \mathfrak{L}_\Lambda(\delta)}{2\alpha_+^2} \Psi_\Lambda + \frac{\delta^{\frac{7}{3}} \mathfrak{L}_y(\delta)}{\sqrt{2}\alpha_+} \Psi_x + \frac{\delta^{\frac{7}{3}} \mathfrak{L}_x(\delta)}{\sqrt{2}\alpha_+} \Psi_y, \end{aligned} \tag{100}$$

where $F_p(z) = \mathcal{O}(z^3)$ (see (18)) and $V(\lambda)$ is the potential given in (19).

First we analyze the potential term. By Lemma 4.3, we have that

$$\begin{aligned} -\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} V(\Psi_\lambda(U)) &= \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \frac{1}{\sqrt{2 + 2 \cos \Psi_\lambda(U)}} + \mathcal{O}(\delta^{\frac{4}{3}}) \\ &= \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \left(9\alpha_+^2 \delta^{\frac{8}{3}} U^{\frac{4}{3}} (1 + g_\Lambda(\delta^2 U))^2 + \mathcal{O} \left(\delta^{\frac{16}{3}} U^{\frac{8}{3}} \right) \right)^{-\frac{1}{2}} \\ &\quad + \mathcal{O}(\delta^{\frac{4}{3}}) = \frac{1}{3U^{\frac{2}{3}}} + \mathcal{O}(\delta^{\frac{4}{3}}). \end{aligned}$$

Then, since $c_1^{-1} \leq |U| \leq c_1$ and $|(W, X, Y)| \leq c_2$, by (100) and Lemma 4.3,

$$\left| \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} R^{\text{eq}} \circ \Psi - \frac{1}{3U^{\frac{2}{3}}} \right| \leq C(c_1, c_2) \delta^{\frac{4}{3}}.$$

3. We deal with the third term of the Hamiltonian H^{in} (see (96)). Since

$$|\Phi(U, W, X, Y) - (\pi, 1, 0, 0)| \leq C(c_1, c_2)\delta^{\frac{4}{3}} \quad \text{and} \quad |\text{Im } \Phi_\lambda(U)| \leq C(c_1)\delta^{\frac{4}{3}},$$

the hypotheses of Lemma 4.1 hold and therefore:

$$\begin{aligned} \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_1^{\text{Poi}, P} \circ \Phi &= -\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \frac{1}{\sqrt{D[\delta^4 - 1] \circ \Phi}} \\ &= -\left(\frac{2\alpha_+}{\delta^{\frac{8}{3}}} (D_0 + D_1 + D_2 + D_{\geq 3}) [\delta^4 - 1] \circ \Phi \right)^{-\frac{1}{2}}. \end{aligned}$$

We compute every term $D_j[\delta^4 - 1]$, $j = 0, 1, 2, \geq 3$.

a) The term $D_0[\delta^4 - 1]$ satisfies

$$\begin{aligned} D_0[\delta^4 - 1](\lambda, L; \delta) &= L^4 + 2(1 - \delta^4)L^2 \cos \lambda + (1 - \delta^4)^2 \\ &= 2(1 + \cos \lambda) + 4(L - 1)(1 + \cos \lambda) \\ &\quad + 2(L - 1)^2(3 + \cos \lambda) + 4(L - 1)^3 \\ &\quad + (L - 1)^4 - 2\delta^4(1 + \cos \lambda) \\ &\quad - 4\delta^4(L - 1) \cos \lambda - 2\delta^4(L - 1)^2 \cos \lambda + \delta^8. \end{aligned}$$

Performing the change Φ , by Lemma 4.4, we have that

$$\begin{aligned} \frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_0[\delta^4 - 1] \circ \Phi &= \frac{2\alpha_+}{\delta^{\frac{8}{3}}} \left(2(1 + \cos \Phi_\lambda) + 4(\Phi_L - 1)^2 + \mathcal{O}(\delta^{\frac{4}{3}}) \right) \\ &= 9U^{\frac{4}{3}} + 4U^{\frac{2}{3}}W^2 - \frac{16}{3}W + \frac{16}{9U^{\frac{2}{3}}} + \mathcal{O}(\delta^{\frac{4}{3}}). \end{aligned}$$

b) Analogously the term $D_1[\delta^4 - 1]$ satisfies

$$\begin{aligned} D_1[\delta^4 - 1] &= \eta \frac{\sqrt{2L^3}}{2} [(-3 - 2e^{-i\lambda} + e^{-2i\lambda}) - 4(L - 1)e^{-i\lambda} \\ &\quad - 2(L - 1)^2e^{-i\lambda} + \delta^4(3 - e^{-2i\lambda})] \\ &\quad + \xi \frac{\sqrt{2L^3}}{2} [(-3 - 2e^{i\lambda} + e^{2i\lambda}) - 4(L - 1)e^{i\lambda} \\ &\quad - 2(L - 1)^2e^{i\lambda} + \delta^4(3 - e^{2i\lambda})] \end{aligned}$$

and therefore

$$\begin{aligned} \frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_1[\delta^4 - 1] \circ \Phi &= \frac{2\alpha_+^2}{\delta^{\frac{4}{3}}} X \left(-4i(\Phi_\lambda - \pi) - 4(\Phi_L - 1) + \mathcal{O}(\delta^{\frac{4}{3}}) \right) \\ &\quad + \frac{2\alpha_+^2}{\delta^{\frac{4}{3}}} Y \left(4i(\Phi_\lambda - \pi) - 4(\Phi_L - 1) + \mathcal{O}(\delta^{\frac{4}{3}}) \right) \\ &= X \left(-12iU^{\frac{2}{3}} + 4U^{\frac{1}{3}} - \frac{8}{3U^{\frac{1}{3}}} \right) \\ &\quad + Y \left(12iU^{\frac{2}{3}} + 4U^{\frac{1}{3}} - \frac{8}{3U^{\frac{1}{3}}} \right) + \mathcal{O}(\delta^{\frac{4}{3}}). \end{aligned}$$

c) The term $D_2[\delta^4 - 1]$ satisfies

$$\begin{aligned} D_2[\delta^4 - 1] &= -\eta^2 \frac{Le^{-i\lambda}}{4} (-1 + 2L^2e^{-i\lambda} - 3e^{-2i\lambda} + \delta^4(1 + 3e^{-2i\lambda})) \\ &\quad - \xi^2 \frac{Le^{-i\lambda}}{4} (-1 + 2L^2e^{i\lambda} - 3e^{2i\lambda} + \delta^4(1 + 3e^{2i\lambda})) \\ &\quad + \eta\xi L(3L^2 - 2\cos\lambda + \delta^4 2\cos\lambda) \end{aligned}$$

and

$$\frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_2[\delta^4 - 1] \circ \Phi = -3X^2 - 3Y^2 + 5XY + \mathcal{O}(\delta^{\frac{4}{3}}).$$

d) By the estimates of $D_{\geq 3}[\delta^4 - 1]$ in (93) and Lemma 4.4,

$$\left| \frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_3[\delta^4 - 1] \circ \Phi \right| \leq C(c_1, c_2)\delta^{\frac{4}{3}}.$$

Collecting these results, we conclude that

$$\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_1^{\text{Poi},P} \circ \Phi = -\frac{1}{3U^{\frac{2}{3}}} \frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} + \mathcal{O}(\delta^{\frac{4}{3}}),$$

where the function \mathcal{J} is given in (40).

4. Proceeding analogously as for $H_1^{\text{Poi},P}$, it can be checked that

$$\left| \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_1^{\text{Poi},S} \circ \Phi \right| \leq C(c_1, c_2)\delta^{\frac{4}{3}}. \quad \square$$

5. Analysis of the inner equation

We split the proof of Theorem 2.7 into two parts. In Section 5.1 we prove the existence of the solutions Z_0^u and Z_0^s and the estimates in (48). In Section 5.2, we provide the asymptotic formula for the difference $\Delta Z_0 = Z_0^u - Z_0^s$ given in (49). For both parts, we follow the approach given in [5].

Throughout this section, we fix the $\beta_0 \in (0, \frac{\pi}{2})$ appearing in the definition of the domains \mathcal{D}_κ^u and \mathcal{D}_κ^s in (46) and \mathcal{E}_κ in (47). We denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3)^T$, unless stated otherwise. In order to simplify the notation, we denote by C any positive constant independent of κ .

5.1. Existence of suitable solutions of the inner equation

From now on we deal only with the analysis for Z_0^u . The analysis for Z_0^s is analogous.

5.1.1. Preliminaries and set up

The invariance equation (43), that is $\partial_U Z_0^u = \mathcal{A}Z_0^u + \mathcal{R}[Z_0^u]$, can be written as $\mathcal{L}Z_0^u = \mathcal{R}[Z_0^u]$ where \mathcal{L} is the linear operator

$$\mathcal{L}\varphi = (\partial_U - \mathcal{A})\varphi. \tag{101}$$

Notice that if we can construct a left-inverse of \mathcal{L} in an appropriate Banach space, we can write (43) as a fixed point equation to be able apply the Banach fixed point theorem.

Given $\nu \in \mathbb{R}$ and $\kappa > 0$, we define the norm

$$\|\varphi\|_\nu = \sup_{U \in \mathcal{D}_\kappa^u} |U^\nu \varphi(U)|,$$

where the domain \mathcal{D}_κ^u is given in (46), and we introduce the Banach space

$$\mathcal{X}_\nu = \{\varphi : \mathcal{D}_\kappa^u \rightarrow \mathbb{C} : \varphi \text{ analytic, } \|\varphi\|_\nu < +\infty\}.$$

Next lemma, proven in [1], gives some properties of these Banach spaces. We use this lemma throughout the section without mentioning it.

Lemma 5.1. *Let $\kappa > 0$ and $\nu, \eta \in \mathbb{R}$. The following statements hold:*

1. *If $\nu > \eta$, then $\mathcal{X}_\nu \subset \mathcal{X}_\eta$ and $\|\varphi\|_\eta \leq (\kappa \cos \beta_0)^{\eta-\nu} \|\varphi\|_\nu$.*
2. *If $\varphi \in \mathcal{X}_\nu$ and $\zeta \in \mathcal{X}_\eta$, then $\varphi\zeta \in \mathcal{X}_{\nu+\eta}$ and $\|\varphi\zeta\|_{\nu+\eta} \leq \|\varphi\|_\nu \|\zeta\|_\eta$.*

In the next lemma, we introduce a left-inverse of the operator \mathcal{L} in (101).

Lemma 5.2. *Consider the operator*

$$\mathcal{G}[\varphi](U) = \left(\int_{-\infty}^U \varphi_1(S) dS, \int_{-\infty}^U e^{-i(S-U)} \varphi_2(S) dS, \int_{-\infty}^U e^{i(S-U)} \varphi_3(S) dS \right)^T.$$

Fix $\eta > 1$, $\nu > 0$ and $\kappa \geq 1$. Then, $\mathcal{G} : \mathcal{X}_\eta \times \mathcal{X}_\nu \times \mathcal{X}_\nu \rightarrow \mathcal{X}_{\eta-1} \times \mathcal{X}_\nu \times \mathcal{X}_\nu$ is a continuous linear operator and is a left-inverse of \mathcal{L} .

Moreover, there exists a constant $C > 0$ such that

1. If $\varphi \in \mathcal{X}_\eta$, then $\mathcal{G}_1[\varphi] \in \mathcal{X}_{\eta-1}$ and $\|\mathcal{G}_1[\varphi]\|_{\eta-1} \leq C \|\varphi\|_\eta$.
2. If $\varphi \in \mathcal{X}_\nu$ and $j = 2, 3$, then $\mathcal{G}_j[\varphi] \in \mathcal{X}_\nu$ and $\|\mathcal{G}_j[\varphi]\|_\nu \leq C \|\varphi\|_\nu$.

Proof. It follows the same lines as the proof of Lemma 4.6 in [1]. \square

Let us then define the fixed point operator

$$\mathcal{F} = \mathcal{G} \circ \mathcal{R}. \tag{102}$$

A solution of $Z_0^u = \mathcal{F}[Z_0^u]$ belonging to $\mathcal{X}_\eta \times \mathcal{X}_\nu \times \mathcal{X}_\nu$ with $\eta, \nu > 0$ satisfies equation (43) and the asymptotic condition (45). Therefore, to prove the first part of Theorem 2.7 and the asymptotic estimates in (48), we look for a fixed point of the operator \mathcal{F} in the Banach space

$$\mathcal{X}_x = \mathcal{X}_{\frac{8}{3}} \times \mathcal{X}_{\frac{4}{3}} \times \mathcal{X}_{\frac{4}{3}},$$

endowed with the norm

$$\|\varphi\|_x = \|\varphi_1\|_{\frac{8}{3}} + \|\varphi_2\|_{\frac{4}{3}} + \|\varphi_3\|_{\frac{4}{3}}.$$

Proposition 5.3. *There exists $\kappa_0 > 0$ such that for any $\kappa \geq \kappa_0$, the fixed point equation $Z_0^u = \mathcal{F}[Z_0^u]$ has a solution $Z_0^u \in \mathcal{X}_x$. Moreover, there exists a constant $b_3 > 0$, independent of κ , such that*

$$\|Z_0^u\|_x \leq b_3.$$

Remark 5.4. Notice that $\mathcal{D}_\kappa^u \subseteq \mathcal{D}_{\kappa_0}^u$ when $\kappa \geq \kappa_0$ (see (46)). Then, for some $\nu \in \mathbb{R}$, if $\zeta \in \mathcal{X}_\nu$ (defined for κ) then $\zeta \in \mathcal{X}_\nu$ (defined for κ_0). This allows us to take κ as big as we need.

5.1.2. Proof of Proposition 5.3

We first state a technical lemma whose proof is postponed until Section 5.3.1. For $\varrho > 0$, we define the closed ball

$$B(\varrho) = \{\varphi \in \mathcal{X}_x : \|\varphi\|_x \leq \varrho\}.$$

Lemma 5.5. *Let \mathcal{R} be the operator defined in (44). Then, for $\varrho > 0$ and for $\kappa > 0$ big enough, there exists a constant $C > 0$ such that, for any $Z_0 \in B(\varrho)$,*

$$\|\mathcal{R}_1[Z_0]\|_{\frac{11}{3}} \leq C, \quad \|\mathcal{R}_j[Z_0]\|_{\frac{4}{3}} \leq C, \quad j = 2, 3,$$

and

$$\begin{aligned} \|\partial_W \mathcal{R}_1[Z_0]\|_3 &\leq C, & \|\partial_X \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} &\leq C, & \|\partial_Y \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} &\leq C, \\ \|\partial_W \mathcal{R}_j[Z_0]\|_{\frac{2}{3}} &\leq C, & \|\partial_X \mathcal{R}_j[Z_0]\|_2 &\leq C, & \|\partial_Y \mathcal{R}_j[Z_0]\|_2 &\leq C, \quad j = 2, 3. \end{aligned}$$

The next lemma gives properties of the operator \mathcal{F} .

Lemma 5.6. *Let \mathcal{F} be the operator defined in (102). Then, for $\kappa > 0$ big enough, there exists a constant $b_4 > 0$ independent of κ such that*

$$\|\mathcal{F}[0]\|_\times \leq b_4.$$

Moreover, for $\varrho > 0$ and $\kappa > 0$ big enough, there exists a constant $b_5 > 0$ independent of κ such that, for any $Z_0 = (W_0, X_0, Y_0)^T$, $\tilde{Z}_0 = (\tilde{W}_0, \tilde{X}_0, \tilde{Y}_0)^T \in B(\varrho) \subset \mathcal{X}_\times$,

$$\begin{aligned} \|\mathcal{F}_1[Z_0] - \mathcal{F}_1[\tilde{Z}_0]\|_{\frac{8}{3}} &\leq b_5 \left(\frac{1}{\kappa^2} \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} + \|X_0 - \tilde{X}_0\|_{\frac{4}{3}} + \|Y_0 - \tilde{Y}_0\|_{\frac{4}{3}} \right), \\ \|\mathcal{F}_j[Z_0] - \mathcal{F}_j[\tilde{Z}_0]\|_{\frac{4}{3}} &\leq \frac{b_5}{\kappa^2} \|Z_0 - \tilde{Z}_0\|_\times, \quad j = 2, 3. \end{aligned}$$

Proof. The estimate for $\mathcal{F}[0]$ is a direct consequence of Lemmas 5.2 and 5.5.

To estimate the Lipschitz constant, we first estimate each component $\mathcal{R}_j[Z_0] - \mathcal{R}_j[\tilde{Z}_0]$ separately for $j = 1, 2, 3$. By the mean value theorem we have

$$\mathcal{R}_j[Z_0] - \mathcal{R}_j[\tilde{Z}_0] = \left[\int_0^1 D\mathcal{R}_j[sZ_0 + (1-s)\tilde{Z}_0] ds \right] (Z_0 - \tilde{Z}_0).$$

Then, for $j = 2, 3$, we have

$$\begin{aligned} \|\mathcal{R}_1[Z_0] - \mathcal{R}_1[\tilde{Z}_0]\|_{\frac{11}{3}} &\leq \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_1[\varphi]\|_1 \\ &\quad + \|X_0 - \tilde{X}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_1[\varphi]\|_{\frac{7}{3}} + \|Y_0 - \tilde{Y}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_1[\varphi]\|_{\frac{7}{3}}, \\ \|\mathcal{R}_j[Z_0] - \mathcal{R}_j[\tilde{Z}_0]\|_{\frac{4}{3}} &\leq \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_j[\varphi]\|_{-\frac{4}{3}} \\ &\quad + \|X_0 - \tilde{X}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_j[\varphi]\|_0 + \|Y_0 - \tilde{Y}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_j[\varphi]\|_0. \end{aligned}$$

Applying Lemma 5.5, we obtain

$$\begin{aligned} \|\mathcal{R}_1[Z_0] - \mathcal{R}_1[\tilde{Z}_0]\|_{\frac{11}{3}} &\leq C \left(\frac{1}{\kappa^2} \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} + \|X_0 - \tilde{X}_0\|_{\frac{4}{3}} + \|Y_0 - \tilde{Y}_0\|_{\frac{4}{3}} \right), \\ \|\mathcal{R}_j[Z_0] - \mathcal{R}_j[\tilde{Z}_0]\|_{\frac{4}{3}} &\leq \frac{C}{\kappa^2} \|Z_0 - \tilde{Z}_0\|_\times, \quad j = 2, 3. \end{aligned}$$

Finally, applying Lemma 5.2, we obtain the estimates in the lemma. \square

Lemma 5.6 shows that, by assuming $\kappa > 0$ big enough, the operators \mathcal{F}_2 and \mathcal{F}_3 have Lipschitz constant less than 1. However, this is not the case for \mathcal{F}_1 . To overcome this problem, we apply a Gauss-Seidel argument and define a new operator

$$\tilde{\mathcal{F}}[Z_0] = \tilde{\mathcal{F}}[(W_0, X_0, Y_0)] = \begin{pmatrix} \mathcal{F}_1[W_0, \mathcal{F}_2[Z_0], \mathcal{F}_3[Z_0]] \\ \mathcal{F}_2[Z_0] \\ \mathcal{F}_3[Z_0] \end{pmatrix},$$

which has the same fixed points as \mathcal{F} and turns out to be contractive in a suitable ball.

End of the proof of Proposition 5.3. We first obtain an estimate for $\|\tilde{\mathcal{F}}[0]\|_\times$. Notice that

$$\tilde{\mathcal{F}}[0] = \mathcal{F}[0] + (\tilde{\mathcal{F}}[0] - \mathcal{F}[0]) = \mathcal{F}[0] + (\mathcal{F}_1[0, \mathcal{F}_2[0], \mathcal{F}_3[0]] - \mathcal{F}_1[0], 0, 0)^T.$$

Then, by Lemma 5.6, $(0, \mathcal{F}_2[0], \mathcal{F}_3[0])^T \in \mathcal{X}_\times$ and

$$\begin{aligned} \|\tilde{\mathcal{F}}[0]\|_\times &\leq \|\mathcal{F}[0]\|_\times + \|\mathcal{F}_1[0, \mathcal{F}_2[0], \mathcal{F}_3[0]] - \mathcal{F}_1[0]\|_{\frac{8}{3}} \\ &\leq \|\mathcal{F}[0]\|_\times + C \|\mathcal{F}_2[0]\|_{\frac{4}{3}} + C \|\mathcal{F}_3[0]\|_{\frac{4}{3}} \leq C \|\mathcal{F}[0]\|_\times. \end{aligned}$$

Thus, we can fix $\varrho > 0$ such that

$$\|\tilde{\mathcal{F}}[0]\|_\times \leq \frac{\varrho}{2}.$$

Now, we prove that the operator $\tilde{\mathcal{F}}$ is contractive in $B(\varrho) \subset \mathcal{X}_\times$. By Lemma 5.6 and assuming $\kappa > 0$ big enough, we have that for $Z_0, \tilde{Z}_0 \in B(\varrho)$,

$$\begin{aligned} \|\tilde{\mathcal{F}}_1[Z_0] - \tilde{\mathcal{F}}_1[\tilde{Z}_0]\|_{\frac{8}{3}} &\leq \frac{C}{\kappa^2} \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} + \sum_{j=2}^3 \|\mathcal{F}_j[Z_0] - \mathcal{F}_j[\tilde{Z}_0]\|_{\frac{4}{3}} \\ &\leq \frac{C}{\kappa^2} \|W_0 - \tilde{W}_0\|_{\frac{8}{3}} + \frac{C}{\kappa^2} \|Z_0 - \tilde{Z}_0\|_\times \leq \frac{C}{\kappa^2} \|Z_0 - \tilde{Z}_0\|_\times, \\ \|\tilde{\mathcal{F}}_j[Z_0] - \tilde{\mathcal{F}}_j[\tilde{Z}_0]\|_{\frac{4}{3}} &\leq \frac{C}{\kappa^2} \|Z_0 - \tilde{Z}_0\|_\times, \quad \text{for } j = 2, 3. \end{aligned}$$

Then, there exists $\kappa_0 > 0$ such that for $\kappa \geq \kappa_0$, the operator $\tilde{\mathcal{F}} : B(\varrho) \rightarrow B(\varrho)$ is well defined and contractive. Thus $\tilde{\mathcal{F}}$ has a fixed point $Z_0^u \in B(\varrho) \subset \mathcal{X}_\times$. \square

5.2. Asymptotic formula for the difference

The strategy to prove the second part of Theorem 2.7 is divided in three steps. In Section 5.2.1 we characterize $\Delta Z_0 = Z_0^u - Z_0^s$ as a solution of a linear homogeneous

equation. In Section 5.2.2, we prove that ΔZ_0 is in fact the unique solution of this linear equation in a suitable Banach space. Finally, in Section 5.2.3, we introduce a Banach subspace of the previous one (with exponential weights) to obtain exponentially small estimates for ΔZ_0 .

5.2.1. *A homogeneous linear equation for ΔZ_0*

By Theorem 2.7, the difference $\Delta Z_0(U) = Z_0^u(U) - Z_0^s(U)$ is well defined for $U \in \mathcal{E}_\kappa$ (see (47)). Since Z_0^u, Z_0^s satisfy the same invariance equation (43), their difference ΔZ_0 satisfies

$$\partial_U \Delta Z_0 = \mathcal{A} \Delta Z_0 + R(U) \Delta Z_0,$$

where

$$R(U) = \int_0^1 D_Z \mathcal{R}[sZ_0^u + (1-s)Z_0^s](U) ds, \tag{103}$$

and \mathcal{A} and \mathcal{R} are given in (41) and (44), respectively. We denote by R_1, R_2 and R_3 , the rows of the matrix R .

By the method of variation of parameters, there exists $c = (c_w, c_x, c_y)^T \in \mathbb{C}^3$ such that

$$\Delta Z_0(U) = e^{\mathcal{A}U} \left(c + \int_{U_0}^U e^{-\mathcal{A}S} R(S) \Delta Z_0(S) dS \right).$$

By Proposition 5.3, $\Delta Z_0 = Z_0^u - Z_0^s$ satisfies $\lim_{\text{Im } U \rightarrow -\infty} \Delta Z_0(U) = 0$. Therefore ΔZ_0 satisfies

$$\Delta Z_0 = \Delta Z_{\text{init}} + \mathcal{I}[\Delta Z_0], \tag{104}$$

where \mathcal{I} is the linear operator

$$\mathcal{I}[\varphi](U) = \begin{pmatrix} \int_{-i\infty}^U \langle R_1(S), \varphi(S) \rangle dS \\ e^{iU} \int_{-i\infty}^U e^{-iS} \langle R_2(S), \varphi(S) \rangle dS \\ e^{-iU} \int_{-i\kappa}^U e^{iS} \langle R_3(S), \varphi(S) \rangle dS \end{pmatrix}, \tag{105}$$

and ΔZ_{init} is the function

$$\Delta Z_{\text{init}}(U) = (0, 0, c_y e^{-iU})^T = (0, 0, e^\kappa \Delta Y_0(-i\kappa) e^{-iU})^T.$$

5.2.2. Characterization of ΔZ_0 as a fixed point

Given $\nu \in \mathbb{R}$ and $\kappa > 0$, we define the norm

$$\|\varphi\|_\nu = \sup_{U \in \mathcal{E}_\kappa} |U^\nu \varphi(U)|,$$

where the domain \mathcal{E}_κ is given in (47), and we introduce the Banach space

$$\mathcal{Y}_\nu = \{\varphi : \mathcal{E}_\kappa \rightarrow \mathbb{C} : \varphi \text{ analytic}, \|\varphi\|_\nu < +\infty\}.$$

Note that \mathcal{Y}_ν satisfy analogous properties as the ones in Lemma 5.1. In this section, we use this lemma without mentioning it.

We state a technical lemma, whose proof is postponed to Section 5.3.2.

Lemma 5.7. *Let \mathcal{I} be the operator defined in (105). Then, for $\kappa > 0$ big enough, there exists a constant $b_6 > 0$ independent of κ such that, for $\Psi \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$,*

$$\begin{aligned} \|\mathcal{I}_1[\Psi]\|_{\frac{8}{3}} &\leq b_6 \left(\frac{1}{\kappa^2} \|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \\ \|\mathcal{I}_j[\Psi]\|_{\frac{4}{3}} &\leq \frac{b_6}{\kappa^2} \left(\|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \quad j = 2, 3. \end{aligned}$$

These estimates characterize ΔZ_0 as the unique solution of (104) in the space $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$.

Lemma 5.8. *For $\kappa > 0$ big enough, ΔZ_0 is the unique solution of equation (104) belonging to $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. In particular,*

$$\Delta Z_0 = \sum_{n \geq 0} \mathcal{I}^n[\Delta Z_{\text{init}}].$$

Proof. By Theorem 2.7, for $\kappa > 0$ big enough, ΔZ_0 is a solution of equation (104) which satisfies $\Delta Z_0 = Z_0^u - Z_0^s \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. Then, it only remains to prove that equation (104) has a unique solution in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. To this end, it is enough to show that the operator \mathcal{I} is contractive with a suitable norm in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. Taking

$$\|\Psi\|_\times = \|\Psi_1\|_{\frac{8}{3}} + \kappa \|\Psi_2\|_{\frac{4}{3}} + \kappa \|\Psi_3\|_{\frac{4}{3}},$$

Lemma 5.7 implies

$$\|\mathcal{I}[\Psi]\|_\times \leq \frac{C}{\kappa} \|\Psi\|_\times$$

and, taking κ big enough, the result is proven. \square

5.2.3. Exponentially small estimates for ΔZ_0

Once we have proved that ΔZ_0 is the unique solution of (104) in the space $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$, we use this equation to obtain exponentially small estimates for ΔZ_0 .

For any $\nu \in \mathbb{R}$, we consider the norm

$$\llbracket \varphi \rrbracket_\nu = \sup_{U \in \mathcal{E}_\kappa} |U^\nu e^{iU} \varphi(U)|,$$

and the associated Banach space

$$\mathcal{Z}_\nu = \{ \varphi : \mathcal{E}_\kappa \rightarrow \mathbb{C} : \varphi \text{ analytic, } \llbracket \varphi \rrbracket_\nu < +\infty \}.$$

Moreover, for $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$, we consider the product space

$$\mathcal{Z}_{\nu_1, \nu_2, \nu_3} = \mathcal{Z}_{\nu_1} \times \mathcal{Z}_{\nu_2} \times \mathcal{Z}_{\nu_3}, \quad \text{with} \quad \llbracket \varphi \rrbracket_{\nu_1, \nu_2, \nu_3} = \sum_{j=1}^3 \llbracket \varphi_j \rrbracket_{\nu_j}.$$

Next lemma, gives some properties of these Banach spaces. It follows the same lines as Lemma 5.1.

Lemma 5.9. *Let $\kappa > 0$ and $\nu, \eta \in \mathbb{R}$. The following statements hold:*

1. *If $\nu > \eta$, then $\mathcal{Z}_\nu \subset \mathcal{Z}_\eta$ and $\llbracket \varphi \rrbracket_\eta \leq (\kappa \cos \beta_0)^{\eta-\nu} \llbracket \varphi \rrbracket_\nu$.*
2. *If $\varphi \in \mathcal{Z}_\nu$ and $\zeta \in \mathcal{Y}_\eta$, then $\varphi \zeta \in \mathcal{Z}_{\nu+\eta}$ and $\llbracket \varphi \zeta \rrbracket_{\nu+\eta} \leq \llbracket \varphi \rrbracket_\nu \|\zeta\|_\eta$.*
3. *If $\varphi \in \mathcal{Z}_\nu$ then $e^{iU} \varphi \in \mathcal{Y}_\nu$ and $\|e^{iU} \varphi\|_\nu = \llbracket \varphi \rrbracket_\nu$.*

The next lemma analyzes how the operator \mathcal{I} acts on the space $\mathcal{Z}_{\frac{4}{3}, 0, 0}$. Its proof is postponed to Section 5.3.2.

Lemma 5.10. *Let \mathcal{I} be the operator defined in (105). For $\kappa > 0$ big enough, there exists a constant $b_7 > 0$ independent of κ such that, for $\Psi \in \mathcal{Z}_{\frac{4}{3}, 0, 0}$,*

$$\llbracket \mathcal{I}_1[\Psi] \rrbracket_{\frac{7}{3}} \leq b_7 \llbracket \Psi \rrbracket_{\frac{4}{3}, 0, 0}, \quad \llbracket \mathcal{I}_2[\Psi] \rrbracket_2 \leq b_7 \llbracket \Psi \rrbracket_{\frac{4}{3}, 0, 0}, \quad \llbracket \mathcal{I}_3[\Psi] \rrbracket_0 \leq \frac{b_7}{\kappa} \llbracket \Psi \rrbracket_{\frac{4}{3}, 0, 0}.$$

Moreover, there exists $\tilde{\Theta}(\kappa) \in \mathbb{C}$ (depending on Ψ), such that

$$\mathcal{I}_3[\Psi] - e^{-iU} \tilde{\Theta}(\kappa) \in \mathcal{Z}_1.$$

End of the proof of the second part of Theorem 2.7. Lemma 5.10 implies that operator $\mathcal{I} : \mathcal{Z}_{\frac{4}{3}, 0, 0} \rightarrow \mathcal{Z}_{\frac{4}{3}, 0, 0}$ is well defined and contractive. Indeed, taking $\kappa > 0$ big enough and $\Psi \in \mathcal{Z}_{\frac{4}{3}, 0, 0}$,

$$\begin{aligned} \llbracket \mathcal{I}[\Psi] \rrbracket_{\frac{4}{3}, 0, 0} &= \llbracket \mathcal{I}_1[\Psi] \rrbracket_{\frac{4}{3}} + \llbracket \mathcal{I}_2[\Psi] \rrbracket_0 + \llbracket \mathcal{I}_3[\Psi] \rrbracket_0 \\ &\leq \frac{C}{\kappa} \llbracket \mathcal{I}_1[\Psi] \rrbracket_{\frac{7}{3}} + \frac{C}{\kappa^2} \llbracket \mathcal{I}_2[\Psi] \rrbracket_2 + \llbracket \mathcal{I}_3[\Psi] \rrbracket_0 \leq \frac{C}{\kappa} \llbracket \Psi \rrbracket_{\frac{4}{3}, 0, 0}. \end{aligned} \tag{106}$$

Therefore, since $\Delta Z_{\text{init}} = (0, 0, c_y e^{-iU})^T \in \mathcal{Z}_{\frac{4}{3},0,0}$, Lemma 5.8 and (106) imply that

$$\Delta Z_0 = (\Delta W_0, \Delta X_0, \Delta Y_0)^T = \sum_{n \geq 0} \mathcal{I}^n [\Delta Z_{\text{init}}] \in \mathcal{Z}_{\frac{4}{3},0,0}.$$

Lemma 5.10 implies $\mathcal{I} : \mathcal{Z}_{\frac{4}{3},0,0} \rightarrow \mathcal{Z}_{\frac{7}{3},2,0} \subset \mathcal{Z}_{\frac{4}{3},0,0}$, which allows to give better estimates for ΔZ_0 . Indeed, we have that $\Delta Z_0 - \Delta Z_{\text{init}} = \mathcal{I}[\Delta Z_0] \in \mathcal{Z}_{\frac{7}{3},2,0}$, which implies

$$\Delta W_0 = \mathcal{I}_1[\Delta Z_0] \in \mathcal{Z}_{\frac{7}{3}}, \quad \Delta X_0 = \mathcal{I}_2[\Delta Z_0] \in \mathcal{Z}_2.$$

Moreover, by the second statement in Lemma 5.10, there exists $\tilde{\Theta}(\kappa)$ such that

$$\Delta Y_0 - c_y e^{-iU} - \tilde{\Theta}(\kappa) e^{-iU} \in \mathcal{Z}_1.$$

Calling $\Theta = c_y + \tilde{\Theta}(\kappa)$ we have that $\Delta Y_0(U) - \Theta e^{-iU} \in \mathcal{Z}_1$, and, therefore $\Theta = \lim_{\text{Im } U \rightarrow -\infty} \Delta Y_0(U) e^{iU}$, which is independent of κ . Then, ΔZ_0 is of the form

$$\Delta Z_0(U) = e^{-iU} \left((0, 0, \Theta)^T + \chi(U) \right), \quad \text{with } \chi \in \mathcal{Y}_{\frac{7}{3}} \times \mathcal{Y}_2 \times \mathcal{Y}_1.$$

Now we prove that, if there exists $U_0 \in \mathcal{E}_\kappa$ such that $\Delta Z_0(U_0) \neq 0$, then $\Theta \neq 0$. This implies $\Delta Z_0(U) \neq 0$ for all $U \in \mathcal{E}_\kappa$, since ΔZ_0 is a solution of an homogeneous linear differential equation. Therefore $c_y \neq 0$. Indeed, $c_y = 0$ would imply $\Delta Z_{\text{init}} = 0$ and, by Lemma 5.8, one could conclude $\Delta Z_0 \equiv 0$.

Thus, it only remains to prove that $c_y \neq 0$ implies $\Theta \neq 0$. By Lemma 5.8,

$$\Delta Z_0 - \Delta Z_{\text{init}} = \sum_{n \geq 1} \mathcal{I}^n [\Delta Z_{\text{init}}].$$

In addition, by the estimate (106), $\llbracket \mathcal{I}_3 \rrbracket_{\frac{4}{3}} \leq \frac{1}{4}$ if $\kappa > 0$ is big enough. Since $\Delta Y_{\text{init}} = c_y e^{-iU}$, we deduce that

$$\llbracket \Delta Y_0 - \Delta Y_{\text{init}} \rrbracket_0 \leq \sum_{n \geq 1} \frac{1}{4^n} \llbracket \Delta Y_{\text{init}} \rrbracket_0 = \frac{1}{3} |c_y|,$$

and, by the definition of the norm $\llbracket \cdot \rrbracket_0$, for any $U \in \mathcal{E}_\kappa$,

$$\left| e^{iU} \Delta Y_{\text{init}}(U) \right| - \left| e^{iU} \Delta Y_0(U) \right| \leq \frac{1}{3} |c_y|.$$

Hence, using that $e^{iU} \Delta Y_0 = \Theta + \chi_3(U)$ with $\chi_3 \in \mathcal{Y}_1$ and $e^{iU} \Delta Y_{\text{init}}(U) = c_y$, we have that for all $U \in \mathcal{E}_\kappa$,

$$\left| e^{iU} \Delta Y_0(U) \right| = |\Theta + \chi_3(U)| \geq \left| e^{iU} \Delta Y_{\text{init}}(U) \right| - \frac{1}{3} |c_y| = \frac{2}{3} |c_y|.$$

Finally, taking $\text{Im}(U) \rightarrow -\infty$, we obtain that $|\Theta| \geq \frac{2}{3} |c_y| > 0$. \square

5.3. Proof of the technical lemmas

We devote this section to prove Lemma 5.5 of Section 5.3.1 and Lemmas 5.7 and 5.10 of Section 5.3.2.

5.3.1. Proof of Lemma 5.5

Fix $\varrho > 0$ and take $Z_0 = (W_0, X_0, Y_0)^T \in B(\varrho) \subset \mathcal{X}_\times$. By the definition of \mathcal{R} in (44),

$$\mathcal{R}[Z_0](U) = \left(\frac{f_1(U, Z_0)}{1 + g(U, Z_0)}, \frac{\tilde{f}_2(U, Z_0)}{1 + g(U, Z_0)}, \frac{\tilde{f}_3(U, Z_0)}{1 + g(U, Z_0)} \right), \tag{107}$$

where

$$\tilde{f}_2(U, Z_0) = f_2(U, Z_0) - iX_0g(U, Z_0), \quad \tilde{f}_3(U, Z_0) = f_3(U, Z_0) + iY_0g(U, Z_0),$$

with $g = \partial_W \mathcal{K}$, $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and \mathcal{K} is the Hamiltonian given in (39) in terms of the function \mathcal{J} (see (40)).

We first estimate \mathcal{J} and its derivatives. For $\kappa > 0$ big enough, we have

$$|\mathcal{J}(U, Z_0)| \leq \frac{C}{U^2}, \quad |1 + \mathcal{J}(U, Z_0)| \geq 1 - \frac{C}{\kappa^2} \geq \frac{1}{2}.$$

Moreover, its derivatives satisfy

$$\begin{aligned} |\partial_U \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^3}, & |\partial_W \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{4}{3}}}, \\ |\partial_X \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{2}{3}}}, & |\partial_Y \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{2}{3}}}, \end{aligned}$$

and

$$\begin{aligned} |\partial_{UW} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{7}{3}}}, & |\partial_{UX} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, & |\partial_{UY} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, \\ |\partial_W^2 \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{2}{3}}}, & |\partial_{WX} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|}, & |\partial_{WY} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|}, \\ |\partial_X^2 \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{4}{3}}}, & |\partial_{XY} \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{4}{3}}}, & |\partial_Y^2 \mathcal{J}(U, Z_0)| &\leq \frac{C}{|U|^{\frac{4}{3}}}. \end{aligned}$$

Using these estimates, we obtain the following bounds for g, f_1, \tilde{f}_2 and \tilde{f}_3 ,

$$|g(U, Z_0)| = \left| -\frac{3}{2}U^{\frac{2}{3}}W_0 + \frac{1}{6U^{\frac{2}{3}}(1 + \mathcal{J})^{\frac{3}{2}}} \right| \leq \frac{C}{|U|^2},$$

$$\begin{aligned}
 |f_1(U, Z_0)| &= \left| \frac{W_0^2}{2U^{\frac{1}{3}}} - \frac{2}{9U^{\frac{5}{3}}} \frac{\mathcal{J}}{\sqrt{1+\mathcal{J}}(1+\sqrt{1+\mathcal{J}})} - \frac{1}{6U^{\frac{2}{3}}} \frac{\partial_U \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} \right| \leq \frac{C}{|U|^{\frac{1}{3}}}, \\
 |\tilde{f}_2(U, Z_0)| &= \left| \frac{i}{6U^{\frac{2}{3}}} \frac{\partial_Y \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - iX_0 g(U, Z_0) \right| \leq \frac{C}{|U|^{\frac{4}{3}}}, \\
 |\tilde{f}_3(U, Z_0)| &= \left| -\frac{i}{6U^{\frac{2}{3}}} \frac{\partial_X \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} + iY_0 g(U, Z_0) \right| \leq \frac{C}{|U|^{\frac{4}{3}}}.
 \end{aligned}$$

Analogously, we obtain estimates for the derivatives,

$$\begin{aligned}
 |\partial_W g(U, Z_0)| &\leq C|U|^{\frac{2}{3}}, & |\partial_X g(U, Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, & |\partial_Y g(U, Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, \\
 |\partial_W f_1(U, Z_0)| &\leq \frac{C}{|U|^3}, & |\partial_X f_1(U, Z_0)| &\leq \frac{C}{|U|^{\frac{7}{3}}}, & |\partial_Y f_1(U, Z_0)| &\leq \frac{C}{|U|^{\frac{7}{3}}}, \\
 |\partial_W \tilde{f}_2(U, Z_0)| &\leq \frac{C}{|U|^{\frac{2}{3}}}, & |\partial_X \tilde{f}_2(U, Z_0)| &\leq \frac{C}{|U|^2}, & |\partial_Y \tilde{f}_2(U, Z_0)| &\leq \frac{C}{|U|^2}, \\
 |\partial_W \tilde{f}_3(U, Z_0)| &\leq \frac{C}{|U|^{\frac{2}{3}}}, & |\partial_X \tilde{f}_3(U, Z_0)| &\leq \frac{C}{|U|^2}, & |\partial_Y \tilde{f}_3(U, Z_0)| &\leq \frac{C}{|U|^2}.
 \end{aligned}$$

Using these results we estimate the components of \mathcal{R} in (107),

$$\|\mathcal{R}_1[Z_0]\|_{\frac{11}{3}} = \left\| \frac{f_1(\cdot, Z_0)}{1+g(\cdot, Z_0)} \right\|_{\frac{11}{3}} \leq C, \quad \|\mathcal{R}_j^{\text{in}}[Z_0]\|_{\frac{4}{3}} = \left\| \frac{\tilde{f}_j(\cdot, Z_0)}{1+g(\cdot, Z_0)} \right\|_{\frac{4}{3}} \leq C,$$

for $j = 2, 3$. Moreover,

$$\begin{aligned}
 \|\partial_W \mathcal{R}_1[Z_0]\|_3 &= \left\| \frac{\partial_W f_1}{1+g} - \frac{f_1 \partial_W g}{(1+g)^2} \right\|_3 \leq C, \\
 \|\partial_X \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} &= \left\| \frac{\partial_X f_1}{1+g} - \frac{f_1 \partial_X g}{(1+g)^2} \right\|_{\frac{7}{3}} \leq C, \\
 \|\partial_W \mathcal{R}_2[Z_0]\|_{\frac{2}{3}} &= \left\| \frac{\partial_W \tilde{f}_2}{1+g} - \frac{\tilde{f}_2 \partial_W g}{(1+g)^2} \right\|_{\frac{2}{3}} \leq C, \\
 \|\partial_X \mathcal{R}_2[Z_0]\|_2 &= \left\| \frac{\partial_X \tilde{f}_2}{1+g} - \frac{\tilde{f}_2 \partial_X g}{(1+g)^2} \right\|_2 \leq C.
 \end{aligned}$$

Analogously, we obtain the rest of the estimates,

$$\begin{aligned}
 \|\partial_Y \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} &\leq C, \quad \|\partial_Y \mathcal{R}_2[Z_0]\|_2 \leq C, \quad \|\partial_Y \mathcal{R}_3^{\text{in}}[Z_0]\|_2 \leq C, \\
 \|\partial_X \mathcal{R}_3[Z_0]\|_2 &\leq C, \quad \|\partial_W \mathcal{R}_3[Z_0]\|_{\frac{2}{3}} \leq C. \quad \square
 \end{aligned}$$

5.3.2. Proof of Lemmas 5.7 and 5.10

Let us introduce, for $\kappa > 0$ and $\alpha \geq 0$, the following linear operators,

$$\begin{aligned} \mathcal{B}_\alpha[\Psi](U) &= e^{i\alpha U} \int_{-i\infty}^U e^{-i\alpha S} \Psi(S) dS, \\ \tilde{\mathcal{B}}[\Psi](U) &= e^{-iU} \int_{-i\kappa}^U e^{iS} \Psi(S) dS. \end{aligned} \tag{108}$$

The following lemma is proven in [1].

Lemma 5.11. Fix $\eta > 1$, $\nu > 0$, $\alpha > 0$ and $\kappa > 1$. Then, the following operators are well defined

$$\mathcal{B}_0 : \mathcal{Y}_\eta \rightarrow \mathcal{Y}_{\eta-1}, \quad \mathcal{B}_\alpha : \mathcal{Y}_\nu \rightarrow \mathcal{Y}_\nu, \quad \tilde{\mathcal{B}} : \mathcal{Y}_\nu \rightarrow \mathcal{Y}_\nu$$

and there exists a constant $C > 0$ such that

$$\|\mathcal{B}_0[\Psi]\|_{\eta-1} \leq C \|\Psi\|_\eta, \quad \|\mathcal{B}_\alpha[\Psi]\|_\nu \leq C \|\Psi\|_\nu, \quad \|\tilde{\mathcal{B}}[\Psi]\|_\nu \leq C \|\Psi\|_\nu.$$

One has $\mathcal{I}_1[\Psi] = \mathcal{B}_0[\langle R_1, \Psi \rangle]$, $\mathcal{I}_2[\Psi] = \mathcal{B}_1[\langle R_2, \Psi \rangle]$ and $\mathcal{I}_3[\Psi] = \tilde{\mathcal{B}}[\langle R_3, \Psi \rangle]$ (see (103) and (105)). Thus, we use this lemma to prove Lemmas 5.7 and 5.10.

Proof of Lemma 5.7. By the definition of the operator R and Lemma 5.5, we have that

$$\begin{aligned} \|R_{1,1}\|_3 &\leq C, & \|R_{1,2}\|_{\frac{7}{3}} &\leq C, & \|R_{1,3}\|_{\frac{7}{3}} &\leq C, \\ \|R_{j,1}\|_{\frac{2}{3}} &\leq C, & \|R_{j,2}\|_2 &\leq C, & \|R_{j,3}\|_2 &\leq C, \quad \text{for } j = 2, 3. \end{aligned} \tag{109}$$

Then, by Lemma 5.11, for κ big enough and $\Psi \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$, we have that

$$\begin{aligned} \|\mathcal{I}_1[\Psi]\|_{\frac{8}{3}} &= \|\mathcal{B}_0[\langle R_1, \Psi \rangle]\|_{\frac{8}{3}} \leq C \|\langle R_1, \Psi \rangle\|_{\frac{11}{3}} \\ &\leq C \left(\|R_{1,1}\|_1 \|\Psi_1\|_{\frac{8}{3}} + \|R_{1,2}\|_{\frac{7}{3}} \|\Psi_2\|_{\frac{4}{3}} + \|R_{1,3}\|_{\frac{7}{3}} \|\Psi_3\|_{\frac{4}{3}} \right) \\ &\leq C \left(\frac{1}{\kappa^2} \|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \end{aligned}$$

which gives the first estimate of the lemma. Analogously, by Lemma 5.11,

$$\begin{aligned} \|\mathcal{I}_2[\Psi]\|_{\frac{4}{3}} &= \|\mathcal{B}_1[\langle R_2, \Psi \rangle]\|_{\frac{4}{3}} \leq C \|\langle R_2, \Psi \rangle\|_{\frac{4}{3}}, \\ \|\mathcal{I}_3[\Psi]\|_{\frac{4}{3}} &= \|\tilde{\mathcal{B}}[\langle R_3, \Psi \rangle]\|_{\frac{4}{3}} \leq C \|\langle R_3, \Psi \rangle\|_{\frac{4}{3}}, \end{aligned}$$

and applying (109), for $j = 2, 3$, we have

$$\begin{aligned} \|(R_j, \Psi)\|_{\frac{4}{3}} &\leq \|R_{j,1}\|_{-\frac{4}{3}} \|\Psi_1\|_{\frac{8}{3}} + \|R_{j,2}\|_0 \|\Psi_2\|_{\frac{4}{3}} + \|R_{j,3}\|_0 \|\Psi_3\|_{\frac{4}{3}} \\ &\leq \frac{C}{\kappa^2} \left(\|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \end{aligned}$$

which gives the second and third estimates of the lemma. \square

Proof of Lemma 5.10. Let us consider $\Psi \in \mathcal{Z}_{\frac{4}{3},0,0}$ and define

$$\Phi(U) = e^{iU} R(U) \Psi(U),$$

in such a way that, by the definition of the operator \mathcal{B}_α in (108),

$$\begin{aligned} e^{iU} \mathcal{I}_1[\Psi](U) &= e^{iU} \int_{-i\infty}^U e^{-iS} \Phi_1(S) dS = \mathcal{B}_1[\Phi_1], \\ e^{iU} \mathcal{I}_2[\Psi](U) &= e^{i2U} \int_{-i\infty}^U e^{-i2S} \Phi_2(S) dS = \mathcal{B}_2[\Phi_2], \\ e^{iU} \mathcal{I}_3[\Psi](U) &= \int_{-i\kappa}^U \Phi_3(S) dS. \end{aligned} \tag{110}$$

Since $e^{iU} \Psi \in \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_0 \times \mathcal{Y}_0$, by the estimates in (109), we have that, for $j = 2, 3$,

$$\begin{aligned} \|\Phi_1\|_{\frac{7}{3}} &\leq \|R_{1,1}\|_1 \|e^{iU} \Psi_1\|_{\frac{4}{3}} + \sum_{k=2,3} \|R_{1,k}\|_{\frac{7}{3}} \|e^{iU} \Psi_k\|_0 \leq C \|\Psi\|_{\frac{4}{3},0,0}, \\ \|\Phi_j\|_2 &\leq \|R_{j,1}\|_{\frac{2}{3}} \|e^{iU} \Psi_1\|_{\frac{4}{3}} + \sum_{k=2,3} \|R_{j,k}\|_2 \|e^{iU} \Psi_k\|_0 \leq C \|\Psi\|_{\frac{4}{3},0,0}. \end{aligned} \tag{111}$$

Therefore, Lemma 5.11 and (110) imply

$$\begin{aligned} \|\mathcal{I}_1[\Psi]\|_{\frac{7}{3}} &= \|\mathcal{B}_1[\Phi_1]\|_{\frac{7}{3}} \leq C \|\Phi_1\|_{\frac{7}{3}} \leq C \|\Psi\|_{\frac{4}{3},0,0}, \\ \|\mathcal{I}_2[\Psi]\|_2 &= \|\mathcal{B}_2[\Phi_2]\|_2 \leq C \|\Phi_2\|_2 \leq C \|\Psi\|_{\frac{4}{3},0,0}. \end{aligned}$$

Now, we deal with operator \mathcal{I}_3 . Notice that, by the definition of the operator \mathcal{B}_α in (108) and (110), we have that

$$e^{iU} \mathcal{I}_3[\Psi](U) = \int_{-i\kappa}^{-i\infty} \Phi_3(S) dS + \int_{-i\infty}^U \Phi_3(S) dS = -\mathcal{B}_0[\Phi_3](-i\kappa) + \mathcal{B}_0[\Phi_3](U).$$

Then, by Lemma 5.11 and using the estimates (111), we obtain

$$\begin{aligned} \|\mathcal{I}_3[\Psi]\|_0 &\leq |\mathcal{B}_0[\Phi_3](-i\kappa)| + \|\mathcal{B}_0[\Phi_3]\|_0 \leq 2 \|\mathcal{B}_0[\Phi_3]\|_0 \\ &\leq \frac{C}{\kappa} \|\mathcal{B}_0[\Phi_3]\|_1 \leq \frac{C}{\kappa} \|\Phi_3\|_2 \leq \frac{C}{\kappa} \|\Psi\|_{\frac{4}{3},0,0}. \end{aligned}$$

Finally, taking $\tilde{\Theta}(\kappa) = -\mathcal{B}_0[\Phi_3](-i\kappa)$, we conclude

$$\left\| \mathcal{I}_3[\Psi](U) - e^{-iU} \tilde{\Theta}(\kappa) \right\|_1 = \|\mathcal{B}_0[\Phi_3]\|_1 \leq C \|\Phi_3\|_2 \leq C \|\Psi\|_{\frac{4}{3},0,0}. \quad \square$$

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