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A rigorous derivation of the asymptotic wavenumber in spiral wave solutions of the complex Ginzburg–Landau equation

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Abstract. In this work, n-armed Archimedean spiral wave solutions of the complex Ginzburg–Landau equation are considered. These solutions are shown to depend on two characteristic parameters, the so-called *twist parameter* q and the *asymptotic wavenumber* k. The existence and uniqueness of the value of $k = k_*(q)$ for which n-armed Archimedean spiral wave solutions exist is a classical result, obtained back in the eighties by Kopell and Howard. In this work, we deal with a different problem, that is, the asymptotic expression of $k_*(q)$ as $q \to 0$. Since the eighties, different heuristic perturbation techniques, like formal asymptotic expansions, have conjectured an asymptotic expression of $k_*(q)$ which is of the form $k_*(q) \sim Cq^{-1}e^{-\frac{\pi}{2n|q|}}$ with a known constant C. However, the validity of this expression has remained opened until now, despite of the fact that it has been widely used for more than 40 years. In this work, using a functional analysis approach, we finally prove the validity of the asymptotic formula for $k_*(q)$, providing a rigorous bound for its relative error, which turns out to be $k_*(q) = Cq^{-1}e^{-\frac{\pi}{2nq}}(1 + \mathcal{O}(|\log|q||^{-1}))$. Moreover, such approach can be used in more general equations such as the celebrated $\lambda - \omega$ systems.

Keywords: spiral Archimedean waves, complex Ginzburg–Landau equation, asymptotic wavenumber.

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Contents

1. Introduction

In a wide range of physical, chemical and biological systems of different interacting species, one usually finds that the dynamics of each species is governed by a diffusion mechanism along with a reaction term, where the interactions with the other species are taken into account. For instance, one finds this type of systems in the modelling of chemical reaction processes as a model for pattern formation mechanisms [9], in the description of some ecological systems [24], in phase transitions in superconductivity [16] or even to describe cardiac muscle cell performance [13], among many others. Mathematically, a reaction-diffusion system is essentially a system of ordinary differential equations to which some diffusion terms have been added,

$$\partial_\tau U = D\Delta U + F(U, a), \tag{1.1}$$

where $U = U(\tau, \vec{x}) \in \mathbb{R}^N$, $\vec{x} = (x, y) \in \mathbb{R}^2$, $\tau \in \mathbb{R}$, D is a diffusion matrix, F is the reaction term, which is usually nonlinear, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator and a is a parameter (for instance, some catalyst concentration in a chemical reaction) or a group of parameters.

In this paper, we deal with a particular type of reaction-diffusion equations which are traditionally called *oscillatory systems*. These are characterized by the fact that they tend to produce oscillations in homogeneous situations (i.e., when the term $D\Delta U$ vanishes). Of particular interest are oscillatory reaction-diffusion systems which tend to produce spatial homogeneous oscillations. These are systems like (1.1) where the dynamical system obtained when one neglects the spatial derivatives (i.e., the Laplace operator) has an asymptotically stable periodic orbit. To be more precise, we refer to dynamical systems that undergo a non-degenerate supercritical Hopf bifurcation at (U_0, a_0). In this case, one can derive an equation for the amplitude of the oscillations, $A \in \mathbb{C}$, by taking $\varepsilon^2 = a - a_0 > 0$ small, $t = \varepsilon^2\tau$ and writing the modulation of local oscillations with frequency ω as solutions of (1.1) of the form

$$U(\tau, \vec{x}, a) = U_0 + \varepsilon[A(t, \vec{x})e^{i\omega\tau}v + \bar{A}(t, \vec{x})e^{-i\omega\tau}\bar{v}] + \mathcal{O}(\varepsilon^2),$$

where bar denotes the complex conjugate. Under generic conditions, performing suitable scalings and upon neglecting the higher order terms in ε (see, for instance, [21, Section 2], [5], or [22]), the amplitude $A(t, \vec{x})$ turns out to satisfy the celebrated complex Ginzburg–Landau equation (CGL)

$$\partial_t A = (1 + i\alpha)\Delta A + A - (1 + i\beta)A|A|^2, \tag{1.2}$$

where $A(t, \vec{x}) \in \mathbb{C}$ and α, β are real parameters (depending on F and D). The universality and ubiquity of CGL have historically produced a large amount of research and it is one of the most studied nonlinear partial differential systems of equations specially among the physics community. The CGL equation is also known to exhibit a rich variety of different pattern solutions whose stability and emergence are still far from being completely understood (see [7, 10, 11, 27, 29, 30] for some of the latest achievements and open problems).

We note that (1.2) has two special features: the solutions are invariant under spatial translations, i.e., if $A(t, \vec{x})$ is a solution, then $A(t, \vec{x} + \vec{x}_0)$ does also satisfy equation (1.2) for any fixed $\vec{x}_0 \in \mathbb{R}^2$, and it also has gauge symmetry, i.e., $\widetilde{A}(t, \vec{x}) = e^{i\phi}A(t, \vec{x})$ is a solution for any $\phi \in \mathbb{R}$.

In this work, we shall focus on some special rigidly rotating solutions of (1.2) called *Archimedean spiral waves*. In order to define these solutions, following [29], we consider first polar coordinates $\vec{x} = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2$ in which equation (1.2) reads

$$\partial_t A = (1 + i\alpha)\Big(\partial_r^2 A + \frac{1}{r}\partial_r A + \frac{1}{r^2}\partial_\varphi^2 A\Big) + A - (1 + i\beta)A|A|^2, \tag{1.3}$$

where, abusing notation, we denote by the same letter $A(t, r, \varphi)$ the solution in polar coordinates. To define spiral waves, let us first consider the one-dimensional CGL equation

$$\partial_t A = (1 + i\alpha)\partial_r^2 A + A - (1 + i\beta)A|A|^2, \quad r \in \mathbb{R}, \tag{1.4}$$

and introduce the notion of *wave train*.

Definition 1.1. A *wave train* of (1.3) is a nonconstant solution $A(t,r)$ of equation (1.4) of the form

$$A(t,r) = A_*(\Omega t - k_* r), \tag{1.5}$$

where the *profile* $A_*(\xi)$ is 2π-periodic, $\Omega \in \mathbb{R}\setminus\{0\}$ is the frequency of the wave train and $k_* \in \mathbb{R}$ is the corresponding (spatial) wavenumber.

The particular case of a single mode wave train, namely $A(t,r) = Ce^{i(\Omega t - k_* r)}$, leads to the well-known relations

$$C = \sqrt{1-k_*^2}, \quad \Omega = \Omega(k_*) = -\beta + k_*^2(\beta - \alpha). \tag{1.6}$$

The last condition on the frequency is the associated *dispersion relation*. Then, for any pair of the parameter values (α,β) there exists a family of single mode wave trains of (1.4) of the form given in (1.5) satisfying conditions (1.6), one for each wavenumber k_*.

Now we define (see Definition 1.2) an *n-armed Archimedean spiral wave* which, roughly speaking, is a bounded solution of (1.3) that asymptotically, as $r \to \infty$, tends to a particular wave train (see Figure 1). Spiral waves actually emerge from points where the amplitude is zero which are usually known as *defects* [5]. By virtue of the translation invariance of (1.2), in spiral wave solutions with a single defect, one can place the defect anywhere in space, in particular at the origin, i.e., $A(t,\vec{0}) = 0$.

The general definition of an n-armed spiral wave solution of the complex Ginzburg–Landau equation is given in [29].

Definition 1.2. Let $n \in \mathbb{N}$. The solution $A(t,r,\varphi)$ is a *rigidly rotating Archimedean n-armed spiral wave* solution of equation (1.3) if it is a bounded solution of form $A(t,r,\varphi) = A_s(r, \Omega t + n\varphi)$, defined for $r \geq 0$ and $\varphi \in [0,2\pi)$ satisfying

$$\lim_{r\to\infty} \max_{\psi\in[0,2\pi]} |A_s(r,\psi) - A_*(\psi - k_* r + \theta(r))| = 0$$

and

$$\lim_{r\to\infty} \max_{\psi\in[0,2\pi]} |\partial_\psi A_s(r,\psi) - A_*'(\psi - k_* r + \theta(r))| = 0,$$

where the profile $A_*(\xi)$ defines a wave train of equation (1.4) through $A_*(\Omega t - k_* r)$, $A_s(r,\cdot)$ is 2π-periodic and θ is a smooth function such that $\lim_{r\to\infty} \theta'(r) = 0$.

The parameter k_* is in this case known as the *asymptotic wavenumber* of the spiral.

Notice that, in a co-rotating frame given by $\psi = \Omega t + n\varphi$ and considering r as the independent variable, spiral wave solutions can be seen as a heteroclinic orbit, as represented in Figure 1, connecting the equilibrium point $A = 0$ with the wave train solution A_*.

We will see in Lemma 2.1 that the Ginzburg–Landau equation only possesses wave trains of a single mode. For this reason, and following the classical literature on spiral waves in reaction-diffusion equations or $\lambda - \omega$ systems (see [6,14,15,20,32]), we consider the following class of Archimedean spiral waves.

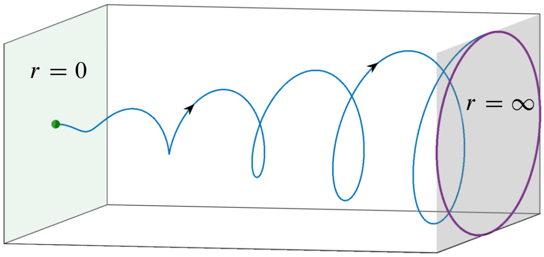


Fig. 1. Representation of the spiral wave solutions of (1.2) as a heteroclinic connection.

Definition 1.3. Let $n \in \mathbb{N}$. The solution $A(t, r, \varphi)$ is a rigidly rotating Archimedean n-armed spiral wave with a unique defect and a single mode if A is a solution of (1.3) of the form

$$A(t, r, \varphi) = \mathbf{f}(r) e^{i(\Omega t + n\varphi + \Theta(r))} \tag{1.7}$$

with $\mathbf{f}$, Θ regular (at least $\mathscr{C}^2$) for $r \geq 0$ satisfying the boundary conditions

$$\mathbf{f}(0) = 0, \quad \lim_{r\to\infty} \mathbf{f}(r) = \sqrt{1 - k_*^2}, \quad \Theta'(0) = 0, \quad \lim_{r\to\infty} \Theta'(r) = -k_*,$$

and Ω, k_* satisfy the dispersion equation in (1.6). The parameter k_* is called, as in Definition 1.2, the *asymptotic wavenumber* of the spiral.

Remark 1.4. The boundary condition $\mathbf{f}(0) = 0$ comes from the fact that we are searching spiral waves with one defect located at $r = 0$ (by the translation property, this is not a restriction), namely $|A(t, 0, \varphi)| = 0$. The boundary conditions, as $r \to \infty$, are consequence of Definition 1.2 and (1.6).

There is no need to impose any boundary condition on Θ at $r = 0$ because of the gauge symmetry of the Ginzburg–Landau solutions. It is a well-known fact [3,14,15,20] that the regularity at $r = 0$ of Θ is equivalent to impose $\Theta'(0) = 0$ (see also Remark 2.6). We keep this redundancy in Definition 1.3 just to emphasize the particular boundary conditions we deal with.

We introduce the so-called *twist parameter* q, depending on α, β,

$$q = q(\alpha, \beta) = \frac{\beta - \alpha}{1 + \alpha\beta} \tag{1.8}$$

which, in particular, is well defined for values of α, β such that $|\alpha - \beta| \ll 1$. As we shall explain in Section 1.1, the shape of the spiral waves strongly depends on this parameter. In fact, when $q = 0$, the solutions of the Ginzburg–Landau equation (1.2) of the form $A(t, \vec{x}) = e^{-i\alpha t} \widehat{A}(t, \vec{x})$ satisfy the "real" Ginzburg–Landau equation

$$\partial_t \widehat{A} = \Delta \widehat{A} + \widehat{A} - \widehat{A}|\widehat{A}|^2, \quad \widehat{A}(t, \vec{x}) \in \mathbb{R}.$$

Our perturbative analysis considers the case in which we are close to the "real" Ginzburg–Landau equation, that is to say, we deal with values of q which are small.

The main result of this paper reads as follows.

Theorem 1.5. *Fix $n \in \mathbb{N}$. Then there exist a constant C_n, only depending on n, $q_0 > 0$ small enough and a unique odd function $\kappa_*\colon (-q_0, q_0) \to \mathbb{R}$ of the form*

$$\kappa_*(q) = \frac{2}{q} e^{-\frac{C_n}{n^2} - \gamma} e^{-\frac{\pi}{2n|q|}} (1 + \mathcal{O}(|\log|q||^{-1})), \quad q \neq 0, \tag{1.9}$$

with the Euler–Mascheroni constant γ, such that the complex Ginzburg–Landau equation (1.3) for $q = q(\alpha, \beta) \in (-q_0, q_0)$ (defined in (1.8)) possesses rigidly rotating Archimedean n-armed spiral wave solutions

$$A(t, r, \varphi; q) = \mathbf{f}(r; q) e^{i(\Omega t + n\varphi + \Theta(r;q))}$$

as in Definition 1.3 if and only if the asymptotic wavenumber of the spiral wave satisfies $k_* = \kappa_*(q)$ as given in (1.9) and the frequency Ω satisfies (1.6).

In addition, for any $q \in (-q_0, q_0)$, we have that $\Theta'(r; q)$ has constant sign, $\mathbf{f}(r; q)$ is an increasing function,

$$\mathbf{f}(r; q) > 0, \quad \text{for } r > 0,$$

and, as a consequence, $\lim_{r\to\infty} \mathbf{f}'(r; q) = 0$.

Remark 1.6. The Ginzburg–Landau equation (1.3) depends on the parameters $\alpha, \beta \in \mathbb{R}$. However, the spiral waves of the form in Definition 1.3 depend only on the twist parameter q given in (1.8).

In the literature, the parameter $q = q(\alpha, \beta)$ is often taken positive due to the fact that if A is a solution of equation (1.3) with parameters α, β, then $\bar{A}$ (complex conjugate) is a solution of (1.3) with parameters $-\alpha, -\beta$. Therefore, if $\beta - \alpha < 0$, then $-\beta - (-\alpha) > 0$. That is, either A or $\bar{A}$ is a solution of a Ginzburg–Landau equation with parameters α, β satisfying $\beta - \alpha \geq 0$.

If $A(t, r) = Ce^{i(\Omega t - k_* r)}$ is a wave train, then $Ce^{i(\Omega t + k_* r)}$ is also a wave train because the dispersion relation (1.6) does not depend on the sign of k_*. That is, if k_* is a (spatial) wavenumber, also $-k_*$ is a wavenumber with the same frequency Ω. By the definition of asymptotic wavenumber, this fact does not imply that k_* and $-k_*$ are both asymptotic wavenumbers of two different Archimedean spiral waves of the same Ginzburg–Landau equation (1.3) with parameters α, β. Instead of this, for spiral waves as in Definition 1.3, if k_* is the asymptotic wavenumber associated to the spiral wave $A(t, r, \varphi; q)$ of equation (1.3) with parameters α, β (and $q = q(\alpha, \beta)$), then $-k_*$ is the asymptotic wavenumber associated to the spiral wave $\bar{A}(t, r, \varphi; -q)$ of equation (1.3) with parameters $-\alpha, -\beta$ (and $q(-\alpha, -\beta) = -q(\alpha, \beta)$).

Remark 1.7. We emphasize that the results of Theorem 1.5 ensure the existence of a constant M (depending on q_0 and n) such that for all $q \in (-q_0, q_0)$, one has

$$\left| \frac{q}{2} e^{\frac{C_n}{n^2} + \gamma} e^{\frac{\pi}{2n|q|}} \kappa_*(q) - 1 \right| \leq \frac{M}{|\log|q||}.$$

That is, we rigorously bound the relative error of $\kappa_*(q)$ with respect to its dominant term.

The simple description of spiral wave patterns of (1.2) clashes with the complexity of obtaining rigorous results on their existence, stability or emergence. In fact, the existence and uniqueness of $\kappa_*(q)$ and, as a consequence, of the rotational frequency of the pattern Ω, is a classical result that was obtained in the eighties by Kopell and Howard in [20]. At the same time, the physics community started showing interest in this type of phenomena and several authors used formal perturbation analysis techniques to describe spiral wave solutions (see, for instance, [6, 14] or [32]). More relevantly, Greenberg in [14] and Hagan in [15] used formal techniques of matched asymptotic expansions to conjecture an asymptotic formula for $k_* = \kappa_*(q)$ when q is small. The conjectured expression (1.9) of the wavenumber $k_*(q)$, has been widely used in the literature and checked numerically in innumerable occasions (see, for instance, [5, 8, 9, 23, 26] or [31]) but it has never been rigorously proven, which is the main purpose of the present paper. Furthermore, and as far as the authors know, in the previous works where expression (1.9) was formally derived, the order of the error was either not mentioned or was considered (without proof) to be $\mathcal{O}(q)$.

The precise computation of the constants in the exponentially small terms arising in (1.9) was already a challenge to overcome when the formal derivation was obtained and, in fact, 30 years later in [4], a new simpler formal asymptotic scheme was used. It is therefore not surprising that it has taken more than 40 years to finally obtain a rigorous proof of expression (1.9) (see Remark 1.7).

The novelty of our approach is to introduce a suitable functional setting which allows us to prove that a necessary and sufficient condition for the spiral waves to exist is that the associated wavenumber k_* has to be exactly $\kappa_*(q)$ as in (1.9). This functional approach has furthermore allowed to provide a very detailed description of the structure of the whole spiral wave solutions, of which several features, such as positivity or monotonicity among many others, have now been rigorously established.

Archimedean spiral wave patterns are present in some other systems. In particular, there is another type of reaction-diffusion systems, the so-called $\lambda - \omega$ *systems*, which have been classically used to investigate rotating spiral wave patterns

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda(f) & -\omega(f) \\ \omega(f) & \lambda(f) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \Delta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1.10}$$

where $u_1(t, \vec{x}), u_2(t, \vec{x}) \in \mathbb{R}$ and $\omega(\cdot), \lambda(\cdot)$ are real functions of the modulus

$$f = \sqrt{u_1^2 + u_2^2}.$$

Actually, this system was first introduced by Kopell and Howard in [18] as a model to describe plane wave solutions in oscillatory reaction diffusion systems. Not much later, the same authors in [17, 19] and [20], under some assumptions on λ, ω, rigorously proved the existence and uniqueness of spiral wave solutions of (1.10) with a single mode. Later, in [3], the authors proved that, in fact, the asymptotic wavenumber $k_* = k_*(q)$ has to be a flat function of the (small) parameter q. The particularity of this system is that the equations satisfied by spiral waves turn out to be exactly the same as the ones for the CGL equation when $\lambda(z) = 1 - z^2$ and $\omega(z) = \Omega + q(1 - k^2 - z^2)$, as we show later in Remark 2.4.

1.1. Spiral patterns

By Definition 1.2 of Archimedean spiral waves, spiral wave solutions of form (1.7) provided by Theorem 1.5, have to tend, for any given $\psi = \Omega t + n\varphi$, as $r \to \infty$, to

$$A_*(\Omega t + n\varphi - k_* r + \theta(r)) = Ce^{i(\Omega t + n\varphi - k_* r + \theta(r))}$$

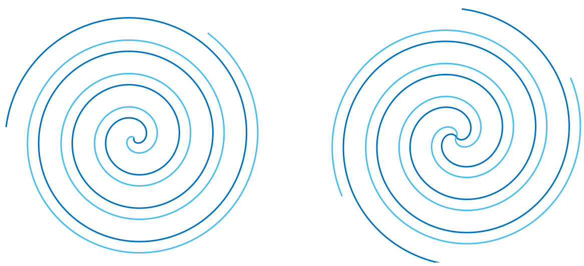
with $A_*(\xi)$ defining a wave train of (1.4) as in Definition 1.1, that is, $C, \Omega \in \mathbb{R}$ satisfying (1.6) and $\theta'(r) \to 0$ as $r \to \infty$. As we have mentioned, we will see in Section 2 that, in fact, these are the only possible wave trains of (1.4), namely, wave trains of equation (1.4) only have one mode. The contour lines of A_*,

$$\mathrm{Re}(A_*(\Omega t + n\varphi - k_* r)e^{-i\Omega t}) = \cos(n\varphi - k_* r) = c$$

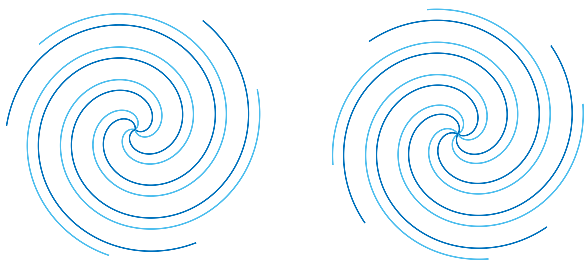
for any real constant c (or equivalently $n\varphi - k_* r = c'$), are Archimedean spirals whose wavelength L (distance between two spiral arms) is given by

$$L = \frac{2\pi n}{|k_*|}.$$

The parameter $n \in \mathbb{Z}$ is known as the *winding number* of the spiral and it represents the number of times the spiral intersects any given circle of radius r_0. In Figure 2, we represent *n-armed* Archimedean spirals for different winding numbers n.



(a) $n = 1$ **(b)** $n = 2$



(c) $n = 3$ **(d)** $n = 4$

Fig. 2. Representation of two Archimedean n-armed spiral waves for different winding numbers n. For a given winding number, these two spirals correspond to the contour lines $\cos(-k_* r + n\varphi) = c \neq \pm 1$ in Cartesian coordinates. When $c = \pm 1$, only one spiral survives.

At this point, we must emphasize the role of the parameter q in (1.8) in the shape of the spiral wave

$$A(t,r,\varphi;q) = \mathbf{f}(r;q)e^{i(\Omega t+n\varphi+\Theta(r;q))}$$

provided in Theorem 1.5. Recall that the asymptotic wavenumber of the spiral wave is $k_* = \kappa_*(q)$ with $\kappa_*(q)$ defined in (1.9). Let A_* be the wave train associated to the spiral wave A as in Definition 1.2. Then, from (1.6),

$$\lim_{r\to\infty} \mathbf{f}(r;q) = \sqrt{1-k_*^2}.$$

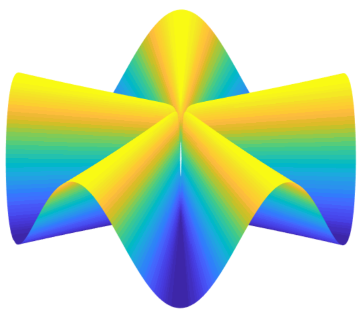
Moreover, (1.9) shows that $\lim_{q\to 0}\kappa_*(q) = 0$, and therefore $\lim_{r\to\infty}\Theta'(r;0) = 0$. In fact, when $q = 0$, that is, $\alpha = \beta$ (see (1.8)), again from the dispersion equation (1.6) one has $C = 1$ and $\Omega = -\beta$. In this case, the solutions of the Ginzburg–Landau equation (1.3) of the form $A(t,r,\varphi) := e^{i\Omega t}\widehat{A}(r,\varphi)$ are such that $\widehat{A}$ satisfies

$$\partial_r^2\widehat{A} + \frac{1}{r}\partial_r\widehat{A} + \frac{1}{r^2}\partial_\varphi^2\widehat{A} + \widehat{A} - \widehat{A}|\widehat{A}|^2 = 0.$$

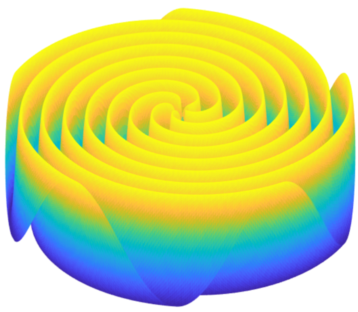
For any $n\in\mathbb{N}$, this equation has a solution of the form $\widehat{A}(r,\varphi) = \mathbf{f}(r)e^{in\varphi}$ with $\mathbf{f}(0) = 0$, $\lim_{r\to\infty}\mathbf{f}(r) = 1$. Indeed, the equation

$$\mathbf{f}'' + \frac{1}{r}\mathbf{f}' - \frac{n^2}{r^2}\mathbf{f} + \mathbf{f} - \mathbf{f}^3 = 0$$

is a particular case of the equation studied in [2], proving that there exists a unique solution satisfying the conditions in Theorem 1.5 when $q = 0$. For instance, plotting $\mathrm{Re}(\widehat{A}(r,\varphi))$ for $n = 5$ with respect to $\vec{x} = (r\cos\varphi, r\sin\varphi)$ for $r \gg 1$ big enough, one obtains the surface depicted in the left image of Figure 3.



(a) $q = 0$



(b) $q \neq 0$

Fig. 3. For $A(t,r,\varphi)$, a spiral wave solution of (1.3) with $n = 5$, the depicted surfaces represent the real part of $\widehat{A}(r,\varphi) = A(t,r,\varphi)e^{i\Omega t}$ with respect to $\vec{x} = (r\cos\varphi, r\sin\varphi)$ if r is big enough. The vertical axis corresponds to $\mathrm{Re}(\widehat{A}(r,\varphi))$ and the core of both surfaces corresponds to $r = r_0$ with r_0 big enough. Observe the arms that can be found emanating from the core of the spirals. Compare with Figure 2.

We note that the contour lines of $\mathrm{Re}(\widehat{A}(r,\varphi))$, namely $\mathbf{f}(r)\cos n\varphi = c$, tend as $r \to \infty$ to be straight lines emanating from the core of the spiral which correspond to different arms: when $c = \pm 1$, we have exactly $n = 5$ straight lines whereas for $c \neq \pm 1$, we have $2n = 10$ of them.

However, if $q \neq 0$, $\Theta(r;q)$ is not constant; the contour lines bend and tend, as $r \to \infty$, to become the already mentioned Archimedean spirals, as the ones depicted in the right image of Figure 3, corresponding to $n = 5$. Again, one can see in the right image of Figure 3 the different arms emanating from the core of the spiral. This is why q is usually denoted as the *twist* parameter of the spiral.

The paper is organized as follows. First, in Section 2 we prove that the only associated wave trains (Definition 1.1) have a single mode (Lemma 2.1) and obtain a system of ordinary differential equations that $\mathbf{f}$ and Θ have to satisfy in order for A, as defined in (1.7), to be a rigidly rotating Archimedean n-armed spiral wave. In addition, we set the boundary conditions which characterize $\mathbf{f}$ and Θ' (see Lemma 2.3). Finally, we state Theorem 2.5, about the existence of such solutions, and we prove Theorem 1.5 as a corollary of Theorem 2.5.

The rest of the paper is devoted to proving Theorem 2.5. First, in Section 3 we explain the strategy we follow to prove Theorem 2.5 as well as some heuristic arguments which motivate the asymptotic expression for the asymptotic wavenumber k_*. Section 4 is devoted to the proof of Theorem 2.5 using rigorous matching methods. For that, Theorems 4.3 and 4.5 prove the existence of families of solutions and, finally, Theorem 4.7 proves the desired formula for the asymptotic wavenumber. The more technical Sections 5 and 6 deal with the proof of Theorems 4.3 and 4.5, respectively.

2. Spiral waves as solutions of ordinary differential equations

The next lemma characterizes the form of the possible wave train solutions of equations (1.4).

Lemma 2.1. *The wave trains associated to* (1.3) *have a unique mode, namely, they are of the form* $A(t,r) = Ce^{i(\Omega t - k_* r)}$ *with* $k_* \in \mathbb{R}$, *and the constants* $C, \Omega \neq 0$ *satisfy relations* (1.6).

Proof. Assume $A_*(\xi) = \sum_{\ell \in \mathbb{Z}} a^{[\ell]} e^{i\ell\xi}$, $a^{[\ell]} \in \mathbb{C}$, and let $A(t,r)$ be the wave train defined through A_*, that is, $A(t,r) = A_*(\widehat{\Omega} t - \widehat{k}_* r)$. Since $A(t,r)$ has to be a solution of (1.4), we have, for all $\ell \in \mathbb{Z}$,

$$i\ell\widehat{\Omega} a^{[\ell]} = -(1+i\alpha)\widehat{k}_*^2 \ell^2 a^{[\ell]} + a^{[\ell]} - (1+i\beta)|A|^2 a^{[\ell]}$$

with $|A|^2 = |A(t,r)|^2 = A(t,r)\overline{A(t,r)}$ the complex modulus. Assume $a^{[\ell_1]}, a^{[\ell_2]} \neq 0$ for some ℓ_1, ℓ_2. Then

$$i\ell_1\widehat{\Omega} = -(1+i\alpha)\widehat{k}_*^2 \ell_1^2 + 1 - (1+i\beta)|A|^2,$$
$$i\ell_2\widehat{\Omega} = -(1+i\alpha)\widehat{k}_*^2 \ell_2^2 + 1 - (1+i\beta)|A|^2.$$

This implies

$$\widehat{\Omega}\ell_1 = -\alpha\widehat{k}_*^2\ell_1^2 - \beta|A|^2, \quad 0 = -\widehat{k}_*^2\ell_1^2 + 1 - |A|^2,$$
$$\widehat{\Omega}\ell_2 = -\alpha\widehat{k}_*^2\ell_2^2 - \beta|A|^2, \quad 0 = -\widehat{k}_*^2\ell_2^2 + 1 - |A|^2$$

and as a consequence $0 = -\widehat{k}_*^2(\ell_1^2 - \ell_2^2)$ so, if $\widehat{k}_* \neq 0$, $\ell_1 = \pm\ell_2$. If $\widehat{k}_* = 0$, then we have $\widehat{\Omega}(\ell_1 - \ell_2) = 0$ so $\ell_1 = \ell_2$ and we are done (recall that $\widehat{\Omega} \neq 0$). If $\ell_1 = -\ell_2$, we deduce $\widehat{\Omega}\ell_1 = \widehat{\Omega}\ell_2 = -\widehat{\Omega}\ell_1$ which implies that $\ell_1 = 0$ and hence $A(t,r)$ is constant which is a contradiction with Definition 1.1. Therefore, $\ell_1 = \ell_2$ and $A(t,r)$ has only one mode indexed by ℓ. Defining $\Omega = \ell\widehat{\Omega}$ and $k_* = \ell\widehat{k}_*$, the wave train is expressed as $A(t,r) = Ce^{i(\Omega t - k_* r)}$. Imposing that $A(t,r)$ is a solution of (1.4), we obtain

$$\Omega = -\alpha k_* - \beta|A|^2, \quad 0 = -k_*^2 + 1 - |A|^2.$$

Using that $|A| = C$, we have $C = \sqrt{1-k_*^2}$ and $\Omega = -\beta + k_*^2(\beta - \alpha)$. ∎

We fix now C, Ω and k_* such that they satisfy the relations in (1.6), namely

$$C^2 = 1 - k_*^2, \quad \Omega = -\beta + k_*^2(\beta - \alpha), \tag{2.1}$$

and the associated wave train is

$$A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}.$$

By Lemma 2.1, in this paper we look for Archimedean n-armed spiral wave with a unique defect and a single mode satisfying Definition 1.3,

$$A(t,r,\varphi) = \mathbf{f}(r;q)e^{i(\Omega t + n\varphi + \Theta(r;q))} \tag{2.2}$$

with

$$\lim_{r\to\infty} \mathbf{f}(r;q) = \sqrt{1-k_*^2}, \quad \lim_{r\to\infty} \Theta'(r;q) = -k_*. \tag{2.3}$$

Remark 2.2. By Definition 1.2, an Archimedean spiral wave, associated to the wave train $A_*(\Omega t - k_* r) = Ce^{i(\Omega t - k_* r)}$, satisfies

$$A(t,r,\varphi) = A_s(r, \Omega t + n\varphi) = \sum_{\ell\in\mathbb{Z}} a^{[\ell]}(r)e^{i\ell(\Omega t + n\varphi)} = \sum_{\ell\in\mathbb{Z}} f^{[\ell]}(r)e^{i\ell(\Omega t + n\varphi) + i\theta_\ell(r)}$$

with $f^{[\ell]}(r) \geq 0$ for all $\ell \in \mathbb{Z}$,

$$\lim_{r\to\infty} |f^{[1]}(r) - C| = \lim_{r\to\infty} |a^{[1]}(r)e^{-i\theta_1(r)} - C| = 0$$

with $\theta_1(r)$ such that $\lim_{r\to\infty} \theta_1'(r) = -k_*$, and, for $\ell \neq 1$,

$$\lim_{r\to\infty} a^{[\ell]}(r) = 0.$$

The spiral waves we are looking for, that is, of the form provided in (2.2) given in Definition 1.3, are the ones where $a^{[\ell]} \equiv 0$, for $\ell \neq 1$. These single mode solutions are the ones studied in previous works of the authors [3, 14, 15, 20].

We look for the equations that $\mathbf{f}$ and Θ have to satisfy in order for $A(t,r,\varphi)$ of the form in (2.2) to be a solution of (1.3). We recall the definition of q provided in (1.8)

$$q = \frac{\beta-\alpha}{1+\alpha\beta}. \tag{2.4}$$

Lemma 2.3. *Assume that* $|\alpha-\beta|<1$. *Let* $\Omega\neq 0$, *let* k_* *be constants satisfying* (2.1) *and* $A(t,r,\varphi;q)=\mathbf{f}(r;q)e^{i(\Omega t+n\varphi+\Theta(r;q))}$ *for some functions* $\mathbf{f}$ *and* Θ. *We introduce*

$$a = \left(\frac{1+\alpha^2}{1-\Omega\alpha}\right)^{\frac{1}{2}} \tag{2.5}$$

and

$$f(r;q) = \left(\frac{1+\alpha\beta}{1-\Omega\alpha}\right)^{\frac{1}{2}}\mathbf{f}(ar;q), \quad \chi(r;q)=\Theta(ar;q).$$

Then $A(t,r,\varphi;q)$ *is a solution of* (1.3) *if and only if* f *and* $v=\chi'$ *satisfy the ordinary differential equations*

$$f''+\frac{f'}{r}-f\frac{n^2}{r^2}+f(1-f^2-v^2)=0, \tag{2.6a}$$
$$fv'+f\frac{v}{r}+2f'v+qf(1-f^2-k^2)=0 \tag{2.6b}$$

with $k\in[-1,1]$ *satisfying the relations*

$$q(1-k^2)=-\frac{\Omega+\alpha}{1-\Omega\alpha}, \quad k_*=\frac{k}{(1-\alpha q(1-k^2))^{\frac{1}{2}}}.$$

Proof. We first note that, for $|\alpha-\beta|<1$, we have $1+\alpha\beta>0$. In addition, $1-\Omega\alpha>0$. Indeed, according to (2.1),

$$1-\Omega\alpha = 1-\alpha(-\beta+k_*^2(\beta-\alpha)) = 1+\alpha\beta-\alpha\beta k_*^2+\alpha^2k_*^2 = 1+\alpha\beta(1-k_*^2)+\alpha^2k_*^2.$$

Therefore, if $\alpha\beta\geq 0$, using that $k_*<1$ (see again (2.1)), we have $1-\Omega\alpha>0$. When $\alpha\beta<0$, since $1+\alpha\beta>0$,

$$1-\Omega\alpha = 1-|\alpha\beta|(1-k_*^2)+\alpha^2k_*^2 > 1-|\alpha\beta| = 1+\alpha\beta>0.$$

Consider the rotating frame with the scalings

$$B(r,\varphi)=\delta e^{-i\Omega t}A(t,ar,\varphi) = f(r;q)e^{i(\pm n\varphi+\chi(r;q))}, \tag{2.7}$$

where $f(r;q)=\delta\mathbf{f}(ar;q)$ and $\chi(r;q)=\Theta(ar;q)$.

Since A is solution of (1.3), B is a solution of

$$\partial_r^2 B+\frac{1}{r}\partial_r B+\frac{1}{r^2}\partial_\varphi^2 B+a^2\frac{1-i\Omega}{1+i\alpha}B-\delta^{-2}a^2\frac{1+i\beta}{1+i\alpha}B|B|^2=0,$$

or equivalently

$$\partial_r^2 B+\frac{1}{r}\partial_r B+\frac{1}{r^2}\partial_\varphi^2 B+a^2\frac{1-\Omega\alpha-i(\Omega+\alpha)}{1+\alpha^2}B-a^2\frac{1+\alpha\beta+i(\beta-\alpha)}{\delta^2(1+\alpha^2)}B|B|^2=0.$$

We define the constants

$$\widehat{\Omega} = -a^2 \frac{\Omega + \alpha}{(1+\alpha^2)} = -\frac{\Omega + \alpha}{1 - \Omega\alpha}, \qquad \delta^2 = a^2 \frac{1+\alpha\beta}{1+\alpha^2} = \frac{1+\alpha\beta}{1-\Omega\alpha},$$

where, in the last equalities, we have used definition (2.5) of a. Then, since

$$a^2 \frac{\beta - \alpha}{\delta^2(1+\alpha^2)} = \frac{\beta-\alpha}{1+\alpha\beta} = q,$$

the function B satisfies the equation

$$\partial_r^2 B + \frac{1}{r}\partial_r B + \frac{1}{r^2}\partial_\varphi^2 B + (1+\widehat{\Omega} i)B - (1+qi)B|B|^2 = 0$$

and, substituting the form of B in (2.7), we obtain that f and χ satisfy the ordinary differential equations

$$\begin{aligned} f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - (\chi')^2) &= 0, \\ 2f'\chi' + f\chi'' + \frac{1}{r} f\chi' + \widehat{\Omega} f - qf^3 &= 0. \end{aligned}$$

Notice that, by (2.1),

$$\widehat{\Omega} = \frac{(\beta-\alpha)}{1-\Omega\alpha}(1-k_*^2)$$

and then $\widehat{\Omega}$ and q have the same sign as $\beta - \alpha$. Introducing $v = \chi'$ and $k \in [-1,1]$ by the relation $\widehat{\Omega} = q(1-k^2)$, the above equations are the ones in (2.6).

To finish, we deduce the relation between k_* and k. First, we note that, using the definition of q,

$$\begin{aligned} 1 - \widehat{\Omega}\alpha = 1 - q\alpha(1-k^2) &= \frac{1 + \alpha\beta - \alpha(\beta-\alpha)(1-k^2)}{1+\alpha\beta} \\ &= \frac{1 + \alpha^2(1-k^2) + \alpha\beta k^2}{1+\alpha\beta} > 0. \end{aligned}$$

Then, since

$$\Omega = -\frac{\alpha + \widehat{\Omega}}{1 - \alpha\widehat{\Omega}} = -\frac{\alpha + q(1-k^2)}{1-\alpha q(1-k^2)},$$

using that $\Omega = -\beta + k_*^2(\beta - \alpha)$,

$$k_*^2(\beta-\alpha) = \frac{\beta - \alpha\beta q(1-k^2) - \alpha - q(1-k^2)}{1-\alpha q(1-k^2)} = \frac{\beta - \alpha - q(1-k^2)(1+\alpha\beta)}{1-\alpha q(1-k^2)}.$$

When $\alpha \neq \beta$, by definition of q, we have

$$k_*^2 = \frac{k^2}{1-\alpha q(1-k^2)}.$$

When $q = 0$, we simply define $k = k_*$ which is consistent with the above definitions. ■

Remark 2.4. Spiral wave solutions of $\lambda - \omega$ systems in (1.10) can be written in terms of a system of ordinary differential equations by writing system (1.10) in complex form. That is, denoting

$$A = u_1 + iu_2,$$

it satisfies

$$\partial_t A = (\lambda(f) + i\omega(f))A + \Delta A.$$

Then considering the change to polar coordinates $\vec{x} = (r\cos\varphi, r\sin\varphi)$ and looking for solutions of the form provided in (1.7) yield the following system of ordinary differential equations:

$$\begin{aligned} f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(\lambda(f) - (\chi')^2) &= 0, \\ f\chi'' + f\frac{\chi'}{r} + 2f'\chi' + f(\omega(f) - \Omega) &= 0. \end{aligned} \tag{2.8}$$

Equations (2.6) correspond to equations (2.8) in the particular case where

$$\lambda(z) = 1 - z^2 \quad \text{and} \quad \omega(z) = \Omega + q(1 - k^2 - z^2).$$

An important observation is that when $q = 0$ (see (2.4) for the definition of q), equation (2.6b) simply reads

$$fv' + f\frac{v}{r} + 2f'v = \frac{(rf^2v)'}{rf} = 0$$

and therefore rf^2v must be constant. Hence, given that the solutions we are looking for must be bounded at $r = 0$, the only possible solution is $v \equiv 0$. Also, substituting in (2.6a) one finds that

$$f(r;0) = f_0(r)$$

is the solution of

$$f_0'' + \frac{f_0'}{r} - f_0\frac{n^2}{2r^2} + f_0(1 - f_0^2) = 0. \tag{2.9}$$

In the previous paper of the first two authors [2] (see also [3]), the existence of solutions of the above differential equation was stated (in fact, a more general set of differential equations was considered) under the boundary conditions

$$f_0(0) = 0, \quad \lim_{r\to\infty} f_0(r) = 1, \tag{2.10}$$

satisfying in addition

$$f_0(r) = 1 - \frac{n^2}{2r^2} + \mathcal{O}(r^{-4}), \quad r \to \infty. \tag{2.11}$$

Using the previous analysis, we will see that Theorem 1.5 is a straightforward consequence of the following result which, moreover, provides more detailed information on the constant C_n.

Theorem 2.5. *Let $n \in \mathbb{N}$. There exist $q_0 > 0$ and a function $\kappa\colon [0, q_0] \to \mathbb{R}$ satisfying $\kappa(0) = 0$, and*

$$\kappa(q) = \frac{2}{q} e^{-\frac{C_n}{n^2} - \gamma} e^{-\frac{\pi}{2n|q|}} (1 + \mathcal{O}(|\log q|^{-1}))$$

with the Euler–Mascheroni constant γ and

$$C_n = \lim_{r\to\infty} \left(\int_0^r \xi f_0^2(\xi)(1 - f_0^2(\xi))\, d\xi - n^2 \log r \right),$$

where f_0 is the solution of (2.9) and (2.10), such that if $k = \kappa(q)$, then system (2.6) subject to the set of boundary conditions

$$f(0; q) = v(0; q) = 0,$$
$$\lim_{r\to\infty} f(r; q) = \sqrt{1 - k^2}, \qquad \lim_{r\to\infty} v(r; q) = -k,$$

has a solution.

In addition, such a solution satisfies that, for $r > 0$, $v(r; q)$ has constant sign, for q fixed, $f(r; q)$ is an increasing function, $f(r; q) > 0$ and, as a consequence,

$$\lim_{r\to\infty} f'(r; q) = 0.$$

Remark 2.6. The extra boundary condition $\lim_{r\to\infty} f'(r; q) = 0$ does not need to be imposed, which, as we will see along the proof of Theorem 2.5, is a consequence of imposing that the solution satisfies $\lim_{r\to\infty}(f(r; q), v(r; q)) = (\sqrt{1 - k^2}, -k)$.

As we claimed in Remark 1.4, if $v(0; q) \in \mathbb{R}$, then $v(0; q) = 0$. Indeed, let us first note that from (2.6a) we have $(rf')' = fn^2r^{-1} - rf(1 - f^2 - v^2)$, and then we deduce that, for $0 < r \ll 1$, $f(r; q) \cdot f'(r; q) > 0$. Therefore, rewriting equation (2.6b) as

$$(rvf^2)' = -qrf^2(q - f^2 - k^2),$$

since v is defined at $r = 0$, we have

$$v(r; q) = -\frac{q}{rf^2(r; q)} \int_0^r \xi f^2(\xi; q)(1 - f^2(\xi; q) - k^2)\, \mathrm{d}\xi.$$

We conclude from l'Hôpital's rule that $v(0; q) = 0$.

Proof of Theorem 1.5 as a corollary of Theorem 2.5. First, emphasize the fact that equations (2.6) remain unaltered when (v, q) is substituted by $(-v, -q)$. That is, $v(r; -|q|) = -v(r; |q|)$. Then, when $q < 0$, we can define $\kappa(q) = -\kappa(|q|)$ and, as a consequence, κ is an odd function on $(-q_0, q_0)$. Therefore, one can consider $q \geq 0$ without loss of generality.

From property (2.11) of f_0 as $r \to \infty$, it is clear that the constant $C_n \in \mathbb{R}$.

By Theorem 2.5 and Lemma 2.3, there exists a spiral wave of form (1.7) satisfying $\lim_{r\to\infty} \mathbf{f}'(r; q) = 0$, $\mathbf{f}(0; q) = \Theta'(0; q) = 0$ and

$$\lim_{r\to\infty} \mathbf{f}(r; q) = \sqrt{1 - \kappa^2(q)} \left(\frac{1 - \Omega\alpha}{1 + \alpha\beta} \right)^{\frac{1}{2}}, \qquad \lim_{r\to\infty} \Theta'(r; q) = -\kappa(q) \left(\frac{1 - \Omega\alpha}{1 + \alpha^2} \right)^{\frac{1}{2}}.$$

By Lemma 2.3,

$$\kappa_*(q) = \kappa(q)(1-\alpha q(1-\kappa(q)))^{-\frac{1}{2}}.$$

Since $\kappa_*(q)$ has the same first-order expression as $\kappa(q)$ provided q is small enough, the expression for $\kappa_*(q)$ in Theorem 1.5 follows from the one for $\kappa(q)$.

To guarantee that $\mathbf{f}$ and Θ satisfy the required asymptotic conditions provided in (2.3), we need to check that $k_* = \kappa_*(q)$ and $k = \kappa(q)$ satisfy

$$1-k_*^2 = (1-k^2)\frac{1-\Omega\alpha}{1+\alpha\beta}, \quad -k_* = -k\left(\frac{1-\Omega\alpha}{1+\alpha^2}\right)^{\frac{1}{2}},$$

where expression (2.1) for Ω has been used to derive the expression for $1-k_*^2$. Indeed, from Lemma 2.3 and using definition (2.4) of q, we have, if $q \neq 0$,

$$1-k^2 = -\frac{1}{q}\frac{\alpha-\beta+k_*^2(\beta-\alpha)}{1-\Omega\alpha} = (1-k_*^2)\frac{(1+\alpha\beta)}{1-\Omega\alpha},$$

and the first equality is proven. With respect to the second one, we have to prove that

$$(1-\Omega\alpha)(1-\alpha q(1-k^2)) = 1+\alpha^2.$$

The equality is satisfied for $\alpha = 0$. When $\alpha \neq 0$, we have to prove

$$0 = -(\Omega + q(1-k^2)) + \alpha(\Omega q(1-k^2)-1) = -(\Omega+\alpha) - q(1-k^2)(1-\Omega\alpha),$$

which from Lemma 2.3 is true.

For the uniqueness of the function $\kappa_*(q)$, we use [20, Theorem 3.1] and [3, Lemma 2.1], related to $\lambda-\omega$ systems as (2.8), with the assumptions $\lambda(1)=0$, $\lambda'(z), \omega'(z)<0$, for $z \in (0,1]$ and $|\omega'(z)| = \mathcal{O}(|q|)$. We note that our case corresponds to $\lambda(z) = 1-z^2$ and $\omega(z) = \Omega + q(1-k^2-z^2)$ satisfying these conditions. The result in [3] says that if system (2.8) has a solution with boundary conditions given by

$$\lim_{r\to\infty} f(r) = f_\infty, \quad \lim_{r\to\infty} f'(r) = 0, \quad \lim_{r\to\infty} v(r) = v_\infty,$$

then f_∞ is such that $\omega(f_\infty) = \Omega$ and $v_\infty^2 = \lambda(f_\infty)$. The result in [20] states that there exists a unique value, $v_\infty(q)$, for q small enough, such that system (2.8) has solution with boundary conditions

$$\lim_{r\to\infty} f(r) = f_\infty, \quad \lim_{r\to\infty} f'(r) = 0, \quad \lim_{r\to\infty} \chi'(r) = v_\infty(q),$$

and f, v regular at $r=0$. Applying these results to our case, we obtain $f_\infty = \sqrt{1-k^2}$ and $v_\infty = -k$ and the results in [20] gives the uniqueness result in Theorem 1.5. ∎

After more than forty years, Theorems 2.5 and 1.5 provide a rigorous proof of the explicit asymptotic expressions widely used for $k=\kappa(q)$ and $k_* = \kappa_*(q)$ as well as rigorous bounds for their relative errors. Furthermore, the rigorous matching scheme used in this paper opens the door to showing without much extra effort the equivalent result for spiral waves in the more general setting of $\lambda-\omega$ systems.

3. Main ideas in the proof of Theorem 2.5

To prove Theorem 2.5, we need to study the existence of solutions of equations (2.6) with boundary conditions

$$\begin{aligned} &f(0;k,q) = v(0;k,q) = 0, \\ &\lim_{r\to\infty} f(r;k,q) = \sqrt{1-k^2}, \quad \lim_{r\to\infty} v(r;k,q) = -k. \end{aligned} \tag{3.1}$$

Observe that the functions (f, f', v) satisfy a system of first-order differential equations of dimension three. It is then natural to expect that no solution exists satisfying the four boundary conditions (3.1), except for a "privileged" value of k. Theorem 2.5 proves that this intuition is true.

The strategy of the proof is as follows. We split the domain $r \geq 0$ in two regions limited by a convenient value $r_0 \gg 1$:

- A far-field (*outer region*) defined as

$$r \in [r_0, \infty), \quad \text{where } \lim_{r\to\infty} f(r;k,q) = \sqrt{1-k^2}, \ \lim_{r\to\infty} v(r;k,q) = -k \tag{3.2}$$

  are the only boundary conditions that are imposed.

- A core-field (*inner region*) defined as

$$r \in [0, r_0], \quad \text{where } f(0;k,q) = v(0;k,q) = 0 \tag{3.3}$$

  are the boundary conditions.

The specific value of $r_0 = r_0(q) = \frac{1}{\sqrt{2}} e^{\frac{\rho}{q}}$ with $\rho = (\frac{q}{|\log q|})^{\frac{1}{3}}$ will be explained in Section 4.3.

We shall obtain two families of solutions (see Theorems 4.3 and 4.5), depending on two free parameters $\mathbf{a}, \mathbf{b} \in \mathbb{R}$, namely:

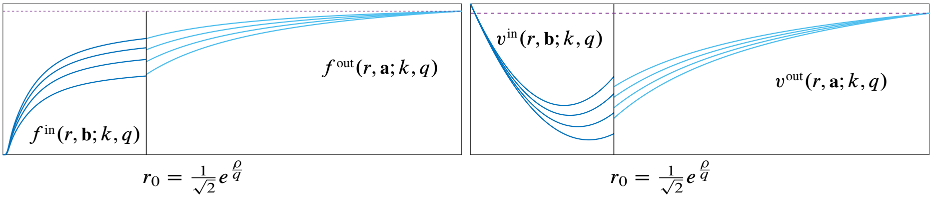
- $f^{\text{out}}(r, \mathbf{a}; k, q)$, $\partial_r f^{\text{out}}(r, \mathbf{a}; k, q)$, $v^{\text{out}}(r, \mathbf{a}; k, q)$ for the *outer region* satisfying (3.2), and
- $f^{\text{in}}(r, \mathbf{b}; k, q)$, $\partial_r f^{\text{in}}(r, \mathbf{b}; k, q)$, $v^{\text{in}}(r, \mathbf{b}; k, q)$ for the *inner region* satisfying (3.3),

which, upon matching them in the common point $r = r_0 = r_0(q)$, provides a system with three equations and three unknowns $(\mathbf{a}, \mathbf{b}, k)$:

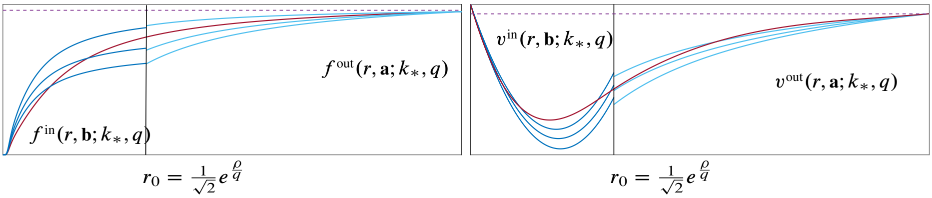
$$\begin{aligned} f^{\text{in}}(r_0, \mathbf{b}; k, q) &= f^{\text{out}}(r_0, \mathbf{a}; k, q), \\ \partial_r f^{\text{in}}(r_0, \mathbf{b}; k, q) &= \partial_r f^{\text{out}}(r_0, \mathbf{a}; k, q), \\ v^{\text{in}}(r_0, \mathbf{b}; k, q) &= v^{\text{out}}(r_0, \mathbf{a}; k, q). \end{aligned}$$

Therefore, having fixed q, this system provides a solution $(\mathbf{a}_*, \mathbf{b}_*, k_*)$. See Figure 4 for a representation of this strategy. Consequently, for the value of $k = k_*$, we have a solution of system (2.6) defined for all $r \geq 0$ as

$$(f(r;k,q), v(r;k,q)) = \begin{cases} (f^{\text{in}}(r, \mathbf{b}_*; k_*, q), v^{\text{in}}(r, \mathbf{b}_*; k_*, q)) & \text{if } r \in [0, r_0], \\ (f^{\text{out}}(r, \mathbf{a}_*; k_*, q), v^{\text{out}}(r, \mathbf{a}_*; k_*, q)) & \text{if } r \geq r_0, \end{cases} \tag{3.4}$$



(a) For a given $k \neq k_*$, the families of solutions $(f^{\text{out}}(r, \mathbf{a}; k, q), v^{\text{out}}(r, \mathbf{a}; k, q))$ (in light blue) and $(f^{\text{in}}(t, \mathbf{b}; k, q), v^{\text{in}}(r, \mathbf{b}; k, q))$ (in dark blue). In magenta the horizontal lines $f = \sqrt{1-k^2}$ (for f-component) and $v = -k$ (for the v-component). Notice that, in general, they do not match in a smooth way at $r = r_0$.



(b) For $k = k_*$, the corresponding families of solutions defined in the *outer* and *inner* regions, labelled by $\mathbf{a}$, $\mathbf{b}$, respectively. The solution of the problem, corresponding to $\mathbf{a}_*$, $\mathbf{b}_*$ (and k_*), is in red.

Fig. 4. A schematic representation of the matching procedure. In blue are depicted several solutions$(f^{\text{in}}(r, \mathbf{b}; k, q), v^{\text{in}}(r, \mathbf{b}; k, q))$ for different values of $\mathbf{b}$, in the *inner region*, $[0, r_0]$, and the counterpart for the *outer region*, namely $[r_0, \infty)$, labelled by $\mathbf{a}$. All of them intersect at $r = r_0$, but there is only one combination of these solutions (in red) that is differentiable at $[0, \infty)$ which corresponds to the selected wavenumber k_*.

satisfying the boundary conditions (3.1). This proves the existence result in Theorem 2.5 taking $\kappa(q) = k_*$.

Before stating the main results which provide Theorem 2.5, in Section 4, in the next subsection we give some intuition about how we obtain the value of $k = \kappa(q)$.

3.1. *The asymptotic expression for $k = \kappa(q)$*

One can find in the literature different heuristic arguments, based on (formal) matched asymptotic expansions techniques, which motivate the particular asymptotic expression for the parameter k,

$$k = \kappa(q) = \frac{\overline{\mu}}{q} e^{-\frac{\pi}{2nq}} (1 + o(1)) \tag{3.5}$$

with $\overline{\mu} \in \mathbb{R}$ a parameter independent of q (see, for instance, [15]). However, in this section we explain the particular deduction that is more consistent with the rigorous proof provided in the present work which we obtain by performing a change of parameter $k = \frac{\mu}{q} e^{-\frac{\pi}{2nq}}$ and finding the value of μ that solves the problem. Furthermore, a novelty of our proof is that it also provides that the relative error in expression (3.5) is in fact $\mathcal{O}(|\log q|^{-1})$.

We begin, as we explained at the beginning of Section 3, by looking for solutions of equations (2.6) which satisfy the boundary conditions (3.2) at $r = \infty$, which we shall denote as the *outer solutions*. We introduce a new parameter

$$\varepsilon = kq \tag{3.6}$$

and perform the scaling

$$R = \varepsilon r, \quad V(R) = k^{-1} v\Big(\frac{R}{\varepsilon}\Big), \quad F(R) = f\Big(\frac{R}{\varepsilon}\Big) \tag{3.7}$$

to equations (2.6). We obtain

$$\varepsilon^2\Big(F'' + \frac{F'}{R} - F\frac{n^2}{R^2}\Big) + F(1 - F^2 - k^2V^2) = 0, \tag{3.8a}$$

$$\varepsilon^2\Big(V' + \frac{V}{R} + 2\frac{VF'}{F} - 1\Big) + q^2(1 - F^2) = 0. \tag{3.8b}$$

If $\varepsilon \neq 0$, one can use the actual value of $1 - F^2$ provided by equation (3.8a) to recombine equations (3.8a) and (3.8b) to obtain the equivalent system

$$\varepsilon^2\Big(F'' + \frac{F'}{R} - F\frac{n^2}{R^2}\Big) + F(1 - F^2 - k^2V^2) = 0, \tag{3.9a}$$

$$V' + \frac{V}{R} + V^2 + q^2\frac{n^2}{R^2} - 1 = \frac{q^2}{F}\Big(F'' + \frac{F'}{R}\Big) - 2V\frac{F'}{F}. \tag{3.9b}$$

By virtue of (3.2), we look for bounded solutions of equations (3.9) satisfying

$$\lim_{R\to\infty} F(R;k,q) = \sqrt{1-k^2}, \qquad \lim_{R\to\infty} V(R;k,q) = -1. \tag{3.10}$$

Following a similar method to that in Proposition 4.2 one can check that the formal asymptotic expansions of bounded solutions when $R \to \infty$ satisfy

$$\begin{aligned} F(R;k,q) &\sim \sqrt{1-k^2} - \frac{k^2}{2R\sqrt{1-k^2}} + \mathcal{O}\Big(\frac{\varepsilon^2}{R^2}\Big) \quad \text{as } R \to \infty, \\ V(R;k,q) &\sim -1 - \frac{1}{2R} + \mathcal{O}\Big(\frac{\varepsilon^2}{R^2}\Big) \quad \text{as } R \to \infty. \end{aligned} \tag{3.11}$$

We note that equation (3.9a) is singular in ε. In particular, if $\varepsilon = 0$, and therefore $k = 0$ (recall (3.6)), either $F = 0$, which is a trivial solution we are not interested in, or $1 - F^2(R) = 0$, which also gives a noninteresting solution. But, if we write equation (3.9a) as

$$\varepsilon^2\Big(F'' + \frac{F'}{R}\Big) + F\Big(-\frac{\varepsilon^2 n^2}{R^2} + 1 - F^2 - k^2V^2\Big) = 0,$$

we observe that the asymptotic expansions (3.11) suggest that the terms $\frac{\varepsilon^2 F'}{R}$ and $\varepsilon^2 F''$ are of higher order in k, and therefore in ε, than the rest. Therefore, we will take as first approximation the solution of

$$-\frac{\varepsilon^2 n^2}{R^2} + 1 - F^2 - k^2V^2 = 0,$$

which gives our candidate to be the main part of the outer solution we are looking for,

$$F_0(R) = F_0(r;k,q) = \sqrt{1 - k^2 V_0^2(R;q) - \varepsilon^2 \frac{n^2}{R^2}}. \tag{3.12}$$

Then, neglecting again the terms depending on F' and F'' in equation (3.9b), a natural definition for V_0 is the solution of the Riccati equation

$$V_0' + \frac{V_0}{R} + V_0^2 + q^2 \frac{n^2}{R^2} - 1 = 0, \quad \text{such that } \lim_{R\to\infty} V_0(R;q) = -1. \tag{3.13}$$

Observe that the boundary condition for V_0 gives

$$\lim_{R\to\infty} F_0(R;k,q) = \sqrt{1-k^2},$$

as expected.

A solution of (3.13) is given by (see, for instance, [1])

$$V_0(R;q) = \frac{K'_{inq}(R)}{K_{inq}(R)} \tag{3.14}$$

with K_{inq} the modified Bessel function of the second kind. It is a well-known fact that (see [1]),

$$K_\nu(R) = \sqrt{\frac{\pi}{2R}} e^{-R}(1 + \mathcal{O}(R^{-1})), \quad \text{as } R \to \infty,$$

for any $\nu \in \mathbb{C}$, where $\mathcal{O}(R^{-1})$ is uniform as $\nu \to 0$. Therefore, the functions (F_0, V_0) satisfy the boundary conditions (3.10).

We go back to our original variables through scaling (3.7) and define

$$\begin{aligned} f_0^{\text{out}}(r;k,q) &= F_0(\varepsilon r;k,q) = F_0(kqr;k,q), \\ v_0^{\text{out}}(r;k,q) &= kV_0(\varepsilon r;q) = kV_0(kqr;q), \end{aligned} \tag{3.15}$$

which satisfy

$$\lim_{r\to\infty} v_0^{\text{out}}(r;k,q) = -k, \quad \lim_{r\to\infty} f_0^{\text{out}}(r;k,q) = \sqrt{1-k^2}. \tag{3.16}$$

The precise properties of the dominant terms f_0^{out}, v_0^{out} will be given in Proposition 4.2.

An important observation if $r \gg 1$, but kr is small enough, is that $v_0^{\text{out}}(r;k,q)$ has the following asymptotic expansion (a rigorous proof of this fact will be done in Proposition 4.2, see (4.3)):

$$v_0^{\text{out}}(r;k,q) = -\frac{n}{r} \tan\Big(nq \log r + nq \log kq + \frac{\pi}{2} - \theta_{0,nq}\Big)[1 + \mathcal{O}(q^2)]$$

with $\theta_{0,nq} = \arg(\Gamma(1+inq)) = -\gamma nq + \mathcal{O}(q^2)$, Γ is the Gamma function, and γ is the Euler–Mascheroni constant.

We now deal with the *inner solutions* of (2.6) departing the origin and satisfying $f(0;k,q) = v(0;k,q) = 0$. For moderate values of r, the *inner problem* is perturbative with respect to the parameter q. For that reason, to define the dominant term of the inner solutions we first consider the case $q = 0$. Let us now recall that in [3] it was proven that, when $q = 0$, system (2.6) has a solution (f, v) with boundary conditions (3.1) if and only if $k = k(0) = 0$. In this case, $v = v(r; 0, 0) = 0$ and $f_0(r) = f(r; 0, 0)$ satisfies the boundary conditions (2.10) and the second order differential equation (2.9),

$$f_0'' + \frac{f_0'}{r} - f_0\frac{n^2}{r^2} + f_0(1 - f_0^2) = 0, \quad f_0(0) = 0, \quad \lim_{r\to\infty} f_0(r) = 1. \tag{3.17}$$

As we already mentioned, the existence and properties of f_0 were studied in the previous work of the first two authors [2].

As $v(r; 0, 0) \equiv 0$, we write $v(r; k, q) = q\hat{v}(r; k, q)$ so system (2.6) reads

$$\begin{aligned} f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - q^2\hat{v}^2) &= 0, \\ f\hat{v}' + f\frac{\hat{v}}{r} + 2\hat{v}f' + f(1 - f^2 - k^2) &= 0. \end{aligned}$$

Let us now consider $(f_0(r), v_0(r; k))$, the unique solution of this system when $q = 0$ satisfying (3.17) and

$$v_0' + \frac{v_0}{r} + 2v_0\frac{f_0'}{f_0} + (1 - f_0^2 - k^2) = 0, \quad v_0(0; k) = 0. \tag{3.18}$$

In [2], it was proven that $f_0(r) > 0$ for $r > 0$ and $f_0(r) \sim \alpha_0 r^n$, as $r \to 0$, thus, the function

$$v_0(r; k) = -\frac{1}{rf_0^2(r)}\int_0^r \xi f_0^2(\xi)(1 - f_0^2(\xi) - k^2)\, d\xi \tag{3.19}$$

satisfies (3.18) and $v_0(0; k) = 0$. We then define the functions, whose properties are stated in Proposition 4.4,

$$f_0^{\text{in}}(r) = f_0(r), \quad v_0^{\text{in}}(r; k, q) = qv_0(r; k). \tag{3.20}$$

In Proposition 4.4, it will be proven that, if $r \gg 1$ but kr is small enough, the function $v_0^{\text{in}}(r; k, q)$ has the following asymptotic expansion, see (4.11):

$$\begin{aligned} v_0^{\text{in}}(r; k, q) = &-q\frac{n^2(1 + k^2)}{r}\log r + \frac{qC_n}{r} - \frac{k^2q}{2}r + q\mathcal{O}(r^{-3}\log r) \\ &+ qk^2\mathcal{O}(r^{-1}) \end{aligned} \tag{3.21}$$

with C_n defined in Theorem 2.5.

As we emphasize, we expect the functions v_0^{out} and v_0^{in} to be the first-order of the functions v^{out} and v^{in} in the outer and inner domains of r. Therefore, a natural request is that they "coincide up to first-order" in some large enough intermediate point r_0 such that kr_0 and $q \log r_0$ are still small enough quantities. With these hypotheses and using the previous asymptotic expansion (3.21), we obtain

$$v_0^{\text{in}}(r_0; k, q) = \frac{q}{r_0}[-n^2 \log r_0 + C_n + \text{HOT}],$$

where the higher-order terms (HOT) are small provided kr_0 is small. With respect to v_0^{out}, using that $\theta_{0,nq} = -\gamma nq + \mathcal{O}(q^2)$, we have

$$v_0^{\text{out}}(r_0; k, q) = \frac{q}{r_0}\Big[-\frac{n}{q}\tan\Big(nq\log r_0 + nq\log kq + \frac{\pi}{2} + nq\gamma + \mathcal{O}(q^2)\Big)[1+\mathcal{O}(q^2)]\Big].$$

Observe that if $nq\log kq + \frac{\pi}{2} = \mathcal{O}(q) = mq$, upon Taylor expanding the tangent function, one obtains

$$v_0^{\text{out}}(r_0; k, q) = -\frac{q}{r_0}[n^2\log r_0 + nm + n^2\gamma + \text{HOT}]$$

and then it is possible to have $v_0^{\text{out}}(r_0) - v_0^{\text{in}}(r_0) = 0$ because the "large" term $n^2\log r_0$ is cancelled.

The last observation of this section is that taking $kq = \mu e^{-\frac{\pi}{2nq}}$ gives $nq\log kq + \frac{\pi}{2} = nq\log\mu = \mathcal{O}(q)$. For this reason, during the proof of Theorem 2.5 in the rest of the paper, we will rewrite the parameter k using the expression

$$kq = \mu e^{-\frac{\pi}{2nq}}, \tag{3.22}$$

and we will prove that, for q small enough, there exists a value of $\overline{\mu}$ independent of q such that, for k given by (3.22) with $\mu = \overline{\mu} + \mathcal{O}(|\log q|^{-1})$, (2.6) has a solution satisfying the required asymptotic conditions (3.1).

4. Proof of Theorem 2.5: Matching argument

In order to prove Theorem 2.5 following the strategy explained in Section 3, we provide the precise statements about the existence of the families of solutions $(f^{\text{out}}, v^{\text{out}})$ in the *outer region* (3.2) (Section 4.1) and $(f^{\text{in}}, v^{\text{in}})$ in the *inner region* (3.3) (Section 4.2). In addition, since our method relies on finding $(f^{\text{out}}, v^{\text{out}})$ and $(f^{\text{in}}, v^{\text{in}})$ near the dominant terms $(f_0^{\text{out}}, v_0^{\text{out}})$ and $(f_0^{\text{in}}, v_0^{\text{in}})$, given in (3.15) and (3.20), respectively, we set all the properties of these dominant terms in Proposition 4.2 and 4.4, respectively. After that, in Sections 4.3 and 4.4, the rigorous matching of the dominant terms is done. Finally, in Section 4.5, we finish the proof of Theorem 2.5.

Let us set some conventions that we will use in the sequel.

- We denote by M a generic constant independent of q, k and consequently on ε (see (3.6)), that can (and will) change its value throughout the text.
- When the notation $\mathcal{O}(\cdot)$ is used, it means that the terms are bounded uniformly everywhere the function is studied. That is, if $\mathbf{h}, h\colon U_0 \subset \mathbf{R}^\ell \to \mathbf{R}^l$, then, for $z \in U_0$,

$$\mathbf{h}(z) = \mathcal{O}(h(z)) \;\Leftrightarrow\; |\mathbf{h}(z)| \le M|h(z)| \tag{4.1}$$

  for some constant M that only depends on U_0. If it is needed, the domain U_0 will be restricted without special mention.
- If $\Lambda = \Lambda(\lambda)$ with Λ, λ real parameters, we will say a function $\mathbf{h}(z,\Lambda)$ continuously depends on λ if $\widehat{\mathbf{h}}(z,\lambda) := \mathbf{h}(z,\Lambda(\lambda))$ is continuous with respect to λ. For instance, we will say that $v_0^{\text{in}}(r; k, q)$ is continuous with respect to μ (see (3.22)).

$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\Big(1 + \frac{4\nu^2 - 1}{8z} + \mathcal{O}\Big(\frac{1}{z^2}\Big)\Big)$	$\nu \in \mathbb{C}$	$\lvert z\rvert \geq z_0$, $\lvert\arg(z)\rvert < \frac{3\pi}{2}$
$I_\nu(z) = \sqrt{\frac{1}{2\pi z}} e^{z}\Big(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\Big)$	$\nu \in \mathbb{C}$	$\lvert z\rvert \geq z_0$, $\lvert\arg(z)\rvert < \frac{\pi}{2}$
$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\Big(\frac{1}{2}z\Big)^{-\nu}$	$\mathrm{Re}\,\nu > 0$	$\lvert z\rvert \leq z_1$, $z \in \mathbb{C}\backslash[-\infty, 0]$
$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\Big(\frac{1}{2}z\Big)^{\nu}$	$-\nu \notin \mathbb{N}$	$\lvert z\rvert \leq z_1$, $z \in \mathbb{C}\backslash[-\infty, 0]$

Tab. 1. Asymptotic expansions of the modified Bessel functions K_ν, I_ν when $z \to \infty$ and $z \to 0$. The values $z_0 \gg 1$ and $0 < z_1 \ll 1$ depend on ν.

The modified Bessel functions I_ν, K_ν, see [1], play an important role in our proofs. We pay special attention to their asymptotic behaviour. Table 1 summarizes the properties we extensively use along the paper.

Remark 4.1. We stress that for $|\nu| \leq \nu_0$ the $\mathcal{O}(\frac{1}{z})$, $\mathcal{O}(\frac{1}{z^2})$ terms in the expansion for K_ν, I_ν, as $z \to \infty$ in Table 1, are bounded by $\frac{M}{|z|}$ for $|z| \geq z_0$, and M, z_0 only depend on ν_0. With respect to the expansions as $z \to 0$, we also have that if $\mathrm{Re}\,\nu > 0$, then $|I_\nu(z)| \leq M|z|^\nu$ for $|z| \leq z_1$ with M, z_1 depending only on ν_0.

4.1. Outer solutions

We begin the proof of Theorem 2.5 by studying the dominant terms f_0^{out}, v_0^{out} (see (3.15)) in the *outer region* (see (3.2)).

Proposition 4.2. *For any $0 < \mu_0 < \mu_1$, there exists $q_0 = q_0(\mu_0, \mu_1) > 0$ such that for any $\mu \in [\mu_0, \mu_1]$ and $q \in (0, q_0]$, the functions $v_0^{\mathrm{out}}(r; k, q)$ and $f_0^{\mathrm{out}}(r; k, q)$ defined in (3.15) with $k = \mu q^{-1} e^{-\frac{\pi}{2nq}}$, satisfy the following properties:*

(i) *There exists $R_0 > 0$ such that for $kqr \geq R_0$,*

$$\begin{aligned} v_0^{\mathrm{out}}(r; k, q) &= -k - \frac{1}{2qr} + k\mathcal{O}\Big(\frac{1}{(kqr)^2}\Big), \\ f_0^{\mathrm{out}}(r, k, q) &= \sqrt{1-k^2}\Big(1 - \frac{k}{2qr(1-k^2)}\Big) + \mathcal{O}\Big(\frac{1}{(qr)^2}\Big). \end{aligned} \tag{4.2}$$

(ii) *For $2e^{-\frac{\pi}{2nq}} \leq kqr \leq (qn)^2$, one has*

$$v_0^{\mathrm{out}}(r; k, q) = -\frac{n}{r} \tan\Big(nq \log r + nq \log\Big(\frac{\mu}{2}\Big) - \theta_{0,nq}\Big)[1 + \mathcal{O}(q^2)] \tag{4.3}$$

with $\theta_{0,nq} = \arg(\Gamma(1 + inq)) = -\gamma nq + \mathcal{O}(q^2)$, where Γ is the Gamma function and γ is the Euler–Mascheroni constant.

(iii) *For* $2e^{-\frac{\pi}{2nq}} \leq kqr$*, one has*

$$\partial_r v_0^{\text{out}}(r;k,q) > 0, \quad v_0^{\text{out}}(r;k,q) < -k, \quad \partial_r f_0^{\text{out}}(r;k,q) > 0.$$

(iv) *Let* $\alpha \in (0,1)$*. There exist* $\overline{q}_0 = \overline{q}_0(\alpha, \mu_0, \mu_1)$ *and a constant* $M = M(\alpha, \mu_0, \mu_1) > 0$ *such that if* $r_{\min}$ *satisfies* $2e^2 e^{-\frac{\pi}{2qn}} \leq kqr_{\min} \leq (kq)^\alpha$*, then, for* $r \geq r_{\min}$*,* v_0^{out} *satisfies*

$$|v_0^{\text{out}}(r;k,q)|,\ |r\partial_r v_0^{\text{out}}(r;k,q)|,\ |r^2\partial_r^2 v_0^{\text{out}}(r;k,q)| \leq M r_{\min}^{-1},$$
$$|r(v_0^{\text{out}}(r;k,q)+k)|,\ |r^2\partial_r v_0^{\text{out}}(r;k,q)|,\ |r^3\partial_r^2 v_0^{\text{out}}(r;k,q)| \leq M q^{-1}.$$

With respect to f_0^{out}*, we have*

$$f_0^{\text{out}}(r;k,q) \geq \frac{1}{2},$$
$$|r^2\partial_r f_0^{\text{out}}(r;k,q)|,\ |r^3\partial_r^2 f_0^{\text{out}}(r;k,q)| \leq M q^{-1} r_{\min}^{-1},$$
$$|1 - f_0^{\text{out}}(r;k,q)|,\ |r\partial_r f_0^{\text{out}}(r;k,q)|,\ |r^2\partial_r^2 f_0^{\text{out}}(r;k,q)| \leq M r_{\min}^{-2}.$$

In addition f_0^{out}*,* v_0^{out} *depend continuously on* $\mu \in [\mu_0, \mu_1]$*.*

The proof of this proposition is postponed to Appendix A, and it involves a careful study of some properties of the Bessel functions K_{inq}.

Once $(f_0^{\text{out}}, v_0^{\text{out}})$ are studied, we look for solutions in the *outer region* satisfying boundary conditions (3.2). This is the content of the following Theorem 4.3 which gives the existence and bounds of a one-parameter family of solutions of equations (2.6), which stay close to the approximate solutions $(f_0^{\text{out}}(r;k,q), v_0^{\text{out}}(r;k,q))$ given in (3.15) for all $r \geq r_2$, r_2 being any number such that $r_2 = \mathcal{O}(\varepsilon^{\alpha-1})$ with $\varepsilon = kq$ defined in (3.6) and $0 < \alpha < 1$ satisfying $q^{-1}\varepsilon^{1-\alpha} \to 0$ when $q \to 0$.

Theorem 4.3. *For any* $\eta > 0$*,* $0 < \mu_0 < \mu_1$*, there exist* $q_0^* = q_0^*(\mu_0, \mu_1) > 0$*,* $0 < q_0 = q_0(\mu_0, \mu_1, \eta) \leq q_0^*(\mu_0, \mu_1)$*,* $e_0 = e_0(\mu_0, \mu_1, \eta) > 0$ *and* $M = M(\mu_0, \mu_1, \eta) > 0$ *such that, for any* $\mu \in [\mu_0, \mu_1]$ *and* $q \in [0, q_0]$*, if we take* $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ *and* $\alpha \in (0,1)$ *satisfying*

$$q^{-1}\varepsilon^{1-\alpha} < e_0, \tag{4.4}$$

taking r_2 *as*

$$r_2 = \varepsilon^{\alpha-1}, \tag{4.5}$$

$k = \mu q^{-1} e^{-\frac{\pi}{2nq}} = \varepsilon q^{-1}$ *and*

$$\mathbf{a} = \hat{\mathbf{a}} r_2^{-\frac{3}{2}} e^{r_2\sqrt{2}}, \quad |\hat{\mathbf{a}}| \leq \eta, \tag{4.6}$$

equations (2.6) *have a family of solutions* $(f^{\text{out}}(r,\mathbf{a};k,q), v^{\text{out}}(r,\mathbf{a};k,q))$ *defined for* $r \geq r_2$ *which are continuous with respect to* $\hat{\mathbf{a}}$*,* μ *and of the form*

$$\begin{aligned} f^{\text{out}}(r,\mathbf{a};k,q) &= f_0^{\text{out}}(r;k,q) + g^{\text{out}}(r,\mathbf{a};k,q), \\ v^{\text{out}}(r,\mathbf{a};k,q) &= v_0^{\text{out}}(r;k,q) + w^{\text{out}}(r,\mathbf{a};k,q), \end{aligned} \tag{4.7}$$

where f_0^{out}, v_0^{out} *are defined in* (3.15). *The functions* g^{out}, w^{out} *satisfy*

$$|r^2 g^{\text{out}}(r, \mathbf{a}; k, q)|, |r^2 \partial_r g^{\text{out}}(r, \mathbf{a}; k, q)| \le M,$$
$$|r^2 w^{\text{out}}(r, \mathbf{a}; k, q)| \le M q^{-1}(\eta + q^{-1}\varepsilon^{1-\alpha}).$$

We can also decompose

$$g^{\text{out}}(r, \mathbf{a}; k, q) = K_0(r\sqrt{2})\mathbf{a} + g_0^{\text{out}}(r; k, q) + g_1^{\text{out}}(r, \mathbf{a}; k, q), \tag{4.8}$$

where K_0 *is the modified Bessel function of the second kind* [1], *and* $g_0^{\text{out}}(r; k, q)$ *is an explicit function independent of* η. *Moreover,*

(i) *there exists* $M_0 = M_0(\mu_0, \mu_1)$ *such that, for* $q \in [0, q_0^*]$,

$$|r^2 g_0^{\text{out}}(r; k, q)|, \ |r^2 \partial_r g_0^{\text{out}}(r; k, q)| \le M_0 \varepsilon^{1-\alpha} q^{-1}, \tag{4.9}$$

(ii) *and for* $q \in [0, q_0]$,

$$|r^2 g_1^{\text{out}}(r, \mathbf{a}; k, q)|, \ |r^2 \partial_r g_1^{\text{out}}(r, \mathbf{a}; k, q)| \le M_1 \varepsilon^{1-\alpha} q^{-1} e^{-r_2\sqrt{2}} r_2^{\frac{3}{2}} |\mathbf{a}|, \tag{4.10}$$

where $M_1 = M_1(\mu_0, \mu_1, \eta)$ *depends on* μ_0, μ_1, *and* η.

As for w^{out}, *it can be decomposed as*

$$w^{\text{out}} = w_0^{\text{out}} + w_1^{\text{out}}$$

satisfying for $q \in [0, q_0]$,

$$|r^2 w_0^{\text{out}}(r, \mathbf{a}; k, q)| \le M_2 q^{-1} e^{-r_2\sqrt{2}} r_2^{\frac{3}{2}} |\mathbf{a}|,$$
$$|r^2 w_1^{\text{out}}(r, \mathbf{a}; k, q)| \le M_2 \varepsilon^{1-\alpha} q^{-2}$$

with $M_2 = M_2(\mu_0, \mu_1, \eta)$.

Theorem 4.3 is proven in Section 5 by performing the scaling (3.7) and studying the solutions of the outer equations (3.9) with boundary conditions (3.10) near the functions F_0, V_0 given in (3.12) and (3.14). The proof is done through a fixed point argument in a suitable Banach space.

We emphasize that, when $r \to \infty$, g^{out} and w^{out} have limit zero, and f_0^{out} and v_0^{out} satisfy (3.16), then f^{out} and v^{out} satisfy the boundary conditions (3.1). With this result in mind, we now proceed with the study of the behaviour of solutions of (2.6) departing $r = 0$, here called *inner solutions*.

4.2. *Inner solutions*

We now deal with the families of solutions of (2.6) departing the origin, satisfying the boundary condition $f(0) = v(0) = 0$ defined for values of r in the *inner region* (see (3.3)).

We first set the properties of f_0^{in}, v_0^{in}, the dominant terms in the *inner region* defined in (3.20), that will mostly be used throughout this proof.

Proposition 4.4. *For any $0 < \mu_0 < \mu_1$, there exists $q_0 = q_0(\mu_0, \mu_1) > 0$ such that for any $\mu \in [\mu_0, \mu_1]$ and $q \in [0, q_0]$, the functions $f_0^{\mathrm{in}}(r), v_0^{\mathrm{in}}(r; k, q)$ defined in (3.20) with $kq = \mu e^{-\frac{\pi}{2nq}}$, satisfy the following properties:*

(i) *For all $r > 0$, we have $f_0^{\mathrm{in}}(r), \partial_r f_0^{\mathrm{in}}(r) > 0$ and there exists $c_f > 0$ such that*

$$f_0^{\mathrm{in}}(r) \sim c_f r^n, \quad r \to 0, \qquad f_0^{\mathrm{in}}(r) = 1 - \frac{n^2}{2r^2} + \mathcal{O}(r^{-4}), \quad r \to \infty,$$
$$\partial_r f_0^{\mathrm{in}}(r) \sim n c_f r^{n-1}, \quad r \to 0, \qquad \partial_r f_0^{\mathrm{in}}(r) = \frac{n^2}{r^3} + \mathcal{O}(r^{-5}), \quad r \to \infty.$$

(ii) *For $0 < r \le \frac{n}{k\sqrt{2}}$, we have $v_0^{\mathrm{in}}(r; k, q) < 0$ and there exists a positive function of k, $c_v(k) = c_v^0 + \mathcal{O}(k^2)$, such that*

$$v_0^{\mathrm{in}}(r; k, q) \sim -q c_v(k) r, \qquad \partial_r v_0^{\mathrm{in}}(r; k, q) \sim -q c_v(k), \quad r \to 0,$$
$$|v_0^{\mathrm{in}}(r; k, q)| \le M q \frac{|\log r|}{r}, \quad |\partial_r v_0^{\mathrm{in}}(r; k, q)| \le M q \frac{\log r}{r^2}, \quad 1 \ll r < \frac{n}{k\sqrt{2}}.$$

(iii) *For $1 \ll r \le \frac{n}{k\sqrt{2}}$, we have*

$$\begin{aligned} v_0^{\mathrm{in}}(r; k, q) = & -q \frac{n^2(1+k^2)}{r} \log r + \frac{q C_n}{r} - \frac{k^2 q}{2} r \\ & + q\mathcal{O}(r^{-3} \log r) + q k^2 \mathcal{O}(r^{-1}) \end{aligned} \tag{4.11}$$

with C_n defined in Theorem 2.5 and

$$\partial_r v_0^{\mathrm{in}}(r; k, q) = q \frac{n^2}{r^2} \log r + q\mathcal{O}(r^{-2}).$$

In addition, v_0^{in} is continuous with respect to $\mu \in [\mu_0, \mu_1]$.

The proof of this proposition is referred to Appendix B and mostly relies on previous works [2,3].

The following theorem, whose proof is provided in Section 6, states that there exists a family of solutions of (2.6), satisfying the boundary conditions at the origin, which remains close to the approximate solutions $(f_0^{\mathrm{in}}(r), v_0^{\mathrm{in}}(r; k, q))$ given in (3.20), for all $r \in [0, r_1]$, where $r_1 = \mathcal{O}(e^{\frac{\rho}{q}})$ for some $\rho > 0$ small enough.

Theorem 4.5. *For any $\eta > 0$, $0 < \mu_0 < \mu_1$, there exist $q_0^* = q_0^*(\mu_0, \mu_1) > 0$, $0 < q_0 = q_0(\mu_0, \mu_1, \eta) \le q_0^*$, $\rho_0 = \rho_0(\mu_0, \mu_1, \eta) > 0$ and $M = M(\mu_0, \mu_1, \eta) > 0$ such that for any $\mu \in [\mu_0, \mu_1]$, $q \in [0, q_0]$ and*

$$\rho \in (0, \rho_0), \tag{4.12}$$

taking r_1 as

$$r_1 = \frac{e^{\frac{\rho}{q}}}{\sqrt{2}}, \tag{4.13}$$

$k = \mu q^{-1} e^{-\frac{\pi}{2nq}}$ and

$$\mathbf{b} = \hat{\mathbf{b}} \rho^2 r_1^{-\frac{3}{2}} e^{-\sqrt{2} r_1}, \quad |\hat{\mathbf{b}}| \le \frac{\eta}{(\sqrt{2})^{\frac{3}{2}}} \frac{q^2}{\rho^2} (\log \sqrt{2} r_1)^2 = \frac{\eta}{(\sqrt{2})^{\frac{3}{2}}}, \tag{4.14}$$

system (2.6) *has a family of solutions* $(f^{\text{in}}(r,\mathbf{b};k,q), v^{\text{in}}(r,\mathbf{b};k,q))$ *defined for* $r \in [0,r_1]$, *which are continuous with respect to* $\widehat{\mathbf{b}}$, μ, *and satisfy the boundary conditions* (3.3), *that is,*

$$f^{\text{in}}(0,\mathbf{b};k,q) = v^{\text{in}}(0,\mathbf{b};k,q) = 0.$$

Moreover, these functions satisfy

$$\begin{aligned} f^{\text{in}}(r,\mathbf{b};k,q) &= f_0^{\text{in}}(r) + g^{\text{in}}(r,\mathbf{b};k,q), \\ v^{\text{in}}(r,\mathbf{b};k,q) &= v_0^{\text{in}}(r;k,q) + w^{\text{in}}(r,\mathbf{b};k,q) \end{aligned} \tag{4.15}$$

with f_0^{in}, v_0^{in} *defined in* (3.20). *The functions* g^{in}, w^{in} *satisfy, for all* $r \in [0,r_1]$,

$$|g^{\text{in}}(r,\mathbf{b};k,q)| \le Mq^2, \quad |w^{\text{in}}(r,\mathbf{b};k,q)| \le Mq^3,$$

for $0 \le r < 1$,

$$\begin{aligned} |g^{\text{in}}(r,\mathbf{b};k,q)| &\le Mq^2 r^n, & |\partial_r g^{\text{in}}(r,\mathbf{b};k,q)| &\le Mq^2 r^{n-1}, \\ |w^{\text{in}}(r,\mathbf{b};k,q)| &\le Mq^3 r, & |\partial_r w^{\text{in}}(r,\mathbf{b};k,q)| &\le Mq^3, \end{aligned}$$

and for $1 \ll r \le r_1$,

$$|g^{\text{in}}(r,\mathbf{b};k,q)| \le Mq^2 \frac{|\log r|^2}{r^2}, \quad |w^{\text{in}}(r,\mathbf{b};k,q)| \le Mq^3 \frac{|\log r|^3}{r}.$$

In addition, there exists a function I *satisfying*

$$\begin{aligned} I'(r_1\sqrt{2})K_n(r_1\sqrt{2}) - I(r_1\sqrt{2})K_n'(r_1\sqrt{2}) &= \frac{1}{r_1\sqrt{2}}, \\ |I(r_1\sqrt{2})|,\ |I'(r_1\sqrt{2})| &\le M_I \frac{1}{\sqrt{r_1}} e^{r_1\sqrt{2}}, \end{aligned} \tag{4.16}$$

for some constant M_I, *and where* K_n *is the modified Bessel function of the second kind (see* [1]*), such that*

$$g^{\text{in}}(r,\mathbf{b};k,q) = I(r\sqrt{2})\mathbf{b} + g_0^{\text{in}}(r;k,q) + g_1^{\text{in}}(r,\mathbf{b};k,q), \tag{4.17}$$

where $g_0^{\text{in}}(r;k,q)$ *is an explicit function which is independent of* η. *Also, for* $1 \ll r \le r_1$,

(i) *there exists* $M_0(\mu_0,\mu_1)$ *such that, for* $q \in [0,q_0^*]$,

$$|g_0^{\text{in}}(r;k,q)|, |\partial_r g_0^{\text{in}}(r;k,q)| \le M_0 q^2 \frac{|\log r|^2}{r^2}, \tag{4.18}$$

(ii) *and for* $q \in [0,q_0]$,

$$|g_1^{\text{in}}(r,\mathbf{b};k,q)|, |\partial_r g_1^{\text{in}}(r,\mathbf{b};k,q)| \le M_1 q^2 \rho^2 \frac{|\log r|^2}{r^2},$$

where $M_1 = M_1(\mu_0,\mu_1,\eta)$ *depends on* μ_0, μ_1, *and* η.

4.3. Matching point and matching equations

Observe that, given $0 < \mu_0 < \mu_1$, the results of Theorems 4.3 and 4.5 are valid for any value of k of the form $k = \frac{\varepsilon}{q} = \frac{\mu}{q} e^{-\frac{\pi}{2nq}}$, $\mu \in [\mu_0, \mu_1]$ and q small enough. To finish the proof of Theorem 2.5, we need to select the value of μ, and therefore of k, which connects an outer solution (given by a particular value of $\mathbf{a}$, and therefore of $\hat{\mathbf{a}}$) with an inner one (given by a particular value of $\mathbf{b}$ and therefore of $\widehat{\mathbf{b}}$). To this end, we need to have a non-empty matching region, for which we shall impose $r_2 = r_1$, that is to say, $\varepsilon^{\alpha-1} = \frac{1}{\sqrt{2}} e^{\frac{\rho}{q}}$. Then, using that $\varepsilon = \mu e^{-\frac{\pi}{2qn}}$, one obtains

$$\alpha = \alpha(\rho, \mu, q) = 1 - \frac{2n\rho}{\pi} \frac{1 - \frac{q \log \sqrt{2}}{\rho}}{1 - \frac{2nq \log \mu}{\pi}}. \tag{4.19}$$

But, according to Theorem 4.3, it is also required that $\frac{\varepsilon^{1-\alpha}}{q} < e_0 \ll 1$, which is equivalent to imposing that q, ρ satisfy

$$q |\log(e_0 q \sqrt{2})| < \rho.$$

Therefore, fixing any $\eta > 0$, since by (4.12), $0 < \rho < \rho_0$, the condition for q, ρ becomes

$$q \left| \log\left(\frac{e_0 q}{\sqrt{2}} \right) \right| < \rho < \rho_0. \tag{4.20}$$

We rename

$$r_0 := r_1 = r_2 = \frac{e^{\frac{\rho}{q}}}{\sqrt{2}} = \varepsilon^{\alpha-1} = \mu^{\alpha-1} e^{\frac{\pi(1-\alpha)}{2qn}}, \tag{4.21}$$

and we take

$$\rho = \left(\frac{q}{|\log q|} \right)^{\frac{1}{3}}, \tag{4.22}$$

which satisfies the required inequalities in (4.20). Therefore, Theorems 4.3 and 4.5 are in particular valid when taking α and ρ as given in (4.19) and (4.22), and $r_1 = r_2$ as given in (4.21), since all these values satisfy conditions (4.4), (4.5), (4.12), and (4.13), if we take any $\mathbf{a}$ and $\mathbf{b}$ satisfying (4.6), (4.14), provided

$$q_0 = q_0(\mu_0, \mu_1, \eta)$$

is small enough (we take the minimum of both theorems).

Once we have chosen the parameters ρ and α and the value of the matching point r_0, the next step is to prove that there exist $\mathbf{a}$, $\mathbf{b}$, k or equivalently, since

$$\mathbf{a} = r_0^{-\frac{3}{2}} e^{\sqrt{2} r_0} \hat{\mathbf{a}}, \quad \mathbf{b} = \rho^2 r_0^{-\frac{3}{2}} e^{-\sqrt{2} r_0} \widehat{\mathbf{b}} \quad \text{and} \quad k = \frac{\varepsilon}{q} = \mu e^{-\frac{\pi}{2qn}},$$

that there exist $\hat{\mathbf{a}}$, $\widehat{\mathbf{b}}$, μ, such that, for q small enough,

$$\begin{aligned} f^{\text{out}}(r_0, \mathbf{a}; k, q) &= f^{\text{in}}(r_0, \mathbf{b}; k, q), \\ \partial_r f^{\text{out}}(r_0, \mathbf{a}; k, q) &= \partial_r f^{\text{in}}(r_0, \mathbf{b}; k, q), \\ v^{\text{out}}(r_0, \mathbf{a}; k, q) &= v^{\text{in}}(r_0, \mathbf{b}; k, q). \end{aligned} \tag{4.23}$$

We stress that the existence results, Theorems 4.3 and 4.5, depend on the set of constants μ_0, μ_1, η not defined yet. We shall fix them, in Section 4.4, as follows:

- First, we match the explicit dominant terms of the outer functions f^{out}, v^{out} (see (4.7) and (4.8)) with dominant terms of the inner functions f^{in}, v^{in} (see (4.15) and (4.17)):

$$\begin{aligned} K_0(r_0\sqrt{2})\mathbf{a}_0 + f_0^{\text{out}}(r_0;k,q) + g_0^{\text{out}}(r_0;k,q) &= I(r_0\sqrt{2})\mathbf{b}_0 + f_0^{\text{in}}(r_0) \\ &\quad + g_0^{\text{in}}(r_0;k,q), \\ v_0^{\text{out}}(r_0;k,q) &= v_0^{\text{in}}(r_0;k,q) \end{aligned} \tag{4.24}$$

  and

$$\begin{aligned} &\sqrt{2}K_0'(r_0\sqrt{2})\mathbf{a}_0 + \partial_r f_0^{\text{out}}(r_0;k,q) + \partial_r g_0^{\text{out}}(r_0;k,q) \\ &\quad = \sqrt{2}I'(r_0\sqrt{2})\mathbf{b}_0 + \partial_r f_0^{\text{in}}(r_0) + \partial_r g_0^{\text{in}}(r_0;k,q). \end{aligned} \tag{4.25}$$

  This is done in Section 4.4, where, in Proposition 4.6, we find

$$\mathbf{a}_0 = \hat{\mathbf{a}}_0 r_0^{-\frac{3}{2}} e^{\sqrt{2}r_0}, \quad \mathbf{b}_0 = \hat{\mathbf{b}}_0 \rho^2 r_0^{-\frac{3}{2}} e^{-\sqrt{2}r_0}$$

  and $\overline{\mu}$ such that, taking the approximate value of

$$k = \overline{\mu} q^{-1} e^{-\frac{\pi}{2qn}},$$

  equations (4.24) and (4.25) are solved. Moreover, we fix two values $0 < \mu_0 < \mu_1$ such that, $\overline{\mu} \in [\mu_0, \mu_1]$.
- The obtained solutions $\mathbf{a}_0$, $\mathbf{b}_0$ satisfy conditions (4.6) and (4.14) for a particular value of η. We will use these values, μ_0, μ_1, η in Theorems 4.3 and 4.5 to obtain families of solutions f^{out}, v^{out}, f^{in}, v^{in} of equations (2.6).

Finally, the existence of the constants $\mathbf{a}$, $\mathbf{b}$ and μ (that will be found to be close to $\mathbf{a}_0$, $\mathbf{b}_0$, $\overline{\mu}$) satisfying the matching conditions (4.23) is provided by means of a Brouwer's fixed point argument in Section 4.5 (see Theorem 4.7).

4.4. Matching the dominant terms: Setting the constants μ_0, μ_1, η

As we explained in the previous section, the purpose of this section is to choose the constants μ_0, μ_1, η which appear in Theorems 4.3 and 4.5 to obtain the families of solutions f^{out}, v^{out}, f^{in}, v^{in} of equations (2.6) satisfying the suitable boundary conditions.

The next proposition gives the existence of solutions of equations (4.24) and (4.25).

Proposition 4.6. *Take $\mu_0 = e^{-\frac{C_n}{n^2}-\gamma}$, $\mu_1 = 3e^{-\frac{C_n}{n^2}-\gamma}$, where C_n and γ are given in Theorem 2.5. Then, there exist $q_1^* = q_1^*(\mu_0,\mu_1)$ and $\widehat{M}(\mu_0,\mu_1)$ such that for $0<q<q_1^*$, equations (4.24) and (4.25) have a solution $(\mathbf{a}_0, \mathbf{b}_0, \overline{\mu})$ satisfying*

$$\overline{\mu} \in [\mu_0,\mu_1], \quad \mathbf{a}_0 = \hat{\mathbf{a}}_0 r_0^{-\frac{3}{2}} e^{r_0\sqrt{2}}, \quad |\hat{\mathbf{a}}_0| \le \widehat{M}\rho^2, \quad \mathbf{b}_0 = \hat{\mathbf{b}}_0 \rho^2 r_0^{-\frac{3}{2}} e^{-r_0\sqrt{2}}, \quad |\hat{\mathbf{b}}_0| \le \widehat{M},$$

where ρ, r_0 are given in (4.22) and (4.21), respectively.

Proof. As we pointed out in (4.1), we will say $\mathbf{h}(q) = \mathcal{O}(h(q))$ if for some $q_0 > 0$ there exists a constant $M > 0$ such that for all $q \in (0, q_0]$, $|\mathbf{h}(q)| \le M|h(q)|$. For instance, by definitions (4.22) and (4.21),

$$\rho = \rho(q) = \mathcal{O}\Big(\Big(\frac{q}{|\log q|}\Big)^{\frac{1}{3}}\Big) \quad \text{and} \quad q\log(r_0) = q\log(r_0(q)) = \mathcal{O}(\rho).$$

We note that, by definitions of r_0 and ρ in (4.21) and (4.22), respectively, we have

$$\Big|nq\log r_0 + nq\log\Big(\frac{\mu}{2}\Big) - \theta_{0,nq}\Big| = \mathcal{O}(\rho) = \mathcal{O}\Big(\Big(\frac{q}{|\log q|}\Big)^{\frac{1}{3}}\Big) \ll 1.$$

Then, using the asymptotic expressions (4.3) and (4.11) for v_0^{out} and v_0^{in} at $r = r_0$ and recalling that $k = \frac{\varepsilon}{q} = \overline{\mu} q^{-1} e^{-\frac{\pi}{2nq}}$, we have

$$\begin{aligned}
&v_0^{\text{in}}(r_0;k,q) - v_0^{\text{out}}(r_0;k,q)\\
&\quad = -qn^2\frac{1+k^2}{r_0}\log r_0 + q\frac{C_n}{r_0} - q\frac{k^2}{2}r_0 + \frac{n}{r_0}\Big(nq\log r_0 + nq\log\Big(\frac{\overline{\mu}}{2}\Big) - \theta_{0,nq}\Big)\\
&\qquad + q\mathcal{O}\Big(\frac{\log r_0}{r_0^3}\Big) + qk^2\mathcal{O}(r_0^{-1}) + \frac{1}{r_0}\mathcal{O}\Big(\Big|nq\log r_0 + nq\log\Big(\frac{\overline{\mu}}{2}\Big) - \theta_{0,nq}\Big|^3, q^2\Big)\\
&\quad = -\frac{n^2k^2\rho}{r_0} + \frac{q}{r_0}\Big(C_n + n^2\log\Big(\frac{\overline{\mu}}{2}\Big) - n\theta_{0,nq}q^{-1}\Big) - q\frac{k^2}{2}r_0 + q^3\mathcal{O}\Big(\frac{(\log r_0)^3}{r_0}\Big)\\
&\qquad + \frac{1}{r_0}\mathcal{O}(q^2, qk^2)\\
&\quad = \frac{q}{r_0}\Big(C_n + n^2\log\Big(\frac{\overline{\mu}}{2}\Big) - n\theta_{0,nq}q^{-1}\Big) + \frac{q}{r_0}\mathcal{O}(|\log q|^{-1}).
\end{aligned} \tag{4.26}$$

Therefore, the only possibility for $\overline{\mu}$ to solve $v_0^{\text{in}}(r_0;k,q) - v_0^{\text{out}}(r_0;k,q) = 0$ is that

$$C_n + n^2\log\Big(\frac{\overline{\mu}}{2}\Big) - n\theta_{0,nq}q^{-1} = \mathcal{O}(|\log q|^{-1}) \Leftrightarrow \overline{\mu} = 2e^{-\frac{C_n}{n^2} - \gamma + \mathcal{O}(|\log q|^{-1})},$$

where we have used $\theta_{0,nq} = -\gamma nq + \mathcal{O}(q^2)$, or equivalently

$$\overline{\mu} = 2e^{-\frac{C_n}{n^2} - \gamma}(1 + \mathcal{O}(|\log q|^{-1})).$$

This last equality suggests that the parameter $\overline{\mu}$ has to belong to $[\mu_0, \mu_1]$ with, for instance,

$$\mu_0 = e^{-\frac{C_n}{n^2} - \gamma}, \quad \mu_1 = 3e^{-\frac{C_n}{n^2} - \gamma}. \tag{4.27}$$

For any $\overline{\mu} \in [\mu_0, \mu_1]$, we introduce now the (independent of η) function

$$\Delta_0(r;k,q) = f_0^{\text{in}}(r) - f_0^{\text{out}}(r;k,q) + g_0^{\text{in}}(r;k,q) - g_0^{\text{out}}(r;k,q). \tag{4.28}$$

Then $\mathbf{a}_0$, $\mathbf{b}_0$ satisfying (4.24) and (4.25) are given by

$$\begin{pmatrix}\mathbf{a}_0\\ \mathbf{b}_0\end{pmatrix} = \frac{1}{d(r_0)}\begin{pmatrix} I'(r_0\sqrt{2})\Delta_0(r_0;k,q) - \frac{1}{\sqrt{2}}I(r_0\sqrt{2})\Delta_0'(r_0;k,q)\\ K_0'(r_0\sqrt{2})\Delta_0(r_0;k,q) - \frac{1}{\sqrt{2}}K_0(r_0\sqrt{2})\Delta_0'(r_0;k,q)\end{pmatrix} \tag{4.29}$$

with $d(r_0) = K_0(r_0\sqrt{2})I'(r_0\sqrt{2}) - K_0'(r_0\sqrt{2})I(r_0\sqrt{2})$.

We first notice that by property (4.16) of the function I and using the asymptotic expansion in Table 1 for $K_0(r)$ and $K_n(r)$ for $r \gg 1$, there exists a constant $\widehat{M}_1$ such that

$$0 < \frac{1}{d(r_0)} = r_0\sqrt{2}\Big(1 + \mathcal{O}\Big(\frac{1}{r_0}\Big)\Big) \le r_0\sqrt{2} + \widehat{M}_1. \tag{4.30}$$

Now we estimate Δ_0. We first note that, by estimate (4.3) of v_0^{out}, if q is small enough,

$$|v_0^{\mathrm{out}}(r_0;k,q)| \le \widehat{M}_2 \frac{\rho}{r_0} \le \frac{1}{4}$$

with a constant $\widehat{M}_2$ only depending on μ_0, μ_1. Then, by item (i) of Proposition 4.4 along with definition (3.15) of f_0^{out}, we have that, for q small enough,

$$\begin{aligned}
&|f_0^{\mathrm{in}}(r_0) - f_0^{\mathrm{out}}(r_0;k,q)| \\
&\quad \le \left|1 - \frac{n^2}{2r_0^2} - \sqrt{1 - (v_0^{\mathrm{out}}(r_0;k,q))^2 - \frac{n^2}{r_0^2}}\right| + \left|f_0^{\mathrm{in}}(r_0) - 1 + \frac{n^2}{2r_0^2}\right| \\
&\quad \le \widehat{M}_3 |v_0^{\mathrm{out}}(r_0,k)|^2 + \frac{\widehat{M}_4}{r_0^4} \le \widehat{M}_5 \frac{\rho^2}{r_0^2}.
\end{aligned}$$

The constant $\widehat{M}_5$ only depends on μ_0, μ_1. Therefore, by bounds (4.9) and (4.18) in Theorems (4.3) and (4.5),

$$\begin{aligned}
|\Delta_0(r_0;k,q)| &\le |f_0^{\mathrm{in}}(r_0) - f_0^{\mathrm{out}}(r_0;k,q)| + |g_0^{\mathrm{in}}(r_0;k,q)| + |g_0^{\mathrm{out}}(r_0;k,q)| \\
&\le \widehat{M}_5 \frac{\rho^2}{r_0^2} + M_0 q^2 \frac{|\log r_0|^2}{r_0^2} + M_0 \frac{\varepsilon^{1-\alpha}}{q r_0^2} \le \widehat{M}_6 \frac{\rho^2}{r_0^2},
\end{aligned} \tag{4.31}$$

where we have used that

$$r_0^{-1} = \varepsilon^{1-\alpha} = e^{-\frac{\rho}{q}} = e^{-\frac{1}{q^{2/3}|\log(q)|^{1/3}}} = \mathcal{O}(q^{\ell}), \quad \text{for any } \ell > 0.$$

Moreover, since, as established in Theorems 4.3 and 4.5, for $0 < q \le q_0^*(\mu_0,\mu_1)$, M_0 only depends on μ_0, μ_1, again, the same happens to $\widehat{M}_6$. Analogously, one can check that if $0 < q \le q_0^*(\mu_0,\mu_1)$, then

$$|\partial_r \Delta_0(r_0;k,q)| \le \widehat{M}_7 \frac{\rho^2}{r_0^2}. \tag{4.32}$$

By using estimates (4.30), (4.31) and (4.32), estimates (4.16) of I and the fact that if $r \gg 1$, one has $|K_0(r\sqrt{2})|, |K_0'(r\sqrt{2})| \le M_K e^{-r\sqrt{2}} r^{-\frac{1}{2}}$, we have, as $k = \overline{\mu} q^{-1} e^{-\frac{\pi}{2nq}}$ with $\overline{\mu} \in [\mu_0,\mu_1]$, the solution $(\mathbf{a}_0,\mathbf{b}_0)$ of (4.29) has to satisfy, for q small enough,

$$\begin{aligned}
|\mathbf{a}_0| &\le \rho^2 \frac{1}{r_0^{\frac{3}{2}}} e^{r_0\sqrt{2}} (\sqrt{2} + \widehat{M}_1 r_0^{-1}) M_I \Big[\widehat{M}_6 + \frac{1}{\sqrt{2}} \widehat{M}_7\Big], \\
|\mathbf{b}_0| &\le \rho^2 \frac{1}{r_0^{\frac{3}{2}}} e^{-r_0\sqrt{2}} (\sqrt{2} + \widehat{M}_1 r_0^{-1}) M_K \Big[\widehat{M}_6 + \frac{1}{\sqrt{2}} \widehat{M}_7\Big].
\end{aligned}$$

Taking q small enough, $M_1 r_0^{-1} \leq \sqrt{2}$ and defining

$$\widehat{M} = 2\sqrt{2}\Big[\widehat{M}_6 + \frac{1}{\sqrt{2}}\widehat{M}_7\Big]\max\{M_I, M_K\},$$

we conclude that there exist $q_1^* = q_1^*(\mu_0, \mu_1)$ and $\widehat{M}(q_1^*)$ such that for $0 < q < q_1^*$,

$$|\mathbf{a}_0| \leq \widehat{M}\rho^2 r_0^{-\frac{3}{2}} e^{r_0\sqrt{2}}, \quad |\mathbf{b}_0| \leq \widehat{M}\rho^2 r_0^{-\frac{3}{2}} e^{-r_0\sqrt{2}},$$

where ρ is given in (4.22). Then, defining $\widehat{\mathbf{a}}_0$ and $\widehat{\mathbf{b}}_0$ as

$$\mathbf{a}_0 = \widehat{\mathbf{a}}_0 r_0^{-\frac{3}{2}} e^{r_0\sqrt{2}}, \quad \mathbf{b}_0 = \widehat{\mathbf{b}}_0 \rho^2 r_0^{-\frac{3}{2}} e^{-r_0\sqrt{2}},$$

we finish the proof. ∎

Proposition 4.6 provides the values $\mathbf{a}_0$, $\mathbf{b}_0$, $\overline{\mu}$ which we expect will be good candidates of the approximated values for the solutions $\mathbf{a}$, $\mathbf{b}$, μ of the matching equations (4.23). In particular, we set the constants μ_0, μ_1 in (4.27).

Now we are going to set the constant η. We note that, since $r_0 = r_1 = r_2$, the constants $\mathbf{a}_0$, $\mathbf{b}_0$, provided by Proposition 4.6 satisfy conditions (4.6) and (4.14) in Theorems 4.3 and 4.5 for any $\eta \geq (\sqrt{2})^{\frac{3}{2}}\widehat{M}$. Since $\mathbf{a}_0$, $\mathbf{b}_0$ have to belong to the set of parameters $\mathbf{a}$, $\mathbf{b}$ for which Theorems 4.3 and 4.5 hold true, and some room for our perturbative analysis is needed, we may set η any value strictly bigger than $(\sqrt{2})^{\frac{3}{2}}\widehat{M}$, for instance,

$$\eta = 2\widehat{M}. \tag{4.33}$$

With this choice of η, the constants $\widehat{\mathbf{a}}_0$, $\widehat{\mathbf{b}}_0$ satisfy

$$|\widehat{\mathbf{a}}_0| \leq \frac{\eta}{2}\rho^2 \leq \frac{\eta}{2}, \quad |\widehat{\mathbf{b}}_0| \leq \frac{\eta}{2}. \tag{4.34}$$

4.5. *Matching the outer and inner solutions: End of the proof of Theorem 2.5*

The main goal of this section is to obtain the parameters $\mathbf{a}$, $\mathbf{b}$ (in fact, $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$) and μ which solve the matching equations (4.23). Having solved these equations, which is the content of next Theorem 4.7, we have a value of μ, and therefore of k as defined in (3.22), for which the original system (2.6) has a solution (f, v) satisfying the required boundary conditions (3.1). Once this result is proven, in order to prove Theorem 2.5 it will only remain to check that f is a positive increasing function and that $v < 0$ (see Proposition 4.8 below).

We begin our construction by considering the families of solutions provided by Theorems 4.3 and 4.5 for the constants μ_0, μ_1, η, fixed in the previous section (Section 4.4) and any values $\mathbf{a}$ and $\mathbf{b}$ satisfying (4.6) and (4.14). Namely, we consider $\mu \in [\mu_0, \mu_1]$, η, r_0, ρ and α as given in (4.27), (4.33), (4.21), (4.22), and (4.19), respectively, and $q \in [0, q_0]$. Along this section, we call q_0 the minimum value provided by all the previous results, that is, Propositions 4.2, 4.4, and 4.6 and Theorems 4.3, 4.5.

Next theorem gives the desired result.

Theorem 4.7. *Take* $\mu_0 = e^{-\frac{C_n}{n^2}-\gamma}$, $\mu_1 = 3e^{-\frac{C_n}{n^2}-\gamma}$, *where* C_n *and* γ *are given in Theorem 2.5 and* η *as given in (4.33). Then, there exists* q^* *such that for* $q \in [0, q^*]$ *equations (4.23) have a solution* $\mathbf{a}(q)$, $\mathbf{b}(q)$, $k(q)$ *satisfying (4.6) and (4.14) and* $k(q) = \mu e^{-\frac{\pi}{2nq}}$ *with* $\mu \in [\mu_0, \mu_1]$.

In addition,

$$|\mathbf{a}(q)| \le \eta\rho^2 e^{r_0\sqrt{2}} r_0^{-\frac{3}{2}}, \quad |\mathbf{b}(q)| \le \eta\rho^2 e^{-r_0\sqrt{2}} r_0^{-\frac{3}{2}},$$

and

$$\mu = \mu(q) = 2e^{-\frac{C_n}{n^2}-\gamma}(1 + \mathcal{O}(|\log q|^{-1})).$$

Proof. We define, as in Theorems 4.3 and 4.5, the parameters

$$\hat{\mathbf{a}} := \mathbf{a}\, e^{-r_0\sqrt{2}} r_0^{\frac{3}{2}}, \quad \hat{\mathbf{b}} := \mathbf{b}\, e^{r_0\sqrt{2}} r_0^{\frac{3}{2}} \rho^{-2}, \tag{4.35}$$

satisfying

$$|\hat{\mathbf{a}}|, |\hat{\mathbf{b}}| \le \eta.$$

We impose that $v^{\text{in}}(r_0, \mathbf{b}; k, q) = v^{\text{out}}(r_0, \mathbf{a}; k, q)$ or equivalently

$$v_0^{\text{in}}(r_0; k, q) - v_0^{\text{out}}(r_0; k, q) = w^{\text{out}}(r_0, \mathbf{a}; k, q) - w^{\text{in}}(r_0, \mathbf{b}; k, q). \tag{4.36}$$

On the one hand, by the results involving w^{out}, w^{in} in Theorems 4.3 and 4.5 we have

$$\begin{aligned} |w^{\text{out}}(r_0, \mathbf{a}; k, q) - w^{\text{in}}(r_0, \mathbf{b}; k, q)| &\le |w^{\text{out}}(r_0; k, q)| + |w^{\text{in}}(r_0; k, q)| \\ &\le M\frac{1}{qr_0^2} + Mq^3\frac{|\log r_0|^3}{r_0} \\ &\le M\frac{1}{qr_0^2} + M\frac{\rho^3}{r_0^2} \\ &\le M\frac{1}{qr_0^2}. \end{aligned}$$

On the other hand, by (4.26),

$$v_0^{\text{in}}(r_0; k, q) - v_0^{\text{out}}(r_0; k, q) = \frac{q}{r_0}\Big(C_n + n^2\log\Big(\frac{\mu}{2}\Big) - n\theta_{0,nq}q^{-1} + \mathcal{O}|\log q|^{-1}\Big).$$

Therefore, since $\theta_{0,nq} = -\gamma nq + \mathcal{O}(q^2)$, $v^{\text{in}}(r_0, \mathbf{b}; k, q) = v^{\text{out}}(r_0, \mathbf{a}; k, q)$ (or equivalently equality (4.36) holds true) if and only if

$$\log\Big(\frac{\mu}{2}\Big) = -\frac{C_n}{n^2} - \gamma + \mathcal{C}_3(\mathbf{a}, \mathbf{b}, k; q), \quad |\mathcal{C}_3(\mathbf{a}, \mathbf{b}, k; q)| \le M|\log q|^{-1},$$

where $\mathcal{C}_3$ contains the remaining terms of $v_0^{\text{in}} - v_0^{\text{out}}$ and $w^{\text{out}} - w^{\text{in}}$.

We recall definition (4.35) of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and introduce the function

$$\mathcal{H}_3(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) = 2e^{-\frac{C_n}{n^2}-\gamma}\big[e^{\mathcal{C}_3(\hat{\mathbf{a}}e^{r_0\sqrt{2}}r_0^{-\frac{3}{2}}, \hat{\mathbf{b}}e^{-r_0\sqrt{2}}r_0^{\frac{3}{2}}\rho^2, \mu q^{-1}e^{-\frac{\pi}{2nq}}; q)} - 1\big]$$

which, from Theorems 4.3 and 4.5, is continuous with respect to $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, μ. It is clear that equation (4.36) is satisfied if and only if

$$\mu = 2e^{-\frac{C_n}{n^2}-\gamma} + \mathcal{H}_3(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q), \quad |\mathcal{H}_3(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q)| \le M \le M|\log q|^{-1}. \tag{4.37}$$

We deal now with the (nonlinear) system,

$$f^{\text{out}}(r_0; k, q) = f^{\text{in}}(r_0; k, q), \quad \partial_r f^{\text{out}}(r_0; k, q) = \partial_r f^{\text{in}}(r_0; k, q),$$

which can be rewritten, using expressions for f^{out}, f^{in} in Theorems 4.3 and 4.5 as

$$\begin{aligned} K_0(r_0\sqrt{2})\mathbf{a} - I(r_0\sqrt{2})\mathbf{b} &= \Delta(r_0, \mathbf{a}, \mathbf{b}; k, q) = \Delta_0(r_0; k, q) + \Delta_1(r_0, \mathbf{a}, \mathbf{b}; k, q), \\ K_0'(r_0\sqrt{2})\mathbf{a} - I'(r_0\sqrt{2})\mathbf{b} &= \frac{1}{\sqrt{2}}\partial_r\Delta(r_0, \mathbf{a}, \mathbf{b}; k, q) \\ &= \frac{1}{\sqrt{2}}(\partial_r\Delta_0(r_0; k, q) + \partial_r\Delta_1(r_0, \mathbf{a}, \mathbf{b}; k, q)) \end{aligned}$$

with Δ_0 defined in (4.28) and

$$\Delta_1(r, \mathbf{a}, \mathbf{b}; k, q) = g_1^{\text{in}}(r, \mathbf{b}; k, q) - g_1^{\text{out}}(r, \mathbf{a}; k, q).$$

Therefore, $\mathbf{a}$, $\mathbf{b}$ satisfy the fixed point equation

$$\begin{aligned} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} &= \begin{pmatrix} \mathcal{C}_1(\mathbf{a}, \mathbf{b}, k; q) \\ \mathcal{C}_2(\mathbf{a}, \mathbf{b}, k; q) \end{pmatrix} \\ &:= \frac{1}{d(r_0)} \begin{pmatrix} I'(r_0\sqrt{2})(\Delta(r_0, \mathbf{a}, \mathbf{b}; k, q)) - \frac{1}{\sqrt{2}} I(r_0\sqrt{2})\partial_r\Delta(r_0, \mathbf{a}, \mathbf{b}; k, q) \\ -K_0'(r_0\sqrt{2})\Delta(r_0, \mathbf{a}, \mathbf{b}; k, q) + \frac{1}{\sqrt{2}} K_0(r_0\sqrt{2})\partial_r\Delta(r_0, \mathbf{a}, \mathbf{b}; k, q) \end{pmatrix}. \end{aligned} \tag{4.38}$$

Using the estimates in Theorems 4.3 and 4.5 for g_1^{out}, g_1^{in}, we obtain

$$|\Delta_1(r_0, \mathbf{a}, \mathbf{b}; k, q)| \le |g_1^{\text{in}}(r_0, \mathbf{b}; k, q)| + |g_1^{\text{out}}(r_0, \mathbf{a}; k, q)| \le M\frac{\rho^4}{r_0^2},$$

and $|r_0^2\partial_r\Delta_1(r_0, \mathbf{a}, \mathbf{b}; k, q)| \le M\rho^4$, for any $\mathbf{a}$ and $\mathbf{b}$ satisfying (4.6) and (4.14).

Recalling $\mathbf{a}_0$, $\mathbf{b}_0$ are defined in (4.29) and using the above bounds for Δ_1 and $\partial_r\Delta_1$ along with (4.16) and Table 1 for I and K_0 and bound (4.30) for $d(r_0)$, gives

$$\begin{aligned} |\mathcal{C}_1(\mathbf{a}, \mathbf{b}, k; q) - \mathbf{a}_0| &\le M e^{r_0\sqrt{2}}\rho^4 r_0^{-\frac{3}{2}}, \\ |\mathcal{C}_2(\mathbf{a}, \mathbf{b}, k; q) - \mathbf{b}_0| &\le M e^{-r_0\sqrt{2}}\rho^4 r_0^{-\frac{3}{2}}. \end{aligned} \tag{4.39}$$

Recalling the definition of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ in (4.35), we introduce

$$\begin{aligned} \mathcal{H}_1(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) &= e^{-r_0\sqrt{2}} r_0^{\frac{3}{2}} \mathcal{C}_1(\hat{\mathbf{a}} e^{r_0\sqrt{2}} r_0^{-\frac{3}{2}}, \hat{\mathbf{b}}\rho^2 e^{-r_0\sqrt{2}} r_0^{-\frac{3}{2}}, \mu q^{-1} e^{-\frac{\pi}{2nq}}; q) - \hat{\mathbf{a}}_0, \\ \mathcal{H}_2(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) &= e^{r_0\sqrt{2}} r_0^{\frac{3}{2}} \rho^{-2} \mathcal{C}_2(\hat{\mathbf{a}} e^{r_0\sqrt{2}} r_0^{-\frac{3}{2}}, \hat{\mathbf{b}}\rho^2 e^{-r_0\sqrt{2}} r_0^{-\frac{3}{2}}, \mu q^{-1} e^{-\frac{\pi}{2nq}}; q) - \hat{\mathbf{b}}_0, \end{aligned}$$

and the fixed point equation (4.38) becomes

$$\begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{a}}_0 + \mathcal{H}_1(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) \\ \hat{\mathbf{b}}_0 + \mathcal{H}_2(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) \end{pmatrix}. \tag{4.40}$$

We note that, by Theorems 4.3 and 4.5, $\mathcal{H}_{1,2}$ are continuous functions with respect to $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, μ. Using bound (4.39) of $\mathcal{C}_1$, $\mathcal{C}_2$

$$|\mathcal{H}_1(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q)| \le M\rho^4, \quad |\mathcal{H}_2(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q)| \le M\rho^4. \tag{4.41}$$

From (4.40) and (4.37), we have that the constants $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and μ must to satisfy the fixed point equation

$$(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu) = H(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) := (\hat{\mathbf{a}}_0, \hat{\mathbf{b}}_0, 2e^{\frac{-C_n}{n^2} - \gamma}) + \mathcal{H}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) \tag{4.42}$$

with $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$. We recall that as defined in (4.22), $\rho^3 = q|\log q|^{-1}$ and the constants μ_0, μ_1 and η were fixed at (4.27) and (4.33), respectively. The function $\mathcal{H}$ satisfies, for $|\hat{\mathbf{a}}|, |\hat{\mathbf{b}}| \le \eta$ and $\mu \in [\mu_0, \mu_1]$,

$$\|\mathcal{H}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q)\| \le \max\{M\rho q|\log q|^{-1}, M|\log q|^{-1}\} = M|\log q|^{-1}.$$

As a consequence, since $\hat{\mathbf{a}}_0$ and $\hat{\mathbf{b}}_0$ satisfy (4.34), for $|\hat{\mathbf{a}}|, |\hat{\mathbf{b}}| \le \eta$ and $\mu \in [\mu_0, \mu_1]$,

$$|H_{1,2}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q)| \le \frac{\eta}{2} + M|\log q|^{-1} \le \eta,$$

and, taking μ_0, μ_1 as defined in (4.27), one finds

$$H_3(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu; q) = 2e^{\frac{-C_n}{n^2} - \gamma} + \mathcal{O}(|\log q|^{-1}) \in [\mu_0, \mu_1].$$

Therefore, there exists q^* small enough such that, if $q \in (0, q^*]$, the map H sends the closed set

$$B = \{(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu) \in \mathbb{R}^3 : |\hat{\mathbf{a}}|, |\hat{\mathbf{b}}| \le \eta,\ \mu \in [\mu_0, \mu_1]\}$$

into itself and is continuous with respect to $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, μ. Therefore, the Brouwer's fixed point theorem provides the existence of the parameters $(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu) = (\hat{\mathbf{a}}(q), \hat{\mathbf{b}}(q), \mu(q))$, defined for any $q \in [0, q^*]$, satisfying the fixed point equation (4.42) and

$$|\hat{\mathbf{a}}| \le \eta, \quad |\hat{\mathbf{b}}| \le \eta, \quad \mu \in [\mu_0, \mu_1].$$

In addition, for this solution, using the bounds in (4.41) and (4.34), we have, for q small enough,

$$|\hat{\mathbf{a}}| \le |\hat{\mathbf{a}}_0| + |\mathcal{H}_1(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mu, q)| \le \frac{\eta}{2}\rho^2 + M\rho^4|\log q| \le \eta\rho^2,$$

and from (4.37),

$$|\mu - 2e^{-\frac{C_n}{n^2} - \gamma}| \le M|\log q|^{-1}.$$

Going back to the original constants $\mathbf{a}$ and $\mathbf{b}$ using (4.35) completes the proof. ■

By Theorem 4.7, we can define the solutions of (2.6) satisfying the boundary conditions (3.1) as in (3.4):

$$\begin{aligned}&(f(r;q),v(r;q))\\&:=\begin{cases}(f^{\mathrm{in}}(r,\mathbf{b}(q);k(q),q),v^{\mathrm{in}}(r,\mathbf{b}(q);k(q),q)), & r\in[0,r_0],\\(f^{\mathrm{out}}(r,\mathbf{a}(q);k(q),q),v^{\mathrm{out}}(r,\mathbf{a}(q);k(q),q)), & r\geq r_0.\end{cases}\end{aligned}\tag{4.43}$$

Therefore, in order to prove Theorem 2.5 it only remains to check the additional properties on the solution (f,v).

Proposition 4.8. *Let* $(f(r;q),v(r;q))$ *be the solution of* (2.6) *defined by* (4.43)*. There exists* q^* *such that, for* $q\in[0,q^*]$ *and* $r>0$*,* $f(r;q)$ *is an increasing function,*

$$0<f(r;q)<\sqrt{1-k^2(q)},\quad v(r;q)<0.$$

Proof. We first prove that $f(r;q)>0$ for $r>0$. We start with the *outer region*. In item (iv) of Proposition 4.2, we proved that $f_0^{\mathrm{out}}(r;k(q),q)\geq\frac{1}{2}$ for $r\geq r_0=\frac{1}{\sqrt{2}}e^{\frac{\rho}{q}}$. Therefore, by Theorem 4.3, when $r\geq r_0$,

$$\begin{aligned}f(r;q)&\geq f_0^{\mathrm{out}}(r;k(q),q)-|g^{\mathrm{out}}(r,\mathbf{a}(q);k(q),q)|\geq\frac{1}{2}-Mr^{-2}\\&\geq\frac{1}{2}-Mr_0^{-2}>0.\end{aligned}\tag{4.44}$$

In the *inner region*, using item (i) of Proposition 4.4 and Theorem 4.5 we deduce that there exists ϱ small enough but independent of q such that if $r\in[0,\varrho]$,

$$f(r;q)=f_0^{\mathrm{in}}(r)+g^{\mathrm{in}}(r,\mathbf{b}(q);k(q),q)=c_f r^n+o(r^n)+q^2\mathcal{O}(r^n)>0$$

since the constant c_f is positive. Then, since f_0^{in} is positive, increasing and independent of q, again using Theorem 4.5, for $\varrho\leq r\leq r_0$,

$$f(r;q)\geq f_0^{\mathrm{in}}(\varrho)-|g^{\mathrm{in}}(r,\mathbf{b}(q);k(q),q)|\geq f_0^{\mathrm{in}}(\varrho)+\mathcal{O}(q^2)>0,$$

if q is small enough. This finishes the proof of f being positive.

Now we check that $f(r;q)<\sqrt{1-k^2(q)}$. We first note that Theorem 4.7 can be used to bound $\mathbf{a}(q)$. Therefore, by (4.8), (4.9) and (4.10) in Theorem 4.3 we have $g(r;q):=f(r;q)-f_0^{\mathrm{out}}(r,\mathbf{a}(q);k(q),q)$ satisfies, for $r\geq r_0$,

$$\begin{aligned}|r^2g(r;q)|&\leq|r^2\mathbf{a}(q)K_0(r)|+M\varepsilon^{1-\alpha}q^{-1}\leq\rho^2\eta e^{-\sqrt{2}(r-r_0)}r^{\frac{3}{2}}r_0^{-\frac{3}{2}}+M\varepsilon^{1-\alpha}q^{-1}\\&\leq M\rho^2,\end{aligned}$$

where we have used that, from definition (4.22) of ρ, $\varepsilon^{1-\alpha}q^{-1}=q^{-1}\sqrt{2}e^{-\frac{\rho}{q}}\ll\rho^2$ and the asymptotic expansion when $r\gg1$ in Table 1 for the Bessel function K_0. Therefore,

$$f(r;q)\leq\sqrt{1-(v_0^{\mathrm{out}}(r;k(q),q))^2-\frac{n^2}{r^2}}+M\rho^2\frac{1}{r^2}\leq\sqrt{1-(v_0^{\mathrm{out}}(r;k(q),q))^2}-M\frac{1}{r^2},$$

where we have used $v_0^{\mathrm{out}}(r;k(q),q) \le M r_0^{-1} = M\varepsilon^{1-\alpha} \ll 1$ and $\rho \ll 1$. Then, $f(r;q) \le \sqrt{1-(v_0^{\mathrm{out}}(r;k(q),q))^2}$ and as a consequence, since $v_0^{\mathrm{out}} \to -k(q)$ as $r \to \infty$ and it is increasing and negative (see item (iii) in Proposition 4.2), we have

$$f(r;q) \le \sqrt{1-k^2(q)}, \quad r \ge r_0.$$

With respect to the *inner region*, namely $r \in [0, r_0]$, using Proposition 4.4 there exists $\varrho \gg 1$ independent on q such that for all $\varrho \le r \le r_0$, $(f_0^{\mathrm{in}})^2(r) \le 1 - \frac{n^2}{2r^2}$. Then, since by Theorem 4.5, $|g^{\mathrm{in}}(r, \mathbf{b}; k, q)| \le Mq^2 |\log r|^2 r^{-2}$ for $\varrho \le r \le r_0$ we have

$$f^2(r;q) \le 1 - \frac{n^2}{2r^2} + M\rho^2 \frac{1}{r^2} \le 1 - \frac{1}{2r_0^2}(n^2 - M\rho^2) \le 1 - M\varepsilon^{2(1-\alpha)},$$

where we have used again definition (4.22) of ρ and that $r_0 = \varepsilon^{\alpha-1} = \frac{1}{\sqrt{2}} e^{\frac{\rho}{q}}$ (see (4.21)). Then, using definition (4.22) of ρ, we conclude that $1 - M\varepsilon^{2(1-\alpha)} \le 1 - k^2(q)$, taking if necessary q small enough. As a consequence, $f(r;q) \le \sqrt{1-k^2(q)}$ if $\varrho \le r \le r_0$. It remains to check the property when $r \in [0, \varrho]$. From the fact that $f_0^{\mathrm{in}}(r)$ is an increasing function and using Theorem 4.5,

$$f(r;q) = f_0^{\mathrm{in}}(r) + g^{\mathrm{in}}(r; \mathbf{b}(q); k(q), q) \le f_0^{\mathrm{in}}(\varrho) + Mq^2 < \sqrt{1-k^2(q)},$$

provided $f_0^{\mathrm{in}}(\varrho) < 1$, ϱ is independent on q, and q is small enough.

The negativeness of $v(r;q) < 0$ for $r > 0$ is straightforward from the previous property, $f(r;q) < \sqrt{1-k^2(q)}$. Indeed, using that $v(0;q) = 0$, from the differential equations (2.6), we have

$$v(r;q) = -q\frac{1}{rf^2(r;q)} \int_0^r \xi f^2(\xi;q)(1 - f^2(\xi;q) - k^2(q))\, \mathrm{d}\xi < 0.$$

To finish, we prove that $\partial_r f(r;q) > 0$. We start with the *inner region*. From Proposition 4.4, there exist $0 < \varrho_0 \ll \varrho_1$ satisfying

$$\partial_r f_0^{\mathrm{in}}(r) \ge \frac{n}{2} c_f r^{n-1} \ \text{ if } r \in [0, \varrho_0] \quad \text{and} \quad \partial_r f_0^{\mathrm{in}}(r) \ge \frac{n^2}{2r^3} \ \text{ if } r \in [\varrho_1, r_0].$$

Let $\overline{\varrho} \in [\varrho_0, \varrho_1]$ be such that $\partial_r f_0^{\mathrm{in}}(r) \ge \partial_r f_0^{\mathrm{in}}(\overline{\varrho}) > 0$ for all $r \in [\varrho_0, \varrho_1]$. Notice that the values of ϱ_0, ϱ_1 and $\overline{\varrho}$ are independent on q. Therefore, using Theorem 4.5, if $r \in [0, \varrho_0]$,

$$\partial_r f(r;q) = \partial_r f_0^{\mathrm{in}}(r) + \partial_r g^{\mathrm{in}}(r, \mathbf{b}(q); k(q), q) \ge \frac{n}{2} c_f r^{n-1} - Mq^2 r^{n-1} > 0.$$

When $r \in [\varrho_0, \varrho_1]$,

$$\partial_r f(r;q) = \partial_r f_0^{\mathrm{in}}(r) + \partial_r g^{\mathrm{in}}(r, \mathbf{b}(q); k(q), q) \ge \partial_r f_0^{\mathrm{in}}(\overline{\varrho}) - Mq^2 > 0,$$

taking, if necessary, q small enough. When $r \ge \varrho_1$, Theorem 4.5 says

$$\partial_r f(r;q) \ge \frac{n^2}{2r^3} - Mq^2 \frac{|\log r|^2}{r^2},$$

that is positive if $\varrho_1 \leq r \leq q^{-2}|\log q|^{-3}$, if q small enough. In conclusion,

$$\partial_r f(r;q) > 0, \quad 0 \leq r \leq \frac{1}{q^2|\log q|^3}.$$

To see that $\partial_r f(r;q) > 0$ for bigger values of r, we first need to check

$$f(r;q) \geq \frac{1}{\sqrt{3}}, \quad r \geq \frac{1}{q^2|\log q|^3}. \tag{4.45}$$

Indeed, if $q^{-2}|\log q|^{-3} \leq r \leq r_0$, that is, when r belongs to the *inner region*, from Theorem 4.5

$$\begin{aligned} f(r;q) &\geq f_0^{\text{in}}(r) - |g^{\text{in}}(r;\mathbf{b}(q);k(q),q)| \geq 1 - \frac{n^2}{2r^2} + \mathcal{O}(r^{-4}) - Mq^2\frac{|\log r|^2}{r^2} \\ &\geq 1 - \mathcal{O}(q^2|\log q|^3) \geq \frac{1}{\sqrt{3}}. \end{aligned}$$

When $r \geq r_0$ (that is, in the *outer region*), by (4.44), $f(r;q) \geq \frac{1}{3}$ and (4.45) is proven.

We finish the argument by proving that f is an increasing function for $r > 0$, by contradiction. Since we have proved that $\partial_r f(r;q) > 0$ for $r > q^{-2}|\log q|^{-3}$ and $f^2(r;q) \leq 1 - k^2(q) = \lim_{r\to\infty} f^2(r;q)$, if f has an extreme at r^*, it has to have a maximum at some point less than r^*. Let $r_* \geq q^{-2}|\log q|^{-3}$ be the minimum value such that $f(r;q)$ has a maximum at $r = r_*$. That is, $\partial_r f(r_*,q) = 0$ and $\partial_r^2 f(r_*;q) \leq 0$. Therefore, since f is a solution of (2.6), we deduce

$$f(r_*;q)\Big[-\frac{n^2}{r_*^2} + (1 - f^2(r_*;q) - v^2(r_*;q))\Big] \geq 0. \tag{4.46}$$

Now we use the following comparison result (see [28]).

Lemma 4.9 ([28]). *Let (a,b) be an interval in $\mathbb{R}$, let $\Omega = \mathbb{R}^2 \times (a,b)$, and let $\mathcal{H} \in \mathcal{C}^1(\Omega,\mathbb{R})$. Suppose $h \in \mathcal{C}^2((a,b))$ satisfies $h''(r) + \mathcal{H}(h'(r),h(r),r) = 0$. If $\partial_h \mathcal{H} \leq 0$ on Ω and if there exist functions $M, m \in \mathcal{C}^2((a,b))$ satisfying*

$$M''(r) + \mathcal{H}(M'(r),M(r),r) \leq 0 \quad \text{and} \quad m''(r) + \mathcal{H}(m'(r),m(r),r) \geq 0,$$

as well as the boundary conditions $m(a) \leq h(a) \leq M(a)$ and $m(b) \leq h(b) \leq M(b)$, then for all $r \in (a,b)$ we have $m(r) \leq h(r) \leq M(r)$.

We set $(a,b) = (r_*,\infty)$ and define

$$\mathcal{H}(h',h,r) = \frac{h'}{r} - h\frac{n^2}{r^2} + h(1 - h^2 - v^2(r;q)), \quad h \geq \frac{1}{\sqrt{3}},$$

with $v(r;q)$ the solution we have already found, and

$$\mathcal{H}(h',h,r) = \frac{h'}{r} - h\frac{n^2}{r^2} - hv^2(r;q) + \frac{2}{3\sqrt{3}}, \quad h \leq \frac{1}{\sqrt{3}}.$$

We have $\mathcal{H} \in \mathcal{C}^1(\Omega, \mathbb{R})$ and $\partial_h \mathcal{H} \leq 0$. According to (4.45), for $r \geq r_* \geq q^{-2}|\log q|^{-3}$, $f(r;q) \geq \frac{1}{\sqrt{3}}$ so that $f(r;q)$ is a solution of $h'' + \mathcal{H}(h'(r), h(r), r) = 0$. Taking $m(r) = f(r_*;q)$, we have

$$\lim_{r\to\infty} m(r) = f(r_*;q) \leq \lim_{r\to\infty} f(r;q) = \sqrt{1-k^2}$$

and

$$\begin{aligned} m'' + \mathcal{H}(m'(r), m(r), r) &= -f(r_*;q)\frac{n^2}{r^2} + f(r_*;q)(1 - f^2(r_*;q) - v^2(r_*;q)) \\ &\geq -f(r_*;q)\frac{n^2}{r_*^2} + f(r_*;q)(1 - f^2(r_*;q) - v^2(r_*;q)) \geq 0, \end{aligned}$$

where we have used the bound in (4.46) for the last inequality. Then Lemma 4.9 concludes that $f(r_*;q) = m(r) \leq f(r;q)$ for $r \geq r_*$. Therefore, r_* is not a maximum and we have a contradiction. ∎

The rest of the work is devoted to proving the results about the existence of families of solutions in the outer and inner regions. From now on, to avoid cumbersome notation, we will skip the dependence on the parameters k, q.

5. Existence result in the outer region. Proof of Theorem 4.3

In this section, we prove Theorem 4.3. To do so, by means of a fixed point equation setting, we look for solutions of equations (3.9) which are written in the *outer variables* introduced in Section 3.1 (see (3.7)). Namely, we look for solutions of equations (3.9) with boundary conditions (3.10) of the form $F_0 + G$, $V_0 + W$ with F_0, V_0 defined in (3.12) and (3.14), respectively, that is, taking $\varepsilon = kq$,

$$V_0(R) = \frac{K'_{inq}(R)}{K_{inq}(R)}, \quad F_0(R) = \sqrt{1 - k^2 V_0^2(R) - \frac{\varepsilon^2 n^2}{R^2}}. \tag{5.1}$$

We first introduce the Banach spaces we will work with. For any given $R_{\min} > 0$, we introduce the Banach spaces:

$$\mathcal{X}_\ell = \{f\colon [R_{\min}, \infty) \to \mathbb{R} \text{ continuous}, \ \|f\|_\ell := \sup_{R\in[R_{\min},\infty)} |R^\ell f(R)| < \infty\}, \tag{5.2}$$

being $\mathcal{X}_0$ the Banach space of continuous bounded functions with the supremum norm.

Notice that $\mathcal{X}_\ell = \mathcal{X}_\ell(R_{\min})$ depends on $R_{\min}$ and so the norm of a function in the space $\mathcal{X}_\ell$ also depends on $R_{\min}$. However, if $R_{\min} \leq R'_{\min}$, $\mathcal{X}_\ell(R_{\min}) \subset \mathcal{X}_\ell(R'_{\min})$ and

$$\sup_{R\in[R_{\min},\infty)} |R^\ell f(R)| \geq \sup_{R\in[R'_{\min},\infty)} |R^\ell f(R)|.$$

This fact allows us to take $R'_{\min} \geq R_{\min}$, if we are working in $\mathcal{X}_\ell(R_{\min})$. We will use this property along the work without any special mention.

5.1. *The fixed point equation*

Our goal in this subsection is to transform equations (3.9a), (3.9b) into a fixed point equation in suitable Banach spaces. For that, the first step is to write such equations in a suitable way.

Let $F = F_0 + G$ and $V = V_0 + W$. The term $F(1 - F^2 - k^2V^2)$ in equation (3.9a) is the following:

$$F(1 - F^2 - k^2V^2) = -2F_0^2G - 3F_0G^2 - G^3 \\ - Wk^2[2V_0F_0 + F_0W + 2V_0G + WG] + (F_0 + G)\frac{n^2\varepsilon^2}{R^2}.$$

Therefore, equation (3.9a) becomes

$$\varepsilon^2\Big(G'' + \frac{G'}{R}\Big) - 2F_0^2(R)G = -\varepsilon^2\Big(F_0''(R) + \frac{F_0'(R)}{R}\Big) + 3F_0(R)G^2 + G^3 \\ + Wk^2[2V_0(R)F_0(R) + F_0(R)W + 2V_0(R)G + WG].$$

In view of (4.2), which in *outer variables* reads as

$$F_0(R) = \sqrt{1-k^2}\Big(1 - \frac{k^2}{2R(1-k^2)} + \mathcal{O}\Big(\frac{k^2}{R^2}\Big)\Big),$$

we introduce

$$F_0^2(R) = 1 + \frac{1}{2}\widehat{F}_0(R). \tag{5.3}$$

Therefore, we may write the above equation for G as

$$G'' + \frac{G'}{R} - G\frac{2}{\varepsilon^2} = -\varepsilon^{-2}\mathcal{N}_1[G, W] \tag{5.4}$$

with

$$\mathcal{N}_1[G, W](R) = \varepsilon^2\Big(F_0''(R) + \frac{F_0'(R)}{R}\Big) - \widehat{F}_0(R)G - 3F_0(R)G^2 - G^3 \\ - Wk^2(2V_0(R)F_0(R) + F_0(R)W + 2V_0(R)G + WG). \tag{5.5}$$

Now we compute the equation for W from (3.9b). We have

$$W' + \frac{W}{R} + 2V_0(R)W + W^2 + V_0'(R) + \frac{V_0(R)}{R} + V_0^2(R) - 1 + \frac{n^2}{R^2}q^2 \\ = \frac{q^2}{F_0(R) + G}\Big(F_0''(R) + \frac{F_0'(R)}{R} + G'' + \frac{G'}{R}\Big) - 2(V_0(R) + W)\frac{F_0'(R) + G'}{F_0(R) + G}.$$

We recall that V_0 is a solution of (3.13). Then

$$W' + \frac{W}{R} + 2V_0W = -\mathcal{N}_2(G, W)(R) \tag{5.6}$$

with

$$\mathcal{N}_2[G,W](R) = W^2 - \frac{q^2}{F_0(R)+G}\Big(F_0''(R) + \frac{F_0'(R)}{R} + G'' + \frac{G'(R)}{R}\Big) + 2(V_0(R)+W)\frac{F_0'(R)+G'}{F_0(R)+G}. \tag{5.7}$$

We define the linear operators

$$\mathcal{L}_1[G](R) = G'' + \frac{G'}{R} - G\frac{2}{\varepsilon^2}, \quad \mathcal{L}_2[W](R) = W' + \frac{W}{R} + 2V_0(R)W,$$

and rewrite equations (5.4) and (5.6) as

$$\mathcal{L}_1[G] = -\varepsilon^{-2}\mathcal{N}_1[G,W], \quad \mathcal{L}_2[W] = -\mathcal{N}_2[G,W]. \tag{5.8}$$

The strategy to prove the existence of solutions of (5.8) is to write them as fixed point equation and to prove that the fixed point theorem can be applied in suitable Banach spaces. For this, first, we need to compute a right inverse of $\mathcal{L}_1$, $\mathcal{L}_2$.

We start with $\mathcal{L}_1$. Assume we have

$$\mathcal{L}_1[G](R) = -h(R), \tag{5.9}$$

where h satisfies some conditions that we will specify later. We are interested in solutions of this equation such that $\lim_{R\to\infty} G(R) = 0$.

Just for doing computations, we perform the scaling

$$s = \frac{R}{\varepsilon}\sqrt{2}, \quad g(s) = G\Big(\frac{s\varepsilon}{\sqrt{2}}\Big),$$

and (5.9) is transformed into

$$g'' + \frac{g'}{s} - g = -\frac{\varepsilon^2}{2}h\Big(\frac{s\varepsilon}{\sqrt{2}}\Big). \tag{5.10}$$

The homogeneous linear system associated with equation (5.10) has a fundamental matrix

$$\begin{pmatrix} K_0(s) & I_0(s) \\ K_0'(s) & I_0'(s) \end{pmatrix},$$

where K_0, I_0 are the modified Bessel functions [1] of the second and first kind. The Wronskian is given by $W(K_0(s), I_0(s)) = s^{-1}$ so that the solutions of (5.10) are given by

$$g(s) = K_0(s)\Big[\mathbf{a} + \frac{\varepsilon^2}{2}\int_{s_0}^{s} \xi I_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\, \mathrm{d}\xi\Big] + I_0(s)\Big[\mathbf{b} - \frac{\varepsilon^2}{2}\int_{s_0}^{s} \xi K_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\, \mathrm{d}\xi\Big].$$

It is well known that $K_0(s) \to 0$ and $I_0(s) \to \infty$ as $s \to \infty$ (see Table 1). Then, in order to have solutions bounded as $s \to \infty$, we have to impose

$$\mathbf{b} - \frac{\varepsilon^2}{2}\int_{s_0}^{\infty} \xi K_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\, \mathrm{d}\xi = 0.$$

Therefore,

$$g(s) = K_0(s)\Big[\mathbf{a} + \frac{\varepsilon^2}{2}\int_{s_0}^{s} \xi I_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\,\mathrm{d}\xi\Big] + \frac{\varepsilon^2}{2} I_0(s)\int_s^{\infty} \xi K_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\,\mathrm{d}\xi,$$

and, proceeding in the same way,

$$g'(s) = K_0'(s)\Big[\mathbf{a} + \frac{\varepsilon^2}{2}\int_{s_0}^{s} \xi I_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\,\mathrm{d}\xi\Big] + \frac{\varepsilon^2}{2} I_0'(s)\int_s^{\infty} \xi K_0(\xi) h\Big(\frac{\xi\varepsilon}{\sqrt{2}}\Big)\,\mathrm{d}\xi.$$

Now we undo the change of variables, that is, $R = \frac{s\varepsilon}{\sqrt{2}}$ and $G(R) = g(\frac{R\sqrt{2}}{\varepsilon})$. We obtain the solution of (5.9)

$$\begin{aligned} G(R) = {} & K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\Big[\mathbf{a} + \int_{R_{\min}}^{R} \xi I_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big) h(\xi)\,\mathrm{d}\xi\Big] \\ & + I_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\int_R^{\infty} \xi K_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big) h(\xi)\,\mathrm{d}\xi \end{aligned}$$

with $R_{\min} = \frac{s_0\varepsilon}{\sqrt{2}}$ to be determined later.

We introduce the linear operator

$$\begin{aligned} \mathcal{S}_1[h](R) = {} & K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\int_{R_{\min}}^{R} \xi I_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big) h(\xi)\,\mathrm{d}\xi \\ & + I_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\int_R^{\infty} \xi K_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big) h(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{5.11}$$

We have proven the following lemma.

Lemma 5.1. *For any* $\mathbf{a} \in \mathbb{R}$, *we define*

$$\mathbf{G}_0(R) = K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\mathbf{a}. \tag{5.12}$$

Then, if G *is a solution of* (5.4) *satisfying* $G(R) \to 0$ *as* $R \to \infty$, *then there exists a constant* $\mathbf{a}$ *such that*

$$G = \mathbf{G}_0 + \mathcal{S}_1[\varepsilon^{-2}\mathcal{N}^{-1}[G, W]].$$

Now we compute the right inverse of $\mathcal{L}_2$. We consider the linear equation

$$\mathcal{L}_2[W] = W' + W\Big(\frac{1}{R} + 2V_0\Big) = h. \tag{5.13}$$

Since $V_0(R) = K_{inq}'(R)K_{inq}(R)$, the solutions are given by

$$W(R) = \frac{1}{RK_{inq}^2(R)}\Big(c_0 + \int_{R_0}^{R} \xi K_{inq}^2(\xi) h(\xi)\,\mathrm{d}\xi\Big)$$

for any constant c_0. In order for W to be bounded as $R \to \infty$, it is required that

$$c_0 + \int_{R_0}^{\infty} \xi K_{inq}^2(\xi) h(\xi)\,\mathrm{d}\xi = 0.$$

Therefore,

$$W(R) = \frac{1}{RK_{inq}^2(R)} \int_\infty^R \xi K_{inq}^2(\xi) h(\xi)\,\mathrm{d}\xi.$$

As a result, we have the following lemma.

Lemma 5.2. *Any solution of* (5.13) *bounded as* $R \to \infty$ *is of the form* $W = \mathcal{S}_2[h]$ *with*

$$\mathcal{S}_2[h] = \frac{1}{RK_{inq}^2(R)} \int_\infty^R \xi K_{inq}^2(\xi) h(\xi)\, d\xi. \tag{5.14}$$

From Lemmas 5.1 and 5.2, we can rewrite (5.8) as a fixed point equation $(G, W) = \mathcal{F}[G, W]$ defined by

$$\begin{aligned} G &= \mathcal{F}_1[G, W] := \mathbf{G}_0 + \mathcal{S}_1[\varepsilon^{-2}\mathcal{N}_1[G, W]], \\ W &= \mathcal{F}_2[G, W] := -\mathcal{S}_2[\mathcal{N}_2[G, W]], \end{aligned}$$

where $\mathbf{G}_0$ linearly depends on a constant $\mathbf{a}$ (see (5.12)). Notice that the nonlinear operator $\mathcal{N}_2$ defined in (5.7) involves the derivatives G', G''. In order to avoid working with norms involving derivatives, we will take advantage of the differential properties of $\mathcal{F}_1$, and using that $G = \mathcal{F}_1[G, W]$ we rewrite the fixed point equation as

$$\begin{aligned} G &= \mathcal{F}_1[G, W] := \mathbf{G}_0 + \mathcal{S}_1[\varepsilon^{-2}\mathcal{N}_1[G, W]], \\ W &= \mathcal{F}_2[G, W] := -\mathcal{S}_2[\mathcal{N}_2[\mathcal{F}_1[G, W], W]], \end{aligned} \tag{5.15}$$

where $\mathcal{S}_1$ is defined in (5.11), $\mathcal{S}_2$ in (5.14), $\mathcal{N}_1$ in (5.5) and $\mathcal{N}_2$ in (5.7).

In Section 5.2, we study the linear operators $\mathcal{S}_1$ and $\mathcal{S}_2$ (see (5.11) and (5.14)) and prove that they are bounded operators in $\mathcal{X}_\ell$ for $\ell \geq 0$.

Our goal is now to prove the following result which is a reformulation of Theorem 4.3.

Theorem 5.3. *Let* $\eta > 0$, $0 < \mu_0 < \mu_1$ *and take* $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ *with* $\mu_0 \leq \mu \leq \mu_1$. *There exist*

$$q_0 = q_0(\mu_0, \mu_1, \eta) > 0, \quad e_0 = e_0(\mu_0, \mu_1, \eta) > 0 \quad \text{and} \quad M = M(\mu_0, \mu_1, \eta) > 0$$

such that, for any $q \in [0, q_0]$, $\alpha \in (0, 1)$ *satisfying*

$$q^{-1}\varepsilon^{1-\alpha} < e_0,$$

and for any constant $\mathbf{a}$ *satisfying*

$$\mathbf{a} = \varepsilon^{\frac{3}{2}}(\varepsilon^\alpha)^{-\frac{3}{2}} e^{\frac{\sqrt{2}}{\varepsilon^{1-\alpha}}} \hat{\mathbf{a}}, \quad |\hat{\mathbf{a}}| \leq \eta, \tag{5.16}$$

there exists a family of solutions $(G(R, \mathbf{a}), W(R, \mathbf{a}))$ *of the fixed point equation* (5.15) *defined for* $R \geq R^*_{\min} = \varepsilon^\alpha$ *which satisfy*

$$\|G\|_2 + \varepsilon\|G'\|_2 + \varepsilon\|W\|_2 \leq M\varepsilon^2.$$

Moreover, $G(R,\mathbf{a}) = G^0(R) + G^1(R,\mathbf{a})$ *and* $W(R,\mathbf{a}) = W^0(R,\mathbf{a}) + W^1(R,\mathbf{a})$ *are continuous with respect to* μ, $\hat{\mathbf{a}}$ *and they satisfy the following properties:*

(i) *there exist* $q_0^* = q_0^*(\mu_0,\mu_1) > 0$ *and* $M_0 = M_0(\mu_0,\mu_1)$ *such that, for* $q \in [0, q_0^*]$,

$$\|G^0\|_2 + \varepsilon\|(G^0)'\|_2 \le M_0 \varepsilon^{3-\alpha} q^{-1},$$

(ii) *for* $q \in [0, q_0]$*, we can decompose* $G^1(R,\mathbf{a}) = K_0(\frac{R\sqrt{2}}{\varepsilon})\mathbf{a} + \widehat{G}^1(R,\mathbf{a})$ *with*

$$\|\widehat{G}^1\|_2 + \varepsilon\|(\widehat{G}^1)'\|_2 \le M\frac{\varepsilon^{1-\alpha}}{q}\Big\|K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\Big\|_2|\mathbf{a}| \le M_1\varepsilon^2,$$

(iii) *and for* $q \in [0, q_0]$,

$$\varepsilon\|W^0\|_2 \le M\Big\|K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\Big\|_2|\mathbf{a}| \le M_1\varepsilon^2, \quad \varepsilon\|W^1\|_2 \le M_1\frac{\varepsilon^{3-\alpha}}{q},$$

where $M_1 = M_1(\mu_0,\mu_1,\eta)$ *depends on* μ_0*,* μ_1*, and* η*.*

The rest of this section is devoted to proving this theorem. In Section 5.2, we prove that the linear operators $\mathcal{S}_1$ and $\mathcal{S}_2$, defined in (5.11) and (5.14), are bounded in $\mathcal{X}_\ell$, $\ell \ge 0$. In Section 5.3, we study $\mathcal{F}[0,0]$ and in Section 5.4, we check that the operator $\mathcal{F}$ is Lipschitz in a suitable ball. Finally, in order to find the suitable decomposition of G, we refine the previous results in Section 5.5.

It is worth mentioning that the more technical part in this procedure comes from the study of the function V_0 (and K_{inq}) done in Proposition 4.2.

From now on, we fix η, μ_0, μ_1, we will take ε, q as small as needed, and $\mathbf{a}$ satisfying (5.16). We also will denote by M any constant independent of ε, q.

5.2. *The linear operators*

We prove that $\mathcal{S}_1$, $\mathcal{S}_2$ are bounded operators in the Banach spaces $\mathcal{X}_\ell$ defined in (5.2) along with important properties of such operators.

5.2.1. The operator $\mathcal{S}_1$. In this subsection, we prove that $\mathcal{S}_1 \colon \mathcal{X}_\ell \to \mathcal{X}_\ell$ is a bounded operator. In addition we also provide bounds for $(\mathcal{S}_1[h])'$, $(\mathcal{S}_1[h])''$.

Lemma 5.4. *Take* $R_{\min} \ge \frac{\varepsilon z_0}{\sqrt{2}}$ *with* z_0 *given in Table 1 corresponding to* K_0*,* I_0*, and* $\ell \ge 0$*. Then, if* ε *is small enough, the linear operator* $\mathcal{S}_1 \colon \mathcal{X}_\ell \to \mathcal{X}_\ell$ *defined in* (5.11) *is a bounded operator. Moreover, there exists a constant* $M > 0$ *such that for* $h \in \mathcal{X}_\ell$,

$$\|\mathcal{S}_1[h]\|_\ell \le M\varepsilon^2\|h\|_\ell.$$

Proof. Since $R_{\min}$ is such that $\frac{R_{\min}\sqrt{2}}{\varepsilon} > z_0$, by the asymptotic expansion in Table 1, for any $R \ge R_{\min}$,

$$K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big) = \sqrt{\frac{\pi\varepsilon}{2\sqrt{2}R}}e^{-\frac{R\sqrt{2}}{\varepsilon}}\Big(1 + \mathcal{O}\Big(\frac{\varepsilon}{R}\Big)\Big), \tag{5.17}$$

and

$$I_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big) = \sqrt{\frac{\varepsilon}{2\sqrt{2}R\pi}}\, e^{\frac{R\sqrt{2}}{\varepsilon}}\Big(1+\mathcal{O}\Big(\frac{\varepsilon}{R}\Big)\Big).$$

Let now $h \in \mathcal{X}_\ell$, that is, $|h(\xi)| \le \xi^{-\ell}\|h\|_\ell$. Then

$$\begin{aligned}
&|R^\ell \mathcal{S}_1[h](R)| \\
&\quad \le CR^{\ell-\frac{1}{2}}\Big(\frac{\varepsilon}{\sqrt{2}}\Big)\|h\|_\ell\Big[e^{-\frac{R\sqrt{2}}{\varepsilon}}\int_{R_{\min}}^{R}\frac{e^{\frac{\xi\sqrt{2}}{\varepsilon}}}{\xi^{\ell-\frac{1}{2}}}\,\mathrm{d}\xi + e^{\frac{R\sqrt{2}}{\varepsilon}}\int_R^\infty \frac{e^{\frac{-\xi\sqrt{2}}{\varepsilon}}}{\xi^{\ell-\frac{1}{2}}}\,\mathrm{d}\xi\Big] \\
&\quad \le C\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)^{\ell-\frac{1}{2}}\Big(\frac{\varepsilon}{\sqrt{2}}\Big)^2\|h\|_\ell\Big[e^{-\frac{R\sqrt{2}}{\varepsilon}}\int_{z_0}^{\frac{R\sqrt{2}}{\varepsilon}}\frac{e^t}{t^{\ell-\frac{1}{2}}}\,\mathrm{d}t + e^{\frac{R\sqrt{2}}{\varepsilon}}\int_{\frac{R\sqrt{2}}{\varepsilon}}^\infty \frac{e^{-t}}{t^{\ell-\frac{1}{2}}}\,\mathrm{d}t\Big] \\
&\quad = C\Big(\frac{\varepsilon}{\sqrt{2}}\Big)^2\|h\|_\ell\,\mathcal{M}\Big(\frac{R\sqrt{2}}{\varepsilon}\Big),
\end{aligned}$$

for some constant C, where

$$\mathcal{M}(z) = z^{\ell-\frac{1}{2}}\Big[e^{-z}\int_{z_0}^{z}\frac{e^t}{t^{\ell-\frac{1}{2}}}\,\mathrm{d}t + e^z\int_z^\infty \frac{e^{-t}}{t^{\ell-\frac{1}{2}}}\,\mathrm{d}t\Big]$$

and one can easily see that $\lim_{z\to\infty}\mathcal{M}(z) = 1$. Therefore, there exists a constant $M > 0$ such that $|\mathcal{M}(z)| \le M$ for $z \ge z_0$ and consequently,

$$|R^\ell \mathcal{S}_1[h](R)| \le CM\varepsilon^2\|h\|_\ell. \qquad \blacksquare$$

Corollary 5.5. *Let* $R_{\min} \ge \frac{1}{\sqrt{2}}\varepsilon z_0$ *and* $\ell \ge 0$. *Then for* ε *small enough and* $h \in \mathcal{X}_\ell$, *the function* $\mathcal{S}_1[h]$ *belongs to* $\mathcal{C}^2([R_{\min},\infty))$. *In addition, there exists a constant* $M > 0$ *such that*

$$\|(\mathcal{S}_1[h])'\|_\ell \le M\varepsilon\|h\|_\ell, \quad \|(\mathcal{S}_1[h])''\|_\ell \le M\|h\|_\ell.$$

Proof. Let $\varphi = \mathcal{S}_1(h)$. We have

$$\begin{aligned}
\varphi'(R) = \frac{\sqrt{2}}{\varepsilon}\Big[&K_0'\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\int_{R_{\min}}^{R}\xi I_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big)h(\xi)\,\mathrm{d}\xi \\
&+ I_0'\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\int_R^\infty \xi K_0\Big(\frac{\xi\sqrt{2}}{\varepsilon}\Big)h(\xi)\,\mathrm{d}\xi\Big]
\end{aligned}$$

which implies that φ is differentiable if h is continuous (by definition). Moreover, since $K_0'(z)$, $I_0'(z)$ have the same asymptotic expansions as K_0, I_0 (in Table 1) performing the same computations as in the proof of Lemma 5.4, we obtain the result for φ'.

We note that φ' is differentiable if h is continuous (again simply by definition). Then φ is $\mathcal{C}^2$. Moreover,

$$\varphi'' + \frac{\varphi'}{R} - 2\frac{\varphi}{\varepsilon^2} = -h,$$

and therefore

$$|R^\ell\varphi''(R)| \le M\|h\|_\ell\Big(3+\frac{\varepsilon}{R}\Big) \le M\|h\|_\ell. \qquad \blacksquare$$

5.2.2. The operator $\mathcal{S}_2$. Let us first provide a technical lemma.

Lemma 5.6. *There exists $q_0 > 0$, such that for any $0 < q < q_0$, if $R \geq 2e^2 e^{-\frac{\pi}{2qn}}$,*

$$\frac{1}{K_{inq}^2(R)} \int_R^\infty K_{inq}^2(\xi)\, d\xi \leq \frac{1}{2}.$$

Proof. The proof is straightforward from item (iii) of Proposition 4.2. Indeed, we first recall that $V_0(R) = v_0^{\text{out}}(\frac{R}{\varepsilon})$ and hence $V_0(R) < -1$. Then, we consider the function $\psi(R) = \int_R^\infty K_{inq}^2(\xi)\, \mathrm{d}\xi - \frac{1}{2}K_{inq}^2(R)$ and point out that we just need to prove $\psi(R) \leq 0$ if $R \geq 2e^2 e^{-\frac{\pi}{2nq}}$. We have

$$\begin{aligned}\psi'(R) &= -K_{inq}^2(R) - K_{inq}(R)K_{inq}'(R) = -K_{inq}^2(R)\Big[1 + \frac{K_{inq}'(R)}{K_{inq}(R)}\Big] \\ &= -K_{inq}^2(R)[1 + V_0(R)].\end{aligned}$$

Therefore, since $V_0(R) < -1$ for $R \geq 2e^2 e^{-\frac{\pi}{2nq}}$, then $\psi'(R) > 0$ and using that $\psi(R) \leq \lim_{R\to\infty} \psi(R) = 0$ the result is proven. ∎

The following lemma, provides bounds for the norm of the linear operator $\mathcal{S}_2$, defined in (5.14).

Lemma 5.7. *There exists $q_0 > 0$ such that for any $0 < q < q_0$, taking $R_{\min} \geq 2e^2 e^{-\frac{\pi}{2qn}}$, the operator $\mathcal{S}_2 \colon \mathcal{X}_\ell \to \mathcal{X}_\ell$, defined in (5.14), is bounded for all $\ell \geq 1$. Moreover, if $h \in \mathcal{X}_\ell$, $\ell = 1, 2$,*

$$\|\mathcal{S}_2[h]\|_\ell \leq \frac{1}{2}\|h\|_\ell.$$

In addition, when $h \in \mathcal{X}_3$,

$$\|\mathcal{S}_2[h]\|_2 \leq \|h\|_3. \tag{5.18}$$

Proof. Let $\ell \geq 1$ and $h \in \mathcal{X}_\ell$. Then, by Lemma 5.6

$$|R^\ell \mathcal{S}_2[h](R)| \leq \frac{R^{\ell-1}\|h\|_\ell}{K_{inq}^2(R)} \int_R^\infty \frac{K_{inq}^2(\xi)}{\xi^{\ell-1}}\, \mathrm{d}\xi \leq \frac{\|h\|_\ell}{K_{inq}^2(R)} \int_R^\infty K_{inq}^2(\xi)\, \mathrm{d}\xi \leq \frac{1}{2}\|h\|_\ell.$$

When $h \in \mathcal{X}_3$, then since $K_{inq} > 0$ and decreasing,

$$|R^2 \mathcal{S}_2[h](R)| \leq \frac{R\|h\|_3}{K_{inq}^2(R)} \int_R^\infty \frac{K_{inq}^2(\xi)}{\xi^2}\, \mathrm{d}\xi \leq \|h\|_3 R \int_R^\infty \frac{1}{\xi^2}\, \mathrm{d}\xi \leq \|h\|_3. \qquad ∎$$

Because in the definition of the operator $\mathcal{N}_2$ (see (5.7)), there are some derivatives involved, we need a more accurate control on how the operator $\mathcal{S}_2$ acts on a special type of functions. In particular, we shall need to control $\mathcal{S}_2[hV_0]$, where we recall that $V_0 = K_{inq}'(R)(K_{inq}(R))^{-1}$. For this reason, we study first the auxiliary linear operator defined by

$$\mathcal{A}[h](R) = \mathcal{S}_2[hV_0](R) = \frac{1}{RK_{inq}^2(R)} \int_\infty^R \xi h(\xi) K_{inq}'(\xi) K_{inq}(\xi)\, \mathrm{d}\xi. \tag{5.19}$$

Lemma 5.8. *With the same hypotheses as in Lemma 5.7, for any $h \in \mathcal{X}_\ell$,*

$$\|\mathcal{A}[h]\|_\ell \leq \frac{1}{2}\|h\|_\ell.$$

Proof. Let $h \in \mathcal{X}_\ell$. Then

$$\begin{aligned}|R^\ell \mathcal{A}[h](R)| &\leq \frac{R^{\ell-1}\|h\|_\ell}{K_{inq}^2(R)} \int_R^\infty (-K_{inq}'(\xi)K_{inq}(\xi)) \frac{1}{\xi^{\ell-1}}\, d\xi \\ &\leq \frac{\|h\|_\ell}{K_{inq}^2(R)} \int_R^\infty (-K_{inq}'(\xi)K_{inq}(\xi))\, d\xi = \frac{1}{2}\|h\|_\ell. \qquad \blacksquare\end{aligned}$$

Lemma 5.9. *Let h_1, h_2 be bounded differentiable functions. Then*

$$\mathcal{S}_2[h_1 h_2'](R) = h_1(R)h_2(R) - \mathcal{S}_2[h_1' h_2] - \mathcal{S}_2[\widehat{h}](R) - 2\mathcal{A}[h_1 h_2](R),$$

where $\widehat{h}(R) = h_1(R)h_2(R)R^{-1}$. If $(\xi\widehat{h}_1)' = \xi\widehat{h}_2$ and h is a differentiable bounded function, then

$$\mathcal{S}_2[\widehat{h}_2 h](R) = \widehat{h}_1(R)h(R) - \mathcal{S}_2[h'\widehat{h}_1](R) - 2\mathcal{A}[\widehat{h}_1 h](R).$$

Proof. We prove both properties by integrating by parts. Indeed, since h_1, h_2 are bounded functions

$$\begin{aligned}&\int_\infty^R \xi h_1(\xi)h_2'(\xi)K_{inq}^2(\xi)\, d\xi \\ &\quad = Rh_1(R)h_2(R)K_{inq}^2(R) - \int_\infty^R h_2(\xi)\big[h_1(\xi)K_{inq}^2(\xi) + \xi h_1'(\xi)K_{inq}^2(\xi) \\ &\qquad\qquad + 2\xi h_1(\xi)K_{inq}'(\xi)K_{inq}(\xi)\, d\xi\big].\end{aligned}$$

Therefore,

$$\mathcal{S}_2[h_1 h_2'](R) = \frac{1}{RK_{inq}^2(R)} \int_\infty^R \xi h_1(\xi)h_2'(\xi)K_{inq}^2(\xi)\, d\xi$$

satisfies the statement.

With respect to the second equality, again by doing an integration by parts,

$$\begin{aligned}&\mathcal{S}_2[\widehat{h}_2 h](R) \\ &\quad = \frac{1}{RK_{inq}^2(R)} \int_\infty^R (\xi\widehat{h}_1(\xi))' h(\xi)K_{inq}^2(\xi)\, d\xi \\ &\quad = \widehat{h}_1(R)h(R) - \frac{1}{RK_{inq}^2(R)} \int_\infty^R \xi\widehat{h}_1(\xi)[h'(\xi)K_{inq}^2(\xi) + 2h(\xi)K_{inq}'(\xi)K_{inq}(\xi)]\, d\xi. \qquad \blacksquare\end{aligned}$$

5.3. *The independent term*

We study now the independent term of the fixed point equation (5.15), that is, $\mathcal{F}[0,0] = (\mathcal{F}_1[0,0], \mathcal{F}_2[0,0])$. We recall that

$$\mathcal{F}_1[0,0] = \mathbf{G}_0 + \mathcal{S}_1[\varepsilon^{-2}\mathcal{N}_1[0,0]], \quad \mathcal{F}_2[0,0] = -\mathcal{S}_2[\mathcal{N}_2[\mathcal{F}_1[0,0],0]] \tag{5.20}$$

and $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathbf{G}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ are defined in (5.5) and (5.7), (5.12), (5.11), (5.14), respectively.

Before starting with the study of $\mathcal{F}[0,0]$ in (5.20), we state a straightforward corollary of items (iii) and (iv) of Proposition 4.2 about the behaviour of F_0, V_0 (see (5.1)).

Corollary 5.10. *Let $R_{\min} = \varepsilon^\alpha$ with $\alpha \in (0,1)$. Then there exist $q_0 > 0$ and a constant $M > 0$ such that for any $0 < q < q_0$ and $R \in [R_{\min}, +\infty)$, $V_0'(R) > 0$, $V_0(R) < -1$,*

$$|kV_0(R)|,\ |kV_0'(R)R|,\ |kV''(R)R^2| \le M\varepsilon^{1-\alpha}$$

with $k = \varepsilon q^{-1}$, and

$$|R(V_0(R)+1)|,\ |R^2V_0'(R)|,\ |R^3V_0''(R)| \le M.$$

With respect to F_0, we have $F_0(R) \ge \frac{1}{2}$, $F_0'(R) > 0$ and

$$|F_0'(R)R^2|,\ |F_0''(R)R^3| \le Ck\varepsilon^{1-\alpha}, \quad |1-F_0(R)|,\ |F_0'(R)R|,\ |F_0''(R)R^2| \le C\varepsilon^{2(1-\alpha)}.$$

From now on, we then take $R_{\min} = \varepsilon^\alpha$ with $0 < \alpha < 1$ satisfying $\frac{\varepsilon^{1-\alpha}}{q}$ small enough. These conditions will ensure that $\frac{\varepsilon}{R_{\min}} \ll 1$. The following proposition provides the size of $\mathcal{F}[0,0]$ in (5.20).

Lemma 5.11. *Let $0 < \mu_0 < \mu_1$ and take $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ with $\mu_0 \le \mu \le \mu_1$. There exist $q_0^* = q_0^*(\mu_0,\mu_1) > 0$, $M = M(\mu_0,\mu_1) > 0$ such that, for any $q \in [0, q_0^*]$ and $\alpha \in (0,1)$ satisfying $\frac{\varepsilon^{1-\alpha}}{q} < 1$, $R_{\min} = \varepsilon^\alpha$, given $\eta > 0$ and $\mathbf{a}$ satisfying (5.16) in the definition of $\mathbf{G}_0$ provided in (5.12), we have*

(1) *Let $G_0 = \mathcal{F}_1[0,0]$. Then the following bound holds:*

$$\|G_0\|_2 + \varepsilon\|G_0'\|_2 + \varepsilon^2\|G_0''\|_2 \le \|\mathbf{G}_0\|_2 + M\varepsilon^{4-2\alpha} \le M(1+\eta)\varepsilon^2. \tag{5.21}$$

As a consequence, there exists $q_1^*(\mu_0,\mu_1,\eta) \le q_0^*(\mu_0,\mu_1)$ such that if $q \in [0, q_1^*]$, then $F_0(R) + G_0(R) \ge \frac{1}{4}$.

(2) *Let $W_0 = \mathcal{F}_2[0,0]$. Then there exists $q_2^*(\mu_0,\mu_1,\eta) \le q_1^*(\mu_0,\mu_1,\eta)$ such that for $q \in [0, q_2^*]$,*

$$\|W_0\|_2 \le M\varepsilon^{2-\alpha}q^{-1} + M\eta\varepsilon \le M(1+\eta)\varepsilon.$$

Remark 5.12. Since $M = M(\mu_0,\mu_1)$ does not depend on η, we have that $q_1^*(\mu_0,\mu_1,0)$, $q_2^*(\mu_0,\mu_1,0) > 0$. In other words, Lemma 5.11 can be also applied for $\eta = 0$.

We divide the proof of this lemma into two parts, the first one, in Section 5.3.1, corresponds to the bound for G_0 and the second one, in Section 5.3.2 corresponds to the bound for W_0.

5.3.1. A bound for the norm of G_0 and its derivatives. Recall that $G_0 = \mathcal{F}_1[0,0]$ as given in (5.20). We start bounding $\|\mathbf{G}_0\|_2$, $\|\mathbf{G}_0'\|_2$, $\|\mathbf{G}_0''\|_2$ with $\mathbf{G}_0$ given in (5.12). By (5.17), it is clear that, for $R \ge R_{\min} = \varepsilon^\alpha$,

$$|R^2\mathbf{G}_0(R)| = \left|R^2K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\mathbf{a}\right| \le M|\mathbf{a}|\sqrt{\varepsilon}R_{\min}^{\frac{3}{2}}e^{-\frac{R_{\min}\sqrt{2}}{\varepsilon}} \le M|\mathbf{a}|\sqrt{\varepsilon}(\varepsilon^\alpha)^{\frac{3}{2}}e^{-\frac{\sqrt{2}}{\varepsilon^{1-\alpha}}},$$

if $0 < q < q_0^*$, for $q_0^* = q_0^*(\mu_0, \mu_1)$. Here we have used that $\varepsilon^{1-\alpha} \le Mq$. Therefore, using that **a** satisfies (5.16), we conclude that $\|\mathbf{G}_0\|_2 \le M\eta\varepsilon^2$. In addition, it is clear that $\varepsilon\|\mathbf{G}_0'\|_2 + \varepsilon^2\|\mathbf{G}_0''\|_2 \le M\|\mathbf{G}_0\|_2 \le M\eta\varepsilon^2$, and thus

$$\|\mathbf{G}_0\|_2 + \varepsilon\|\mathbf{G}_0'\|_2 + \varepsilon^2\|\mathbf{G}_0''\|_2 \le M\eta\varepsilon^2. \tag{5.22}$$

To deal with $\mathcal{S}_1[\varepsilon^{-2}\mathcal{N}_1[0,0]]$ (see (5.5)), we first bound

$$\mathcal{F}_0(R) = \mathcal{N}_1[0,0](R) = \varepsilon^2\Big(F_0''(R) + \frac{F_0'(R)}{R}\Big).$$

By Corollary 5.10,

$$|R^2\varepsilon^{-2}\mathcal{F}_0(R)| \le M\varepsilon^{2(1-\alpha)},$$

and applying Lemma 5.4, we obtain $\|\mathcal{S}_1(\varepsilon^{-2}\mathcal{F}_0(R))\|_2 \le C\varepsilon^{4-2\alpha}$, which gives

$$\|G_0\|_2 \le \|\mathbf{G}_0\|_2 + M\varepsilon^{4-2\alpha} \le M(\eta\varepsilon^2 + \varepsilon^{4-2\alpha}) \le M(1+\eta)\varepsilon^2.$$

Using Corollary 5.5, we obtain the bounds for the derivatives,

$$\varepsilon\|G_0'\|_2 + \varepsilon^2\|G_0''\|_2 \le M(\eta\varepsilon^2 + \varepsilon^{4-2\alpha}) \le M(1+\eta)\varepsilon^2, \tag{5.23}$$

and (5.21) is proved.

To finish, we notice that by Corollary 5.10, there exists $q_1^*(\mu_0, \mu_1, \eta)$ such that if $q \in [0, q_1^*]$,

$$F_0(R) + G_0(R) \ge \frac{1}{2} - M(1+\eta)\frac{\varepsilon^2}{R^2} \ge \frac{1}{2} - M(1+\eta)\varepsilon^{2(1-\alpha)} \ge \frac{1}{4}. \tag{5.24}$$

5.3.2. A bound for $\|W_0\|_2$. We recall that $W_0 = \mathcal{S}_2[\mathcal{N}_2[\mathcal{F}_1[0,0],0]] = \mathcal{S}_2[\mathcal{N}_2[G_0,0]]$, where $\mathcal{N}_2$ is defined in (5.7), namely

$$\mathcal{N}_2[G_0,0] = 2V_0\frac{F_0'+G_0'}{F_0+G_0} - q^2\frac{1}{F_0+G_0}\Big(F_0''+G_0''+\frac{F_0'+G_0'}{R}\Big).$$

By definition (5.19) of $\mathcal{A}$,

$$\mathcal{S}_2\Big[V_0\frac{F_0'+G_0'}{F_0+G_0}\Big] = \mathcal{A}\Big[\frac{F_0'+G_0'}{F_0+G_0}\Big].$$

Therefore, for $0 < q < q_1^*(\mu_0, \mu_1, \eta)$, using Lemma 5.8, Corollary 5.10 and bounds (5.24) and (5.23),

$$\begin{aligned}\Big\|\mathcal{S}_2\Big[V_0\frac{F_0'+G_0'}{F_0+G_0}\Big]\Big\|_2 &\le \Big\|\frac{F_0'+G_0'}{F_0+G_0}\Big\|_2 \le M(k\varepsilon^{1-\alpha} + \varepsilon^{3-2\alpha} + \varepsilon\eta)\\ &\le M(\varepsilon^{2-\alpha}q^{-1} + \varepsilon^{3-2\alpha} + \varepsilon\eta) \le M(\varepsilon^{2-\alpha}q^{-1} + \varepsilon\eta),\end{aligned}$$

where we have used that $k\varepsilon^{1-\alpha} = \varepsilon q^{-1}\varepsilon^{1-\alpha} \le \varepsilon$. In the rest of the proof, we will reduce the value of q_1^*, if necessary, without changing the notation. In addition, by Corollary 5.10 since

$$q^2\Big|R^3\frac{1}{F_0+G_0}\Big(F_0''+\frac{F_0'}{R}\Big)\Big| \le Mq^2k\varepsilon^{1-\alpha} = Mq\varepsilon^{2-\alpha},$$

we also have by inequality (5.18) in Lemma 5.7,

$$q^2\Big\|\mathcal{S}_2\Big[\frac{1}{F_0+G_0}\Big(F_0''+\frac{F_0'}{R}\Big)\Big]\Big\|_2 \le Mq\varepsilon^{2-\alpha}.$$

To bound the last term in $\mathcal{W}_0$, we use the second statement of Lemma 5.9 with

$$h=\frac{1}{F_0+G_0},\qquad \widehat{h}_2=G_0''+\frac{G_0'}{R},\qquad \widehat{h}_1=G_0'.$$

Then

$$\Big\|\mathcal{S}_2\Big[\frac{1}{F_0+G_0}\Big(G_0''+\frac{G_0'}{R}\Big)\Big]\Big\|_2 \le \Big\|\frac{G_0'}{F_0+G_0}\Big\|_2+\|\mathcal{S}_2[h'G_0']\|_2+2\Big\|\mathcal{A}\Big[\frac{G_0'}{F_0+G_0}\Big]\Big\|_2.$$

By bounds (5.23) and (5.24),

$$\Big\|\frac{G_0'}{F_0+G_0}\Big\|_2 \le M\eta\varepsilon+M\varepsilon^{3-2\alpha},$$

and as a consequence, by Lemma 5.8,

$$\Big\|\mathcal{A}\Big[\frac{G_0'}{F_0+G_0}\Big]\Big\|_2 \le M\eta\varepsilon+M\varepsilon^{3-2\alpha}.$$

By bound (5.24) and since $R\ge R_{\min}=\varepsilon^{\alpha}$,

$$\begin{aligned}|G_0'(R)h'(R)| &\le |G_0'(R)|\,\frac{|F_0'(R)|+|G_0'(R)|}{|F_0(R)+G_0(R)|^2}\\ &\le M\Big(\frac{\varepsilon^{3-2\alpha}+\eta\varepsilon}{R^2}\Big)\frac{\varepsilon^{2-\alpha}q^{-1}+\varepsilon^{3-2\alpha}+\eta\varepsilon}{R^2}\le\frac{M}{R^3}(\varepsilon^{4-3\alpha}+\eta\varepsilon^{2-\alpha}),\end{aligned}$$

where we have used that $\frac{\varepsilon^{1-\alpha}}{q}\le 1$. Then, using Lemma 5.7, $\|\mathcal{S}_2[h'G_0']\|_2\le\|h'G_0'\|_3$ and therefore, $\|q^2\mathcal{S}_2[h'G_0']\|_2\le M(q^2\varepsilon^{4-3\alpha}+q^2\eta\varepsilon^{2-\alpha})$. We conclude that

$$\|W_0\|_2\le M\varepsilon^{2-\alpha}q^{-1}+M\eta\varepsilon\le M(1+\eta)\varepsilon.$$

5.4. *The contraction mapping*

In Lemma 5.11, we have proven that the independent term $(G_0,W_0)=\mathcal{F}[0,0]$ (defined in (5.20)) satisfies $\|G_0\|_2+\varepsilon\|W_0\|_2\le M(1+\eta)\varepsilon^2$. In other words, the independent term belongs to the Banach space $\mathcal{X}_2\times\mathcal{X}_2$ endowed with the norm

$$[\![(G,W)]\!]=\|G\|_2+\varepsilon\|W\|_2.$$

Let

$$\kappa_0=\kappa_0(\mu_0,\mu_1,\eta)=[\![(G_0,W_0)]\!]\varepsilon^{-2}, \tag{5.25}$$

then we get the following assertion.

Lemma 5.13. *Let* $\eta > 0$, $0 < \mu_0 < \mu_1$ *and take* $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ *with* $\mu_0 \le \mu \le \mu_1$. *Take* $\kappa \ge 2\kappa_0$, *where* κ_0 *is defined in* (5.25), *and* $\mathbf{a}$ *satisfying condition* (5.16). *There exist* $q_0 = q_0(\mu_0, \mu_1, \eta) > 0$ *and* $M = M(\mu_0, \mu_1, \eta) > 0$ *such that, for any* $q \in [0, q_0]$ *and* $\alpha \in (0, 1)$ *satisfying* $q^{-1}\varepsilon^{1-\alpha} < 1$, *taking* $R_{\min} \ge \varepsilon^\alpha$, *if* $(G_1, W_1), (G_2, W_2) \in \mathcal{X}_2 \times \mathcal{X}_2$ *with* $\llbracket (G_1, W_1) \rrbracket, \llbracket (G_2, W_2) \rrbracket \le \kappa\varepsilon^2$, *then*

$$\llbracket \mathcal{F}[G_1, W_1] - \mathcal{F}[G_2, W_2] \rrbracket \le M\varepsilon^{1-\alpha} q^{-1} \llbracket (G_1, W_1) - (G_2, W_2) \rrbracket, \tag{5.26}$$

where the operator $\mathcal{F}$ *is defined in* (5.15).

If moreover $\|G_1'\|_2, \|G_2'\|_2 \le \kappa\varepsilon$, *then*

$$\begin{aligned} &\varepsilon \| \mathcal{S}_2[\mathcal{N}_2[G_1, W_1]] - \mathcal{S}_2[\mathcal{N}_2[G_2, W_2]] \|_2 \\ &\qquad \le M\varepsilon^{2-\alpha} \|W_1 - W_2\|_2 + M\varepsilon^{1-\alpha} \|G_1 - G_2\|_2 + M\varepsilon \|G_1' - G_2'\|_2 \end{aligned} \tag{5.27}$$

with $\mathcal{S}_2$ *defined in* (5.14) *and* $\mathcal{N}_2$ *in* (5.7). *Also,*

$$\varepsilon \| (\mathcal{F}_1[G_1, W_1] - \mathcal{F}_1[G_2, W_2])' \|_2 \le M\varepsilon^{1-\alpha} q^{-1} \llbracket (G_1, W_1) - (G_2, W_2) \rrbracket. \tag{5.28}$$

Next subsection is devoted to proving Theorem 5.3 from the above results and Lemma 5.13. We postpone the proof of this lemma to Section 5.6.

5.5. *Proof of Theorem* 5.3

Lemma 5.13, for $0 < q < q_0$, gives us the Lipschitz constant of $\mathcal{F}$ with the norm $\llbracket \cdot \rrbracket$ on $\mathcal{B}_{\kappa\varepsilon^2}$, the closed ball of $\mathcal{X}_2 \times \mathcal{X}_2$ of radius $\kappa\varepsilon^2$. Indeed, the Lipschitz constant is $M\varepsilon^{1-\alpha} q^{-1} \le \frac{1}{2}$ if $\varepsilon^{1-\alpha} q^{-1} < e_0 := \frac{1}{2M}$. Then the operator $\mathcal{F}$ is a contraction. Moreover, recalling the definition of κ_0 given in (5.25), if $(G, W) \in \mathcal{B}_{\kappa\varepsilon^2}$, it is clear that

$$\begin{aligned} \llbracket \mathcal{F}[G, W] \rrbracket &\le \llbracket \mathcal{F}[G, W] - \mathcal{F}[0, 0] \rrbracket + \llbracket \mathcal{F}[0, 0] \rrbracket \\ &\le \frac{1}{2} \llbracket (G, W) \rrbracket + \kappa_0 \varepsilon^2 \le \frac{1}{2}\kappa\varepsilon^2 + \frac{\kappa}{2}\varepsilon^2 \le \kappa\varepsilon^2. \end{aligned}$$

Then, the existence of a solution of the fixed point equation (5.15), namely $(G, W) = \mathcal{F}[G, W]$, belonging to $\mathcal{B}_{\kappa\varepsilon^2}$ is guaranteed by the Banach fixed point theorem.

Moreover, as

$$\|G\|_2 = \|\mathcal{F}_1[G, W]\|_2 \le \kappa\varepsilon^2,$$

using (5.28) and Lemma 5.11 to bound the norm of $(\mathcal{F}_1[0, 0])'$, one can easily see, for some constant M,

$$\|G'\|_2 = \|(\mathcal{F}_1[G, W])'\|_2 \le \|(\mathcal{F}_1[G, W] - \mathcal{F}_1[0, 0])'\|_2 + \|(\mathcal{F}_1[0, 0])'\|_2 \le M\varepsilon.$$

The continuity with respect to μ and $\hat{\mathbf{a}} = \varepsilon^{-\frac{3}{2}} (\varepsilon^\alpha)^{\frac{3}{2}} e^{-\frac{\sqrt{2}}{\varepsilon^{1-\alpha}}} \mathbf{a}$ can be proven as follows. It is clear that from definition (5.12) of $\mathbf{G}_0$ and Table 1, we have

$$\mathbf{G}_0(R) = K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\mathbf{a} = K_0\Big(\frac{R\sqrt{2}}{\varepsilon}\Big)\varepsilon^{\frac{3}{2}} (\varepsilon^\alpha)^{-\frac{3}{2}} e^{\frac{\sqrt{2}}{\varepsilon^{1-\alpha}}} \hat{\mathbf{a}} =: K(R)\hat{\mathbf{a}}$$

with $|K(R)| \leq M$ if $R \geq R^*_{\min} = \varepsilon^\alpha$. Moreover, by construction,

$$(G, W) = \lim_{k\to\infty} (G^{(k)}, W^{(k)}), \quad (G^{(k)}, W^{(k)}) = \mathcal{F}[G^{(k-1)}, W^{(k-1)}],$$
$$(G^{(0)}, W^{(0)}) = (0, 0).$$

Therefore, using that the operator $\mathcal{F}$ defined in (5.15) is continuous with respect to $\mu \in [\mu_0, \mu_1]$ and depends on $\hat{\mathbf{a}}$ through $\mathbf{G}_0$, so the operator $\mathcal{F}$ is also continuous with respect to $\hat{\mathbf{a}} \in [-\eta, \eta]$, we deduce that (G, W) is continuous with respect to $\hat{\mathbf{a}}$, μ since the operator $\mathcal{F}$ is a contraction uniformly on these parameters.

We introduce now the auxiliary operator

$$\widehat{\mathcal{F}}[G, W] = (\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2)[G, W] := (\varepsilon^{-2}\mathcal{S}_1[\mathcal{N}_1[G, W]], -\mathcal{S}_2[\mathcal{N}_2[\widehat{\mathcal{F}}_1[G, W], W]]).$$

Observe that $\widehat{\mathcal{F}}[G, W] = \mathcal{F}[G, W]$ for $\mathbf{a} = 0$. We denote by (G^0, W^0) the solution of the fixed point equation $(G, W) = \widehat{\mathcal{F}}[G, W]$, and we emphasize that, since $\mathbf{a} = 0$, $\mathbf{G}_0 \equiv 0$ (see (5.12)). We point out that, applying Lemma 5.11 with $\eta \to 0$ (see Remark 5.12) and recalling that $\varepsilon^{1-\alpha} \leq \frac{q}{2M}$, for $0 < q \leq q_0^*(\mu_0, \mu_1)$, we have

$$\llbracket \widehat{\mathcal{F}}[0, 0] \rrbracket \leq M(\varepsilon^{4-2\alpha} + \varepsilon^{3-\alpha} q^{-1}) \leq M\varepsilon^{3-\alpha} q^{-1}.$$

Therefore, in this case, $\overline{\kappa}_0 = \kappa_0(\mu_0, \mu_1, 0) = \varepsilon^{-2}\llbracket \widehat{\mathcal{F}}[0, 0] \rrbracket \leq M\varepsilon^{1-\alpha} q^{-1}$ with κ_0 defined in (5.25), and this implies

$$\llbracket (G^0, W^0) \rrbracket \leq 2\overline{\kappa}_0 \varepsilon^2 \leq 2M\varepsilon^{3-\alpha} q^{-1}.$$

Denoting by $M_0 = 2M$ (which only depends on μ_0, μ_1), the proof of first item of Theorem 5.3 is done.

Let now (G, W) be the solution for a given $\mathbf{a}$ satisfying (5.16). We have

$$\begin{aligned} G &= \mathcal{F}_1[G, W] = \mathbf{G}_0 + \widehat{\mathcal{F}}_1[G, W], \\ W &= \mathcal{F}_2[G, W] = -\mathcal{S}_2[\mathcal{N}_2[\mathcal{F}_1[G, W], W]] \\ &= -\mathcal{S}_2[\mathcal{N}_2[\mathbf{G}_0 + \widehat{\mathcal{F}}_1[G, W], W]] + \mathcal{S}_2[\mathcal{N}_2[\widehat{\mathcal{F}}_1[G, W], W]] - \widehat{\mathcal{F}}_2[G, W]. \end{aligned}$$

Therefore, using that $(G^0, W^0) = \widehat{\mathcal{F}}[G^0, W^0]$, we have, using (5.26) and (5.27),

$$\begin{aligned} \llbracket (G, W) - (G^0, W^0) \rrbracket \leq{}& \|\mathbf{G}_0\|_2 + \llbracket \widehat{\mathcal{F}}[G, W] - \widehat{\mathcal{F}}[G^0, W^0] \rrbracket \\ &+ \varepsilon \|\mathcal{S}_2[\mathcal{N}_2[\mathbf{G}_0 + \widehat{\mathcal{F}}_1[G, W], W]] - \mathcal{S}_2[\mathcal{N}_2[\widehat{\mathcal{F}}_1[G, W], W]]\|_2 \\ \leq{}& \|\mathbf{G}_0\|_2 + M\varepsilon^{1-\alpha} q^{-1} \llbracket (G, W) - (G^0, W^0) \rrbracket \\ &+ M\varepsilon^{1-\alpha} \|\mathbf{G}_0\|_2 + M\varepsilon \|\mathbf{G}_0'\|_2 \\ \leq{}& M\|\mathbf{G}_0\|_2 + M\varepsilon \|\mathbf{G}_0'\|_2 + M\varepsilon^{1-\alpha} q^{-1} \llbracket (G, W) - (G^0, W^0) \rrbracket. \end{aligned}$$

As a consequence, using that, by (5.22), $\|\mathbf{G}_0\|_2 + \varepsilon\|\mathbf{G}_0'\|_2 \leq M\varepsilon^2$, we obtain

$$\llbracket (G, W) - (G^0, W^0) \rrbracket \leq M\varepsilon^2.$$

Then

$$\|G - \mathbf{G}_0 - G^0\|_2 = \|\widehat{\mathcal{F}}_1[G, W] - \widehat{\mathcal{F}}_1[G^0, W^0]\|_2 \leq M\varepsilon^{1-\alpha} q^{-1} \|(G, W) - (G^0, W^0)\| \\ \leq M\varepsilon^{3-\alpha} q^{-1}.$$

The bounds for $\|(G^0)'\|_2$ and $\|G' - (G^0)' - \mathbf{G}_0'\|_2$ follow from bound (5.28) and an analogous expression for $\widehat{F}_1$, along with expression (5.23). Denoting by $\widehat{G}^1 = G - G^0 - \mathbf{G}_0$, Theorem 5.3 is proven.

5.6. *Proof of Lemma* 5.13

The proof of Lemma 5.13 is divided into two parts. In Section 5.6.1, we prove inequality (5.26) and (5.28). In Section 5.6.2, we prove (5.27).

5.6.1. The Lipschitz constant of $\mathcal{F}_1$. Let $(G_1, W_1), (G_2, W_2) \in \mathcal{X}_2 \times \mathcal{X}_2$ belonging to the closed ball of radius $\kappa\varepsilon^2$, that is, $\|(G_1, W_1)\|, \|(G_1, W_1)\| \leq \kappa\varepsilon^2$. We have, using Lemma 5.4,

$$\|\mathcal{F}_1[G_1, W_1] - \mathcal{F}_1[G_2, W_2]\|_2 = \varepsilon^{-2}\|\mathcal{S}_1[\mathcal{N}_1[G_1, W_1] - \mathcal{N}_1[G_2, W_2]]\|_2 \\ \leq M\|\mathcal{N}_1(G_1, W_1) - \mathcal{N}_1(G_2, W_2)\|_2. \tag{5.29}$$

Then to compute the Lipschitz constant of $\mathcal{F}_1$, it is enough to deal with the Lipschitz constant of $\mathcal{N}_1$.

Now we write $\eta(\lambda) = (1 - \lambda)(G_1, W_1) + \lambda(G_2, W_2)$ and, for any $R \geq R_{\min} = \varepsilon^\alpha$,

$$\mathcal{N}_1[G_2, W_2](R) - \mathcal{N}_1[G_1, W_1](R) = \int_0^1 \partial_G \mathcal{N}_1[\eta(\lambda)](R)(G_2(R) - G_1(R))\, d\lambda \\ + \int_0^1 \partial_W \mathcal{N}_1[\eta(\lambda)](R)(W_2(R) - W_1(R))\, d\lambda.$$

Then, since $\|\eta(\lambda)\|_2 \leq \kappa\varepsilon^2$, to bound the Lipschitz constant of $\mathcal{N}_1$, it is enough to bound $|\partial_G \mathcal{N}_1[G, W]|$ and $|\partial_W \mathcal{N}_1[G, W]|$ for $\|(G, W)\|_2 \leq \kappa\varepsilon^2$.

We now recall that $\widehat{F}_0$ in (5.3) is defined as $F_0^2 = 1 + \frac{\widehat{F}_0}{2}$. Then, since by Corollary 5.10 $|kV_0(R)| \leq M\varepsilon^{1-\alpha}$ and $F_0^2 = 1 - k^2V_0^2 - \varepsilon^2 n^2 R^{-2}$, we have, using that $R \geq R_{\min} = \varepsilon^\alpha$,

$$|\widehat{F}_0(R)| \leq Mk^2|V_0^2(R)| + M\varepsilon^2 R^{-2} \leq M\varepsilon^{2-2\alpha}. \tag{5.30}$$

Then, if $|G(R)| \leq \kappa\varepsilon^2 R^{-2} \leq M\varepsilon^{2-2\alpha}$, using $\varepsilon^{1-\alpha} \leq Mq$,

$$|F_0(R) + G(R)| \leq \sqrt{1 + \frac{\widehat{F}_0(R)}{2}} + |G(R)| \leq 1 + \mathcal{O}(\varepsilon^{2-2\alpha}) \leq 1 + \mathcal{O}(q^2) \leq 2 \tag{5.31}$$

if q is small enough.

We claim that if $\|(G, W)\|_2 \leq \kappa\varepsilon^2$, then

$$|\partial_G \mathcal{N}_1[G, W](R)| \leq M\varepsilon^{2-2\alpha}, \quad |\partial_W \mathcal{N}_1[G, W](R)| \leq Mk\varepsilon^{1-\alpha}. \tag{5.32}$$

Indeed, we have

$$\partial_G \mathcal{N}_1(G,W) = -\widehat{F}_0 - 6F_0 G - 3G^2 - 2k^2 W V_0 - k^2 W^2,$$

where $\mathcal{N}_1$ is given in (5.5). Then, using (5.30), $|G(R)| \le \kappa\varepsilon^2 R^{-2}$ and $|W(R)| \le \kappa\varepsilon R^{-2}$, we get

$$\begin{aligned}
|\partial_G \mathcal{N}_1[G,W]| &\le M\Big(\varepsilon^{2-2\alpha} + \kappa\frac{\varepsilon^2}{R^2} + \kappa^2\frac{\varepsilon^4}{R^4} + \kappa k^2 |V_0(R)|\frac{\varepsilon}{R^2} + \kappa^2 k^2 \frac{\varepsilon^2}{R^4}\Big) \\
&\le M(\varepsilon^{2-2\alpha} + \kappa\varepsilon^{2-2\alpha} + \kappa^2\varepsilon^{4-4\alpha} + \kappa k\varepsilon^{-\alpha}\varepsilon^{2-2\alpha} + \kappa^2 k^2 \varepsilon^{-2\alpha}\varepsilon^{2-2\alpha}) \\
&\le M\varepsilon^{2-2\alpha}\Big(1 + \kappa + \kappa^2\varepsilon^{2-2\alpha} + \kappa\frac{\varepsilon^{1-\alpha}}{q} + \kappa^2 q^{-2}\varepsilon^{2-2\alpha}\Big) \le M\varepsilon^{2-2\alpha},
\end{aligned}$$

where we have used again that $\frac{\varepsilon^{1-\alpha}}{q} \le 1$. With respect to $\partial_W \mathcal{N}_1[G,W]$, we have

$$\partial_W \mathcal{N}_1[G,W] = -2k^2 V_0(F_0+G) - 2k^2 W(F_0+G).$$

Then, using (5.31),

$$|\partial_W \mathcal{N}_1[G,W]| \le M\Big(k\varepsilon^{1-\alpha} + k^2\frac{\varepsilon}{R^2}\Big) \le M\Big(k\varepsilon^{1-\alpha} + k^2\varepsilon^{1-2\alpha}\Big) \le Mk\varepsilon^{1-\alpha}\Big(1 + \frac{\varepsilon^{1-\alpha}}{q}\Big),$$

provided $\frac{\varepsilon^{1-\alpha}}{q} < 1$, and (5.32) is proven.

Finally, using bounds (5.32) of $\partial_W \mathcal{N}_1$, $\partial_G \mathcal{N}_2$,

$$\begin{aligned}
&|\mathcal{N}_1[G_2,W_2](R) - \mathcal{N}_1[G_1,W_1](R)| \\
&\qquad \le M\varepsilon^{2-2\alpha}|G_1(R) - G_2(R)| + Mk\varepsilon^{1-\alpha}|W_1(R) - W_2(R)|,
\end{aligned}$$

and therefore, recalling that $k = \varepsilon q^{-1}$,

$$\begin{aligned}
\|\mathcal{N}_1[G_2,W_2] - \mathcal{N}_1[G_1,W_1]\|_2 &\le M\varepsilon^{2-2\alpha}\|G_1 - G_2\|_2 + Mk\varepsilon^{1-\alpha}\|W_1 - W_2\|_2 \\
&\le M\varepsilon^{1-\alpha}q^{-1}[\![(G_1,W_1) - (G_2,W_2)]\!].
\end{aligned}$$

This bound and (5.29) lead to the Lipschitz constant of $\mathcal{F}_1$, which is $M\frac{\varepsilon^{1-\alpha}}{q}$.

From these computations, we also deduce expression (5.28) using Corollary 5.5.

5.6.2. The Lipschitz constant of $\mathcal{F}_2$. Now we deal with $\mathcal{F}_2[G,W]$ which is defined by

$$\mathcal{F}_2[G,W] = \mathcal{S}_2(\mathcal{N}_2[\mathcal{F}_1[G,W],W]).$$

Recall that $\mathcal{N}_2$ was introduced at (5.7),

$$\begin{aligned}
\mathcal{N}_2[G,W](R) = W^2 &- \frac{q^2}{F_0(R)+G(R)}\Big(F_0''(R) + G''(R) + \frac{F_0'(R)+G'(R)}{R}\Big) \\
&+ 2(V_0(R)+W)\frac{F_0'(R)+G'(R)}{F_0(R)+G(R)},
\end{aligned}$$

We have to deal with each term of the difference

$$\mathcal{S}_2[\mathcal{N}_2[\mathcal{F}_1[G_1, W_1], W_1] - \mathcal{N}_2[\mathcal{F}_1[G_2, W_2], W_2]]$$

separating in a similar way as we did for computing the norm of W_0 in Lemma 5.11. In this proof, we will use without special mention the first item of Lemma 5.13 (already proven) and the bounds in (5.28).

Take $(G_1, W_1), (G_2, W_2) \in \mathcal{X}_2 \times \mathcal{X}_2$ satisfying $\llbracket (G_1, W_1) \rrbracket, \llbracket (G_2, W_2) \rrbracket \le \kappa\varepsilon^2$ and $\|G_1'\|_2, \|G_2'\|_2 \le \kappa\varepsilon$. We first prove

$$\begin{aligned} &\varepsilon\|\mathcal{S}_2[\mathcal{N}_2[G_1, W_1]] - \mathcal{S}_2[\mathcal{N}_2[G_2, W_2]]\|_2 \\ &\qquad \le M\varepsilon\varepsilon^{1-\alpha}\|W_1 - W_2\|_2 + Mq\varepsilon\|G_1' - G_2'\|_2 + Mq^2\varepsilon^{1-\alpha}\|G_1 - G_2\|_2. \end{aligned} \tag{5.33}$$

We define $G_\lambda = (1-\lambda)G_2 + \lambda G_1$ and $W_\lambda = (1-\lambda)W_2 + \lambda W_1$, and we notice that the operator $\mathcal{N}_2$ can be written as

$$\mathcal{N}_2[G, W] = \widetilde{\mathcal{N}}_2[G, G', G'', W].$$

By the mean's value theorem,

$$\begin{aligned} \mathcal{N}_2[G_1, W_1] - \mathcal{N}_2[G_2, W_2] &= (W_1 - W_2)\int_0^1 \partial_W \widetilde{\mathcal{N}}_2[G_\lambda, G_\lambda', G_\lambda'', W_\lambda]\,\mathrm{d}\lambda \\ &\quad + (G_1 - G_2)\int_0^1 \partial_G \widetilde{\mathcal{N}}_2[G_\lambda, G_\lambda', G_\lambda'', W_\lambda]\,\mathrm{d}\lambda \\ &\quad + (G_1' - G_2')\int_0^1 \partial_{G'} \widetilde{\mathcal{N}}_2[G_\lambda, G_\lambda', G_\lambda'', W_\lambda]\,\mathrm{d}\lambda \\ &\quad + (G_1'' - G_2'')\int_0^1 \partial_{G''} \widetilde{\mathcal{N}}_2[G_\lambda, G_\lambda', G_\lambda'', W_\lambda]\,\mathrm{d}\lambda \\ &=: N_1 + N_2 + N_3 + N_4. \end{aligned}$$

We start with $\varepsilon\mathcal{S}_2[N_1]$. We have $\partial_W \widetilde{\mathcal{N}}_2[G, G', G'', W] = 2W + 2\frac{F_0'+G'}{F_0+G}$ and therefore, using the bounds for F_0, F_0' in Corollary 5.10,

$$\begin{aligned} \varepsilon|N_1(R)| &\le \varepsilon\|W_1 - W_2\|_2\left(\frac{\varepsilon M}{R^4} + \frac{\varepsilon^{1-\alpha}k}{R^4}\right) \le \varepsilon\|W_1 - W_2\|_2\left(\frac{\varepsilon M}{R^4} + \frac{\varepsilon^{2-\alpha}q^{-1}}{R^4}\right) \\ &\le M\varepsilon\|W_1 - W_2\|_2\frac{\varepsilon^{1-\alpha}}{R^3}, \end{aligned}$$

where we have used that $\frac{\varepsilon^{1-\alpha}}{q} < 1$. Then, by Lemma 5.7,

$$\varepsilon\|\mathcal{S}_2[N_1]\|_2 \le \varepsilon M\|N_1\|_3 \le M\varepsilon\varepsilon^{1-\alpha}\|W_1 - W_1\|_2.$$

We follow with N_2. It is clear that

$$\begin{aligned} &\varepsilon|\partial_G \widetilde{\mathcal{N}}_2[G, G', G'', W](R)| \\ &\quad = \frac{\varepsilon}{(F_0(R) + G(R))^2}\left|q^2 F_0''(R) + G''(R) + q^2\frac{F_0'(R) + G'(R)}{R}\right. \\ &\qquad\qquad \left. - 2(V_0(R) + W(R))(F_0'(R) + G'(R))\right|. \end{aligned}$$

We use now that $k\varepsilon^{1-\alpha} \leq \varepsilon$ and $\varepsilon R^{-2} \leq \varepsilon^{1-2\alpha} \leq \varepsilon^{1-\alpha}k^{-1}$ and obtain

$$\begin{aligned}
&\varepsilon|R^2\partial_G \widetilde{\mathcal{N}}_2[G,G',G'',W](R)| \\
&\quad \leq M\varepsilon[q\varepsilon^{2(1-\alpha)} + q^2 + \varepsilon^{1-\alpha}q^2 + \varepsilon^{2-2\alpha} + q\varepsilon^{1-\alpha} + q^{-1}\varepsilon^{3-3\alpha} + \varepsilon^{2-2\alpha}] \\
&\quad \leq M\varepsilon q^2,
\end{aligned}$$

where again we have used that $\varepsilon^{1-\alpha} \leq q$. This gives

$$\varepsilon|R\partial_G \widetilde{\mathcal{N}}_2[G,G',G'',W](R)| \leq M\varepsilon^{1-\alpha}q^2.$$

Therefore,

$$\varepsilon|R^3N_2(R)| \leq Mq^2\varepsilon^{1-\alpha}\|G_1 - G_2\|_2,$$

and we obtain $\varepsilon|||\mathcal{S}_2[N_2]|||_2 \leq \varepsilon\|N_2\|_3 \leq M\varepsilon^{1-\alpha}q^2\|G_1 - G_2\|_2$.

With respect to N_3, we have

$$\begin{aligned}
\partial N_\lambda(R) &:= \partial_{G'}\widetilde{\mathcal{N}}_2[G_\lambda, G'_\lambda, G''_\lambda, W_\lambda](R) - 2\frac{V_0(R)}{F_0(R) + G_\lambda(R)} \\
&= -\frac{q^2}{R(F_0(R) + G_\lambda(R))} + 2\frac{W_\lambda(R)}{F_0(R) + G_\lambda(R)}.
\end{aligned}$$

Then

$$\varepsilon|\partial N_\lambda(R)| \leq \frac{Mq^2\varepsilon}{R} + \frac{M\varepsilon^2}{R^2} \leq M\varepsilon(q^2 + \varepsilon^{1-\alpha})\frac{1}{R} \leq M\varepsilon q\frac{1}{R},$$

which implies that

$$\varepsilon|R^3\partial N_\lambda(R)||G'_1(R) - G'_2(R)| \leq M\varepsilon q\|G'_1 - G'_2\|_2,$$

and therefore

$$\begin{aligned}
\varepsilon\left\|\mathcal{S}_2\left[(G'_1 - G'_2)\int_0^1 \partial N_\lambda \, \mathrm{d}\lambda\right]\right\|_2 &\leq \varepsilon\left\|(G'_1 - G'_2)\int_0^1 \partial N_\lambda \, \mathrm{d}\lambda\right\|_3 \\
&\leq M\varepsilon q\|G'_1 - G'_2\|_2. \qquad (5.34)
\end{aligned}$$

We point out that

$$\mathcal{S}_2\left[V_0(R)(G'_1 - G'_2)\int_0^1 \frac{1}{F_0 + G_\lambda}\,\mathrm{d}\lambda\right] = \mathcal{A}\left[(G'_1 - G'_2)\int_0^1 \frac{1}{F_0 + G_\lambda}\,\mathrm{d}\lambda\right],$$

and then

$$\varepsilon\left\|\mathcal{S}_2\left[V_0(R)(G'_1 - G'_2)\int_0^1 \frac{1}{F_0 + G_\lambda}\,\mathrm{d}\lambda\right]\right\|_2 \leq \varepsilon\|G'_1 - G'_2\|_2. \qquad (5.35)$$

Bounds (5.34) and (5.35) imply $\varepsilon\|\mathcal{S}_2[N_3]\|_2 \leq M\varepsilon\|G'_1 - G'_2\|_2$.

Finally, we deal with N_4. Using Lemma 5.9 with

$$h(R) = \int_0^1 \frac{\mathrm{d}\lambda}{F_0 + G_\lambda}, \quad \hat{h}_2(R) = G''_1 - G''_2, \quad \hat{h}_1 = G'_1 - G'_2,$$

we have

$$\varepsilon\|\mathcal{S}_2[N_4]\|_2 \le \varepsilon q^2\|h\widehat{h}_1\|_2 + \varepsilon q^2\||\mathcal{S}_2[h'\widehat{h}_1]\|_2 + 2\varepsilon q^2\|\mathcal{A}[\widehat{h}_1 h]\|_2.$$

Then, we obtain

$$\varepsilon q^2\|h\widehat{h}_1\|_2 \le M\varepsilon q^2\|G_1' - G_2'\|_2,$$

and by Lemma 5.8,

$$\varepsilon q^2\|\mathcal{A}[\widehat{h}_1 h]\|_2 \le M\varepsilon q^2\|G_1' - G_2'\|_2.$$

In addition,

$$\begin{aligned}\varepsilon|h'(R)\widehat{h}_1(R)| &\le \varepsilon|G_1'(R) - G_2'(R)| \int_0^1 \frac{|F_0'(R)| + |G_\lambda'(R)|}{|F_0(R) + G_\lambda(R)|^2}\,\mathrm{d}\lambda \\ &\le M\varepsilon\frac{k\varepsilon^{1-\alpha} + \varepsilon}{R^4}\|G_1' - G_2'\|_2 \le M\varepsilon q\frac{1}{R^3}\|G_1' - G_2'\|_2.\end{aligned}$$

Then, using Lemma 5.7, $\|\mathcal{S}_2[h'\widehat{h}_1]\|_2 \le \|h'\widehat{h}_1\|_3$, and we obtain

$$\varepsilon\|\mathcal{S}_2[N_4]\|_2 \le M\varepsilon q\|G_1' - G_2'\|_2,$$

which finishes the proof of bound (5.33).

Now we define $\varphi_1 = \mathcal{F}_1[G_1, W_1]$, $\varphi_2 = \mathcal{F}_1[G_2, W_2]$. By bound (5.33), using that the Lipschitz constant of $\mathcal{F}_1$ is $M\frac{\varepsilon^{1-\alpha}}{q}$ and also (5.28), we have

$$\begin{aligned}&\varepsilon\|\mathcal{S}_2[\mathcal{N}_2[\varphi_1, W_1]] - \mathcal{S}_2[\mathcal{N}_2[\varphi_2, W_2]]\|_2 \\ &\quad\le M\varepsilon^{1-\alpha}\llbracket(G_1, W_1) - (G_2, W_2)\rrbracket + \varepsilon^{1-\alpha}\|\varphi_1 - \varphi_2\|_2 + \varepsilon\|\varphi_1' - \varphi_2'\|_2 \\ &\quad\le M\varepsilon^{1-\alpha}\llbracket(G_1, W_1) - (G_2, W_2)\rrbracket + \varepsilon^{1-\alpha}q^{-1}\llbracket(G_1, W_1) - (G_2, W_2)\rrbracket,\end{aligned}$$

and the proof of Lemma 5.13 is finished.

6. Existence result in the inner region. Proof of Theorem 4.5

We want to find solutions of equations (2.6) departing the origin that remain close to $(f_0^{\mathrm{in}}(r), v_0^{\mathrm{in}}(r)) = (f_0(r), qv_0(r))$ defined by (3.20), where we recall that $f_0(r)$ is the unique solution of (3.17) and $v_0(r)$ is the solution of (3.18),

$$\begin{aligned}f_0'' + \frac{f_0'}{r} - f_0\frac{n^2}{r^2} + f_0(1 - f_0^2) = 0, \quad f_0(0) = 0, \quad \lim_{r\to\infty} f_0(r) = 1, \\ v_0' + \frac{v_0}{r} + 2v_0\frac{f_0'}{f_0} + (1 - f_0^2 - k^2) = 0, \quad v_0(0) = 0.\end{aligned} \tag{6.1}$$

Then v_0 can be expressed (see (3.19)) as a function of $f_0(r)$ by writing

$$v_0(r) = -\frac{1}{rf_0^2(r)}\int_0^r \xi f_0^2(\xi)(1 - f_0^2(\xi) - k^2)\,\mathrm{d}\xi.$$

The asymptotic and regularity properties of f_0, v_0 are given in Proposition 4.4 and will be used along the proof of Theorem 4.5. Again, as in Section 5, the proof of such result relies on a fixed point argument.

Let us now introduce the Banach spaces we shall be working with. For any $0 < s_1$ and $c > 0$, we define $w(s) = f_0'(\frac{s}{\sqrt{2}}) > 0$, $w_0(s) = v_0^2(s) f_0(s) > 0$ and

$$\mathcal{X} = \left\{\psi\colon [0,s_1] \to \mathbb{R},\ \psi \in \mathcal{C}^0([0,s_1]),\ \sup_{s\in[0,s_1]} \left|\frac{\psi(s)}{w(s)+cw_0(s)}\right| < \infty\right\}, \tag{6.2}$$

endowed with the norm

$$\|\psi\| = \sup_{s\in[0,s_1]} \left|\frac{\psi(s)}{w(s)+cw_0(s)}\right|.$$

We stress that in $\mathcal{X}$, the norm $\|\cdot\|$ and

$$\|\psi\|_{\text{aux}} = \sup_{s\in[0,s_*]} \frac{|\psi(s)|}{s^{n-1}} + \sup_{s\in[s_*,s_1]} \left(\frac{1}{s^3} + c\frac{|\log s|^2}{s^2}\right)^{-1} |\psi(s)|,$$

for any given $s_* \in (0,s_1)$ are equivalent (see Lemma 4.4). We also introduce the Banach space

$$\mathcal{Y} = \{\psi\colon [0,s_1] \to \mathbb{R},\ \psi \in \mathcal{C}^0([0,s_1]),\ \|\psi\|_n < \infty\},$$

where the norm $\|\cdot\|_n$ is defined by

$$\|\psi\|_n = \sup_{s\in[0,s_*]} \frac{|\psi(s)|}{s^{n}} + \sup_{s\in[s_*,s_1]} \left(\frac{1}{s^3} + c\frac{|\log s|^2}{s^2}\right)^{-1} |\psi(s)|,$$

which satisfies that $\mathcal{Y} \subset \mathcal{X}$.

Finally, for any fixed $m, l, \nu > 0$, we define

$$\mathcal{Z}_m^{l,\nu} = \{\psi\colon [0,s_1] \to \mathbb{R},\ \psi \in \mathcal{C}^0([0,s_1]),\ \|\psi\|_m^{l,\nu} < \infty\},$$

and the norm

$$\|\psi\|_m^{l,\nu} = \sup_{s\in[0,s_*]} \frac{|\psi(s)|}{s^m} + \sup_{s\in[s_*,s_1]} \frac{|\psi(s)|s^l}{|\log s|^\nu}.$$

From now on, we will fix s_* (independent of q and k) as the minimum value which guarantees that, for $s \geq s_*$, $f_0(s) \geq \frac{1}{2}$ and the asymptotic expression for K_n, I_n as $s \to \infty$ in Table 1 is satisfied for $s \geq s_*$, namely

$$\begin{aligned} K_n(s) &= \sqrt{\frac{\pi}{2s}} e^{-s}\left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right), \\ I_n(s) &= \sqrt{\frac{1}{2\pi s}} e^{s}\left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right) \end{aligned} \tag{6.3}$$

with $s \geq s_*$.

6.1. The fixed point equation

We denote by $\widehat{v} = \frac{v}{q}$ and we shall derive a system of two coupled fixed point equations equivalent to

$$f'' + \frac{f'}{r} - f\frac{n^2}{r^2} + f(1 - f^2 - q^2\widehat{v}^2) = 0, \tag{6.4a}$$

$$f\widehat{v}' + f\frac{\widehat{v}}{r} + 2\widehat{v}f' + f(1 - f^2 - k^2) = 0. \tag{6.4b}$$

We thus start by noting that since q is small, we may write $(f, \widehat{v})$ as a perturbation around $(f_0(r), v_0(r))$ of the form $(f, \widehat{v}) = (f_0(r) + g, v_0(r) + w)$. Therefore, using that f_0, v_0 are solutions of (6.1), equation (6.4a) can be expressed as

$$g'' + \frac{g'}{r} - g\frac{n^2}{r^2} + g(1 - 3f_0^2(r)) = \widehat{H}[g, w], \tag{6.5}$$

with

$$\widehat{H}[g, w](r) = g^3 + 3g^2 f_0(r) + q^2(v_0(r) + w)^2(g + f_0(r)),$$

along with the initial condition $g(0) = 0$. We also have that equation (6.4b) can be written like

$$w' + \frac{w}{r} + w\frac{f_0'}{f_0} = g(g + 2f_0) - \frac{v_0 + w}{f_0(f_0 + g)}(f_0 g' - f_0' g), \tag{6.6}$$

along with $w(0) = 0$.

We now write the differential equations (6.5) and (6.6) as a fixed point equation. We start by pointing out that, equivalently to what happens for the outer equations, one cannot explicitly solve the homogeneous linear problem associated to (6.5). However, we shall conveniently modify equation (6.5) to obtain a set of dominant linear terms at the left-hand side for which we will have explicit solutions.

We first note that, as shown in [2], $f_0(r)$ very rapidly approaches the value of 1. Inspired by this, we define

$$\widehat{\mathcal{E}}[g] := g'' + \frac{g'}{r} - g\frac{n^2}{r^2} + 3g(1 - f_0^2(r)),$$

and therefore, equation (6.5) reads $\widehat{\mathcal{E}}[g] - 2g = \widehat{H}[g, w](r)$, which motivates to perform the change

$$g = -\frac{\widehat{H}[0, 0]}{2} + \Delta g$$

into (6.5). Denoting by

$$h_0 = -\frac{\widehat{H}[0, 0]}{2} = \frac{1}{2}q^2 v_0^2 f_0,$$

Δg is found to satisfy

$$\Delta g'' + \frac{\Delta g'}{r} - \Delta g\frac{n^2}{r^2} - 2\Delta g = \widehat{H}[h_0 + \Delta g] - \widehat{H}[0, 0] - \widehat{\mathcal{E}}[h_0] - 3\Delta g(1 - f_0^2(r)),$$

along with $\Delta g(0) = 0$. Now we perform the change $s = \sqrt{2}r$ and we denote by $\delta g(s) = \Delta g(\frac{s}{\sqrt{2}})$, $\delta v(s) = w(\frac{s}{\sqrt{2}})$, $\widetilde{f}_0(s) = f_0(\frac{s}{\sqrt{2}})$, $\widetilde{v}_0(s) = v_0(\frac{s}{\sqrt{2}})$ and $\widetilde{h}_0(s) = h_0(\frac{s}{\sqrt{2}})$. Therefore,

$$\delta g'' + \frac{\delta g'}{s} - \delta g\Big(1 + \frac{n^2}{s^2}\Big) = \mathcal{N}_1[\delta g, \delta v], \tag{6.7}$$

where

$$\mathcal{N}_1[\delta g, \delta v](s) = -\frac{3}{2}(1 - \widetilde{f}_0^2(s))\delta g + \frac{1}{2}(H[\delta g + \widetilde{h}_0, \delta v] - H[0,0]) - \frac{1}{2}\mathcal{E}[\widetilde{h}_0] \tag{6.8}$$

with

$$H[g, \delta v](s) = g^3 + 3g^2 \widetilde{f}_0(s) + q^2(\widetilde{v}_0(s) + \delta v)^2(\widetilde{f}_0(s) + g),$$
$$\mathcal{E}[\widetilde{h}_0](s) = \widehat{\mathcal{E}}[h_0](s\sqrt{2}) = 2\Big(\widetilde{h}_0'' + \frac{\widetilde{h}_0'}{s} - \frac{n^2}{s^2}\widetilde{h}_0\Big) + 3\widetilde{h}_0(1 - \widetilde{f}_0^2(s)).$$

The homogeneous linear equation associated to (6.7), namely

$$\delta g'' + \frac{\delta g'}{s} - \delta g\Big(1 + \frac{n^2}{s^2}\Big) = 0,$$

has solutions K_n, I_n, the modified Bessel functions. They satisfy the property that their Wronskian is $\frac{1}{s}$. Therefore, equation (6.7) may also be written, for any $s_1 > 0$, like

$$\begin{aligned}
\delta g(s) = {} & K_n(s)\Big(c_1 + \int_{s_1}^{s} \xi I_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi\Big) \\
& + I_n(s)\Big(c_2 - \int_{s_1}^{s} \xi K_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi\Big), \\
\delta g'(s) = {} & K_n'(s)\Big(c_1 + \int_{s_1}^{s} \xi I_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi\Big) \\
& + I_n'(s)\Big(c_2 - \int_{s_1}^{s} \xi K_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi\Big),
\end{aligned}$$

where c_1, c_2 are so far free parameters. It is well known (see expansions in Table 1 as $s \to 0$) that $K_n(s) \to \infty$ and $I_n(s)$ is zero as $s \to 0$, if $n \geq 1$. Then, in order to have solutions bounded at $s = 0$, we have to impose

$$c_1 - \int_0^{s_1} \xi I_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi = 0.$$

Therefore,

$$\delta g(s) = K_n(s)\int_0^{s} \xi I_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi + I_n(s)\Big(c_2 + \int_s^{s_1} \xi K_n(\xi)\mathcal{N}_1[\delta g, \delta v](\xi)\,\mathrm{d}\xi\Big).$$

For any $s_1 > 0$, we introduce the linear operator

$$\widehat{\mathcal{S}}_1[\psi](s) = K_n(s)\int_0^{s} \xi I_n(\xi)\psi(\xi)\,\mathrm{d}\xi + I_n(s)\int_s^{s_1} \xi K_n(\xi)\psi(\xi)\,\mathrm{d}\xi.$$

We have proven the following result.

Lemma 6.1. *For any* $\mathbf{b} \in \mathbb{R}$*, we define*

$$\delta\hat{\mathbf{g}}_0(s) = I_n(s)\mathbf{b}.$$

Then, if δg *is a solution of* (6.7) *satisfying* $\delta g(0) = 0$*, there exists* $\mathbf{b}$ *such that*

$$\delta g = \delta\hat{\mathbf{g}}_0 + \hat{\mathcal{S}}_1 \circ \mathcal{N}_1[\delta g, \delta v]. \tag{6.9}$$

We emphasize that $\mathcal{N}_1$, given in (6.8), has linear terms in δg. In fact, we decompose

$$\mathcal{N}_1[\delta g, \delta v] = \mathcal{L}[\delta g] + \mathcal{R}_1[\delta g, \delta v]$$

with

$$\begin{aligned} \mathcal{L}[\delta g](s) &= -\frac{3}{2}(1 - \tilde{f}_0^2(s))\delta g(s), \\ \mathcal{R}_1[\delta g, \delta v](s) &= \frac{1}{2}(H[\delta g + \tilde{h}_0, \delta v] - H[0,0]) - \frac{1}{2}\mathcal{E}[\tilde{h}_0]. \end{aligned} \tag{6.10}$$

Therefore, equation (6.9) is rewritten as

$$\delta g = \delta\hat{\mathbf{g}}_0 + \hat{\mathcal{S}}_1 \circ \mathcal{L}[\delta g] + \hat{\mathcal{S}}_1 \circ \mathcal{R}_1[\delta g, \delta v] \tag{6.11}$$

with $\delta\hat{\mathbf{g}}_0$ defined in Lemma 6.1.

Lemma 6.2. *There exist* $0 < c,\ L \le 1$ *such that for any* $0 < s_* < s_1$*, the linear operator* $\mathcal{T} := \hat{\mathcal{S}}_1 \circ \mathcal{L}$ *satisfies* $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ *with* $0 < c < 1$ *the constant defining the norm* $\|\cdot\|$ *of* $\mathcal{X}$ *(see* (6.2)*) and* $\|\mathcal{T}\| \le L < 1$*. As a consequence,* $\mathrm{Id} - \mathcal{T}$ *is invertible.*

Proof. In [3], it is proven that the linear operator

$$\begin{aligned} \tilde{\mathcal{T}}[h](s) := \frac{3}{2}K_n(s)\int_0^s \xi I_n(\xi)(1 - \tilde{f}_0^2(\xi))h(\xi)\,\mathrm{d}s \\ + \frac{3}{2}I_n(s)\int_s^\infty \xi K_n(\xi)(1 - \tilde{f}_0^2(\xi))h(\xi)\,\mathrm{d}s, \end{aligned}$$

is contractive in the Banach space defined by

$$\tilde{\mathcal{X}} = \Big\{\psi : [0,\infty) \to \mathbb{R},\ \psi \in C^0[0,\infty),\ \|\psi\|_w := \sup_{s \ge 0} \frac{|\psi(s)|}{w(s)} < \infty\Big\}.$$

The proof is based on the fact that

$$\begin{aligned} |\tilde{\mathcal{T}}[h](s)| \le \frac{3}{2}K_n(s)\int_0^s \xi I_n(\xi)(1 - \tilde{f}_0^2(\xi))\|h\|_w w(\xi)\,\mathrm{d}\xi \\ + \frac{3}{2}I_n(s)\int_s^\infty \xi K_n(\xi)(1 - \tilde{f}_0^2(\xi))\|h\|_w w(\xi)\,\mathrm{d}\xi \le \|h\|_w T(s), \end{aligned}$$

where the function T is defined by

$$\begin{aligned} T(s) := \frac{3}{2}K_n(s)\int_0^s \xi I_n(\xi)(1 - \tilde{f}_0^2(\xi))w(\xi)\,\mathrm{d}\xi \\ + \frac{3}{2}I_n(s)\int_s^\infty \xi K_n(\xi)(1 - \tilde{f}_0^2(\xi))w(\xi)\,\mathrm{d}\xi \end{aligned}$$

and satisfies $\|T\|_w = \tilde{L} < 1$.

Let now $h \in \mathcal{X}$,

$$\begin{aligned}|\mathcal{T}[h](s)| &\leq \frac{3}{2} K_n(s)\|h\| \int_0^s \xi I_n(\xi)(1-\tilde{f}_0^2(\xi))(w(\xi)+cw_0(\xi))\,\mathrm{d}\xi \\ &\quad + \frac{3}{2} I_n(s)\|h\| \int_s^{s_1} \xi K_n(\xi)(1-\tilde{f}_0^2(\xi))(w(\xi)+cw_0(\xi))\,\mathrm{d}\xi \\ &\leq \|h\|(T(s)+R(s)),\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}R(s) = \frac{3c}{2} K_n(s) \int_0^s \xi I_n(\xi)(1-\tilde{f}_0^2(\xi))w_0(\xi)\,\mathrm{d}\xi \\ + \frac{3c}{2} I_n(s) \int_s^{s_1} \xi K_n(\xi)(1-\tilde{f}_0^2(\xi))w_0(\xi)\,\mathrm{d}\xi.\end{aligned}$$

When $s \in [0, s_*]$,

$$R(s) \leq cM\left(s^{-n}\int_0^s \xi^{2n+3}\,\mathrm{d}\xi + s^n \int_s^{s_*} \xi^2\,\mathrm{d}\xi + s^n \int_{s_*}^{\infty} \xi K_n(\xi)\frac{|\log\xi|^2}{\xi^2}\,d\xi\right) \leq cMs^n.$$

For $s \in [s_*, s_1]$, using $1 - f_0^2(s) = \mathcal{O}(s^{-2})$,

$$\begin{aligned}R(s) &\leq cM\left(\frac{e^{-s}}{\sqrt{s}}\int_0^{s*} \xi^{2n+3}\,\mathrm{d}\xi + \frac{e^{-s}}{\sqrt{s}}\int_{s_*}^{s} e^{\xi}\frac{|\log\xi|^2}{\xi^{\frac{7}{2}}}\,\mathrm{d}\xi + \frac{e^{s}}{\sqrt{s}}\int_s^{s_1} e^{-\xi}\frac{|\log\xi|^2}{\xi^{\frac{7}{2}}}\,\mathrm{d}\xi\right) \\ &\leq cM\left(\frac{e^{-s}}{\sqrt{s}} + \frac{|\log s|^2}{s^4}\right) \leq cM\frac{1}{s^3} \leq cM(w(s)+cw_0(s)).\end{aligned}$$

Therefore, using (6.12) one obtains $\|\mathcal{T}[h]\| \leq \|h\|(\tilde{L} + cb_0)$, where b_0 is a constant which is independent of c.

Taking $c \leq \min\{1, \frac{1-\tilde{L}}{2b_0}\}$ so that $L := \tilde{L} + cb_0 \leq \frac{\tilde{L}+1}{2} < 1$, the proof is finished. ∎

As a consequence of this lemma, equation (6.11) can be expressed as

$$\begin{aligned}\delta g &= \delta\mathbf{g}_0 + \mathcal{S}_1 \circ \mathcal{R}_1[\delta g, \delta v], \\ \delta\mathbf{g}_0 &:= (\mathrm{Id} - \mathcal{T})^{-1}[\delta\hat{\mathbf{g}}_0], \quad \mathcal{S}_1 := (\mathrm{Id} - \mathcal{T})^{-1} \circ \hat{\mathcal{S}}_1,\end{aligned} \tag{6.13}$$

and we recall that $\delta\hat{\mathbf{g}}_0$ was defined in Lemma 6.1.

Lemma 6.3. *There exists a function $I(s)$ satisfying*

$$I'(s_1)K_n(s_1) - I(s_1)K_n'(s_1) = \frac{1}{s_1}, \quad |I(s_1)|, |I'(s_1)| \leq M\frac{1}{\sqrt{s_1}}e^{s_1},$$

such that $\delta\mathbf{g}_0(s) = I(s)\mathbf{b}$.

Proof. Recall that (see Lemma 6.2 for the definition of $\mathcal{T}$)

$$\begin{aligned}\mathcal{T}[h](s) = -\frac{3}{2}K_n(s)\int_0^s \xi I_n(\xi)(1-\tilde{f}_0(\xi))h(\xi)\,\mathrm{d}\xi \\ - \frac{3}{2}I_n(s)\int_s^{s_1} \xi K_n(\xi)(1-\tilde{f}_0(\xi))h(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Since $\delta\mathbf{g}_0 = (\mathrm{Id} - \mathcal{T})^{-1}[\delta\hat{\mathbf{g}}_0]$, by definition of the operator $\mathcal{T}$ it is clear that

$$\delta\mathbf{g}_0(s) = \sum_{m\geq 0} \mathcal{T}^m[\delta\hat{\mathbf{g}}_0](s),$$

and therefore, using that the operator $\mathcal{T}$ is linear and $\delta\hat{\mathbf{g}}_0(s) = I_n(s)\mathbf{b}$, we conclude

$$\delta\mathbf{g}_0(s) = \sum_{m\geq 0} \mathcal{T}^m[\delta\hat{\mathbf{g}}_0](s) = \Big(\sum_{m\geq 0} \mathcal{T}^m[I_n](s)\Big)\mathbf{b} =: I(s)\mathbf{b}.$$

Notice that if $\mathbf{b} = 0$, one can take $I(s) = I_n(s)$ and we are done. Assume then that $\mathbf{b} \neq 0$. Then, from $\delta\mathbf{g}_0 - \mathcal{T}[\delta\mathbf{g}_0] = \delta\hat{\mathbf{g}}_0 = I_n(s)\mathbf{b}$, one deduces that

$$\begin{aligned} I_n(s) &= I(s) - \frac{3}{2}K_n(s)\int_0^s \xi I_n(\xi)(1-\tilde{f}_0(\xi))I(\xi)\,\mathrm{d}\xi \\ &\quad - \frac{3}{2}I_n(s)\int_s^{s_1} \xi K_n(\xi)(1-\tilde{f}_0(\xi))I(\xi)\,\mathrm{d}\xi, \\ I_n'(s) &= I'(s) - \frac{3}{2}K_n'(s)\int_0^s \xi I_n(\xi)(1-\tilde{f}_0(\xi))I(\xi)\,\mathrm{d}\xi \\ &\quad - \frac{3}{2}I_n'(s)\int_s^{s_1} \xi K_n(\xi)(1-\tilde{f}_0(\xi))I(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Therefore,

$$I'(s_1)K_n(s_1) - I(s_1)K_n'(s_1) = s_1^{-1}.$$

To finish, we observe that $\|\delta\mathbf{g}_0\| \leq M\|\delta\hat{\mathbf{g}}_0\| = M\|I_n\|\mathbf{b}$ for some positive constant M. That is, $\|I\| \leq M\|I_n\|$. Then, from the asymptotic expression of I_n in (6.3), we deduce that $I_n(s)(w(s)+cw_0(s))^{-1}$ is an increasing function and then we have $\|I_n\| = (w(s_1) + cw_0(s_1))^{-1}I_n(s_1)$ and then

$$|(w(s_1)+cw_0(s_1))^{-1}I(s_1)| \leq M(w(s_1)+cw_0(s_1))^{-1}I_n(s_1),$$

which implies $|I(s_1)| \leq MI_n(s_1) \leq Ms_1^{-\frac{1}{2}}e^{s_1}$. The bound for $|I'(s_1)|$ comes from (6.3) and the fact that $I'(s_1) = [s_1^{-1} + I(s_1)K_n'(s_1)](K_n(s_1))^{-1}$. ■

Now we deal with equation (6.6) which, along with the initial condition $w(0) = 0$, is equivalent to

$$w(r) = \frac{1}{rf_0^2(r)}\int_0^r \xi f_0^2(\xi)\Big[g(g+2f_0) - \frac{v_0+w}{f_0(f_0+g)}(f_0g' - f_0'g)\Big]\,\mathrm{d}\xi.$$

Therefore, recalling that $g = h_0 + \Delta g$, the function $\delta v(s) = w(\frac{s}{\sqrt{2}})$ satisfies

$$\delta v(s) = \mathcal{S}_2 \circ \mathcal{R}_2[\delta g, \delta v] \tag{6.14}$$

with

$$\mathcal{S}_2[\psi](s) = \frac{\sqrt{2}}{2s\tilde{f}_0^2(s)}\int_0^s \xi\tilde{f}_0^2(\xi)\psi(\xi)\,\mathrm{d}\xi \tag{6.15}$$

and

$$\mathcal{R}_2[\delta g, \delta v](s) = (\widetilde{h}_0 + \delta g)(2\widetilde{f}_0 + \widetilde{h}_0 + \delta g) + \frac{\widetilde{v}_0 + \delta v}{\widetilde{f}_0(\widetilde{f}_0 + \widetilde{h}_0 + \delta g)}[\widetilde{f}_0(\widetilde{h}_0' + \delta g') - \widetilde{f}_0'(\widetilde{h}_0 + \delta g)]. \tag{6.16}$$

In view of (6.13) and (6.14), we are looking for solutions of the associated fixed point equation. However, as for the *outer region*, for several technical reasons, we consider instead the equivalent version of the fixed point equation given by

$$(\delta g, \delta v) = \mathcal{F}[\delta g, \delta v], \tag{6.17}$$

where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ with

$$\mathcal{F}_1 = \delta \mathbf{g_0} + \mathcal{S}_1 \circ \mathcal{R}_1, \quad \mathcal{F}_2[\delta g, \delta v] = \mathcal{S}_2 \circ \mathcal{R}_2[\mathcal{F}_1[\delta g, \delta v], \delta v]. \tag{6.18}$$

Remark 6.4. This strategy of using $\mathcal{F}_1$, the first component of the fixed point operator, to compute $\mathcal{F}_2$ reminds of the Gauss-Seidel method for solving linear systems. One could say then that the operator $\mathcal{F}$ is a *Gauss-Seidel* fixed point operator.

Remark 6.5. We note that we need to guarantee that $\widetilde{f}_0(s) + \widetilde{h}_0(s) + \delta g(s) > 0$ for $s \in [0, s_1]$, in order for the operator $\mathcal{R}_2$ to be well defined. The following bounds, which are a straightforward consequence of Proposition 4.4, will be crucial to guarantee the well-posedness of $\mathcal{R}_2$:

$$|\widetilde{h}_0(s)| \le Mq^2 s^{n+2}, \ \ s \to 0, \qquad |\widetilde{h}_0(s)| \le Mq^2 \frac{|\log s|^2}{s^2}, \ \ s \gg 1,$$
$$\mathcal{E}[\widetilde{h}_0](s) \sim Mq^2 s^n, \ \ s \to 0, \qquad |\mathcal{E}[\widetilde{h}_0](s)| \le Mq^2 \frac{|\log s|^2}{s^4} \le Mq^2 \frac{1}{s^3}, \ \ s \gg 1.$$

Moreover, $|\widetilde{h}_0'(s)| \le Mq^2 |\log s|^2 s^{-3}$ for $s \gg 1$.

In what follows, we simplify the notation by dropping the symbol $\widetilde{\ }$ of $\widetilde{f}_0$, $\widetilde{v}_0$ and $\widetilde{h}_0$. Now we reformulate Theorem 4.5 to adapt it to the fixed point setting.

Theorem 6.6. *Let $\eta > 0$, $0 < \mu_0 < \mu_1$ and take $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ with $\mu_0 \le \mu \le \mu_1$. There exist $q_0 = q_0(\mu_0, \mu_1, \eta) > 0$ and $\rho_0 = \rho_0(\mu_0, \mu_1, \eta) > 0$, $M = M(\mu_0, \mu_1, \eta) > 0$ such that, for any $q \in [0, q_0]$ and*

$$0 < \rho < \rho_0,$$

taking s_1 as

$$s_1 = e^{\frac{\rho}{q}},$$

if $\mathbf{b}$ satisfies

$$\mathbf{b} = s_1^{-\frac{3}{2}} e^{-s_1} \rho^2 \widehat{\mathbf{b}}, \quad |\widehat{\mathbf{b}}| \le \eta, \tag{6.19}$$

then there exists a family of solutions $(\delta g(s, \mathbf{b}), \delta v(s, \mathbf{b}))$ of the fixed point equation (6.17) which is continuous with respect to μ and $\widehat{\mathbf{b}}$, defined for $0 \le s \le s_1$ and satisfies

$$\|\delta g\| + \|\delta g'\| + \|\delta v\|_1^{1,3} \le Mq^2.$$

The function δg can be decomposed as

$$\delta g(s,\mathbf{b}) = \delta g_0(s,\mathbf{b}) + \delta g_1(s,\mathbf{b})$$

with $\delta g_0(s,\mathbf{b}) = I(s)\mathbf{b} + \widetilde{\delta g}_0(s)$ *and* $I(s)$ *satisfies* $I'(s_1)K_n(s_1) - I(s_1)K_n'(s_1) = s_1^{-1}$. *Moreover,*

(i) *there exist* $q_* = q_*(\mu_0,\mu_1)$ *and* $M_0 = M_0(\mu_0,\mu_1)$ *such that for* $q \in [0, q_0*]$,

$$\|\widetilde{\delta g}_0\| + \|\widetilde{\delta g}_0'\| \le M_0 q^2,$$

(ii) *and for* $q \in [0, q_0]$,

$$\|\delta g_1\|,\ \|\delta g_1'\| \le M q^2 \rho^2.$$

As we did in the *outer region*, we prove this proposition in three main steps. We first study the continuity of the linear operators $\mathcal{S}_1$, $\mathcal{S}_2$ in Section 6.2 in the defined Banach spaces. After that, in Section 6.3 we study $\mathcal{F}[0,0]$ and finally, in Section 6.4 we prove that the operator $\mathcal{F}$ is Lipschitz.

From now on, we fix η, μ_0, μ_1, we will take q_0, ρ_0 as small as we need and $\mathbf{b}$ a constant satisfying (6.19). As a convention, in the proof there appear a number of different constants, depending on η, μ_0, μ_1 but independent of q which, to simplify the notation, will all be simply denoted as M.

6.2. *The linear operators*

The following results provide bounds and differentiability properties of the linear operators $\mathcal{S}_1$, $\mathcal{S}_2$ defined in (6.13) and (6.15).

Lemma 6.7. *Let s_1, c be such that $0 < s_* < s_1$ and $0 < c \le 1$, and let $\psi \in \mathcal{X}$. Then, the function $\mathcal{S}_1[\psi]$ is a differentiable function in $(0,s_1)$ such that $\mathcal{S}_1[\psi] \in \mathcal{Y} \subset \mathcal{X}$, $\mathcal{S}_1[\psi]' \in \mathcal{X}$ and*

$$\|\mathcal{S}_1[\psi]\|_n \le M\|\mathcal{S}_1[\psi]\| \le M\|\psi\|, \quad \|\mathcal{S}_1[\psi]'\| \le M\|\psi\|$$

for M a constant independent of s_1, s_0, c.

Proof. Let $\psi \in \mathcal{X}$. One has

$$|\mathcal{S}_1[\psi](s)| \le M\|\psi\|\Big[K_n(s)\int_0^s \xi I_n(\xi)(w(\xi)+cw_0(\xi))\,\mathrm{d}\xi \\ + I_n(s)\int_s^{s_1} \xi K_n(\xi)(w(\xi)+cw_0(\xi))\,\mathrm{d}\xi\Big],$$

where we have used that $\|(\mathrm{Id}-\mathcal{T})^{-1}\| \le M$. If $s \in [0,s_*]$, then

$$|K_n(s)| \le M s^{-n}, \quad |I_n(s)| \le M s^n, \quad w(s) + cw_0(s) \le M s^{n-1},$$

and therefore,

$$|\mathcal{S}_1[\psi](s)| \le M\|\psi\|\left(s^{-n}\int_0^s \xi^{2n}\,d\xi + s^n\int_s^{s_*} 1\,\mathrm{d}\xi + s^n \int_{s_*}^{s_1} \xi K_n(\xi)\,\mathrm{d}\xi\right) \le M\|\psi\|s^n,$$

where we have used that

$$\int_{s_*}^{s_1} \xi K_n(\xi)\,\mathrm{d}\xi \le \int_{s_*}^{\infty} \xi K_n(\xi)\,\mathrm{d}\xi \le M.$$

When $s \in [s_*, s_1]$,

$$\begin{aligned} |\mathcal{S}_1[\psi](s)| &\le M\|\psi\|\Big(\frac{e^{-s}}{\sqrt{s}}\int_0^{s_*} \xi^{2n}\,\mathrm{d}\xi + \frac{e^{-s}}{\sqrt{s}}\int_{s_*}^{s} \sqrt{\xi}e^{\xi}\Big(\frac{1}{\xi^3} + c\frac{(\log\xi)^2}{\xi^2}\Big)\,\mathrm{d}\xi \\ &\quad + \frac{e^{s}}{\sqrt{s}}\int_{s}^{s_1} \sqrt{\xi}e^{-\xi}\Big(\frac{1}{\xi^3} + c\frac{(\log\xi)^2}{\xi^2}\Big)\,\mathrm{d}\xi\Big) \\ &\le M\|\psi\|\Big(\frac{1}{s^3} + c\frac{|\log s|^2}{s^2}\Big) \le M\|\psi\|(w(s) + c w_0(s)), \end{aligned}$$

which easily follows upon using that for any $\nu, l \in \mathbb{N}$,

$$\int_{s_*}^{s} e^{\xi}\frac{|\log\xi|^l}{\xi^\nu}\,\mathrm{d}\xi \le Me^{s}\frac{|\log s|^l}{s^\nu}, \qquad \int_{s}^{s_1} e^{-\xi}\frac{|\log\xi|^l}{\xi^\nu}\,\mathrm{d}\xi \le Me^{-s}\frac{|\log s|^l}{s^\nu}.$$

Therefore, $\|\mathcal{S}_1[\psi]\|_n \le M\|\psi\|$.

As for $\mathcal{S}_1[\psi]'$, we notice that

$$(\mathrm{Id} - \mathcal{T}) \circ \mathcal{S}_1[\psi]'(s) = K_n'(s)\int_0^s \xi I_n(\xi)\psi(\xi)\,\mathrm{d}\xi + I_n'(s)\int_s^{s_1} \xi K_n(\xi)\psi(\xi)\,\mathrm{d}\xi,$$

and so analogous computations as the ones for $\mathcal{S}_1[\psi]$ lead to the result. ∎

Lemma 6.8. *Let us fix s_1 such that $0 < s_* < s_1$. Then if $\psi \in \mathcal{Z}_0^{2,l}$, the function $\mathcal{S}_2[\psi]$, defined in (6.15), is a differentiable function in $(0, s_1)$ such that $\mathcal{S}_2[\psi] \in \mathcal{Z}_1^{1,l+1}$ and*

$$\|\mathcal{S}_2[\psi]\|_1^{1,l+1} \le M\|\psi\|_0^{2,l}.$$

In addition, if $\psi \in \mathcal{Z}_0^{\nu,l}$ with $\nu > 2$, the function $\mathcal{S}_2[\psi]$ is a differentiable function in $(0, s_1)$ such that $\mathcal{S}_2[\psi] \in \mathcal{Z}_1^{1,0}$ and

$$\|\mathcal{S}_2[\psi]\|_1^{1,0} \le M\|\psi\|_0^{\nu,l}.$$

The constant $M > 0$ does not depend on s_1.

Proof. Let $\psi \in \mathcal{Z}_0^{2,l}$. When $s \in [0, s_*]$, we have

$$|\mathcal{S}_2[\psi](s)| \le \frac{\sqrt{2}}{2sf_0^2(s)}\int_0^s \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi \le M\|\psi\|_0^{2,l}\frac{1}{s^{2n+1}}\int_0^s \xi^{2n+1}\,\mathrm{d}\xi \le M\|\psi\|_0^{2,l}s.$$

When $s \in [s_*, s_1]$,

$$\begin{aligned} |\mathcal{S}_2[\psi](s)| &\le \frac{1}{sf_0^2(s)}\int_0^{s_*} \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi + \frac{1}{sf_0^2(s)}\int_{s_*}^{s} \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi \\ &\le \frac{M}{s}\|\psi\|_0^{2,l} + \frac{M}{s}\|\psi\|_0^{2,l}\int_{s_*}^{s}\frac{(\log\xi)^l}{\xi}\,\mathrm{d}\xi \le M\|\psi\|_0^{2,l}\Big(\frac{1}{s} + \frac{|\log s|^{l+1}}{s}\Big). \end{aligned}$$

Finally, let $\psi \in \mathcal{Z}_0^{\nu,l}$ with $\nu > 2$. Then for $s \in [0, s_*]$,

$$
\begin{aligned}
|\mathcal{S}_2[\psi](s)| &\le \frac{1}{s f_0^2(s)} \int_0^s \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi \le M\|\psi\|_0^{\nu,l} \frac{1}{s^{2n+1}} \int_0^s \xi^{2n+1}\,\mathrm{d}\xi \\
&\le M\|\psi\|_0^{\nu,l} s,
\end{aligned}
$$

and if $s \in [s_*, s_1]$,

$$
\begin{aligned}
|\mathcal{S}_2[\psi](s)| \le& \frac{1}{s f_0^2(s)} \int_0^{s_*} \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi + \frac{1}{s f_0^2(s)} \int_{s_*}^s \xi f_0^2(\xi)|\psi(\xi)|\,\mathrm{d}\xi \\
\le& \frac{M}{s}\|\psi\|_0^{\nu,l} + \frac{M}{s}\|\psi\|_0^{\nu,l} \int_{s_*}^s \frac{(\log\xi)^l}{\xi^{\nu-1}}\,\mathrm{d}\xi \le \|\psi\|_0^{\nu,l} \frac{M}{s}.
\end{aligned}
$$

∎

6.3. *The independent term*

We now deal with the first iteration of the fixed point procedure given by equation (6.17), namely we study $\mathcal{F}[0,0]$.

Lemma 6.9. *Let $0 < c \le 1$ as in Lemma 6.2, let $0 < \mu_0 < \mu_1$, and take $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ with $\mu_0 \le \mu \le \mu_1$. There exist $q^* = q^*(\mu_0, \mu_1) > 0$ and $M = M(\mu_0, \mu_1) > 0$ such that, for any $q \in [0, q^*]$ and $0 < \rho < \frac{\pi}{2n}$, for $0 < s_* < s_1 \le e^{\frac{\rho}{q}}$, given $\eta > 0$ and $\mathbf{b}$ satisfying (6.19), the function $(\delta g_0, \delta v_0) = \mathcal{F}[0,0]$ belongs to $\mathcal{X} \times \mathcal{Z}_1^{1,3}$, δg_0 is a differentiable function belonging to $\mathcal{X}$ and*

$$
\|\delta g_0'\|,\ \|\delta g_0\| \le M(1+\eta)q^2, \quad \|\delta v_0\|_1^{1,3} \le M(1+\eta)q^2.
$$

Furthermore, $\delta g_0 \in \mathcal{Y}$ with $\|\delta g_0\|_n \le M(1+\eta)q^2$, and $\delta v_0 \in \mathcal{Z}_1^{1,1}$ with $\|\delta v_0\|_1^{1,1} \le M\rho^2$.

Proof. Notice that $s_1 k < 1$ if q is small enough. We have $\delta g_0 = \delta\mathbf{g}_0 + \mathcal{S}_1 \circ \mathcal{R}_1[0,0]$. We recall that

$$
\delta\mathbf{g}_0(s) = (\mathrm{Id} - \mathcal{T})^{-1}[\delta\hat{\mathbf{g}}_0],
$$

where $\delta\hat{\mathbf{g}}_0(s) = I_n(s)\mathbf{b}$. Using that I_n is an increasing positive function, the norms $\|\cdot\|$, $\|\cdot\|_{\mathrm{aux}}$ are equivalent and $I_n(s) = \mathcal{O}(s^n)$ as $s \to 0$,

$$
\begin{aligned}
\|\delta\mathbf{g}_0\|_n &\le M\|\delta\hat{\mathbf{g}}_0\| \le M|\mathbf{b}| I_n(s_1)(w(s_1) + c w_0(s_1))^{-1} \\
&\le M|\mathbf{b}| I_n(s_1)\Big(\frac{1}{s_1^3} + c\frac{|\log s_1|^2}{s_1^2}\Big)^{-1}.
\end{aligned}
$$

Since $s_1 > s_*$, the asymptotic expression (6.3) for $I_n(s_1)$ applies and then, since $\mathbf{b}$ satisfies (6.19), we conclude that $\|\delta\mathbf{g}_0\| \le M\eta q^2$.

We now compute $\mathcal{R}_1[0,0]$ (see (6.10)):

$$
\begin{aligned}
\mathcal{R}_1[0,0] &= -\frac{1}{2}\mathcal{E}[h_0] + \frac{1}{2}(H[h_0,0] - H[0,0]) \\
&= -\frac{1}{2}\mathcal{E}[h_0] + \frac{1}{2}(h_0^3 + 3h_0^2 f_0 + q^2 v_0^2 h_0).
\end{aligned}
$$

Therefore, using the estimates for f_0, v_0, h_0 and $\mathcal{E}[h_0]$ in Proposition 4.4 and Remark 6.5, we have

$$\sup_{s\in[0,s_*]} |\mathcal{R}_1[0,0](s)| \leq Mq^2 s^n, \qquad \sup_{s\in[s_*,s_1]} |\mathcal{R}_1[0,0](s)| \leq M\frac{q^2|\log s|^2}{s^4} + M\frac{q^4|\log s|^4}{s^4}.$$

Using that for any $l \in \mathbb{Z}$, $|\log s|^l s^{-1}$ is bounded if $s \in (2, s_1)$ and $s^{-3} \leq Mw(s)$, we have

$$\sup_{s\in[s_*,s_1]} |\mathcal{R}_1[0,0](s)| \leq Mq^2\frac{1}{s^3} \leq Mq^2(w(s) + cw_0(s)).$$

As a consequence, $\mathcal{R}_1[0,0] \in \mathcal{Y} \subset \mathcal{X}$, $\|\mathcal{R}_1[0,0]\| \leq Cq^2$ and using Lemmas 6.2 and 6.7, we obtain

$$\|\mathcal{S}_1[\mathcal{R}_1[0,0]]\|_n \leq M\|\mathcal{S}_1[\mathcal{R}_1[0,0]]\| \leq M\|\mathcal{R}_1[0,0]\| \leq Mq^2.$$

Moreover, $\|\mathcal{S}_1[\mathcal{R}_1[0,0]]'\| \leq Mq^2$.

We deal now with δv_0. First, we notice that $f_0 + h_0 + \delta g_0 > 0$. Indeed, we have, for $s \in [0, s_*]$, $f_0(s) \geq M|s|^n$ for some positive constant M (see Proposition 4.4). Therefore, if q is small enough,

$$f_0(s) + h_0(s) + \delta g_0(s) \geq Cs^n - Mq^2|s|^{n+2} - Mq^2|s|^n > 0.$$

For $s \geq s_*$, since $f_0(s) \geq \frac{1}{2}$, taking q small enough,

$$f_0(s) + h_0(s) + \delta g_0(s) \geq \frac{1}{2} - Mq^2\frac{|\log s|^2}{s^2} - Mq^2\frac{1}{s^3} - Mq^2\frac{|\log s|^2}{s^2} > \frac{1}{4}.$$

We conclude then that δv_0 is well defined. Now we are going to prove that it belongs to $\mathcal{Z}_1^{1,3}$. By definition, $\delta v_0 = \mathcal{F}_2[0,0] = \mathcal{S}_2 \circ \mathcal{R}_2[\delta g_0, 0]$ with $\mathcal{R}_2$ defined by (6.16),

$$\mathcal{R}_2[\delta g_0, 0] = (h_0 + \delta g_0)(2f_0 + h_0 + \delta g_0) + \frac{v_0[f_0(h_0' + \delta g_0') - f_0'(h_0 + \delta g_0)]}{f_0(f_0 + h_0 + \delta g_0)}.$$

Therefore, using that $\delta g_0 \in \mathcal{Y}$, for $s \in [0, s_*]$ we have

$$|\mathcal{R}_2[\delta g_0, 0](s)| \leq M(1+\eta)^2(s^{2n} + 1) \leq M(1+\eta)q^2.$$

On the other hand, for $s \in [s_*, s_1]$,

$$|\mathcal{R}_2[\delta g_0, 0](s)| \leq M(1+\eta)\frac{q^2|\log s|^2}{s^2} + M(1+\eta)\frac{q^2|\log s|^3}{s^3} \leq M(1+\eta)\frac{q^2|\log s|^2}{s^2}.$$

As a consequence, $\mathcal{R}_2[\delta g_0, 0] \in \mathcal{Z}_0^{2,2}$ with norm $\|\mathcal{R}_2[\delta g_0, 0]\|_0^{2,2} \leq M(1+\eta)q^2$. Therefore, by Lemma 6.8 $\delta v_0 \in \mathcal{Z}_1^{1,3}$ with norm $\|\delta v_0\|_1^{1,3} \leq M(1+\eta)q^2$, and thus, for $s \leq s_1 \leq e^{\frac{\rho}{q}}$

$$|\delta v_0(s)| \leq M(1+\eta)q^2\frac{|\log s|^3}{s} \leq M(1+\eta)\rho^2\frac{|\log s|}{s}. \qquad \blacksquare$$

6.4. *The contraction mapping*

In what follows, we shall show that the fixed point equation (6.17) is a contraction in a suitable Banach space. We define the norm

$$\llbracket(\delta g, \delta v)\rrbracket = \|\delta g\| + \|\delta v\|_1^{1,3}$$

in the product space $\mathcal{X} \times \mathcal{Z}_1^{1,3}$, and we notice that, under the conditions of Lemma 6.9, we have proved that $\llbracket(\delta g_0, \delta v_0)\rrbracket \le \kappa_0 q^2$, where $\kappa_0 = \kappa_0(\mu_0, \mu_1, \eta)$.

Lemma 6.10. *Let μ_0, μ_1, η, $\mathbf{b}$ and μ be as in Lemma 6.9, and take $\varepsilon = \mu e^{-\frac{\pi}{2nq}}$ with $\mu_0 \le \mu \le \mu_1$. There exist $q_0 = q_0(\mu_0, \mu_1, \eta) > 0$ and $M = M(\mu_0, \mu_1, \eta) > 0$ such that, for any $q \in [0, q_0]$, $0 < \rho < \frac{\pi}{2n}$ and $0 < s_* < s_1 \le e^{\frac{\rho}{q}}$, we have if $(\delta g_1, \delta v_1), (\delta g_2, \delta v_2) \in \mathcal{X} \times \mathcal{Z}_1^{1,3}$ satisfying $\llbracket(\delta g_1, \delta v_1)\rrbracket, \llbracket(\delta g_2, \delta v_2)\rrbracket \le 2\kappa_0 q^2$, then*

(1) *with respect to $\mathcal{F}_1$,*

$$\|\mathcal{F}_1[\delta g_1, \delta v_1] - \mathcal{F}_1[\delta g_2, \delta v_2]\| \le M q^2 \|\delta g_1 - \delta g_2\| + M c^{-1}\rho^2 \|\delta v_1 - \delta v_2\|_1^{1,3},$$

(2) *and for $\mathcal{F}_2$,*

$$\|\mathcal{F}_2[\delta g_1, \delta v_1] - \mathcal{F}_2[\delta g_2, \delta v_2]\| \le M q^2 \|\delta g_1 - \delta g_2\| + M(\rho^2 c^{-1} + q^2)\|\delta v_1 - \delta v_2\|_1^{1,3}.$$

The remaining part of this section is devoted to prove Theorem 6.6 (Section 6.5 below) and Lemma 6.10 whose proof is divided into two technical sections, Sections 6.6.1 and 6.6.2.

6.5. *Proof of Theorem* 6.6

The proof of the result is a straightforward consequence of the previous analysis. We define $\mathcal{B} = \{(\delta g, \delta v) \in \mathcal{X} \times \mathcal{Z}_1^{2,3}, \llbracket(\delta g, \delta v)\rrbracket \le 2\kappa_0 q^2\}$. The Lipschitz constant of $\mathcal{F}$ in $\mathcal{B}$, $\operatorname{lip}\mathcal{F}$, satisfies the inequality

$$\operatorname{lip}\mathcal{F} \le M(\mu_0, \mu_1, \eta) \max\{q^2, c^{-1}\rho^2\} \le \frac{1}{2},$$

provided q is small enough and $c^{-1}\rho^2 < \frac{1}{2}$, so that $\mathcal{F}$ is a contraction. In addition, if $\llbracket(\delta g, \delta v)\rrbracket \le 2\kappa_0 q^2$, then

$$\begin{aligned}\llbracket\mathcal{F}[\delta g, \delta v]\rrbracket &\le \llbracket\mathcal{F}[0,0]\rrbracket + \llbracket\mathcal{F}[\delta g, \delta v] - \mathcal{F}[0,0]\rrbracket \le \kappa_0 q^2 + \frac{1}{2}\llbracket(\delta g, \delta v)\rrbracket \\ &\le \kappa_0 q^2 + \frac{1}{2} 2\kappa_0 q^2 \le 2\kappa_0 q^2.\end{aligned}$$

Therefore, the operator $\mathcal{F}$ sends $\mathcal{B}$ to itself. The fixed point theorem assures the existence of solutions $(\delta g, \delta v) \in \mathcal{B}$, consequently satisfies,

$$\llbracket(\delta g, \delta v)\rrbracket = \llbracket\mathcal{F}[\delta g, \delta v]\rrbracket \le 2\kappa_0 q^2,$$

and, if $(\delta g, \delta v) = \mathcal{F}[\delta g, \delta v]$, then $\delta g_1 = \mathcal{F}_1[\delta g, \delta v] - \mathcal{F}_1[0,0]$ satisfies

$$\|\mathcal{F}_1[\delta g, \delta v] - \mathcal{F}_1[0,0]\| \leq Mq^2\|\delta g\| + Mc^{-1}\rho^2\|\delta v\|_1^{1,3} \leq Mc^{-1}\rho^2 q^2,$$

provided $q \ll \rho$. The bound for $\|\delta g_1'\|$ follows from the previous bound and Lemma 6.7. Therefore, also using Lemma 6.3, to prove Theorem 6.6 only remains to check the continuity with respect to the parameters μ, $\widehat{\mathbf{b}}$ which is proven as follows. From Lemma 6.3 and Table 1,

$$\delta \mathbf{g}_0 = I(s)\mathbf{b} =: \widehat{I}(s)\widehat{\mathbf{b}}$$

with $|\widehat{I}(s)| \leq M$ if $s \in [0, s_1]$. By construction,

$$(\delta g, \delta v) = \lim_{k\to\infty}(\delta g^{(k)}, \delta v^{(k)}), \quad (\delta g^k, \delta v^k) = \mathcal{F}[\delta g^{(k-1)}, \delta v^{(k-1)}]$$

with initial seed $(\delta g^{(0)}, \delta v^{(0)}) = (0,0)$. Using that the operator depends continuously on μ (through $k = \mu e^{-\frac{\pi}{2nq}}$) and on $\widehat{\mathbf{b}}$ (through $\delta\mathbf{g}_0$) along with the fact that it is a contraction which is uniform with respect to $\mu \in [\mu_0, \mu_1]$, $\widehat{\mathbf{b}} \in [-\eta, \eta]$, we conclude that $(\delta g, \delta v)$ is also continuous with respect to μ, $\widehat{\mathbf{b}}$.

6.6. *Proof of Lemma* 6.10

6.6.1. The Lipschitz constant for $\mathcal{F}_1$. Let $(\delta g_1, \delta v_1)$, $(\delta g_2, \delta v_2)$ belonging to $\mathcal{X} \times \mathcal{Z}_1^{1,3}$, be such that $\|\!|(\delta g_1, \delta v_1)|\!\|, \|\!|(\delta g_2, \delta v_2)|\!\| \leq 2\kappa_0 q^2$.

From the definition of $\mathcal{F}_1$ in (6.18), definition of $\mathcal{R}_1$ in (6.10) and by Lemma 6.7, we have

$$\|\mathcal{F}_1[\delta g_1, \delta v_1] - \mathcal{F}_1[\delta g_2, \delta v_2]\| \leq M\|H[h_0 + \delta g_1, \delta v_1] - H[h_0 + \delta g_2, \delta v_2]\|.$$

Let $\delta g(\lambda) = \delta g_2 + \lambda(\delta g_1 - \delta g_2)$ and $\delta v(\lambda) = \delta v_2 + \lambda(\delta v_1 - \delta v_2)$. Using the mean's value theorem,

$$\begin{aligned} &H[h_0 + \delta g_1, \delta v_1](s) - H[h_0 + \delta g_2, \delta v_2](s) \\ &\quad = \int_0^1 \partial_1 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)(\delta g_1(s) - \delta g_2(s))\,\mathrm{d}\lambda \\ &\qquad + \int_0^1 \partial_2 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)(\delta v_1(s) - \delta v_2(s))\,\mathrm{d}\lambda. \end{aligned} \tag{6.20}$$

We have $\|\delta g(\lambda)\| \leq Aq^2$, $\|\delta v(\lambda)\|_1^{1,3} \leq Bq^2$ and

$$\begin{aligned} \partial_1 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s) &= 3(h_0(s) + \delta g(\lambda)(s))^2 + 6(h_0(s) + \delta g(\lambda)(s)) f_0(s) \\ &\quad + q^2(v_0(s) + \delta v(\lambda)(s))^2, \\ \partial_2 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s) &= 2q^2(v_0(s) + \delta v(\lambda)(s))(f_0(s) + h_0(s) + \delta g(s)). \end{aligned}$$

Then, recalling that $\|h_0\|_{n+2}^{2,2} \leq Mq^2$, we obtain, if $s \in [0, s_*]$,

$$\begin{aligned} &|\partial_1 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)| \leq Mq^4 s^{2n-2} + Mq^2 s^{n-1} + Mq^2 s^2 \leq Mq^2, \\ &|\partial_2 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)| \leq Mq^2 s^n, \end{aligned}$$

and for $s \in [s_*, s_1]$, noticing that

$$|v_0(s) + \delta v(\lambda)(s)| \leq M\Big(\frac{|\log s|}{s} + q^2\frac{|\log s|^3}{s}\Big) \leq M\frac{|\log s|}{s}(1+\rho^2) \leq M\frac{|\log s|}{s}.$$

Then

$$\begin{aligned}|\partial_1 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)| &\leq Mq^4\frac{|\log s|^4}{s^4} + Mq^2\frac{|\log s|^2}{s^2} + Mq^2\frac{|\log s|^2}{s^2}\\ &\leq Mq^2\frac{|\log s|^2}{s^2},\\ |\partial_2 H[h_0 + \delta g(\lambda), \delta v(\lambda)](s)| &\leq Mq^2\frac{|\log s|}{s}.\end{aligned}$$

Using all these bounds in (6.20), one finds that, for $s \in [0, s_*]$, denoting $\delta g_{0j} = h_0 + \delta g_j$, $j = 1, 2$,

$$\begin{aligned}&|H[\delta g_{01}, \delta v_1](s) - H[\delta g_{02}, \delta v_2](s)|\\ &\qquad \leq Mq^2 s^{n-1}\|\delta g_1 - \delta g_2\| + Mq^2 s^{n+1}\|\delta v_1 - \delta v_2\|_1^{1,3}\\ &\qquad \leq Mq^2 s^{n-1}(\|\delta g_1 - \delta g_2\| + \|\delta v_1 - \delta v_2\|_1^{1,3})\\ &\qquad \leq Mq^2(\|\delta g_1 - \delta g_2\| + \|\delta v_1 - \delta v_2\|_1^{1,3}),\end{aligned}$$

and for $s \in [s_*, s_1]$, using again the notation $\delta g_{0j} = h_0 + \delta g_j$, $j = 1, 2$,

$$\begin{aligned}&|H[\delta g_{01}, \delta v_1](s) - H[\delta g_{02}, \delta v_2](s)|\\ &\quad \leq Mq^2\frac{|\log s|^2}{s^2}(w(s) + cw_0(s))\|\delta g_1 - \delta g_2\| + Mq^2\frac{|\log s|^4}{s^2}\|\delta v_1 - \delta v_2\|_1^{1,3}.\end{aligned}$$

Notice that, for $s_* < s < s_1$,

$$\begin{aligned}\frac{|\log s|^2}{s^2}(w(s) + cw_0(s))^{-1} &\leq M\frac{|\log s|^2}{s^2}\Big(\frac{1}{s^3} + c\frac{|\log s|^2}{s^2}\Big)^{-1}\\ &\leq M\Big(\frac{1}{s|\log s|^2} + c\Big)^{-1} \leq \frac{M}{c}.\end{aligned}$$

In addition, if $s_1 \leq e^{\frac{\rho}{q}}$, then $q^2|\log s|^2 \leq \rho^2$. Therefore, since $|\log s|^2 \leq Ms^2$,

$$\begin{aligned}&|H[h_0 + \delta g_1, \delta v_1](s) - H[h_0 + \delta g_2, \delta v_2](s)|\\ &\quad \leq M(w(s) + cw_0(s))q^2\|\delta g_1 - \delta g_2\| + Mc^{-1}\rho^2(w(s) + cw_0(s))\|\delta v_1 - \delta v_2\|_1^{1,3},\end{aligned}$$

which proves the first item in Lemma 6.10.

Remark 6.11. As a consequence, using Lemma 6.2, if $\delta g, \delta v \in \mathcal{X} \times \mathcal{Z}_1^{1,3}$ with $\|\delta g\| \leq 2\kappa_0 q^2$, $\|\delta v\|_1^{1,3} \leq 2\kappa_0 q^2$,

$$\begin{aligned}\|\mathcal{F}_1[\delta g, \delta v]\| &\leq \|\mathcal{F}_1[0,0]\| + \|\mathcal{F}_1[\delta g, \delta v] - \mathcal{F}_1[0,0]\|\\ &\leq \kappa_0 q^2 + Mq^2\|\delta g\| + Mc^{-1}\rho\|\delta v\|_1^{1,3} \leq 2\kappa_0 q^2,\end{aligned}$$

if q is small enough. The bound for the derivative is a consequence of Lemma 6.7.

6.6.2. The Lipschitz constant for $\mathcal{F}_2$. We recall that $\mathcal{F}_2[\delta g, \delta v] = \mathcal{S}_2 \circ \mathcal{R}_2[\mathcal{F}_1[\delta g, \delta v], \delta v]$, where the operator $\mathcal{R}_2$ is defined in (6.16). We rewrite $\mathcal{R}_2 = \mathcal{P} + \mathcal{P}_1 \cdot \mathcal{P}_2$ with

$$\begin{aligned}\mathcal{P}[\delta g, \delta v] &= (h_0 + \delta g)(2f_0 + h_0 + \delta g),\\ \mathcal{P}_1[\delta g, \delta v] &= \frac{v_0 + \delta v}{f_0(f_0 + h_0 + \delta g)},\\ \mathcal{P}_2[\delta g] &= f_0(h_0' + \delta g') - f_0'(h_0 + \delta g).\end{aligned}$$

For $(\delta g_1, \delta v_1)$, $(\delta g_2, \delta v_2)$ such that $\llbracket(\delta g_1, \delta v_1)\rrbracket, \llbracket(\delta g_2, \delta v_2)\rrbracket \leq 2\kappa_0 q^2$, we denote $\overline{\delta g}_j = \mathcal{F}_1[\delta g_j, \delta v_j]$, $j = 1, 2$.

We recall that $\|h_0\|_{n+2}^{2,2} \leq Mq^2$, and we shall deal separately with $\mathcal{P}$, $\mathcal{P}_1 \cdot \mathcal{P}_2$. Starting with $\mathcal{P}$,

$$\begin{aligned}&|\mathcal{P}[\overline{\delta g}_1, \delta v_1](s) - \mathcal{P}[\overline{\delta g}_2, \delta v_2](s)|\\ &\qquad\leq [2|\overline{\delta g}_1(s) - \overline{\delta g}_2(s)| \cdot |f_0(s) + h_0(s)| + |\overline{\delta g}_1(s) + \overline{\delta g}_2(s)| \cdot |\overline{\delta g}_1(s) - \overline{\delta g}_2(s)|].\end{aligned}$$

Therefore, when $s \in [0, s_*]$,

$$|\mathcal{P}[\overline{\delta g}_1, \delta v_1](s) - \mathcal{P}[\overline{\delta g}_2, \delta v_2](s)| \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\| s^{2n-2} \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|$$

and for $s \in [s_*, s_1]$

$$\begin{aligned}|\mathcal{P}[\overline{\delta g}_1, \delta v_1](s) - \mathcal{P}[\overline{\delta g}_2, \delta v_2](s)| &\leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|(w(s) + cw_0(s))\\ &\leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|\left(\frac{1}{s^3} + c\frac{|\log s|^2}{s^2}\right).\end{aligned}$$

As a consequence,

$$\|\mathcal{P}[\overline{\delta g}_1, \delta v_1] - \mathcal{P}[\overline{\delta g}_2, \delta v_2]\|_0^{2,2} \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|,$$

and by Lemma 6.8 and the first item of Lemma 6.10,

$$\begin{aligned}&\|\mathcal{S}_2[\mathcal{P}[\overline{\delta g}_1, \delta v_1]] - \mathcal{S}_2[\mathcal{P}[\overline{\delta g}_2, \delta v_2]]\|_1^{1,3}\\ &\qquad\leq Mq^2\|\delta g_1 - \delta g_2\| + Mc^{-1}\rho^2\|\delta v_1 - \delta v_2\|_1^{1,3}.\end{aligned}\tag{6.21}$$

Now we deal with $\widehat{\mathcal{P}} := \mathcal{P}_1 \cdot \mathcal{P}_2$. Using the mean value theorem, as described in (6.20), yields

$$\begin{aligned}&\widehat{\mathcal{P}}[\overline{\delta g}_1, \delta v_1] - \widehat{\mathcal{P}}[\overline{\delta g}_2, \delta v_2]\\ &\quad= \mathcal{P}_1[\overline{\delta g}_1, \delta v_1](\mathcal{P}_2[\overline{\delta g}_1] - \mathcal{P}_2[\overline{\delta g}_2]) + \mathcal{P}_2[\overline{\delta g}_2](\mathcal{P}_1[\overline{\delta g}_1, \delta v_1] - \mathcal{P}_1[\overline{\delta g}_2, \delta v_2])\\ &\quad= \mathcal{P}_1[\overline{\delta g}_1, \delta v_1](f_0(\overline{\delta g}_1' - \overline{\delta g}_2') - f_0'(\overline{\delta g}_1 - \overline{\delta g}_2))\\ &\qquad+ \mathcal{P}_2[\overline{\delta g}_2]\bigg((\overline{\delta g}_1 - \overline{\delta g}_2)\int_0^1 \partial_1\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)]\,d\lambda\\ &\qquad\qquad+ (\delta v_1 - \delta v_2)\int_0^1 \partial_2\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)]\bigg)d\lambda,\end{aligned}\tag{6.22}$$

where we denote by $\overline{\delta g}(\lambda) = \lambda\overline{\delta g}_1 + (1-\lambda)\overline{\delta g}_2$ and analogously for $\delta v(\lambda)$. We emphasize now that $\overline{\delta g}_j$ is a differentiable function since $\overline{\delta g}_j = \mathcal{F}_1[\delta g_j, \delta v_j] = \mathcal{S}_1 \circ \mathcal{R}_1[\delta g_j, \delta v_j]$ and by Lemma 6.7, the linear operator $\mathcal{S}_1$ converts continuous functions into differentiable ones. Moreover, $\overline{\delta g}_j \in \mathcal{Y}$ and this implies that for $s \in [0, s_*]$,

$$f_0(s) + h_0(s) + \overline{\delta g}(s) \geq M s^n,$$

while for $s \in [s_*, s_1]$, using that $f_0(s) \geq \frac{1}{2}$, we have $f_0(s) + h_0(s) + \overline{\delta g}(s) \geq \frac{1}{4}$ if q is small enough. Taking this into account, one can now bound the terms in (6.22).

For $s \in [0, s_*]$,

$$|\mathcal{P}_1[\overline{\delta g}_1, \delta v_1](s) f_0(s)(\overline{\delta g}_1'(s) - \overline{\delta g}_2'(s))| \leq M\|\overline{\delta g}_1' - \overline{\delta g}_2'\| \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|,$$
$$|\mathcal{P}_1[\overline{\delta g}_1, \delta v_1](s) f_0'(s)(\overline{\delta g}_1(s) - \overline{\delta g}_2(s))| \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\|,$$

and

$$\left|\mathcal{P}_2[\overline{\delta g}_2](s)(\overline{\delta g}_1(s) - \overline{\delta g}_2(s)) \int_0^1 \partial_1\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)](s)\,\mathrm{d}\lambda\right| \leq Mq^2\|\overline{\delta g}_1 - \overline{\delta g}_2\|,$$
$$\left|\mathcal{P}_2[\overline{\delta g}_2](s)(\delta v_1(s) - \delta v_2(s)) \int_0^1 \partial_2\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)](s)\,\mathrm{d}\lambda\right| \leq Mq^2\|\delta v_1 - \delta v_2\|_1^{1,3}.$$

Then for $s \in [0, s_*]$, recalling that $\widehat{\mathcal{P}} := \mathcal{P}_1 \cdot \mathcal{P}_2$,

$$|\widehat{\mathcal{P}}[\overline{\delta g}_1, \delta v_1](s) - \widehat{\mathcal{P}}[\overline{\delta g}_2, \delta v_2](s)| \leq M\|\overline{\delta g}_1 - \overline{\delta g}_2\| + Mq^2\|\delta v_1 - \delta v_2\|_1^{1,3}. \quad (6.23)$$

When $s \in [s_*, s_1]$, using that $s_1 = e^{\frac{\rho}{q}}$ and

$$|\delta v_j(s)| \leq 2\kappa_0 q^2 |\log s|^3 s^{-1} \leq 2\kappa_0\rho^2|\log s| s^{-1},$$

we obtain

$$\begin{aligned}|\mathcal{P}_1[\overline{\delta g}_1, \delta v_1](s) f_0(s)(\overline{\delta g}_1'(s) - \overline{\delta g}_2'(s))| &\leq M\frac{|\log s|^3}{s^3}\|\overline{\delta g}_1' - \overline{\delta g}_2'\| \\ &\leq M\frac{|\log s|^3}{s^3}\|\overline{\delta g}_1 - \overline{\delta g}_2\|, \\ |\mathcal{P}_1[\overline{\delta g}_1, \delta v_1](s) f_0'(s)(\overline{\delta g}_1(s) - \overline{\delta g}_2(s))| &\leq M\frac{|\log s|^3}{s^6}\|\overline{\delta g}_1 - \overline{\delta g}_2\|,\end{aligned}$$

and

$$\begin{aligned}&\left|\mathcal{P}_2[\overline{\delta g}_2](s)(\overline{\delta g}_1(s) - \overline{\delta g}_2(s)) \int_0^1 \partial_1\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)](s)\,\mathrm{d}\lambda\right| \\ &\quad\leq Mq^2\frac{|\log s|^5}{s^5}\|\overline{\delta g}_1 - \overline{\delta g}_2\|, \\ &\left|\mathcal{P}_2[\overline{\delta g}_2](s)(\delta v_1(s) - \delta v_2(s)) \int_0^1 \partial_2\mathcal{P}_1[\overline{\delta g}(\lambda), \delta v(\lambda)](s)\,\mathrm{d}\lambda\right| \\ &\quad\leq Mq^2\frac{|\log s|^5}{s^3}\|\delta v_1 - \delta v_2\|_1^{1,3}.\end{aligned}$$

Then for $s \in [s_*, s_1]$, increasing s_*, if necessary

$$
\begin{aligned}
&|\widehat{\mathcal{P}}[\overline{\delta g}_1, \delta v_1](s) - \widehat{\mathcal{P}}[\overline{\delta g}_2, \delta v_2](s)| \\
&\quad \le \frac{M}{s^{\frac{5}{2}}} \|\overline{\delta g}_1 - \overline{\delta g}_2\| + Mq^2 \frac{1}{s^{\frac{5}{2}}} \|\delta v_1 - \delta v_2\|_1^{1,3}.
\end{aligned} \tag{6.24}
$$

By bounds (6.23) and (6.24), we have

$$
\begin{aligned}
&\|\widehat{\mathcal{P}}[\mathcal{F}_1[\delta g_1, \delta v_1], \delta v_1] - \widehat{\mathcal{P}}[\mathcal{F}_1[\delta g_2, \delta v_2], \delta v_2]\|_0^{\frac{5}{2},0} \\
&\quad \le M\|\overline{\delta g}_1 - \overline{\delta g}_2\| + Mq^2\|\delta v_1 - \delta v_2\|.
\end{aligned}
$$

We use now Lemma 6.8, that $\|\cdot\|_1^{1,3} \le \|\cdot\|_1^{1,0}$ and again the first item in Lemma 6.10 to conclude

$$
\begin{aligned}
&\|\mathcal{S}_2[\widehat{\mathcal{P}}[\mathcal{F}_1[\delta g_1, \delta v_1], \delta v_1]] - \mathcal{S}_2[\widehat{\mathcal{P}}[\mathcal{F}_1[\delta g_2, \delta v_2], \delta v_2]]\|_1^{1,3} \\
&\quad \le Mq^2\|\delta g_1 - \delta g_2\| + M(\rho^2 c^{-1} + q^2)\|\delta v_1 - \delta v_2\|_1^{1,3}.
\end{aligned}
$$

Finally, also by the bound in (6.21), since $\mathcal{R}_2 = \mathcal{P} + \widehat{\mathcal{P}}$, the second item of Lemma 6.10 is proven.

Appendix A. The dominant solutions in the outer region. Proof of Proposition 4.2

Along this appendix, we will work with *outer variables* (see (3.7)) namely $R = kqr$ and, according to definition (3.15) and (3.6),

$$
F_0(R) = F_0(R; k, q) = f_0^{\text{out}}\Big(\frac{R}{\varepsilon}\Big), \quad V_0(R) = V_0(R; q) = k^{-1} v_0^{\text{out}}\Big(\frac{R}{\varepsilon}\Big), \quad \varepsilon = kq.
$$

We also recall that, $V_0(R) = \frac{K'_{inq}(R)}{K_{inq}(R)}$ (see (3.14)), and F_0 was defined in (3.12).

Remark A.1. Since $k = \mu q^{-1} e^{-\frac{\pi}{2nq}}$ with $\mu \in [\mu_0, \mu_1]$, where $\mu_1 > \mu_0 > 0$, the continuity of v_0^{out}, f_0^{out} with respect to μ directly follows from the continuity of K_{inq}, K'_{inq} with respect to R.

The proof of Proposition 4.2 requires a thorough analysis, among other things, of the Bessel function K_{inq}. We separate it into different subsections which correspond to the different items in the proposition.

A.1. The asymptotic behaviour of the dominant outer solution

This short appendix corresponds to the first item in Proposition 4.2. That is, we study f_0^{out}, v_0^{out} for $kqr \gg 1$. Consider $q < \frac{1}{2}$, using the asymptotic expansions when $R \to \infty$ in Table 1 for K_{inq}, we have, as $R \to \infty$,

$$
V_0(R) = \frac{K'_{inq}(R)}{K_{inq}(R)} = -\frac{1 + \frac{c_1}{R} + \mathcal{O}(\frac{1}{R^2})}{1 + \frac{\overline{c}_1}{R} + \mathcal{O}(\frac{1}{R^2})} = -1 - \frac{c_1}{R} + \frac{\overline{c}_1}{R} + \mathcal{O}\Big(\frac{1}{R^2}\Big) \tag{A.1}
$$

with

$$\overline{c}_1 - c_1 = \frac{4(inq)^2 - 1}{8} - \frac{4(inq)^2 + 3}{8} = -\frac{1}{2},$$

and the claim is proved. This expansion is valid for $R \geq R_0$ with R_0 independent of q. The expansion for F_0 is

$$\begin{aligned} F_0(R) &= \sqrt{1 - k^2 V_0^2 - \varepsilon^2 \frac{n^2}{R^2}} = \sqrt{1 - k^2\Big(1 + \frac{1}{R} + \mathcal{O}\Big(\frac{1}{R^2}\Big)\Big) - \varepsilon^2 \frac{n^2}{R^2}} \\ &= \sqrt{1-k^2}\sqrt{1 - \frac{k^2}{R(1-k^2)} + \mathcal{O}\Big(\frac{k^2}{R^2}\Big)} \\ &= \sqrt{1-k^2}\Big(1 - \frac{k^2}{2R(1-k^2)} + \mathcal{O}\Big(\frac{k^2}{R^2}\Big)\Big), \end{aligned}$$

where we have also used that $k = \frac{\varepsilon}{q}$ is small (compare with (3.11)). Going back to the original variables, we obtain the result.

A.2. The behaviour of v_0^{out} in an intermediate region

Now we deal with the asymptotic expression of v_0^{out} in (4.3) (item (ii)) for values of r satisfying $e^{-\frac{n}{2nq}} \leq kqr \leq (qn)^2$. In *outer variables*, it reads as

$$V_0(R) = \frac{nq}{R}\cot\Big(nq\log\Big(\frac{R}{2}\Big) - \theta_{0,nq}\Big)[1 + \mathcal{O}(q^2)], \quad 2e^{-\frac{\pi}{2nq}} \leq R \leq q^2 n^2, \tag{A.2}$$

with $\theta_{0,nq} = \arg\Gamma(1 + inq) = -\gamma nq + \mathcal{O}(q^2)$ and γ the Euler–Mascheroni constant.

Let $\nu = nq$. We first recall some properties of $K_{i\nu}$ with $\nu > 0$, see [12,25]. For $x \in \mathbb{R}$ (in fact, the formula is also satisfied for some complex domains), we have

$$\begin{aligned} K_{i\nu}(x) &= -\frac{\pi i}{2\sinh(\nu\pi)}[I_{-i\nu}(x) - I_{i\nu}(x)], \\ I_\eta(x) &= \Big(\frac{x}{2}\Big)^\eta \sum_{k\geq 0}\Big(\frac{x^2}{2}\Big)^k \frac{1}{k!\Gamma(\eta + k + 1)}, \end{aligned} \tag{A.3}$$

where $\Gamma(z)$ is the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t.$$

Using

$$\Gamma(1 + k + i\nu) = (k + i\nu)\cdots(1 + i\nu)\Gamma(1 + i\nu), \quad |\Gamma(1 \pm i\nu)|^2 = \frac{\pi\nu}{\sinh(\pi\nu)} \tag{A.4}$$

and denoting $\theta_{k,\nu} = \arg(\Gamma(1 + k + i\nu))$, from (A.3) we deduce that

$$K_{i\nu}(x) = -\frac{1}{\nu}\Big(\frac{\nu\pi}{\sinh\pi\nu}\Big)^{\frac{1}{2}} \sum_{k\geq 0}\Big(\frac{x^2}{2}\Big)^k \frac{\sin(\nu\log(\frac{x}{2}) - \theta_{k,\nu})}{k![(k^2 + \nu^2)\cdots(1 + \nu^2)]^{\frac{1}{2}}}. \tag{A.5}$$

By convention, when $k = 0$, $k![(k^2 + \nu^2)\cdots(1 + \nu^2)]^{\frac{1}{2}} = 1$.

By formula (A.4), we have

$$\arg(\Gamma(1+k+\nu i)) = \sum_{l=1}^{k} \arg(l+\nu i) + \arg(\Gamma(1+\nu i)).$$

Now we notice that

$$-\theta_{0,\nu} = -\arg\Gamma(1+\nu i) = \gamma\nu + \mathcal{O}(\nu^2), \tag{A.6}$$

being the Euler–Mascheroni constant γ. Indeed, it is well known (see [1]) that

$$\log\Gamma(1+z) = -\log(1+z) + z(1-\gamma) + \mathcal{O}(z^2).$$

Then

$$\begin{aligned}\Gamma(1+i\nu) &= \frac{1}{1+i\nu} e^{i\nu(1-\gamma)+\mathcal{O}(\nu^2)} = (1-i\nu+\mathcal{O}(\nu^2))(1+i\nu(1-\gamma)+\mathcal{O}(\nu^2))\\ &= 1-\gamma i\nu + \mathcal{O}(\nu^2),\end{aligned}$$

and henceforth, $\arg\Gamma(1+i\nu) = -\gamma\nu + \mathcal{O}(\nu^2)$ as we wanted to check.

We use expansion (A.5) for $K_{i\nu}$ which has a decomposition

$$K_{i\nu}(x) = \frac{1}{\nu}\left[\frac{\nu\pi}{\sinh\nu\pi}\right]^{\frac{1}{2}} \left\{-\sin\left(\nu\log\left(\frac{x}{2}\right) - \theta_{0,\nu}\right) + h(x)\right\} \tag{A.7}$$

with $h(x)$ satisfying $|h(x)| \le C|x|^2$, $|h'(x)| \le C|x|$ and $|h''(x)| \le C$. Therefore,

$$K'_{i\nu}(x) = \left[\frac{\nu\pi}{\sinh\nu\pi}\right]^{\frac{1}{2}} \left\{-\frac{1}{x}\cos\left(\nu\log\left(\frac{x}{2}\right) - \theta_{0,\nu}\right) + \frac{h'(x)}{\nu}\right\}, \tag{A.8}$$

and as a consequence

$$V_0(R) = \frac{nq}{R}\,\frac{\cos(nq\log(\frac{R}{2}) - \theta_{0,nq}) - (nq)^{-1}Rh'(R)}{\sin(nq\log(\frac{R}{2}) - \theta_{0,nq}) - h(R)}$$

with $|h(x)|, |xh'(x)| \le Cx^2$ and $\theta_{0,nq} = \arg\Gamma(1+inq) = -nq\gamma + \mathcal{O}(q^2)$.

We notice now that when $2e^{-\frac{\pi}{2\nu}} \le x \le \nu^2$,

$$-\frac{\pi}{2} + \nu\gamma + \mathcal{O}(\nu^2) \le \nu\log\left(\frac{x}{2}\right) - \theta_{0,\nu} \le -2\nu|\log\nu|(1+\mathcal{O}(|\log\nu|^{-1})).$$

Then, taking $\nu = nq$ we deduce that, for $2e^{-\frac{\pi}{2nq}} \le R \le (qn)^2$,

$$\begin{aligned} a(R) &:= \frac{Rh'(R)}{nq\cos(nq\log(\frac{R}{2}) - \theta_{0,nq})} \le C(nq)^4\frac{1}{(nq)^2\gamma}(1+\mathcal{O}(q^2))\\ &\le C(nq)^2,\\ |b(R)| &:= \left|\frac{h(R)}{\sin(nq\log(\frac{R}{2}) - \theta_{0,nq})}\right| \le C(nq)^4\frac{1}{q|\log q|}(1+\mathcal{O}(|\log q|^{-1}))\\ &\le C(nq)^3 \end{aligned} \tag{A.9}$$

and therefore

$$V_0(R) = \frac{nq}{R}\operatorname{cotan}\Big(nq\log\Big(\frac{R}{2}\Big) - \theta_{0,nq}\Big)\frac{1-a(R)}{1-b(R)}. \tag{A.10}$$

The result in (A.2) (and consequently item (ii) of Proposition 4.2) follows from (A.10) and (A.9).

A.3. Monotonicity of the dominant outer solution

This appendix is devoted to proving item (iii) in Proposition 4.2. First, we will see that v_0^{out}, f_0^{out} are increasing functions for $2e^{-\frac{n}{2nq}} \leq kqr$. It is equivalent to prove that $V_0'(R)$, $F_0'(R)$ are increasing functions in the corresponding domain $2e^{-\frac{n}{2nq}} \leq R$.

We begin with V_0. Using the expansion for K_{inq} in Table 1 and the corresponding expansions for K_{inq}', K_{inq}'', we have, for $R \gg 1$,

$$V_0'(R) = \frac{1}{2R^2} + \mathcal{O}\Big(\frac{1}{R^3}\Big),$$

so that $V_0'(R) > 0$ if $R \gg 1$.

Assume then that there exists $R_* > 2e^{-\frac{n}{2nq}}$ such that $V_0'(R_*) = 0$ and take the larger R_* critical point. That is, $V_0'(R) \neq 0$ if $R > R_*$. Notice that, using that $V_0(R) \to -1$ as $R \to \infty$ and $V_0'(R) > 0$ if $R \gg R_*$ we deduce that $V_0(R_*) < -1$ and $V_0''(R_*) \geq 0$, indeed, if $V_0''(R_*) < 0$, it should be a maximum which is a contradiction. Then, since V_0 is solution of (3.13),

$$\frac{V_0(R_*)}{R_*} + V_0^2(R_*) + \frac{q^2n^2}{R_*^2} - 1 = 0,$$

or equivalently

$$\begin{aligned} V_0(R_*) = v_\pm(R_*) &:= \frac{1}{2}\Big[-\frac{1}{R_*} \pm \sqrt{\frac{1}{R_*^2} + 4\Big(1 - \frac{q^2n^2}{R_*^2}\Big)}\Big] \\ &= \frac{1}{2}\Big[-\frac{1}{R_*} \pm \sqrt{\frac{1}{R_*^2}(1-4q^2n^2) + 4}\Big]. \end{aligned}$$

Note that, when q is small enough, $v_\pm(R)$ are defined for all $R > 0$, and

$$\lim_{R\to 0} v_\pm(R) = -\infty, \qquad \lim_{R\to\infty} v_+(R) = 1, \qquad \lim_{R\to\infty} v_-(R) = -1,$$

$v_-(R) < v_+(R)$ for all $R > 0$. We also have

$$V_0(R) < -1, \quad v_-(R) < V_0(R) < v_+(R) \quad \text{if } R \gg 1.$$

We emphasize that, differentiating equation (3.13), we obtain

$$V_0''(R) + \frac{V_0'(R)}{R} - \frac{V_0(R)}{R^2} + 2V_0V_0' - 2\frac{q^2n^2}{R^3} = 0.$$

Evaluating at $R = R_*$, we have

$$V_0''(R_*) - \frac{V_0(R_*)}{R_*^2} - 2\frac{q^2n^2}{R_*^3} = 0 \Leftrightarrow V_0''(R_*) = \frac{V_0(R_*)}{R_*^2} + 2\frac{q^2n^2}{R_*^3}.$$

That is, assuming that $V_0(R_*) = v_-(R_*)$, we obtain

$$V_0''(R_*) = \frac{1}{2R_*^2}\left[-\frac{1}{R_*} - \sqrt{\frac{1}{R_*^2} + 4\Big(1 - \frac{q^2n^2}{R_*^2}\Big)}\right] + 2\frac{q^2n^2}{R_*^3},$$

and it is clear that, if q is small enough, $V_0''(R_*) < 0$, and therefore we have a contradiction with the fact that R_* cannot be a maximum. We conclude then that $V_0(R_*) = v_+(R_*)$. In this case, $V_0(R_*) < -1$ if and only if

$$-1 + \frac{1}{2R_*} > \frac{1}{2}\sqrt{\frac{1}{R_*^2} + 4\Big(1 - \frac{q^2n^2}{R_*^2}\Big)},$$

which implies that $R_* < \frac{1}{2}$ and

$$1 - \frac{1}{R_*} > 1 - \frac{q^2n^2}{R_*^2} \Leftrightarrow R_* < q^2n^2.$$

We recall that $V_0''(R_*) > 0$. Therefore, using again

$$V_0''(R_*) = \frac{V_0(R_*)}{R_*^2} + 2\frac{q^2n^2}{R_*^3} > 0 \Rightarrow V_0(R_*) > -2\frac{q^2n^2}{R_*}. \tag{A.11}$$

Since $2e^{-\frac{\pi}{2nq}} < R_* < q^2n^2$, using (A.10), we rewrite $V_0(R_*)$ as

$$V_0(R_*) = \frac{nq}{R_*}\frac{\cos(nq\log(\frac{R_*}{2}) - \theta_{0,nq})}{\sin(nq\log(\frac{R_*}{2}) - \theta_{0,nq})}\frac{1 + a(R_*)}{1 + b(R_*)}.$$

Using (A.9) and that the function $\frac{\cos(x)}{\sin(x)}$ is a decreasing function if $x \in [-\frac{\pi}{2}, 0]$, we have

$$\begin{aligned} V_0(R_*) &\le \frac{nq}{R_*}\frac{1 + a(R_*)}{1 + b(R_*)}\frac{\cos(nq\log(\frac{(nq)^2}{2}) - \theta_{0,nq})}{\sin(nq\log(\frac{(nq)^2}{2}) - \theta_{0,nq})} \\ &= \frac{nq}{R_*}\frac{1}{2nq\log(nq)}(1 + \mathcal{O}(q^2|\log q|^2)) = -\frac{1}{2R_*|\log(nq)|}(1 + \mathcal{O}(q^2|\log q|^2)) \end{aligned}$$

which is a contradiction with (A.11). Then we conclude that $V_0'(R) > 0$ for $2e^{-\frac{\pi}{2nq}} < R$.

Note that since we have proved that $V_0'(R) > 0$ for $R \ge 2e^{-\frac{\pi}{2\nu}}$, then by (A.1) $V_0(R) = -1 - \frac{1}{2R} + \mathcal{O}(\frac{1}{R^2})$ if $R \gg 1$ which implies that $V_0(R) \to -1$ when $R \to \infty$ and hence $V_0(R) < -1$ in the same domain.

Differentiating the expression for F_0 (see, for instance, A.1) and using that $V_0'(R) > 0$, we easily obtain $F_0'(R) > 0$.

Going back to the original variables, item (iii) of Proposition 4.2 is proven.

A.4. Bounding the dominant outer solutions

This appendix is devoted to proving the bounds for v_0^{out} and f_0^{out} and its derivatives given in item (iv) of Proposition 4.2.

Let us first provide a technical lemma whose proof is postponed to the end of this section.

Lemma A.2. *There exists $q_0 > 0$ such that if $0 < q < q_0$, the modified Bessel function $K_{inq}(R)$ satisfies*

$$K_{inq}(R) > 0, \quad K'_{inq}(R) < 0, \quad K''_{inq}(R) > 0, \quad \text{for all } R \geq 2e^2 e^{-\frac{\pi}{2qn}}.$$

We point out that, in *outer variables*, in order to prove the bounds in items (iii) and (iv), it is enough to prove the following result (see also Corollary 5.10).

Lemma A.3. *Let $\alpha \in (0,1)$. There exist $q_0 = q_0(\alpha) > 0$ and a constant $M > 0$ such that for any $0 < q < q_0$ and $R \in [R_{\min}, +\infty)$ with $R_{\min}$ satisfying $2e^2 e^{-\frac{\pi}{2qn}} \leq R_{\min} \leq \varepsilon^\alpha$, where $\varepsilon = kq$, one has*

$$|kV_0(R)|,\ |kV_0'(R)R|,\ |kV''(R)R^2| \leq M\varepsilon R_{\min}^{-1},$$

and

$$|R(V_0(R)+1)|,\ |R^2 V_0'(R)|,\ |R^3 V_0''(R)| \leq M.$$

With respect to F_0, we have $F_0(R) \geq \frac{1}{2}$ and

$$|F_0'(R)R^2|, |F_0''(R)R^3| \leq Mk\varepsilon R_{\min}^{-1}, \quad |1-F_0(R)|, |F_0'(R)R|, |F_0''(R)R^2| \leq M\varepsilon^2 R_{\min}^{-2}.$$

Proof. Because of item (iii) of Proposition 4.2, one deduces that V_0 is an increasing and negative function on $[2e^{-\frac{\pi}{2qn}}, \infty]$ and therefore in $[R_{\min}, \infty)$. Therefore, we have $|kV_0(R)| \leq k|V_0(R_{\min})|$. We notice that, from (A.7) and (A.8),

$$V_0(R_{\min}) = \frac{K'_{inq}(R_{\min})}{K_{inq}(R_{\min})} = -\frac{-\frac{1}{R_{\min}}\cos(nq\log(\frac{R_{\min}}{2}) - \theta_{0,nq}) + \frac{h'(R_{\min})}{nq}}{\frac{1}{nq}\{-\sin(nq\log(\frac{R_{\min}}{2}) - \theta_{0,nq}) + h(R_{\min})\}}$$
$$= \frac{nq}{R_{\min}} \frac{\cos(nq\log(\frac{R_{\min}}{2}) - \theta_{0,nq}) - R_{\min}\frac{h'(R_{\min})}{nq}}{\sin(nq\log(\frac{R_{\min}}{2}) - \theta_{0,nq}) - h(R_{\min})}$$

with $h(R)$ satisfying $|h(R)| \leq M|R|^2$ and $|h'(R)| \leq M|R|$. We recall that $\varepsilon = kq = \mu e^{-\frac{\pi}{2nq}}$ and $-\theta_{0,nq} = \gamma nq + \mathcal{O}(q^2)$. Then, since $R_{\min} \leq \varepsilon^\alpha \ll q$,

$$|kV_0(R_{\min})| \leq k\frac{nq}{R_{\min}} \frac{1+\mathcal{O}(R_{\min})}{|\sin(-\frac{\pi\alpha}{2} + \mathcal{O}(q))| + \mathcal{O}(R_{\min}^2)} \leq M\varepsilon R_{\min}^{-1}.$$

Define now the function $g(R) = RV_0(R)$. We want to see that, for $R \geq R_{\min}$, $g'(R) \neq 0$. Assume that, for some R_*, the function has a critical point, namely, $g'(R_*) = R_* V_0'(R_*) + V_0(R_*) = 0$. Then using equation (3.13) satisfied by V_0, we get

$$V_0^2(R_*) - 1 + \frac{q^2n^2}{R_*^2} = 0 \Leftrightarrow V_0^2(R_*) = 1 - \frac{q^2n^2}{R_*^2},$$

which is a contradiction with the fact that $V_0(R) < -1$. Therefore, $g'(R) = RV_0'(R) + V_0(R) \neq 0$ for $R \geq R_{\min}$.

Recall that, for $R \gg 1$,

$$V_0(R) = -1 - \frac{1}{2R} + \mathcal{O}\Big(\frac{1}{R^2}\Big).$$

As a consequence,

$$g'(R) = -1 - \mathcal{O}(R^{-2}) \to -1, \quad \text{as } R \to \infty$$

and therefore, $g'(R) < 0$ for all $R \geq R_{\min}$.

Then $g(R_1) \leq g(R_2)$ if $R_1 \geq R_2$, and using that $g(R) < 0$, we conclude that $|g(R_2)| \leq |g(R_1)|$ when $R_1 \geq R_2$. On the other hand, $|R(V_0(R) + 1)| \leq M$ when $R \geq R_0$ if R_0 is big enough (but independent of q). Thus, if $R_{\min} \leq R \leq R_0$,

$$|R(V_0(R) + 1)| = |RV_0(R)| - R \leq R|V_0(R)| \leq R_0|V_0(R_0)| \leq M\varepsilon R_{\min}^{-1}.$$

With respect to $V_0'(R)$, we have $|R^2 V_0'(R)| \leq M$ if $R \geq R_0$ with R_0 big enough. Take now $R_{\min} \leq R \leq R_0$. We recall that

$$V_0(R) = \frac{K_{inq}'(R)}{K_{inq}(R)} < 0,$$

and we notice that

$$0 < V_0'(R) = \frac{K_{inq}''(R)}{K_{inq}(R)} - \Big(\frac{K_{inq}'(R)}{K_{inq}(R)}\Big)^2 \leq \frac{K_{inq}''(R)}{K_{inq}(R)}.$$

The modified Bessel function K_{inq} satisfies the linear differential equation

$$K_{inq}'' + \frac{K_{inq}'(R)}{R} - K_{inq}(R)\Big(1 - \frac{n^2q^2}{R^2}\Big) = 0.$$

Then, using that, by Lemma A.2, for $R \geq R_{\min} \geq 2e^2 e^{-\frac{\pi}{2nq}}$ we know that $K_{inq}(R) > 0$, $K_{inq}'(R) < 0$ and $K_{inq}''(R) > 0$ and therefore

$$0 < K_{inq}''(R) = -\frac{K_{inq}'(R)}{R} + K_{inq}(R)\Big(1 - \frac{n^2q^2}{R^2}\Big) \leq -\frac{K_{inq}'(R)}{R} + K_{inq}(R).$$

Hence, if $R_{\min} \leq R \leq R_0$,

$$\begin{aligned} |R^2 V_0'(R)| = R^2 V_0'(R) &\leq -R\frac{K_{inq}'(R)}{K_{inq}(R)} + R^2 = R|V_0(R)| + R^2 \\ &\leq R_0|V_0(R_0)| + R_0^2 \leq M. \end{aligned}$$

In addition, using that V_0 satisfies equation (3.13), we have

$$0 < kRV_0'(R) = -kV_0(R) - kR(V_0^2(R) - 1) - k\frac{q^2n^2}{R} \leq -kV_0(R) \leq M\varepsilon R_{\min}^{-1}.$$

Now we deal with $V_0''(R)$. We have, when $R \geq R_0$ with R_0 big enough (but independent of q), $|R^3 V_0''(R)| \leq M$. For $R_{\min} \leq R \leq R_0$,

$$V_0''(R) = \frac{V_0}{R^2} - \frac{V_0'}{R} + 2V_0 V_0'(R) + \frac{n^2 q^2}{R^3}.$$

Therefore, using that $|RV_0(R)|$ and $|R^2 V_0'(R)| \leq M$ for $R_{\min} \leq R \leq R_0$, we obtain

$$|R^3 V_0''(R)| \leq M.$$

Moreover, using that $kqR_{\min}^{-1} \leq \varepsilon^{1-\alpha}$,

$$|kR^2 V_0''(R)| \leq |kV_0(R)| + |kRV_0'(R)| + 2k|V_0(R)||R^2 V_0'(R)| + k\frac{n^2 q^2}{R} \leq M\varepsilon R_{\min}^{-1}.$$

Now we deal with the properties of F_0 and its derivatives. Since $|kV_0(R)| \leq M\varepsilon^{1-\alpha}$ and $F_0(R) = \sqrt{1 - k^2 V_0^2 - \varepsilon^2 n^2 R^{-2}}$, we have

$$F_0(R) = 1 - \sum_{n\geq 1} a_n B_0(R)^n, \quad a_n > 0,$$

with

$$B_0(R) = k^2 V_0^2(R) + \frac{\varepsilon^2 n^2}{R^2}.$$

Then

$$F_0'(R) = -\sum_{n\geq 1} n a_n B_0^{n-1}(R) B_0'(R),$$
$$F_0''(R) = -\sum_{n\geq 1} n a_n [(n-1) B_0^{n-2}(R)(B_0'(R))^2 + B_0^{n-1}(R) B_0''(R)].$$

Using the properties for V_0, we deduce from the above expression, the corresponding ones for F_0. ∎

To finish the proof of Proposition 4.2, we prove Lemma A.2.

Proof of Lemma A.2. We take $\nu = nq \leq \nu_0$. Besides expression (A.5) of $K_{i\nu}$, we also have the integral expression

$$K_{i\nu}(x) = \int_0^\infty e^{-x\cosh t} \cos(\nu t)\, dt, \quad x > 0, \tag{A.12}$$

from which we deduce that $K_{i\nu}(x)$ is real if $x > 0$.

Notice that, from Remark 4.1, there exists x_0 only depending on ν_0 such that $\forall x \geq x_0$,

$$K_{i\nu}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) > 0,$$
$$K_{i\nu}'(x) = -\sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) < 0, \tag{A.13}$$

therefore, we only need to prove that $K_{i\nu}''(x) > 0$.

We first claim that $K''_{i\nu}(x) > 0$ if $x \geq \nu^2$ and $\nu > 0$. Indeed, differentiating twice expression (A.12),

$$K''_{i\nu}(x) = \int_0^\infty e^{-x\cosh t} \cosh^2 t \cos(\nu t)\, \mathrm{d}t.$$

For $0 \leq \nu t \leq \frac{\pi}{4}$, we have $\cos(\nu t) \geq \frac{\sqrt{2}}{2}$ and then, also using that $e^t \leq 2\cosh t \leq e^t + 1 \leq 2e^t$ for $t \geq 0$, we obtain

$$\begin{aligned} K''_{i\nu}(x) &\geq \frac{\sqrt{2}}{2} \int_0^{\frac{\pi}{4\nu}} e^{-x\cosh t} \cosh^2 t\, \mathrm{d}t - \int_{\frac{\pi}{4\nu}}^\infty e^{-x\cosh t} \cosh^2 t\, \mathrm{d}t \\ &\geq \frac{\sqrt{2}}{8} \int_0^{\frac{\pi}{4\nu}} e^{-x\frac{e^t+1}{2}} e^{2t}\, \mathrm{d}t - \int_{\frac{\pi}{4\nu}}^\infty e^{-x\frac{e^t}{2}} e^{2t}\, \mathrm{d}t \\ &= \frac{\sqrt{2}}{8} e^{-\frac{x}{2}} \int_0^{\frac{\pi}{4\nu}} e^{-\frac{x}{2}e^t} e^{2t}\, \mathrm{d}t - \int_{\frac{\pi}{4\nu}}^\infty e^{-\frac{x}{2}e^t} e^{2t}\, \mathrm{d}t. \end{aligned}$$

Note that, performing the obvious change $u = e^t$,

$$\begin{aligned} \int e^{-x\frac{e^t}{2}} e^{2t}\, \mathrm{d}t &= \int e^{-\frac{x}{2}u} u\, \mathrm{d}u = -\frac{2}{x} e^{-\frac{x}{2}u} u + \frac{2}{x} \int e^{-\frac{x}{2}u}\, \mathrm{d}u \\ &= -\frac{2}{x} e^{-\frac{x}{2}u} u - \frac{4}{x^2} e^{-\frac{x}{2}u} = -\frac{2}{x} e^{-\frac{x}{2}e^t} e^t - \frac{4}{x^2} e^{-\frac{x}{2}e^t} \\ &= -\frac{2}{x^2} e^{-\frac{x}{2}e^t} [x e^t + 2] =: -F(t). \end{aligned}$$

We obtain then

$$K''_{i\nu}(x) \geq \left[F(0) - F\left(\frac{\pi}{4\nu}\right)\right] \frac{\sqrt{2}}{8} e^{-\frac{x}{2}} - F\left(\frac{\pi}{4\nu}\right).$$

In order to check that $K''_{i\nu}(x) > 0$, we have to prove the inequality

$$F(0) > F\left(\frac{\pi}{4\nu}\right)\left[1 + \frac{8}{\sqrt{2}} e^{\frac{x}{2}}\right].$$

Since $x \geq 0$, it is enough to check that

$$2 > e^{-\frac{x}{2}(e^{\frac{\pi}{4\nu}} - 1)} (x e^{\frac{\pi}{4\nu}} + 2)\left(1 + \frac{8}{\sqrt{2}} e^{\frac{x}{2}}\right).$$

On one hand, $x \geq \nu^2$ with ν small enough, implies $2 \leq \nu^2 e^{\frac{\pi}{4\nu}} \leq x e^{\frac{\pi}{4\nu}}$. On the other hand, it is clear that $1 \leq e^{\frac{x}{2}}$ if $x > 0$ and $x \leq e^x$. Therefore, the above inequality is satisfied if

$$2 > 6 \frac{8}{\sqrt{2}} e^{-\frac{x}{2}(e^{\frac{\pi}{4\nu}} - 1)} e^x e^{\frac{\pi}{4\nu}} e^{\frac{x}{2}} \Leftrightarrow \frac{\sqrt{2}}{24} > e^{-\frac{x}{2}(e^{\frac{\pi}{4\nu}} - 4) + \frac{\pi}{4\nu}},$$

for all $x \geq \nu^2$. Thus, we need ν to satisfy

$$\frac{\sqrt{2}}{24} > e^{-\frac{\nu^2}{2}(e^{\frac{\pi}{4\nu}}-4)+\frac{\pi}{4\nu}},$$

which is true if ν is small enough.

In conclusion, we have proven that, for $\nu > 0$ small enough and $x \geq \nu^2$, the function K_ν satisfies $K''_{i\nu}(x) > 0$. It remains to prove that $K''_{i\nu}(x) > 0$ if $x \leq \nu^2$. From (A.7) and (A.8), we have

$$K''_{i\nu}(x) = \left[\frac{\nu\pi}{\sinh\nu\pi}\right]^{\frac{1}{2}}\left\{\frac{\nu}{x^2}\sin\left(\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu}\right)+\frac{1}{x^2}\cos\left(\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu}\right) + \frac{h''(x)}{\nu}\right\}. \tag{A.14}$$

For $2e^2e^{-\frac{\pi}{2\nu}} \leq x \leq \nu^2$, it is clear from (A.6)

$$\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu} < 2\nu\log\nu + (\gamma-\log 2)\nu + \mathcal{O}(\nu^2) < 0,$$
$$\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu} > 2\nu - \frac{\pi}{2} + \gamma\nu + \mathcal{O}(\nu^2) > -\frac{\pi}{2}$$

if we take ν small enough. Therefore, if ν is small enough,

$$\begin{aligned}\cos\left(\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu}\right) &\geq \cos\left(-\frac{\pi}{2}+2\nu+\gamma\nu+\mathcal{O}(\nu^2)\right)\\ &= \sin((2+\gamma)\nu+\mathcal{O}(\nu^2)) \geq \left(1+\frac{\gamma}{2}\right)\nu,\\ \sin\left(\nu\log\left(\frac{x}{2}\right)-\theta_{0,\nu}\right) &\geq -1.\end{aligned}$$

Then, from expression (A.14) of $K''_{i\nu}(x)$,

$$K''_{i\nu}(x) \geq \left[\frac{\nu\pi}{x^4\sinh\nu\pi}\right]^{\frac{1}{2}}\left\{\left(1+\frac{\gamma}{2}\right)\nu - \nu - C\frac{x^2}{\nu}\right\} \geq \left[\frac{\nu\pi}{x^4\sinh\nu\pi}\right]^{\frac{1}{2}}\left\{\frac{\gamma}{2}\nu - C\nu^3\right\} > 0$$

if ν is small enough. Therefore, we have just shown that $K''_{i\nu}(x) \geq 0$ if $x \geq 2e^2e^{-\frac{\pi}{2\nu}}$. This result along with the asymptotic expressions (A.13) provides the sign for $K'_{i\nu}$ and $K_{i\nu}$. ■

Appendix B. The dominant solutions in the inner region. Proof of Proposition 4.4

We now prove the asymptotic properties of f_0^{in}, v_0^{in} defined in (3.20). As we have already pointed out, the properties of $f_0 = f_0^{\mathrm{in}}$ and $\partial_r f_0^{\mathrm{in}}$ are all provided in [2]. With respect to the properties of $v_0^{\mathrm{in}}(r) = qv_0(r)$, with v_0 in (3.19), in the second item, in [3] the function

$$\overline{v_0}(r) = -\frac{1}{rf_0^2(r)}\int_0^r \xi f_0^2(\xi)(1-f_0^2(\xi))\,\mathrm{d}\xi,$$

was considered and the same asymptotic properties of $\overline{v_0}$ was considered as the ones stated in the second item but for all $r > 0$. We introduce

$$\Delta v_0(r;k) := v_0(r;k) - \overline{v_0}(r) = k^2\frac{1}{rf_0^2(r)}\int_0^r \xi f_0^2(\xi)\,\mathrm{d}\xi.$$

Remark B.1. Since $v_0^{\text{in}}(r;k) = qv_0(r;k) = q\overline{v_0}(r) + q\Delta v_0(r;k)$ and $k = \mu q^{-1} e^{-\frac{\pi}{2nq}}$, the continuity with respect to $\mu \in [\mu_0, \mu_1]$ is clear.

Note that, if $0 < r \ll 1$, using that $f_0(r) \sim \alpha_0 r^n$,

$$\Delta v_0(r;k) \sim \frac{1}{2n+2} k^2 r, \quad \partial_r \Delta v_0(r;k) \sim k^2 c,$$

for some constant c. Then it is clear that, for $0 < r \ll 1$, the properties of $v_0^{\text{in}}(r;k,q)$ are deduced from the analogous ones for $\overline{v_0}(r)$ proven in [3].

When $kr \le \frac{n}{\sqrt{2}}$ and $r \gg 1$, we have $\frac{1}{2} \le f_0(r) \le 1$. Then

$$|\Delta v_0(r;k)| \le M k^2 r.$$

As a consequence, $|\Delta v_0(r;k)| \le M \frac{n^2}{2r} \le M |\log r| r^{-1}$ if $kr \le \frac{n}{\sqrt{2}}$. In [2], it was already proved that $|\overline{v_0}(r)| \le M |\log r| r^{-1}$. Therefore, this property (and analogously the one for v_0') is satisfied.

It only remains to check that $v_0 < 0$. From its definition (3.19), it is enough to check the inequality $1 - k^2 - f_0^2(r) > 0$ for $0 \le r \le \frac{n}{k\sqrt{2}}$. We first notice that there exists $r_0 \gg 1$ such that

$$1 - f_0^2(r) \ge \frac{n^2}{2r^2}, \quad r \gg r_0.$$

Therefore, for $kr \le \frac{n}{\sqrt{2}}$ and $r \gg r_0$, we have $1 - k^2 - f_0^2(r) \ge 0$. Since f_0 is an increasing function, we have $1 - k^2 - f_0^2(r) \ge 0$ for all $r \ge 0$ such that $kr \le \frac{n}{\sqrt{2}}$.

Now we prove the third item. We first deal with the asymptotic expression of $v_0^{\text{in}} = qv_0$. We use the asymptotic expressions of $f_0(r)$ already proven in the first item, namely

$$f_0(r) = 1 - \frac{n^2}{2r^2} + \mathcal{O}(r^{-4}) \quad \text{as } r \to \infty.$$

We write

$$\begin{aligned} v_0(r) &= -\frac{1}{r f_0^2(r)} \int_0^r \xi f_0^2(\xi)(1 - f_0^2(\xi))\, \mathrm{d}\xi + \frac{k^2}{r f_0^2(r)} \int_0^r \xi f_0^2(\xi)\, \mathrm{d}\xi \\ &=: v_0^1(r) + v_0^2(r). \end{aligned}$$

We take $r_* \gg 1$. It is clear that

$$\frac{k^2}{r f_0^2(r)} \int_0^r \xi f_0^2(\xi)\, \mathrm{d}\xi = \frac{k^2}{r f_0(r)} \int_0^{r_*} \xi f_0^2(\xi)\, \mathrm{d}\xi + \frac{k^2}{r f_0(r)} \int_{r_*}^r \xi f_0^2(\xi)\, \mathrm{d}\xi.$$

Notice that

$$\frac{k^2}{r f_0(r)} \int_0^{r_*} \xi f_0^2(\xi)\, \mathrm{d}\xi = k^2 \mathcal{O}(r^{-1}),$$

and, using $f_0^2(r) = 1 - \frac{n^2}{r^2} + \mathcal{O}(r^{-4})$ if $r, r_* \gg 1$,

$$\frac{k^2}{r f_0(r)} \int_{r_*}^r \xi f_0^2(\xi)\, \mathrm{d}\xi = k^2 \frac{r^2 - r_*^2}{2r} - \frac{k^2 n^2 \log r}{r} + k^2 \mathcal{O}(r^{-1}).$$

Consider now $r_* \gg 1$ and let us define

$$\begin{aligned}\Delta v_0(r,r_*) &:= v_0^1(r) + \frac{n^2}{rf_0^2(r)}\log\Big(\frac{r}{r_*}\Big) + \frac{1}{rf_0^2(r)}\int_0^{r_*} \xi f_0^2(\xi)(1-f_0^2(\xi))\,\mathrm{d}\xi \\ &= \frac{1}{rf_0^2(r)}\int_r^{r_*} \xi f_0^2(\xi)(1-f_0^2(\xi))\,\mathrm{d}\xi + \frac{n^2}{rf_0^2(r)}\log\Big(\frac{r}{r_*}\Big).\end{aligned}$$

It is clear, using again that $f_0^2(r) = 1 - \frac{n^2}{2r^2} + \mathcal{O}(r^{-4})$

$$\begin{aligned}\Delta v_0(r,r_*) &= \frac{1}{rf_0^2(r)}\int_r^{r_*} \frac{n^2}{\xi} + \mathcal{O}\Big(\frac{1}{\xi^3}\Big)\,\mathrm{d}\xi + \frac{n^2}{rf_0^2(r)}\log\Big(\frac{r}{r_*}\Big) \\ &= \mathcal{O}(r^{-3}) + \mathcal{O}(r^{-1}r_*^{-2}).\end{aligned}$$

Therefore, taking $r_* \to \infty$, we have

$$\begin{aligned}\mathcal{O}(r^{-3}) &= v_0^1(r) + \frac{n^2}{rf_0^2(r)}\log r \\ &\quad + \frac{1}{rf_0^2(r)}\lim_{r_*\to\infty}\Big(-n^2\log r_* + \int_0^{r_*} \xi f_0^2(\xi)(1-f_0^2(\xi))\,\mathrm{d}\xi\Big) \\ &= v_0^1(r) + \frac{1}{rf_0^2(r)}(n^2\log r + C_n) = v_0^1(r) + \frac{1}{r}(n^2\log r + C_n)(1+\mathcal{O}(r^{-2})) \\ &= v_0^1(r) + \frac{n^2}{r}\log r + \frac{C_n}{r} + \mathcal{O}(r^{-3}\log r)\end{aligned}$$

with C_n as defined in Theorem 2.5. Collecting all these estimates, the proof of (4.11) is complete.

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