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ONE DIMENSIONAL INVARIANT MANIFOLDS OF GEVREY TYPE IN REAL-ANALYTIC MAPS

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Abstract. In this paper we study the basic questions of existence, uniqueness, differentiability, analyticity and computability of one dimensional parabolic manifolds of degenerate fixed points, i.e. invariant manifolds tangent to the eigenspace of 1, which is assumed to be a simple eigenvalue. We use the parameterization method, reducing the dynamics on the parabolic manifold to a polynomial. We prove that the asymptotic expansions of the parabolic manifold are of Gevrey type. Moreover, under suitable hypothesis, we also prove that the asymptotic expansions correspond to a real-analytic parameterization of an invariant curve that goes to the fixed point. The parameterization is Gevrey at the fixed point, hence C^∞.

Dedicated to Carles Simó for his 60th birthday

1. **Introduction.** Center manifold theory is very important in the analysis of degenerate fixed points and in bifurcation theory. The questions of existence, uniqueness, smoothness of the center manifold, and its applications, have been studied by many authors, among them [25, 22, 20, 8, 28]. These works show up the puzzling properties of center manifolds. In the study of degenerate fixed points, it is important to know the dynamical properties of the center manifold, what is known as reduction of the dynamics to the center manifold. In numerical applications, one can approximate the center manifold through power series expansions whose coefficients are recursively computed (see, for instance [32, 16, 21]). In order to bound the error of finite order approximations, it is important to know the rate of growth of the coefficient of the asymptotic expansions [33]. As far as we know, rigorous results on Gevrey estimates of the expansions appearing in center manifold theory or in normal form theory are still scarce. We however mention the paper [4] that appears in this issue.

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The present paper considers the previous topics for real-analytic (local) diffeomorphisms with a fixed point whose linearization has 1 as a simple eigenvalue. Thus, we look for a one dimensional invariant manifold having the fixed point at the boundary where it is tangent to the eigenspace of 1. We will refer to this branch of manifold as a *parabolic manifold*, a concept coming from the field of holomorphic dynamics, see [1, 7]. A particular case is when the fixed point is parabolic-hyperbolic, that is, the rest of eigenvalues are away of the unit circle, and the parabolic manifold is a branch of the center manifold. But in case there were other eigenvalues of modulus 1, we are also interested on finding these invariant branches, that would be included inside the center manifold.

In this setting, the goals of this paper are:

(a) To give a parameterization of the parabolic manifold for which the reduced dynamics is "simple". In particular, we show that it can be reduced to a polynomial.
(b) To describe the asymptotic properties of the expansions of the parabolic manifold. We prove that the expansions are of the Gevrey type, i.e., the coefficients of the expansions grow as a power of a factorial.
(c) To prove, under suitable hypotheses, that the expansions correspond to a real-analytic invariant manifold that goes to the fixed point (a parabolic manifold). This is proved under the hypothesis that the dynamics tangent to the manifold is attracting and the dynamics transversal to the manifold is not linearly attracting. We also prove that the parabolic manifold is of Gevrey type at the fixed point, hence C^∞.
(d) To give sufficient conditions of uniqueness of the parabolic manifold. This is proved under the assumption that the dynamics on the manifold is attracting and the transversal dynamics is repelling.

Notice that the map on the parabolic manifold is tangent to the identity. Several authors have considered either conjugacy or normal form problems for maps that are tangent to the identity. These authors find that one can reduce the dynamics to a polynomial. For instance Takens [34] studied the C^∞-conjugation between C^∞ maps in the real line. Voronin [35] dealt with the problem of formal and conformal conjugation between analytic maps in the complex plane. See also [13]. Hence, instead of considering the parabolic manifold as a graph, it is natural to consider an adapted parameterization of the manifold so that its dynamics is a polynomial. The formal construction of the parameterization is stated in Theorem 2.3. We emphasize that the information regarding the dynamics on the manifold is given by this polynomial, which is of the form $R(t) = t - at^N + bt^{2N-1}$, with $N \geq 2$ and $a \neq 0$ (in fact, doing scalings one can obtain $a = \pm 1$). The dynamics on the parabolic manifold can be stable, unstable or semi-stable, depending on the sign of a and the parity of N.

The idea of the parameterization method was developed in [9, 10, 11], for invariant manifolds associated to non-resonant spectral components of the linearization at the fixed point. With this method one finds simultaneously a parameterization of the invariant manifold and the reduction (normal form) of the dynamics on it. This methodology has already been used to prove the existence of C^r one dimensional branches of weak stable manifolds of tangent to the identity maps, see [6]. (See also [23, 14, 5, 27, 12] for different approaches and aspects of this problem, including applications to Celestial Mechanics. See [29, 30, 31] for studies on the stability around a parabolic fixed point of a real-analytic area preserving map). We also note that the parameterization method has been applied to compute invariant manifolds attached to invariant tori [18, 17, 19].

It is well known that the center manifold of an analytic map can be non analytic at the fixed point, even it could be non C^∞, see for instance [28]. But one always can find a formal power series expansion of the manifold by matching terms of the same order in the corresponding invariance equation. Thus, we are able to compute an approximation of the center manifold. Hence, expansions are useful in numeric calculations, so it is crucial to control the growth of these coefficients [33]. In this paper we will prove that the formal expansion of the one-dimensional parabolic manifold is Gevrey of order $\alpha = \frac{1}{N-1}$, that is the coefficients (indexed by n) do not grow more than $CK^n(n!)^\alpha$ for some constants C, K. This result is stated in Theorem 2.3.

As Poincaré already pointed out, formal expansions, even if they are not convergent, are very useful since they give information about the functions that they represent. See for instance [26, 3]. We will use some of these asymptotic techniques to prove that, under suitable assumptions, the formal expansion of the parabolic manifold corresponds to a real-analytic function defined in a complex sector whose vertex is the origin, and this function is Gevrey at the origin. See Theorem 2.4. We emphasize that the standard techniques in the literature of center manifold theory, such as cut-off functions, are out of question here since we work in the analytic category.

Finally, the way we consider the uniqueness problem is quite standard in the literature. See for instance [25, 23, 5]. Assuming that the dynamics on the parabolic manifold is attracting, and the transversal dynamics is repelling, one constructs a cone such that the points on the parabolic manifold and its iterates belong to this cone and tend to the origin. Moreover, the distance between two different points in a fiber transversal to the parabolic manifolds experience a growth when iterating. As a result, such a fiber can only intersect one parabolic manifold, that is a weak stable manifold (a branch of the center manifold). We present this result in Theorem 2.6. See [28] for a different approach.

As a corollary of the results of this paper, we come back to the conjugacy problem of tangent to the identity maps on the real line mentioned above. We prove that a real-analytic map of this type is α-Gevrey conjugated to a polynomial of the form $R(t) = t - at^N + bt^{2N-1}$, with $\alpha = \frac{1}{N-1}$. See Corollary 2.7.

The paper is organized as follows. In Section 2 we introduce the problem and the notation used, and we state the main three theorems of this paper. The first theorem corresponds to the formal approach and the Gevrey estimates of the expansions, developed in Section 3. The second theorem establishes the analyticity and differentiability properties of the parabolic manifold, proved in Section 4. The third theorem is the uniqueness result, which is proved in Section 5. In Appendix A we provide useful properties on Gevrey functions.

2. **The problem and the results.**

2.1. **The parameterization method.**

In this paper we will consider a real-analytic map

$$\begin{array}{rlcl} F: & U \subset \mathbb{R} \times \mathbb{R}^d & \longrightarrow & \mathbb{R} \times \mathbb{R}^d \\ & z = (x, y) & \longrightarrow & F(x, y) = (f(x, y), g(x, y)) \end{array} \tag{2.1}$$

defined in an open neighborhood $\mathcal{U}$ of $0 = (0, 0)$, giving a discrete dynamical system of the form

$$\left\{ \begin{array}{lll} \bar{x} = & x - ax^N & + \hat{f}_N(x, y) + f_{\geq N+1}(x, y), \\ \bar{y} = & Ay & + g_{\geq 2}(x, y), \end{array} \right. \tag{2.2}$$

where:

- the constant a is non-zero;
- 1 is not in the spectrum of A;
- $N \geq 2$ is an integer number;
- $\hat{f}_N(x, y)$ is an homogeneous polynomial of degree N such that $\hat{f}_N(x, 0) = 0$. We denote $v = (N-1)!\partial_x^{N-1}\partial_y \hat{f}(0, 0) \in \mathbb{R}^d$, so that $\partial_y \hat{f}_N(x, 0) = x^{N-1}v^\top$. We will also write $f_N(x, y) = -ax^N + \hat{f}_N(x, y)$;
- $f_{\geq N+1}$ has order $N + 1$ (all its derivatives up to order N vanish at $(0, 0)$);
- $g_{\geq 2}$ has order 2 (that is $g_{\geq 2}(0, 0) = 0$ and $Dg_{\geq 2}(0, 0) = 0$).

By "real-analytic" we mean that F can be extended to a holomorphic function defined in a complex neighborhood $\mathcal{U}_\mathbb{C}$ of $\mathcal{U}$, that is $F_\mathbb{C} : \mathcal{U}_\mathbb{C} \subset \mathbb{C} \times \mathbb{C}^d \longrightarrow \mathbb{C} \times \mathbb{C}^d$. For the sake of simplicity, we will also use the notation F for its complexification $F_\mathbb{C}$.

Remark 2.1. A natural question is to characterize the maps that are (locally) conjugated to a map of the form (2.2). Assume that a map defines a dynamical system with a fixed point whose linearization has 1 as a simple eigenvalue. After a translation of the fixed point to the origin of the coordinate system, and a linear change of variables, we can write the equations as

$$\left\{ \begin{array}{ll} \bar{x} = & x + \hat{f}(x, y), \\ \bar{y} = & Ay + \hat{g}(x, y), \end{array} \right. \tag{2.3}$$

where $\hat{f}$ and $\hat{g}$ have order 2. Let N be the smaller integer such that the $\frac{\partial^N \hat{f}}{\partial x^N}(0, 0) \neq 0$, that is the coefficient of x^N in the expansion of $\hat{f}$ is non-zero. We would like to eliminate all the terms of $\hat{f}$ of order lower than N using changes of variables. This can be done using standard normal form techniques, under suitable non-resonance conditions that we now describe.

Let $\lambda_1, \ldots \lambda_d \in \mathbb{C}$ be the eigenvalues of the matrix A. Then, the map (2.3) is (locally) conjugated to a map of the form (2.2) in the following cases:

- If $N = 2$, obviously;
- If $N > 2$, and $\lambda_1^{k_1} \ldots \lambda_d^{k_d} \neq 1$ for all $(k_1, \ldots, k_d) \in \mathbb{N}^d$ such that $1 \leq k_1 + \cdots + k_d < N$

Remark 2.2. In particular, notice that having a parabolic-hyperbolic fixed point (i.e., 1 is the only eigenvalue in the unit circle) and $N = 2$ is a degeneracy of codimension 1.

Notice that if the matrix A is hyperbolic, it is then clear that the dynamics near the fixed point is dominated by the lower order terms

$$L(x, y) = \begin{pmatrix} x - ax^N \\ Ay \end{pmatrix}. \tag{2.4}$$

In this case the fixed point has a center manifold (possibly non-unique) tangent to the x-axis, whose dynamics depends on the sign of a and the parity of N. Thus, the dynamics on the center manifold can be stable, unstable of semi-stable, see the examples in Section 2.2. Since the center manifold is one dimensional and tangent to the x-axis, the parabolic manifold we defined corresponds to the left branch or to the right branch of the center manifold. If A is non-hyperbolic, the parabolic manifold we are looking for is an invariant curve inside the center manifold, either (locally) included on $\{x \geq 0\}$ (right branch) or on $\{x \leq 0\}$ (left branch).

Let us focus on the right parabolic manifold, since similar arguments can be made for the left branch. The goal is to find an adapted parameterization of the parabolic manifold, $K : [0, \rho) \to \mathbb{R}^{1+d}$ with $K(0) = (0, 0)$ and $DK(0) = (1, 0)^\top$, in such a way that the invariance equation

$$F \circ K(t) = K \circ R(t) \tag{2.5}$$

is satisfied for a suitable polynomial $R(t)$. Notice that in such a case the parabolic manifold

$$\mathcal{W} = \{K(t) \mid t \in [0, \rho)\} \tag{2.6}$$

is invariant under (2.2), and that the information about its dynamics is given by the polynomial $R(t)$.

We can deal with (2.5) at different levels. Either we can consider (2.5) as a functional equation in a suitable Banach space of functions, or we can consider (2.5) in spaces of formal power series.

In both cases, a main ingredient will be the so called Faa-di-Bruno formula, which we now recall. If $f = f(w)$ and $g = g(z)$ are two composible functions, that for the sake of simplicity we assume are C^∞, we can compute the l derivative of $f \circ g$ by

$$\frac{D^l(f \circ g)(z)}{l!} = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1 \leq l_i}} \frac{D^k f(g(z))}{k!} \frac{[D^{l_1} g(z), \cdots, D^{l_k} g(z)]}{l_1! \cdots l_k!}. \tag{2.7}$$

If $f(0) = 0$ and $g(0) = 0$, denoting $f_k = \frac{1}{k!} D^k f(0)$ and $g_k = \frac{1}{k!} D^k g(0)$, formula (2.7) at $z = 0$ reads

$$(f \circ g)_l = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1 \leq l_i}} f_k[g_{l_1}, \cdots, g_{l_k}]. \tag{2.8}$$

Notice that f_k and g_k are k-multilinear symmetric maps (that can be identified with homogeneous polynomials of order k, and we will write $f_k w^k = f_k[w, \overset{k}{\ldots}, w]$, etc.).

If we think now $\hat{f}(w) = \sum_{l \geq 1} f_l w^l$ and $\hat{g}(z) = \sum_{l \geq 1} g_l z^l$ as formal power series, then the l order term of the formal composition $f \circ g$ is given by (2.8). We emphasize that $(\hat{f} \circ \hat{g})_l$ depends only on $\hat{f}_{\leq l}(w) = \sum_{k=1}^{l} f_k w^k$ and $\hat{g}_{\leq l}(z) = \sum_{k=1}^{l} g_k z^k$ (we will use along the paper notations such as $f_{<l}$, $f_{\geq l+1}$, etc. without more mention). Moreover, the only term of $(\hat{f} \circ \hat{g})_l$ in which f_l appears is $f_l g_1^l$, and the only term in which g_l appears is $f_1 g_l$. This remark is important when doing induction arguments.

As a result, when one considers (2.5) in the sense of composition of formal power series, one looks for a formal expansion $\hat{K}(t) = \sum_{l \geq 1} K_l t^l \in \mathbb{R}[[t]]^{1+d}$ and a polynomial $R(t) = R_1 t + \cdots + R_m t^m$ of unknown degree (to be found) such that

$$\sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1 \leq l_i}} F_k[K_{l_1}, \cdots, K_{l_k}] = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1 \leq l_i}} K_k R_{l_1} \cdots R_{l_k} \tag{2.9}$$

for all $l \geq 1$.

Let us finish this introductory section with several notational conventions that we use throughout the paper. We denote the projection over the x-component by π^x, and the projection over the y-components by π^y. If $W \in \mathbb{C}^{1+d}$ (or if W is a map taking values in $\mathbb{C}^{1+d}$, or a power series with coefficients in $\mathbb{C}^{1+d}$), we write $W^x = \pi^x W$ and $W^y = \pi^y W$.

2.2. **Examples.** The dynamical properties of one-dimensional center manifolds, and the puzzling questions about its existence, uniqueness, differentiability and analyticity can be grasped with the following simple but, we hope, illuminating examples.

The first example is the time-1 map of the autonomous planar vector field

$$\left\{ \begin{array}{rl} \dot{x} = & -ax^N, \\ \dot{y} = & \lambda y, \end{array} \right. \tag{2.10}$$

that is

$$\left\{ \begin{array}{rl} \bar{x} = & x\left(1 + (N-1)ax^{N-1}\right)^{-1/(N-1)} \\ \bar{y} = & e^{\lambda} y, \end{array} \right. \tag{2.11}$$

Map (2.11) is of the form (2.2) with $A = e^{\lambda}$. The dynamical properties of the fixed point, and the uniqueness and dynamical properties of both branches (left and right) of the center manifold are summarized below:

- $a > 0$, $\lambda < 0$ $(0 < A < 1)$
  - N even: saddle-node, unique left branch (weak unstable manifold), non-unique right branch;
  - N odd: attracting node, non-unique left and right branches;
- $a > 0$, $\lambda > 0$ $(1 < A)$
  - N even: saddle-node, non-unique left branch, unique right branch (weak stable manifold);
  - N odd: saddle, unique left and right branches (weak stable manifold);
- $a < 0$, $\lambda < 0$ $(0 < A < 1)$
  - N even: saddle-node, non-unique left branch, unique right branch (weak unstable manifold);
  - N odd: saddle, unique left and right branches (weak unstable manifold);
- $a < 0$, $\lambda > 0$ $(1 < A)$
  - N even: saddle-node, unique left branch (weak stable manifold), non unique right branch ;
  - N odd: repelling node, non-unique left and right branches;

Let us consider now the analytical properties of the center manifold. Notice that if we represent it as a graph $y = \psi(x)$, then it is

$$y = c \exp\left(\frac{\lambda}{a(N-1)} \frac{1}{x^{N-1}}\right) \tag{2.12}$$

where c is a constant, and the right and the left branch is defined if $\lim_{x\to 0^+} \psi(x) = 0$ and $\lim_{x\to 0^-} \psi(x) = 0$, respectively. In particular, $y = 0$ is a center manifold which is analytic, but the rest of the branches are C^∞ in the origin. Moreover, the difference between two any branches is exponentially small. Notice also that all the branches of center manifold have the same asymptotic expansion at the origin: it is a formal power series with all the coefficients equal to zero.

The second example appears in [8] with $N = 3$. It is the autonomous planar vector field

$$\left\{ \begin{array}{ll} \dot{x} = & -x^N, \\ \dot{y} = & -y + x^2, \end{array} \right. \tag{2.13}$$

that can also easily be solved by quadratures. With respect to the asymptotic expansion of the center manifold, represented as a graph, we obtain

$$y = \psi(x) = \sum_{l \geq 0} 2(N+1)\dots(2+(N-1)(l-1))x^{2+(N-1)l}. \tag{2.14}$$

Using Stirling's formula, it is easy to see that the coefficient ψ_k of x^k in (2.14), with $k = 2 + (N-1)l$, can be compared with $k!^\alpha$, with $\alpha = \frac{1}{N-1}$, in such a way that

$$\frac{\psi_k}{k!^\alpha} \sim \frac{(2\pi)^{\frac{1-\alpha}{2}}}{\Gamma(2\alpha)} \; \alpha^{2\alpha - \frac{1}{2}} \; k^{-\frac{1}{2} - \frac{1}{2}\alpha}$$

where Γ is the Gamma function. That is, the coefficients ψ_k grow as the power α of $k!$. In this case, one says that the power series (2.14) is α-Gevrey. We emphasize again that the Gevrey order has to do with the order of the dominant term in the center manifold: $\alpha = \frac{1}{N-1}$. We also emphasize that the center manifold is not analytic.

Let us finish this section with one example in which our results do not apply directly. Let us consider the 2-dimensional map

$$\left\{ \begin{array}{ll} \bar{x} = & x - xy + x^3, \\ \bar{y} = & 2y + x^3. \end{array} \right. \tag{2.15}$$

It is clear that, although (2.15) is not of the form (2.2), the map has a center manifold which is tangent to the x-axis. Notice, moreover, that the lower order terms constitute a map

$$\left\{ \begin{array}{ll} \bar{x} = & x - xy, \\ \bar{y} = & 2y, \end{array} \right. \tag{2.16}$$

which is too degenerate (it has a line of fixed points). We can not know from the lower order terms what is the dynamics on the center manifold. To do so, one uses the standard reduction principle to the center manifold. In order to apply our results, notice that one can eliminate the term xy in the first component using a normal form analysis. See Remark (2.1).

2.3. **The results.** Along this paper, we denote $\alpha = \frac{1}{N-1}$.

In the following theorems, we consider the dynamical system (2.2).

The first result is related to the formal solution of the invariance condition (2.5). We prove that there exists a formal solution of (2.5) being R a polynomial of degree $2N-1$. Moreover, the expansions are α-Gevrey.

Theorem 2.3. *Assume that $1 \notin \operatorname{Spec} A$.*

Then, there exist a unique polynomial $R(t) = t - at^N + bt^{2N-1}$ and a formal power series $\hat{K}(t) = \sum_{n \geq 0} K_n t^n \in \mathbb{R}[[t]]^{1+d}$ with $K_0 = (0,0)$ and $K_1 = (1,0)^\top$ such that

$$F \circ \hat{K} = \hat{K} \circ R$$

(in the sense of formal composition). Moreover, the expansion is α-Gevrey, that is there exist constants $C, M > 0$ such that

$$\|K_n\| \leq C M^n n!^\alpha,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^{1+d}$.

We emphasize that any C^∞ function $K : [0,\rho) \to \mathbb{R}^{1+d}$ satisfying the invariant equation $F \circ K = K \circ R$, has the Taylor expansion $\hat{K}$ which, by Theorem 2.3, is an α-Gevrey formal power series. Notice also that the dynamics on a C^∞ parabolic manifold can be reduced (via C^∞ transformation) to a polynomial R thanks to the results in [34].

The following result deals with the existence of a real-analytic parameterization K of a parabolic manifold which it turns to be α-Gevrey at 0. We state the theorem for the case of a (weak) stable invariant right branch, i.e. $a > 0$, under the assumption that the dynamics in the complementary directions is not linearly asymptotically stable. We emphasize that those complementary directions can contain eigenvalues of modulus 1. In such a case, the theorem constructs a parabolic manifold which is a weak stable invariant manifold inside the center manifold. An analogous result is enunciated for a (weak) unstable invariant branch. In the theorem we state both results for the right branch, but they also hold for a left branch with minor changes.

Theorem 2.4. *Assume that $a > 0$ and $\mathrm{Spec} A \subset \{\mu \in \mathbb{C} \mid |\mu| \geq 1\} \setminus \{1\}$. Then, for any $0 < \beta < \alpha\pi$ there exist $\rho > 0$ small enough and a real-analytic function $K : (0,\rho) \to \mathbb{R}^{1+d}$ which can be holomorphically extended to a complex sector*

$$S = S(\beta,\rho) = \{t = re^{\mathrm{i}\varphi} \in \mathbb{C} \mid 0 < r < \rho,\ |\varphi| < \beta/2\}$$

such that

$$F \circ K = K \circ R$$

in the sector, where R is the polynomial produced in Theorem 2.3. Moreover the parameterization K is asymptotic α-Gevrey to the expansion $\hat{K}$ produced in Theorem 2.3, which implies that

$$\lim_{S \ni t \to 0} \frac{1}{n!} D^n K(t) = K_n.$$

In particular, K can be extended to a C^∞ function at 0. Moreover, the dynamics in the local invariant manifold

$$\mathcal{W}^{ws} = \{K(t) \mid t \in [0,\rho)\}$$

is (weak) asymptotically stable at 0.

Analogously, if we assume $a < 0$ and $\mathrm{Spec} A \subset \{\mu \in \mathbb{C} \mid |\mu| \leq 1\} \setminus \{1, 0\}$, then there exists a real-analytic function K satisfying $F \circ K = K \circ R$, that is the parameterization of a (weak) asymptotically unstable manifold $\mathcal{W}^{wu}$ at 0.

As a result, in both cases, either $\mathcal{W}^{ws}$ or $\mathcal{W}^{wu}$ are parabolic manifolds.

Remark 2.5. We emphasize that the opening of the complex sector $S(\beta,\rho)$, which is the angle β, is bounded by $\pi/(N-1)$. That is, the opening is related with the degree of degeneracy of the fixed point.

Notice that in the previous theorem the result about the existence of a (weak) unstable branch can also be obtained solving the functional equation $F^{-1} \circ K = K \circ \tilde{R}$. This is just to apply the first part of Theorem 2.4 to F^{-1}.

In the following result we consider the uniqueness problem of the (weak) stable manifold constructed in the previous theorem. We prove that it is unique under the assumption that the complementary directions are linearly unstable. Again, an analogous result is stated for the uniqueness of a (weak) unstable manifold.

Theorem 2.6. *(a) Assume that $a > 0$ and $\operatorname{Spec} A \subset \{\mu \in \mathbb{C} \mid |\mu| > 1\}$. Then, there is a unique right branch of center manifold, and it is (locally) $\mathcal{W}^{ws}$, the parabolic manifold produced in Theorem 2.4.*

(b) Analogously, if we assume $a < 0$ and $\operatorname{Spec} A \subset \{\mu \in \mathbb{C} \mid |\mu| < 1\} \setminus \{0\}$, there is a unique right branch of center manifold, and it is (locally) $\mathcal{W}^{wu}$, the parabolic manifold produced in Theorem 2.4.

We notice that if we consider the holomorphic extension of the real-analytic diffeomorphism F to a complex neighborhood of 0 in $\mathbb{C}^{1+d}$, the complex domain of the parabolic manifold we find vaguely resembles one of the so-called petals appearing in the Leau-Fatou flower theorem (see, e.g., the reviews [24, 1, 7]). In this context of holomorphic dynamics we also emphasize that, in some sense, Theorem 2.6 complements results of Hakim [**?**] (see again [1, 7] for related results), in which it is proved that there exist attracting domains for semi-attractive holomorphic transformations of $\mathbb{C}^{1+d}$ (case (b) above). What we prove is that in such a case there exist a (weak) unstable manifold, which is of course outside these basins of attraction. It should be interesting to extend our real-analytic results to the context of holomorphic dynamics.

Finally we present a corollary about the conjugation of tangent to the identity real-analytic one dimensional maps. We prove that the conjugacy is α-Gevrey and analytic in a complex sector which does not include the origin. This result is related with the C^∞ results given by [34]. In [35] the problem is studied in the holomorphic category for the case $N = 2$. Related results appear in [13].

Corollary 2.7. *Let $f(x) = x - ax^N + \hat{f}(x)$ be a real-analytic map in a neighborhood of 0 in $\mathbb{R}$, where $a \neq 0$ and all the derivatives of $\hat{f}$ up to order N vanish at 0. Then, f is (locally) α-Gevrey conjugated to a polynomial map $R(t) = t - at^N + bt^{2N-1}$, and the conjugacy is real-analytic except possibly in 0, and it is analytic in a complex bisector $-S(\beta, \rho) \cup S(\beta, \rho)$ with $\beta < \alpha\pi$. In particular, the conjugacy is C^∞ at 0.*

Remark 2.8. We observe that this conjugation result says that formal conjugacy is equivalent to real-analytic conjugacy in bisectors of the form $-S(\beta, \rho) \cup S(\beta, \rho)$. We recall that an analytic map f of the form $f(x) = x - ax^N + \hat{f}(x)$ as above, is formally conjugated to $g(x) = x + x^N + \beta x^{2N-1}$, see for instance [34, 2].

Proof. Notice that there exist a formal power series $\hat{K}^x(t)$ and a unique polynomial $R(t)$ such that $f \circ \hat{K}^x(t) = \hat{K}^x \circ R(t)$.

For the conjugacy in the right branch, if $a > 0$ we take a constant $A > 1$ and if $a < 0$ we take a constant $0 < A < 1$. In both cases, we consider the 2D map F given by $F(x, y) = (f(x), Ay)$. Applying Theorem 2.6 to F, we get a parameterization of the right branch of the center manifold $K_+(t) = (K_+^x(t), 0)$ for $t \in S(\beta, \rho)$. In particular, we have $f \circ K_+^x = K_+^x \circ R$, and therefore f is conjugated to the polynomial R in a sector $S(\beta, \rho)$.

For the left branch, we perform a change of variables $x \to -x$ and we repeat the previous argument to the function $\tilde{f}(x) = -f(-x)$, obtaining a conjugacy $\tilde{K}_+^x$ defined in a sector $S(\beta, \rho)$. The reduced polynomial we obtain is $\tilde{R}(t) = -R(-t)$. Hence, $f \circ K_-^x(t) = K_-^x \circ R(t)$ for $t \in -S(\beta, \rho)$, where $K_-^x(t) = -\tilde{K}_+^x(-t)$.

The conjugacy φ in then defined by $\varphi(t) = K_+^x(t)$ if $t \in S(\beta, \rho)$ and by $\varphi(t) = K_-^x(t)$ if $t \in -S(\beta, \rho)$. Both branches of φ are asymptotic α-Gevrey to the expansion $\hat{K}^x$, and hence they are C^∞ at 0. Moreover, the left and right derivatives at all order coincide: $D^l K_+^x(0) = D^l K_-^x(0) = l! K_l^x$. In summary, φ satisfies the properties stated in the corollary. $\square$

3. **The formal solution of $F \circ K - K \circ R = 0$.** In this section we will prove Theorem 2.3. First of all, in subsection 3.1, we prove that there exists a formal solution of the invariance equation $F \circ K = K \circ R$ being R a suitable polynomial. To do that we match powers in t. We also give a recurrence formula to compute the coefficients K_l of the formal solution $\hat{K} = \sum_{n\geq 1} K_l t^l$. The main key to obtain this recurrence formula will be the Faa-di-Bruno for formal power series formula given in (2.8). Later we will prove that the formal expansion is actually α-Gevrey. This is done in subsection 3.2.

It will be useful for the arguments to separate the leading term $L(x, y)$, see (2.4), from the higher order terms $G(x, y) = F(x, y) - L(x, y)$.

3.1. **Construction of the formal solution.** The goal of this section is to prove the following proposition:

Proposition 3.1. *There exist a unique $b \in \mathbb{R}$ such that for any $c \in \mathbb{R}$ there exist a unique formal power series $\hat{K} = \sum_{l=1}^{\infty} K_l t^l$, $K_l \in \mathbb{R}^{1+d}$ with $K_1 = (1, 0)^\top$ and $K_N^x = c$, such that $R(t) = t - at^N + bt^{2N-1}$ and $\hat{K}$ satisfies formally the equation $F\circ\hat{K} - \hat{K}\circ R = 0$.*

Moreover, the coefficients of K and R can be computed inductively. In the step $l > 1$,

- *If $l \neq N$: $K_l^y = -(A - Id)^{-1} E_l^y$, $K_l^x = \dfrac{-1}{a(l-N)}(E_{l+N-1}^x + v^\top K_l^y)$, $R_{l+N-1} = 0$;*
- *If $l = N$: $K_N^y = -(A - Id)^{-1} E_N^y$, $K_N^x = c$, $b = R_{2N-1} = E_{2N-1}^x + v^\top K_N^y$;*

where

$$E_l^y = \sum_{k=2}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1\leq l_i\leq l-1}} G_k^y[K_{l_1}, \cdots, K_{l_k}] - \sum_{k=2}^{l-N+1} K_k^y \sum_{\substack{l_1+\cdots+l_k=l \\ l_i\geq 1}} \prod_{i=1}^{k} R_{l_i} \tag{3.1}$$

and

$$\begin{aligned} E_{l+N-1}^x = &- a \sum_{\substack{l_1+\cdots+l_N=l+N-1 \\ 1\leq l_i\leq l-1}} \prod_{i=1}^{N} K_{l_i}^x + \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\leq l_i\leq l-1}} G_k^x[K_{l_1}, \cdots, K_{l_k}] \\ &- \sum_{k=2}^{l-1} K_k^x \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ l_i\geq 1}} \prod_{i=1}^{k} R_{l_i}. \end{aligned} \tag{3.2}$$

Proof. We will prove first that the error E^l in the order l approximation $K_{\leq l}$ is

$$E^l(t) = F \circ K_{\leq l}(t) - K_{\leq l} \circ R(t) = \begin{pmatrix} O(t^{N+l}) \\ O(t^{l+1}) \end{pmatrix}. \tag{3.3}$$

First notice that $K_{\leq 1}(t) = (t, 0)$, $R_{\leq N}(t) = t - at^N$ satisfies

$$E^1(t) = F \circ K_{\leq 1}(t) - K_{\leq 1} \circ R_{\leq N}(t) = \begin{pmatrix} O(t^{N+1}) \\ O(t^2) \end{pmatrix},$$

so (3.3) holds for $l = 1$.

Now we proceed by induction. Let $l \geq 2$, and assume that there exist polynomials $K_{<l}$ of degree $l-1$ and $R_{<l+N-1}$ of degree $l+N-2$ such that the error in the step $l-1$ is

$$E^{l-1} = F \circ K_{<l} - K_{<l} \circ R_{<l+N-1} = \begin{pmatrix} O(t^{N+l-1}) \\ O(t^l) \end{pmatrix}. \tag{3.4}$$

We want to find $K_l \in \mathbb{R}^{1+d}$ and $R_{l+N-1} \in \mathbb{R}$ such that $K_{\leq l} = K_{<l} + K_l t^l$ and $R_{\leq l+N-1} = R_{<l+N-1} + R_{l+N-1}t^{l+N-1}$ satisfy (3.3). For that we introduce $H_l(t) = K_l t^l$ and $S_{l+N-1}(t) = R_{l+N-1}t^{l+N-1}$ and we compute

$$\begin{aligned} &F \circ K_{\leq l} - K_{\leq l} \circ R_{\leq l+N-1} \\ &= E^{l-1} + (F\circ K_{\leq l} - F\circ K_{<l} - DF(K_{<l})H_l) + DF(K_{<l})H_l \\ &\quad - (K_{\leq l}\circ R_{\leq l+N-1} - K_{\leq l}\circ R_{<l+N-1}) - H_l\circ R_{<l+N-1} \end{aligned} \tag{3.5}$$

up to order $l+N-1$ in the x-components and up to order l in the y-component.

Now we are going to compute the different terms in (3.5). We have that

$$E^{l-1}(t) = \begin{pmatrix} E^x_{l+N-1}t^{l+N-1} \\ E^y_l t^l \end{pmatrix} + \begin{pmatrix} O(t^{l+N}) \\ O(t^{l+1}) \end{pmatrix},$$

$$F\circ K_{\leq l}(t) - F\circ K_{<l}(t) - DF(K_{<l}(t))H_l(t) = \begin{pmatrix} O(t^{l+N}) \\ O(t^{l+1}) \end{pmatrix},$$

$$DF(K_{<l}(t))H_l(t) = \begin{pmatrix} (1-aNt^{N-1})K^x_l t^l + v^\top K^y_l t^{l+N-1} \\ AK^y_l t^l \end{pmatrix} + \begin{pmatrix} O(t^{l+N}) \\ O(t^{l+1}) \end{pmatrix},$$

$$K_{\leq l} \circ R_{\leq l+N-1}(t) - K_{\leq l} \circ R_{<l+N-1}(t) = \begin{pmatrix} R_{l+N-1}t^{l+N-1} \\ 0 \end{pmatrix} + \begin{pmatrix} O(t^{l+N}) \\ O(t^{l+N}) \end{pmatrix},$$

and

$$H_l \circ R_{<l+N-1}(t) = (t - at^N + O(t^{N+1}))^l K_l = K_l t^l - alK_l t^{l+N-1} + O(t^{l+N}).$$

Henceforth we obtain that

$$E^l(t) = \begin{pmatrix} \left(E^x_{l+N-1} + a(l-N)K^x_l + v^\top K^y_l - R_{l+N-1}\right) t^{l+N-1} \\ \left(E^y_l + (A-\mathrm{Id})K^y_l\right) t^l \end{pmatrix} + \begin{pmatrix} O(t^{l+N}) \\ O(t^{l+1}) \end{pmatrix}.$$

Then, in order to satisfy (3.3), we take

- If $l \neq N$: $K^y_l = -(A-\mathrm{Id})^{-1}E^y_l$, $K^x_l = \dfrac{-1}{a(l-N)}(E^x_{l+N-1} + v^\top K^y_l)$, $R_{l+N-1} = 0$;
- If $l = N$: $K^y_N = -(A-\mathrm{Id})^{-1}E^y_N$, $K^x_N = c$, $R_{2N-1} = E^x_{l+N-1} + v^\top K^y_l$.

We denote $b = R_{2N-1}$ and then $R(t) = t - at^N + bt^{2N-1}$. We also emphasize that in the step N the term K^x_N is free, and we fix it equal to c.

We will prove now the formulae for E^y_l and E^x_{l+N-1}. First, notice that E^y_l is the term of order l of $\pi_y E^{l-1} = F^y\circ K_{<l} - K^y_{<l}\circ R_{<l+N-1}$, so that $E^y_l = D^l \pi^y E^{l-1}(0)/l!$. Applying Faa-di-Bruno formula we obtain:

$$E^y_l = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1\leq l_i\leq l-1}} F^y_k[K_{l_1}, \cdots, K_{l_k}] - \sum_{k=1}^{l-1} K^y_k \sum_{\substack{l_1+\cdots+l_k=l \\ 1\leq l_i\leq l+N-2}} \prod_{i=1}^{k} R_{l_i}. \tag{3.6}$$

In the first term, if $k = 1$ the summatory vanishes, and notice that for $k \geq 2$ $F^y_k = G^y_k$. In the second term, $K^y_1 = 0$. Then, for $k \geq 2$, if $l_k > l + N - 2$ we would have $l > l + N - 2 + (k-1) = l + N + k - 3$, which is false. So, we have:

$$\sum_{\substack{l_1+\cdots+l_k=l \\ 1\leq l_i\leq l+N-2}} \prod_{i=1}^{k} R_{l_i} = \sum_{\substack{l_1+\cdots+l_k=l \\ 1\leq l_i}} \prod_{i=1}^{k} R_{l_i} := R_{k,l}$$

Notice that $R_{k,l}$ is the coefficient of t^l in

$$R(t)^k = (t - at^N + bt^{2N-1})^k = t^k - akt^{N+k-1} + \dots,$$

and in particular, $R_{k,l} = 0$ if $k < l < N + k - 1$. As a result, $2 \leq k \leq l - N + 1$ in the second term of (3.6). So, we have proved (3.1).

It only remains to prove (3.2). We proceed in a similar way as before. Again, notice that E^x_{l+N-1} is the term of order $l+N-1$ of $\pi_x E^{l-1} = F^x \circ K_{<l} - K^x_{<l} \circ R_{<l+N-1}$. Applying again Faa-di-Bruno formula, we obtain

$$\begin{aligned} &E^x_{l+N-1} \\ &= \sum_{k=1}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\leq l_i\leq l-1}} F^x_k[K_{l_1}, \cdots, K_{l_k}] - \sum_{k=1}^{l-1} K^x_k \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\leq l_i\leq l+N-2}} \prod_{i=1}^{k} R_{l_i} \end{aligned} \tag{3.7}$$

Since $F^x(x,y) = x - ax^N + G^x(x,y)$, then for $k = 1, \dots N-1$ the summatory in the first term of (3.7) vanishes, and the whole term can be replaced by

$$-a \sum_{\substack{l_1+\cdots+l_N=l+N-1 \\ 1\leq l_i\leq l-1}} \prod_{i=1}^{N} K^x_{l_i} + \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\leq l_i\leq l-1}} G^x_k[K_{l_1}, \cdots, K_{l_k}].$$

Finally, in the second term of (3.7), notice that if $k = 1$, then $l_1 = l + N - 1 > l + N - 2$, so the corresponding summatory vanishes. We obtain then (3.2).

With these lines we finish the proof of Proposition 3.1. ☐

3.2. **Gevrey estimates.** In this section we prove that the formal expansion $\hat{K}$ given in Proposition 3.1 is α-Gevrey. First we perform some change of variables and scalings to get some suitable conditions.

3.2.1. *Preliminary changes of variable and scalings.* The next lemma provide us a change of coordinates in such a way that the new parameterization of the (formal) parabolic manifold is flatter than the original one.

Lemma 3.2. *We define the change of variables*

$$(x,y) = H(u,v) := K_{\leq N-1}(u) + (0,v)$$

where $K_{\leq N-1}(u) = \sum_{j=1}^{N-1} K_j u^j$. *In these new variables:*

1. $\bar{F} = H^{-1} \circ F \circ H$ *has the same form* (2.2) *of* F.
2. *The formal solutions of* $\bar{F} \circ \bar{K} - \bar{K} \circ \bar{R} = 0$ *obtained applying Proposition 3.1 to* $\bar{F}$ *satisfy*

$$\bar{K}(t) = (t,0)^T + O(t^N), \qquad \text{and} \qquad \bar{R}(t) = t - at^N + \bar{b}t^{2N-1}$$

3. $\bar{b} = b$, *so* $\bar{R}(t) = R(t)$.

The proof of this lemma is straightforward.

Remark 3.3. It is clear that, if $\bar{K}(t) = \sum_{j=1}^{\infty} \bar{K}_j t^j$ is α-Gevrey, then $\hat{K}(t) = \sum_{j=1}^{\infty} K_j t^j$ is also α-Gevrey.

Now we are going to perform adequate scalings in order to get the constants a and b small enough. We also obtain a simpler Gevrey condition for the first terms of the series $\bar{K}$.

From now on, for $\delta > 0$, we denote by $\overline{B}(\delta)$ the closed ball of radius δ centered at the origin of the complex plane.

Lemma 3.4. *Let $\bar{\delta}$ be such that $\overline{B}(\bar{\delta})$ is contained in the complex domain $\bar{\mathcal{U}}_{\mathbb{C}}$ of $\bar{F}$. Let $\bar{G} = \bar{F} - L$ and $\bar{M} = \max_{(x,y)\in\overline{B}(\bar{\delta})} \|\bar{G}(x,y)\|$.*

For all $l_0 \in \mathbb{N}$, $\alpha_0 > 0$, $\delta_0 > 0$ and $\varepsilon > 0$, there exists $\lambda := \lambda(l_0, \alpha_0, \delta_0, \varepsilon, \bar{\delta}) > 0$ such that the functions $\tilde{F}(x,y) = \lambda\overline{F}(\lambda^{-1}x, \lambda^{-1}y)$, $\tilde{G}(x,y) = \lambda\overline{G}(\lambda^{-1}x, \lambda^{-1}y)$, $\tilde{R}(t) = \lambda\overline{R}(\lambda^{-1}t)$ and $\tilde{K}(t) = \lambda\overline{K}(\lambda^{-1}t)$ satisfy the following properties:

1. *$\tilde{F}$ has the form (2.2), and its domain contains a ball $\overline{B}(\tilde{\delta})$, with $\tilde{\delta} = \lambda\bar{\delta} > \delta_0$.*
2. *Let $\tilde{M} = \max\limits_{(x,y)\in\overline{B}(\tilde{\delta})} \|\tilde{G}(x,y)\|$. Then $\tilde{M} = \lambda\bar{M}$ and hence $\|\tilde{G}_l\| \leq \lambda\bar{M}\tilde{\delta}^{-k}$ for all $k \geq 0$.*
3. *$\|\tilde{K}_l\| \leq (l!)^{\alpha_0}$ for all $N \leq l \leq l_0$. Moreover, $\tilde{K}_l = 0$ if $2 \leq l \leq N-1$.*
4. *$\tilde{R}(t) = t - \tilde{a}t^N + \tilde{b}t^{2N-1}$ with $\tilde{a} = \lambda^{-N+1}a \neq 0$ and $\tilde{b} = \lambda^{-2N+2}b$. Moreover $|\tilde{a}| \leq \varepsilon$.*
5. *Formally we have that*

$$\tilde{F} \circ \tilde{K} - \tilde{K} \circ \tilde{R} = 0. \tag{3.8}$$

We note that $\sigma := \frac{\tilde{b}}{\tilde{a}^2}$ is invariant under scalings like the given in Lemma 3.4

In order to prove that the formal power series $\hat{K}$ is α-Gevrey at 0, we will check that $\tilde{K}$ satisfies such condition. To do that we will apply Proposition 3.1 to the map $\tilde{F}$ to get an inductive formula for the coefficients $\tilde{K}_l$.

First of all we provide some technical lemma which are given in section below.

3.2.2. *Preliminary bounds.* We define

$$R_{k,\nu} = \sum_{\substack{l_1+\cdots+l_k=\nu \\ l_i\geq 1}} \prod_{i=1}^{k} \tilde{R}_{l_i}.$$

We note that in formulae of Proposition 3.1 for $\tilde{K}_l^x$ and $\tilde{K}_l^y$ are involved sums of the form $\tilde{K}_k^x R_{k,\nu}$ and $\tilde{K}_k^y R_{k,\nu}$, with $\nu = l + N - 1$ and $\nu = l$, respectively. The following lemma, give us a bound of these sums if we assume that $\|\tilde{K}_k\| \leq (k!)^\alpha$ (which will be the case when we will proceed by induction).

Lemma 3.5. *Let*

$$R_{k,\nu} = \sum_{\substack{l_1+\cdots+l_k=\nu \\ l_i\geq 1}} \prod_{i=1}^{k} \tilde{R}_{l_i} \ , \ J_{k,\nu}^1 = (k!)^\alpha R_{k,\nu}.$$

We have that:

$$|J^1_{k,\nu}| \le ((\nu - N + 1)!)^\alpha (\nu - mN + 1)|\tilde a|^m (1 + |\sigma|)^{m/2}, \quad \textit{if} \quad m := \frac{\nu - k}{N-1} \in \mathbb{N},$$

where $\sigma = \frac{\tilde b}{\tilde a^2}$. *Otherwise* $J^1_{k,\nu} = 0$.

Proof. Since $R_{k,\nu}$ is the coefficient of t^ν in $(t - \tilde a t^N + \tilde b t^{2N-1})^k$, then

$$R_{k,\nu} = \sum_{\substack{m_1+m_2+m_3=k \\ m_1+Nm_2+(2N-1)m_3=\nu}} \frac{k!}{m_1!m_2!m_3!} (-\tilde a)^{m_2} \tilde b^{m_3}.$$

The indices m_2, m_3 in the formula has to satisfy $(N-1)m_2 + 2(N-1)m_3 = l-k$, that is $m_2 + 2m_3 = m = (\nu - k)/(N-1) \in \mathbb{N}$. Henceforth $R_{k,\nu} = 0$ if $(\nu - k)/(N-1) \notin \mathbb{N}$ and otherwise

$$R_{k,\nu} = \sum_{m_3=0}^{[\frac{m}{2}]} \frac{(\nu - (N-1)m)!}{(\nu - mN + m_3)!(m - 2m_3)!m_3!} (-\tilde a)^{m-2m_3} \tilde b^{m_3}.$$

Then, since

$$\frac{(\nu - (N-1)m)!}{(\nu - Nm + m_3)!} \le \frac{(\nu - (N-1)m)!}{(\nu - Nm)!} \le (\nu - (N-1)m)^{m-1}(\nu - Nm + 1)$$

and

$$\begin{aligned} &\sum_{m_3=0}^{[\frac{m}{2}]} \frac{|\tilde a|^{m-2m_3} |\tilde b|^{m_3}}{(m-2m_3)!m_3!} \\ \le &|\tilde a|^m \sum_{m_3=0}^{[\frac{m}{2}]} \frac{1}{([m/2] - m_3)!m_3!} \left| \frac{\tilde b}{\tilde a^2} \right|^{m_3} \\ \le &\frac{1}{[\frac{m}{2}]!} |\tilde a|^m (1 + |\sigma|)^{m/2}, \end{aligned}$$

therefore,

$$|R_{k,\nu}| \le \frac{1}{[\frac{m}{2}]!} (\nu - (N-1)m)^{m-1} (\nu - Nm + 1) |\tilde a|^m (1 + |\sigma|)^{m/2}. \tag{3.9}$$

We are now to bound $J^1_{k,\nu}$. Notice that, since $k = \nu - (N-1)m$, then

$$|J^1_{k,\nu}| \le \frac{1}{[\frac{m}{2}]!} (\nu - (N-1)m)!^\alpha (\nu - (N-1)m)^{m-1} (\nu - Nm + 1) |\tilde a|^m (1 + |\sigma|)^{m/2}$$

The proof of Lemma 3.5 follows from

$$\begin{aligned} \frac{((\nu - (N-1)m)!)^\alpha (\nu - (N-1)m))^{m-1}}{((\nu - N + 1)!)^\alpha} &= \frac{(\nu - (N-1)m)^{m-1}}{[(\nu - N + 1) \cdots (\nu - (N-1)m + 1)]^\alpha} \\ &\le \frac{(\nu - (N-1)m)^{m-1}}{(\nu - (N-1)m + 1)^{(m-1)(N-1)\alpha}} \le 1, \end{aligned}$$

where we use that $\alpha = \frac{1}{N-1}$. $\square$

In order to make estimates of the norms of $\tilde{K}_l^x$ and $\tilde{K}_l^y$ computed in the l step of the construction given in Proposition 3.1, we have to estimate

$$\sum_{\substack{l_1+\cdots+l_k=\nu\\ 1\le l_i\le \nu-1}} \|\tilde{K}_{l_1}\|\dots\|\tilde{K}_{l_k}\|,$$

where, again, $\nu = l+N-1$ and $\nu = l$. In the induction arguments, we have to estimate such a sum assuming that $\|\tilde{K}_{l_i}\| \le (l_i!)^\alpha$. Notice also that we assume that $\tilde{K}_2 = \dots \tilde{K}_{N-1} = 0$, by Lemma 3.2.

Lemma 3.6. *We denote*

$$M_{k,\nu} = \sum_{l_1+\cdots+l_k=\nu,\ l_i\ge N} (l_1!\cdot\dots\cdot l_k!)^\alpha.$$

We have that:

$$M_{k,\nu} \le ((\nu-k+1)!)^\alpha N^{k-1} \quad \text{if } kN\le \nu.$$

Otherwise $M_{k,\nu}=0$.

Proof. Obviously, if $kN>\nu$, $M_{\nu,k}=0$. Let us assume that $kN\le\nu$. It is easy to see that, if $a,b,c\in\mathbb{N}$ with $b\le c$, then $(a+b)!c!\le b!(a+c)!$. We fix $l_1,l_2,\cdots,l_k\ge N$ such that $l_1+\cdots+l_k=\nu$ and we use the previous property to bound $l_1!l_2!$ in such a way that, since $l_i\ge N$,

$$l_1!l_2! = (l_1-N+N)!l_2! \le N!(l_1+l_2-N)!.$$

Analogously,

$$l_1!l_2!l_3! \le N!(l_1+l_2-N)!l_3! = N!(l_1+l_2-2N+N)!l_3! \le (N!)^2(l_1+l_2+l_3-2N)!$$

and applying this procedure recursively we get

$$l_1!l_2!\cdots l_k! \le (N!)^{k-1}(l_1+\cdots+l_k-(k-1)N)! = (N!)^{k-1}(\nu-(k-1)N)!.$$

On the other hand it is clear that

$$\begin{aligned}
&\#\{l_1+\cdots+l_k=\nu,\ l_i\ge N\}\\
=&\#\{m_1+\cdots+m_k=\nu-kN,\ m_i\ge 0\} = \binom{\nu-kN+k-1}{k-1}.
\end{aligned}$$

Henceforth

$$\begin{aligned}
M_{k,\nu} &\le (N!)^{\alpha(k-1)}((\nu-(k-1)N)!)^\alpha \binom{\nu-kN+k-1}{k-1}\\
&\le N^{k-1}((\nu-(k-1)N)!)^\alpha(\nu-kN+1)^{k-1}\\
&\le N^{k-1}((\nu-k+1)!)^\alpha \frac{(\nu-kN+1)^{k-1}}{(\nu-(k-1)N+1)^{\alpha(N-1)(k-1)}}\\
&\le N^{k-1}((\nu-k+1)!)^\alpha,
\end{aligned}$$

and the proof is complete. ∎

Lemma 3.7.

$$J^2_{k,\nu} := \sum_{\substack{l_1+\cdots+l_k=\nu\\ l_i=1\ \text{or}\ l_i\ge N}} (l_1!\cdot\dots\cdot l_k!)^\alpha \le \frac{1}{N}(1+N)^k((\nu-k+1)!)^\alpha. \tag{3.10}$$

Proof. For $k=\nu$, $J^2_{k,\nu}=1$ and the bound is obvious. Assume that $0<k<\nu$. Then,

$$J^2_{k,\nu}=\sum_{i=0}^{k-1}\binom{k}{i}M_{k-i,\nu-i}\le\sum_{i=0}^{k-1}\binom{k}{i}(\nu-k+1)!^\alpha N^{k-i-1}\le(\nu-k+1)!^\alpha\frac{(1+N)^k}{N},$$

and the proof is over. ☐

3.2.3. *The formal solution is α-Gevrey.* We prove Proposition 3.8 below which finish the proof of Theorem 2.3.

Proposition 3.8. *Let $\delta_0=2(1+N)$,*

$$\varepsilon=\min\left\{\frac{1}{4\sqrt{1+|\sigma|}\ \|(A-Id)^{-1}\|},\frac{1}{4(1+|\sigma|)}\right\}$$

and

$$l_0\ge\max\{\left(4\bar M\bar\delta^{-1}\|(A-Id)^{-1}\|\right)^{N-1},N+\frac{2}{N}(1+N)^N\left(1+\frac{2\bar M}{|a|\bar\delta^N}\right)\}\tag{3.11}$$

(where the constants $\bar\delta,\bar M$ are defined in Lemma 3.4).

The formal solution $\tilde K=\sum_{j=1}^\infty\tilde K_jt^j$ of the equation $\tilde F\circ\tilde K-\tilde K\circ R=0$ with $\tilde F$ the map of Lemma 3.4 with constants $l_0,\alpha_0=\alpha,\delta_0,\varepsilon$, satisfy that $\|\tilde K_j\|\le(j!)^\alpha$ for all $j\ge0$.

Proof. By Lemma 3.4, $\|\tilde K_l\|\le(l!)^\alpha$ for all $l\le l_0$ with l_0 satisfying the condition (3.11) of Proposition 3.8. We proceed now by induction. Let $l>l_0$ and assume that for all $j\le l-1$ we have that $\|\tilde K_j\|\le(j!)^\alpha$.

First we deal with $\tilde K^y_l$. From Proposition 3.1, $\tilde K^y_l=(A-\mathrm{Id})^{-1}(H^1_l-H^2_l)$, where

$$H^1_l=\sum_{k=N}^{l-N+1}\tilde K^y_kR_{k,l}\ ,\ \ H^2_l=\sum_{k=2}^{l}\sum_{\substack{l_1+\cdots+l_k=l\\1\le l_i\le l-1}}\tilde G^y_k[\tilde K_{l_1},\cdots,\tilde K_{l_k}].$$

(Notice that $\tilde K^y_2=\cdots=\tilde K^y_{N-1}=0$, by Lemma 3.4.) On the one hand, by Lemma 3.5,

$$\begin{aligned}\|H^1_l\|&\le\sum_{k=N}^{l-N+1}(k!)^\alpha|R_{k,l}|\le((l-N+1)!)^\alpha\sum_{m=1}^{\left[\frac{l-N}{N-1}\right]}(l-mN+1)(|\tilde a|\sqrt{1+|\sigma|})^m\\&\le((l-N+1)!)^\alpha(l-N+1)2|\tilde a|\sqrt{1+|\sigma|},\end{aligned}$$

where we assume that $|\tilde a|\le\varepsilon$ with $\sqrt{1+|\sigma|}\ \varepsilon\le\frac12$, which is implied by the hypothesis $\varepsilon\le\frac{1}{4(1+|\sigma|)}$ of Proposition 3.8. Moreover, since $\alpha=1/(N-1)$, we have that

$$((l-N+1)!)^\alpha(l-N+1)\le(l!)^\alpha\frac{l-N+1}{(l-N+2)^{\alpha(N-1)}}\le(l!)^\alpha.$$

On the other hand, by Lemma 3.7 and Lemma 3.4

$$\begin{aligned}\|H_l^2\| &\le \lambda \bar{M} \sum_{k=2}^{l} \tilde{\delta}^{-k} J_{l,k}^2 \le \lambda \bar{M} \sum_{k=2}^{l} \tilde{\delta}^{-k} ((l-k+1)!)^\alpha \frac{1}{N}(1+N)^k \\ &\le \lambda \bar{M} \frac{1}{N} 2 \left(\frac{1+N}{\tilde{\delta}}\right)^2 l^{-\alpha} (l!)^\alpha \le 2 \frac{\bar{M}}{\bar{\delta}} l_0^{-\alpha} (l!)^\alpha\end{aligned}$$

where we assume that $\frac{1+N}{\tilde{\delta}} \le \frac{1}{2}$, and use that $\tilde{\delta} = \lambda \bar{\delta}$. In summary,

$$(l!)^{-\alpha} \|\tilde{K}_l^y\| \le 2\|(A-\mathrm{Id})^{-1}\| \left(\sqrt{1+|\sigma|}\ \varepsilon + \frac{\bar{M}}{\bar{\delta} l_0^\alpha}\right) \le 1,$$

because $2\|(A-\mathrm{Id})^{-1}\|\sqrt{1+|\sigma|}\ \varepsilon \le \frac{1}{2}$ and $2\|(A-\mathrm{Id})^{-1}\|\bar{M}\bar{\delta}^{-1} l_0^{-\alpha} \le \frac{1}{2}$ by the hypotheses of Proposition 3.8.

Now, we deal with $\tilde{K}_l^x$. Again from Proposition 3.1, $\tilde{K}_l^x = C_l^1 + C_l^2 - C_l^3$ with

$$C_l^1 = \frac{1}{l-N} \sum_{\substack{l_1+\cdots+l_N = l+N-1 \\ 1\le l_i \le l-1}} \prod_{i=1}^{N} \tilde{K}_{l_i}^x\ ,\ C_l^2 = \frac{1}{\tilde{a}(l-N)} \sum_{k=N}^{l-1} \tilde{K}_k^x R_{k,l+N-1}$$

$$C_l^3 = \frac{1}{\tilde{a}(l-N)} \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k = l+N-1 \\ 1\le l_i \le l}} \tilde{G}_k^x[\tilde{K}_{l_1}, \cdots, \tilde{K}_{l_k}],$$

where in C_l^3 we use that $\tilde{K}_2^x = \cdots = \tilde{K}_{N-1}^x = 0$ by Lemma 3.4 and that $\tilde{K}_l^y$ is already known. We notice that, using Lemma 3.7,

$$|C_l^1| \le \frac{1}{l-N} \sum_{\substack{l_1+\cdots+l_N = l+N-1 \\ l_i = 1 \text{ or} N \le l_i \le l-1}} (l_1! \cdots l_N!)^\alpha \le \frac{1}{l_0 - N} \frac{1}{N} (1+N)^N (l!)^\alpha.$$

To bound $|C_l^2|$ we use Lemma 3.5 and we get

$$\begin{aligned}|C_l^2| \le& \frac{1}{|\tilde{a}|(l-N)} \sum_{k=N}^{l-1} (k!)^\alpha R_{k,l-N+1} \\ \le& \frac{1}{|\tilde{a}|(l-N)} \sum_{m=2}^{\left[\frac{l-1}{N-1}\right]} (l!)^\alpha (l-N(m-1)) |\tilde{a}|^m (1+|\sigma|)^{m/2} \\ \le& \frac{1}{|\tilde{a}|} 2|\tilde{a}|^2 (1+|\sigma|)(l!)^\alpha = 2|\tilde{a}|(1+|\sigma|)(l!)^\alpha,\end{aligned}$$

where we assume $\sqrt{1+|\sigma|}\varepsilon \le \frac{1}{2}$. Finally,

$$\begin{aligned} |C_l^3| \le & \frac{\lambda \bar{M}}{|\tilde{a}|(l-N)} \sum_{k=N}^{l+N-1} \tilde{\delta}^{-k} J_{k,l+N-1}^2 \\ \le & \frac{\lambda \bar{M}}{|\tilde{a}|(l-N)} \sum_{k=N}^{l+N-1} ((l+N-k)!)^\alpha \frac{1}{N} \left(\frac{1+N}{\tilde{\delta}}\right)^k \\ \le & \frac{\lambda \bar{M}}{|\tilde{a}|(l-N)} (l!)^\alpha \frac{1}{N} 2 \left(\frac{1+N}{\rho}\right)^N \le \frac{2\bar{M}}{|a|N(l_0-N)} \left(\frac{1+N}{\bar{\delta}}\right)^N (l!)^\alpha \end{aligned}$$

where we assume that $\frac{1+N}{\tilde{\delta}} \le \frac{1}{2}$ and we use that $\tilde{a} = \lambda^{1-N} a$ and $\tilde{\delta} = \lambda \bar{\delta}$. In summary, we obtain the bound

$$(l!)^{-\alpha} \|\tilde{K}_l^x\| \le 2(1+|\sigma|)\varepsilon + \frac{1}{N}(1+N)^N \left(1 + \frac{2\bar{M}}{|a|\bar{\delta}^N}\right) \frac{1}{l_0-N} \le 1,$$

because $2(1+|\sigma|)\varepsilon \le \frac{1}{2}$ and $\frac{1}{N}(1+N)^N \left(1+\frac{2\bar{M}}{|a|\delta^N}\right)\frac{1}{l_0-N} \le \frac{1}{2}$ by the hypotheses of Proposition 3.8.

With these lines we are done with the proof of Proposition 3.8, and so the proof of Theorem 2.3. ☐

4. **The solution of $F \circ K - K \circ R$.** In this section we will prove Theorem 2.4. In fact, we will give all the details of the proof for the stable case in which $a > 0$ and the eigenvalues of A are all of them of modulus not smaller than 1. At the end of this section, we will indicate the minor changes to prove the Theorem in the unstable case.

We will see that the formal solution $\hat{K}$ obtained in Theorem 2.3 is the α-Gevrey asymptotic expansion of a real-analytic function K defined in a sector S, and K is a parameterization of a one dimensional invariant manifold of F. As a result, K is C^∞ in a interval $[0,r)$, and real-analytic in $(0,r)$.

We recall also that $\alpha = \frac{1}{N-1}$. We also denote by $\mathcal{U}_{\mathbb{C}}$ the domain of the analytic extension of F to the complex numbers.

4.1. **The action of R on sectors.** In this short section we are going to study how R maps sectors of small enough opening β (see Appendix A) of the complex plane. For the sake of completeness, we will consider a more general case.

Lemma 4.1. *Let $\mathcal{R}(t) = t - at^N + b(t)t^{N+1}$ be an analytic function defined in a neighborhood of 0 in $\mathbb{C}$, where $a > 0$, $N \ge 2$ is an integer number and $b(t)$ is analytic. Let $\beta < \alpha\pi$ be an opening. Then, for all ρ small enough the function $\mathcal{R}$ maps the sector $S = S(\beta,\rho)$ into itself. Moreover, for all $t \in S$,*

$$|\mathcal{R}(t)| \le |t| \sqrt{1 - a\cos\lambda |t|^{N-1}}, \tag{4.1}$$

where $\lambda = (N-1)\frac{\beta}{2}$. In fact, $\mathcal{R}$ maps any closed subsector $\overline{S}_1 \subset S$ into itself.

Proof. Let us write $\mathcal{R}(t) = t\hat{\mathcal{R}}(t)$, with $\hat{\mathcal{R}}(t) = 1 - at^{N-1} + b(t)t^N$. In order to obtain (4.1) we have just to bound $\hat{\mathcal{R}}$ for points $t = re^{\mathrm{i}\varphi}$ with $|t| = r < \rho$ and $|(N-1)\varphi| < \lambda$. We write $\hat{\mathcal{R}}(t) = \hat{r}e^{\mathrm{i}\hat{\varphi}}$, so that $\mathcal{R}(t) = r\hat{r}e^{\mathrm{i}(\varphi+\hat{\varphi})}$.

Hence

$$\hat{r}^2 = 1 - 2a\cos((N-1)\varphi) r^{N-1} + O(r^{2(N-1)}).$$

By taking ρ small enough (depending on a, b and λ), we obtain $\hat{r}^2 \leq 1 - a\cos\lambda r^{N-1}$ and the bound (4.1).

We also obtain $\tan\hat{\varphi} = -a\sin((N-1)\varphi)r^{N-1}(1+O(r^{N-1}))$, so

$$\hat{\varphi} = -a(N-1)\varphi r^{N-1}(1+O(r^{N-1})).$$

Again, by taking ρ small enough $\hat{\varphi}\varphi \leq 0$ and $|\hat{\varphi}| \leq |\varphi|$.

In summary, for ρ small enough, the points of the sector $S = S(\beta,\rho)$ are mapped into itself. In fact, any closed subsector $\overline{S}_1 \subset S$ gets mapped into itself. □

4.2. **A quasi solution.** Let $\hat{K}$ the formal solution obtained in the previous section.

Proposition 4.2. *Let $\beta < \alpha\pi$ be an opening. For all ρ small enough, there exists an analytic function $K_e : S = S(\beta,\rho) \to \mathcal{U}_{\mathbb{C}} \subset \mathbb{C}^{1+d}$ such that*

(a) *$\hat{K}$ is the α-Gevrey asymptotic expansion of K_e;*
(b) *The error function $E = F \circ K_e - K_e \circ R$ is exponentially small in S of order α.*

That is, for a given norm $\|\cdot\|$ in $\mathbb{C}^{1+d}$, for every closed subsector $\overline{S}_1 \subset S$ there exist positive constants C, M and c, κ such that

(a') *for any $n \geq 0$ and $t \in \overline{S}_1$, $\|K_e(t) - K_{<n}(t)\| \leq CM^n n!^\alpha |t|^{-n}$;*
(b') *for any $t \in \overline{S}_1$, $\|E(t)\| \leq c\exp\left(-\kappa|t|^{-(N-1)}\right)$.*

Proof. The existence of a function K_e such that $K_e \cong_\alpha \hat{K}$ in a sector $S = S(\beta,\rho)$ is guaranteed by the Borel-Ritt-Gevrey Theorem (see Theorem A.4). Notice also that, by Proposition A.2,

$$\lim_{S\ni t\to 0} K_e^{(n)}(t) = n!K_n. \tag{4.2}$$

In particular, $K_e(0) = 0 \in \mathcal{U}_{\mathbb{C}}$. So, making ρ small enough, we can also assure that the image set of K_e is included in the (complex) domain of F, and that R maps the sector S to itself, which is the domain of K_e.

An straightforward application of Faa-di-Bruno formula assures that the function E is α-Gevrey in S (in fact, is a well known result that the composition of Gevrey functions is also Gevrey). By the formal construction in Theorem 2.3 and (4.2) we obtain that

$$\lim_{S\ni t\to 0} E^{(n)}(t) = 0.$$

Again by Proposition A.2 we obtain that $E \cong_\alpha \hat{0}$, where here $\tilde{0}$ means the formal series with all the coefficients equal to 0. The exponentially small estimate of E comes from Proposition A.5. □

4.3. **The invariance equation.** We will solve first the invariance equation in the stable case. So, let us assume $a > 0$ and the spectral radius of A^{-1} is not greater than 1.

We will fix now a closed sector $\overline{S}_1 \subset S(\beta,\rho)$, so the conclusions (a') and (b') of Proposition 4.2 are satisfied, in particular that there exist positive constants c, κ so that $\|E(t)\| \leq c\exp\left(-\kappa|t|^{-(N-1)}\right)$ in $\overline{S}_1$. We emphasize that κ does not depend on the norm $\|\cdot\|$.

Since K_e is a quasi-solution, we will look for a "flat" and real-analytic function $H : \overline{S}_1 \to \mathbb{C}^{1+d}$ such that

$$F \circ (K_e + H) - (K_e + H) \circ R = 0. \tag{4.3}$$

Let

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

By writing $N(z) = F(z) - \hat{A}z$, notice that (4.3) is equivalent to the fixed-point equation

$$H = -\hat{A}^{-1}(E + N \circ (K_e + H) - N \circ K_e - H \circ R) \tag{4.4}$$

The Banach space in which we will consider (4.4) is

$$\mathcal{X} = \{H : \overline{S}_1 \cup \{0\} \to \mathbb{C}^{1+d} \mid \text{ continuous, real-analytic in } S_1 \text{ and } \|H\|_{\mathcal{X}} < \infty\}, \tag{4.5}$$

where

$$\|H\|_{\mathcal{X}} = \sup_{t \in \overline{S}_1} \|\exp(\kappa |t|^{-(N-1)}) H(t)\|. \tag{4.6}$$

In order to prove that the RHS $\mathcal{F}(H)$ of (4.4) is contracting, we have to control all the terms. In particular, since $N(0) = 0$ and $DN(0) = 0$, we can make $\|N \circ (K_e + H) - N \circ K_e\|$ very small compared with H. The crux point in then to control $H \circ R$, which is provided by the following estimate.

Lemma 4.3.

$$\|H \circ R\|_{\mathcal{X}} \leq e^{-\frac{1}{2} a \kappa (N-1) \cos \lambda} \|H\|_{\mathcal{X}}.$$

Proof. From

$$\|H \circ R\|_{\mathcal{X}} = \sup_{t \in \overline{S}_1} \left(e^{\kappa |t|^{-(N-1)}} \|H \circ R(t)\| \right) = \sup_{t \in \overline{S}_1} \left(e^{\kappa \left(|t|^{-(N-1)} - |R(t)|^{-(N-1)} \right)} \|H\|_{\mathcal{X}} \right)$$

and, using Lemma 4.1,

$$\begin{aligned} |R(t)|^{-(N-1)} &\geq |t|^{-(N-1)} \left(1 - a \cos \lambda |t|^{(N-1)} \right)^{-\frac{N-1}{2}} \\ &\geq |t|^{-(N-1)} \left(1 + \frac{N-1}{2} a \cos \lambda |t|^{(N-1)} \right) \\ &= |t|^{-(N-1)} + \frac{1}{2}(N-1) a \cos \lambda \end{aligned}$$

we obtain the estimate of Lemma 4.3. □

Proposition 4.4. *Taking the radius r of $\overline{S}_1$ small enough, there exists $H \in \mathcal{X}$ satisfying the equation* (4.4).

Proof. In a given closed sector $\overline{S}_1 \subset S(\beta, \rho)$, we recall that the error E of the quasi-solution K_e is exponentially small of Gevrey order $\alpha = \frac{1}{N-1}$ and constant κ (see Proposition 4.2). We emphasize that κ does not depend on the norm chosen to make the estimates.

Since the spectral radius of $\hat{A}^{-1}$ is 1, we can find a norm $\|\cdot\|$ in $\mathbb{C}^{1+d}$ so that

$$L := \|\hat{A}^{-1}\| e^{-\frac{1}{2} a \kappa (N-1) \cos \lambda} < 1, \tag{4.7}$$

which makes contracting the term $\hat{A}^{-1} H \circ R$ of (4.4).

Since $N(0) = 0$ and $DN(0) = 0$, there exists $\delta > 0$ small enough so that for all $z \in \mathbb{C}^{1+d}$ with $\|z\| \leq \delta$, $z \in \mathcal{U}_{\mathbb{C}}$ and

$$\|\hat{A}^{-1}\| \; \|DN(z)\| \leq \frac{1-L}{2}. \tag{4.8}$$

Let us also define

$$\eta = \|\hat{A}^{-1}\| \; \|E\|_{\mathcal{X}} \; , \; s = \frac{2\eta}{1-L}. \tag{4.9}$$

Finally, let us take also the radius r of the $\overline{S}_1$ so small that:

- for all $t \in \overline{S}_1$, $\|K_e(t)\| \leq \frac{\delta}{2}$;

- $s\ \exp(-\kappa r^{-(N-1)}) \leq \frac{\delta}{2}$.

With this election of the radius r, we claim that the operator

$$\begin{array}{rlcl} \mathcal{F}: & \overline{B}_{\mathcal{X}}(s) & \longrightarrow & \overline{B}_{\mathcal{X}}(s) \\ & H & \longrightarrow & -\hat{A}^{-1}(E + N\circ(K_e+H) - N\circ K_e - H\circ R) \end{array} \tag{4.10}$$

is well-defined and contracting in the closed ball of radius s and centered in the origin of $\mathcal{X}$, $\overline{B}_{\mathcal{X}}(s)$.

First, notice that for $H \in \overline{B}_{\mathcal{X}}(s)$, and for all $t \in \overline{S}_1$,

$$\|H(t)\| \leq \exp(-\kappa|t|^{-(N-1)})\|H\|_{\mathcal{X}} \leq \exp(-\kappa|r|^{-(N-1)})s \leq \frac{\delta}{2}.$$

So, we can make the compositions involved in the definition of $\mathcal{F}(H)$.

Moreover, for all $H_1, H_2 \in \overline{B}_{\mathcal{X}}(s)$,

$$\begin{aligned} &\|\mathcal{F}(H_2) - \mathcal{F}(H_1)\|_{\mathcal{X}} \\ \leq &\|\hat{A}^{-1}\| \left(\|N\circ(K_e+H_2) - N\circ(K_e+H_1)\|_{\mathcal{X}} + \|(H_2-H_1)\circ R\|_{\mathcal{X}}\right) \\ \leq &\left(\|\hat{A}^{-1}\| \sup_{\|z\|\leq\delta}\|DN(z)\| + \|\hat{A}^{-1}\|e^{-\frac{1}{2}a\kappa(N-1)\cos\lambda}\right)\|H_2-H_1\|_{\mathcal{X}} \\ \leq &\frac{1}{2}(1+L)\|H_2-H_1\|_{\mathcal{X}}. \end{aligned}$$

In particular, for all $H \in \overline{B}_{\mathcal{X}}(s)$,

$$\begin{aligned} \|\mathcal{F}(H)\|_{\mathcal{X}} \leq &\|\mathcal{F}(0)\|_{\mathcal{X}} + \|\mathcal{F}(H) - \mathcal{F}(0)\|_{\mathcal{X}} \\ &\leq \eta + \frac{1}{2}(1+L)\|H\|_{\mathcal{X}} \leq \frac{1-L}{2}s + \frac{1+L}{2}s = s, \end{aligned}$$

so $\mathcal{F}(H) \in \overline{B}_{\mathcal{X}}(s)$.

Hence, we have proved the claim that $\mathcal{F}$ maps the closed ball $\overline{B}_{\mathcal{X}}(s)$ into itself, and that it is a contraction there, with Lipschitz constant $\frac{1}{2}(1+L)$. The fixed point H satisfies (4.4). ∎

With the proof of Proposition 4.4 we are done with the proof of Theorem 2.4 in the stable case.

Let us consider briefly now the unstable case, that is $a < 0$ and the spectral radius of A is not greater than one. It is clear that the formal power series $\hat{K}$ satisfies (formally) $F^{-1}\circ\hat{K} = \hat{K}\circ R^{-1}$. By Lemma 4.1 the function R^{-1} maps a sector $S(\beta,\rho)$ into itself. Hence, following previous arguments, one can deduce that there exists an analytic function K_e in a sector $S(\beta,\rho)$, such that $\hat{K}$ is its asymptotic α-Gevrey expansion and the error $E = F^{-1}\circ K_e - K_e\circ R^{-1}$ is exponentially small (Proposition 4.2). From now, all the arguments in the stable case apply now in the unstable case, changing F by F^{-1} and R by R^{-1}.

5. **The uniqueness of the parabolic manifold.** In this section we will prove Theorem 2.6. We only deal with the stable case, being the unstable case analogous. In particular, we will prove that the parabolic manifold, the right branch of center manifold, is uniquely determined by the parameterization K founded in the previous section. The main assumption is that the dynamics in the y-direction is strongly repelling, that is $\mathrm{spec} A \subset \{\mu \in \mathbb{C} : |\mu| > 1\}$. Hence we fix a norm in $\mathbb{R}^d$ such that $\|A\| > 1$ and $\|A^{-1}\| < 1$. We define the norm in $\mathbb{R}^{1+d}$ by $\|(x,y)\| = \max\{|x|, \|y\|\}$.

We will follow the scheme presented in [23, 5] to prove that the (weak) stable invariant manifold is actually the graph of a suitable function.

For $h, p > 0$, we define the cone

$$C(h,p) = \{z = (x,y) \in \mathbb{R}^{1+d} : 0 < x < h,\ \|y\| \le px\},$$

which is a convex subset of $\mathbb{R}^{1+d}$. We also define the sector

$$S = \{\zeta = (\xi,\eta) \in \mathbb{R}^{1+d} : |\xi| \le \|\eta\|\}.$$

From now on we will take $p \le 1$ so that $\|(x,y)\| = |x|$ if $(x,y) \in C(h,p)$. Now we are going to prove a technical lemma, which will be used as an induction step.

Lemma 5.1. *For all h, p small enough, the cone $C(h,p)$ satisfies the following properties:*

1. *There exists a constant $M > 0$ such that for all $z = (x,y) \in C(h,p)$*
$$0 < \pi^x F(x,y) \le x(1 - Mx^{N-1}).$$
2. *Let $z_1, z_2 \in C(h,p)$ such that $z_2 - z_1 \in S$. Then*
$$F(z_2) - F(z_1) \in S \quad \text{and} \quad \|\pi^y(F(z_2) - F(z_1))\| \ge \|\pi^y(z_2 - z_1)\|.$$

Proof. We recall that the map F can be expressed as

$$F(x,y) = \begin{pmatrix} x + f_N(x,y) + f_{\ge N+1}(x,y) \\ Ay + g_{\ge 2}(x,y) \end{pmatrix}$$

with

$$f_N(x,y) = -ax^N + \hat{f}_N(x,y), \quad a > 0,\ \hat{f}_N(x,0) = 0.$$

Since $\hat{f}_N$ is an homogeneous polynomial of degree N and $\hat{f}_N(x,0) = 0$, there exists a positive constant C such that $\hat{f}_N(x,y) \le C\|y\|\ \|(x,y)\|^{N-1}$. Moreover, since all the derivatives up to order N of $f_{\ge N+1}$ vanish at zero, for any $\varepsilon > 0$ there exists $h > 0$ small enough so that $|f_{\ge N+1}(x,y)| \le \varepsilon\|(x,y)\|^N$ for $\|(x,y)\| < h$. Hence, for all points (x,y) in the cone $C(h,p)$, with $p \le 1$, we have

$$\pi^x F(x,y) = x - ax^N + \hat{f}_N(x,y) + f_{\ge N+1}(x,y)$$

with

$$|\hat{f}_N(x,y)| \le Cp|x|^N \text{ and } |f_{\ge N+1}(x,y)| \le \varepsilon|x|^N$$

The first item is proved taking p and ε small enough.

Now we deal with the second item. Let $z_1, z_2 \in C(h,p)$ such that $\zeta = z_2 - z_1 \in S$. By the mean's value theorem, we have that

$$F(z_2) - F(z_1) = \int_0^1 DF(z(t))(z_2 - z_1)\,dt,$$

where $z(t) = z_1 + t(z_2 - z_1) \in C(h,p)$ for all $t \in [0,1]$. Notice that,

$$DF(z) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} O(\|z\|^{N-1}) & O(\|z\|^{N-1}) \\ O(\|z\|) & O(\|z\|) \end{pmatrix}.$$

Moreover, since $\zeta = (\xi,\eta)$ satisfies $|\xi| \le \|\eta\|$, $\|\zeta\| = \|\eta\|$. Then, using that $\|z(t)\| = |\pi^x z(t)| < h$, there exists a constant C satisfying

$$\left|\int_0^1 \pi^x DF(z(t))\zeta\,dt\right| \le \int_0^1 \left(|\xi| + C\|z(t)\|^{N-1}\|\zeta\|\right)dt \le \left(1 + Ch^{N-1}\right)\|\eta\|.$$

$$\left\|\int_0^1 \pi^y DF(z(t))\zeta\,dt\right\| \ge \|A\eta\| - \int_0^1 \|\pi^y DF(z(t))\zeta - A\eta\|\,dt \ge \left(\|A^{-1}\|^{-1} - Ch\right)\|\eta\|.$$

Henceforth, taking h small enough $F(z_1) - F(z_2) \in S$ provided that $\|A^{-1}\| < 1$. Moreover,

$$\|\pi^y(F(z_2) - F(z_1))\| \geq \left(\|A^{-1}\|^{-1} - Ch\right)\|\eta\| \geq \|\pi^y(z_2 - z_1)\|.$$

□

The next result follows from induction arguments.

Lemma 5.2. *Let $C(h,p)$ be the cone of Lemma 5.1. We have that:*

1. *If z belongs to the parabolic manifold then $F^n(z) \in C(h,p)$ for all $n \geq 0$.*
2. *If $z \in C(h,p)$ and for all $n \geq 0$ $F^n(z) \in C(h,p)$, then $\lim F^n(z) = 0$.*
3. *If $z_1, z_2 \in C(h,p)$ such that $z_2 - z_1 \in S$, and for all $n \geq 0$ $F^n(z_1), F^n(z_2) \in C(h,p)$, then $z_1 = z_2$.*

Proof. We prove that if (x,y) belongs to the center manifold, then $(x,y) \in C(h,p)$. Since

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

the origin has an unstable manifold which is tangent at the origin to the vector $(0,1)^\top$ and the center manifold which is tangent at the origin to the vector $(1,0)^\top$ (of course the center manifold could not be unique). It is clear that the parabolic manifold is one of the branches of the center manifold, hence it is tangent at the origin to the vector $(1,0)^\top$ and the claim follows trivially. Finally we note that, if z belongs to the center manifold, then $F^n(x,y)$ also satisfies this condition, for all $n \geq 0$ and henceforth, by the previous claim, $F^n(x,y) \in C(h,p)$ for all $n \geq 0$.

For 2., let us consider the sequence $z_n = (x_n, y_n) = F^n(z) \in C(h,p)$. For 1. of Lemma 5.1, the sequence of positive numbers x_n is strictly decreasing and then it has a non-negative limit, say x_∞. Moreover, from the fact that $x_{n+1} \leq x_n(1 - Mx_n^{N-1})$ it follows than the limit is $x_\infty = 0$. Since for all n we have $\|y_n\| \leq px_n$, then the sequence y_n goes to zero when n goes to ∞. In summary, $\lim F^n(z) = 0$.

For 3., notice that both sequences $F^n(z_1)$ and $F^n(z_2)$ go to the origin of $\mathbb{R}^{1+d}$. Notice also that from 2. of Lemma 5.1 we obtain that for all $n \geq 0$ $F^n(z_2) - F^n(z_1) \in S$ and that the sequence $\|\pi^y(F^n(z_2) - F^n(z_1))\|$ is increasing (and converges to 0!). So $\pi^y z_2 = \pi^y z_1$. Finally, since $z_2 - z_1 \in S$, then $|\pi^x(z_2 - z_1)| \leq \|\pi^y(z_2 - z_1)\| = 0$, so $z_1 = z_2$. □

We are now ready to prove the following uniqueness result, which is a corollary of the previous lemma.

Proposition 5.3. *There is only one right branch of center manifold, Moreover, this parabolic manifold is a (weak) stable manifold of the origin.*

Proof. We observe that applying Theorem 2.4, we already know that for any $x \in (0,r)$ there exists at least one $y_1 \in \mathbb{R}^n$ such that (x,y_1) belongs to the center manifold. Indeed, this is due to the fact that $\pi^x K$ is invertible, hence there exists t such that $x = \pi^x K(t)$ and therefore $y_1 = \pi^y K(t)$ satisfies that (x,y_1) belongs to the center manifold.

Let us assume that there exists $x \in (0,h)$ such that there are $y_1 \neq y_2$ satisfying that $(x,y_1), (x,y_2)$ belong to the center manifold. Let $z_1 = (x,y_1)$ and $z_2 = (x,y_2)$.

We notice that by 1. of Lemma 5.2, $F^n(z_1), F^n(z_2) \in C(h,p)$ for all $n \geq 0$. Moreover, by 2. of Lemma 5.2 $\lim F^n(z_1) = 0$. Obviously, we also have $\lim F^n(z_2) = 0$.

Since $z_2 - z_1 = (0, y_2 - y_1) \in S$ and for all $n \geq 0$ $F^n(z_1), F^n(z_2) \in C(h,p)$ then, by 3. of Lemma 5.2, $z_1 = z_2$. So $y_1 = y_2$ for all $0 < x < h$, and both branches coincide. □

Remark 5.4. We emphasize that the uniqueness result stated in Proposition 5.3 holds under the assumption that F is C^{r+1}, with $r \geq N$.

Appendix A. **Some elementary facts on Gevrey asymptotics.** In this paper, we deal with asymptotic expansions of real-analytic functions which can be extended to analytic functions in sectorial regions of complex numbers. In this appendix we review some definitions and results on Gevrey asymptotics, adapted to the purposes of this paper. See, for instance, [3].

A *sector* of radius $\rho > 0$ and *opening* $\gamma \in (0, 2\pi]$ is the set

$$S(\gamma, \rho) = \{z = re^{\mathrm{i}\varphi} \in \mathbb{C} \mid 0 < r < \rho,\ |\varphi| < \gamma/2\}.$$

A *closed sector* is given by

$$\overline{S}(\gamma, \rho) = \{z = re^{\mathrm{i}\varphi} \in \mathbb{C} \mid 0 < r \leq \rho,\ |\varphi| \leq \gamma/2\}.$$

In the following, $\alpha \in (0, 1]$.

We say that a *formal power series* $\hat{f} = \sum_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$ is α-Gevrey iff there exist positive constants C, K such that for every non-negative integer n

$$|f_n| \leq CK^n n!^{\alpha}.$$

The set of α-Gevrey power series is denoted by $C[[z]]_{\alpha}$.

We say that an analytic function f in a sector S is α-Gevrey iff for every closed subsector $\overline{S}_1 \subset S$ there exist positive constants C, K such that for every non-negative integer n and every $z \in \overline{S}_1$,

$$\frac{1}{n!}|f^{(n)}(z)| \leq CK^n n!^{\alpha}\ .$$

The set of α-Gevrey functions in a sector S is denoted by $G_{\alpha}(S)$.

We say that an analytic function f in a sector S is asymptotic α-Gevrey to a formal power series $\hat{f}$, or that $\hat{f}$ *is the* α*-Gevrey asymptotic expansion of* f, in short $f \cong_{\alpha} \hat{f}$, iff for every closed subsector $\overline{S}_1 \subset S$ there exist positive constants C, K such that for every non-negative integer n and every $z \in \overline{S}_1$,

$$|r_f(z, n)| \leq CK^n n!^{\alpha}\ ,$$

where r_f is the *residue*

$$r_f(z, n) = z^{-n}\left(f(z) - \sum_{k=0}^{n-1} f_k z^k\right).$$

The set of asymptotic α-Gevrey functions in a sector S is denoted by $A_{\alpha}(S)$.

The following propositions relate the notions introduced above.

Proposition A.1. *Let f be an analytic function in a sector S, asymptotic α-Gevrey to a formal power series $\hat{f}$. Then, $\hat{f}$ is α-Gevrey.*

Proof. Let z be any point in S. Take $\overline{S}_1$ a closed subsector of S containing z. Then,

$$|r_f(z, n) - f_n| = |z||r_f(z, n+1)| \leq |z|CK^{n+1}(n+1)!^{\alpha}.$$

Therefore

$$\lim_{S \ni z \to 0} r_f(z, n) = f_n.$$

Since, in any closed subsector $\overline{S}_1$ we have $|r_f(z,n)| \le CK^n n!^\alpha$ for suitable constants C, K, we obtain the same bound when z goes to zero. So, $|f_n| \le CK^n n!^\alpha$. ☐

Proposition A.2. *Let f be an analytic function in a sector S and $\hat{f}(z) = \sum_{n\ge 0} f_n z^n$ be a formal power series. Then, f is asymptotic α-Gevrey to $\hat{f}$ if and only if f is α-Gevrey and for every non-negative integer n*

$$\lim_{S\ni z\to 0} f^{(n)}(z) = n! f_n. \tag{A.1}$$

Proof. Let us assume first that $f \cong_\alpha \hat{f}$. Notice that we can consider the termwise l-derivative formal series of $\hat{f}$,

$$\hat{f}^{(l)}(z) = \sum_{n\ge 0} f_n^{(l)} z^n = \sum_{n\ge 0} (n+1)\dots(n+l) f_{n+l} z^n,$$

and the corresponding residues of $f^{(l)}$, $r_{f^{(l)}}(z,n)$.

Let $\overline{S}_1 \subset S$ be a closed subsector, and take $\overline{S}_2 \subset S$ another closed subsector such that $\overline{S}_1 \subsetneqq \overline{S}_2$. Let $\delta > 0$ be small enough so that for every $z \in \overline{S}_1$ we have $B(z, |z|\delta) \subset \overline{S}_2$. Let C_2, K_2 be the α-Gevrey constants of f in the sector $\overline{S}_2$.

Let $z \in \overline{S}_1$, and n, l non-negative integers. From Cauchy's theorem

$$z^n r_{f^{(l)}}(z,n) = \frac{d^l}{dz^l}(z^{n+l} r_f(z, n+l)) = \frac{l!}{2\pi \mathrm{i}} \int_{|w-z|=\delta|z|} \frac{w^{n+l} r_f(w, n+l)}{(w-z)^{l+1}}\, dw,$$

and $(n+l)! \le 2^{n+l} n! l!$, we obtain

$$|r_{f^{(l)}}(z,n)| \le C_2 \left(2^\alpha (1+\delta)\delta^{-1} K_2\right)^l l!^{1+\alpha} \left(2^\alpha(1+\delta)K_2\right)^n n!^\alpha,$$

incidentally proving that $f^{(l)} \cong_\alpha \hat{f}^{(l)}$ [3]. Moreover, we have

$$|f^{(l)}(z)| = |r_{f^{(l)}}(z,0)| \le C_2 \left(2^\alpha(1+\delta)\delta^{-1}K_2\right)^l l!^{1+\alpha},$$

proving that f is α-Gevrey. Finally,

$$\lim_{S\ni z\to 0} f^{(l)}(z) = \lim_{S\ni z\to 0} r_{f^{(l)}}(z,0) = f_0^{(l)} = l! f_l.$$

Conversely, assume that f is α-Gevrey at the sector S and that the limit (A.1) exists. We consider the Taylor residues

$$\begin{aligned} R_f(z,u,n) &= z^{-n}\Big(f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(u)}{k!}(z-u)^k\Big) \\ &= \frac{z^{-n}(z-u)^n}{(n-1)!} \int_0^1 f^{(n)}(u + \lambda(z-u))\, (1-\lambda)^{n-1}\, d\lambda, \end{aligned}$$

for every $z, u \in S$ and $n \ge 0$. We observe that

$$r_f(z,n) = \lim_{S\ni u\to 0} R_f(z,u,n). \tag{A.2}$$

We fix now a closed subsector $\bar{S}_1 \subset S$, and let C, K be the corresponding α-Gevrey constants. For $z, u \in \bar{S}_1$, $|f^{(n)}(u+\lambda(z-u))| \le CK^n (n!)^{1+\alpha}$, so

$$|R_f(z,u,n)| \le |z|^{-n} |z-u|^n CK^n n!^\alpha.$$

Therefore, using (A.2) we have that

$$|r_f(z,N)| \le \lim_{\bar{S}\ni u\to 0} |z|^{-N}|z-u|^N CK^n (n!)^\alpha = CK^n n!^\alpha,$$

and the proof is complete. □

Notice that Proposition A.2 incidentally proves that the asymptotic α-Gevrey expansion $\hat{f}$ of an asymptotic α-Gevrey function f in a sector S is unique, by (A.1). We can then define a map $J : A_\alpha(S) \to C[[z]]_\alpha$ mapping each f to its asymptotic expansion $\hat{f}$.

The following proposition is straightforward.

Proposition A.3. *Let S be a sector. The sets $C[[z]]_\alpha$, $G_\alpha(S)$ and $A_\alpha(S)$, under natural operations, are differential algebras. Moreover, $J : A_\alpha(S) \to C[[z]]_\alpha$ is a morphism of differential algebras.*

So, for an asymptotic α-Gevrey function in a sector S, there is a unique α-Gevrey asymptotic expansion $\hat{f} = Jf$. The following is a Borel-Ritt-Gevrey theorem, which states that J is surjective if the opening of S is "small".

Theorem A.4. *Let $\hat{f} \in C[[z]]_\alpha$ and a sector S of opening $\beta < \alpha\pi$. Then, there exists a function f, analytic in S, so that $f \cong_\alpha \hat{f}$.*

A natural question is then how much do two asymptotic α-Gevrey functions with the same asymptotic expansion differ. The answer is in the following proposition.

Proposition A.5. *Let S be a sector of opening $\beta < \alpha\pi$, and let f be analytic in S with $f \cong_\alpha \hat{0}$, where $\hat{0}$ denotes the zero power series. Then, for every closed subsector $\bar{S}_1 \subset S$ there exist $c, \kappa > 0$ so that*

$$|f(z)| \leq c \exp\left(-\kappa |z|^{-\frac{1}{\alpha}}\right)$$

for all $z \in \bar{S}_1$. That is, f is exponentially small in S of Gevrey order α and constant κ.

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