DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **37**, Number **8**, August **2017**

doi:10.3934/dcds.2017177

pp. **4159–4190**

GEVREY ESTIMATES FOR ONE DIMENSIONAL PARABOLIC INVARIANT MANIFOLDS OF NON-HYPERBOLIC FIXED POINTS

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(Communicated by Chongchun Zeng)

Abstract. We study the Gevrey character of a natural parameterization of one dimensional invariant manifolds associated to a *parabolic direction* of fixed points of analytic maps, that is, a direction associated with an eigenvalue equal to 1. We show that, under general hypotheses, these invariant manifolds are Gevrey with type related to some explicit constants. We provide examples of the optimality of our results as well as some applications to celestial mechanics, namely, the Sitnikov problem and the restricted planar three body problem.

1. **Introduction.** Let us consider a dynamical system defined through a map $F: \mathbb{R}^m \to \mathbb{R}^m$ with a fixed point at the origin. To each invariant subspace E of $DF(0)$ one can try to identify its corresponding counterpart for F, that is, a manifold tangent to E at the origin invariant by F, if it exists. Of course, these invariant manifolds need not be unique, or even if they do exist, they can be less regular than the map F, depending on the resonance relations in $\operatorname{Spec} DF(0)_{|E}$. In the case that F is analytic or C^∞, one can even ask if there exists a *formal* invariant manifold tangent to E, that is, a formal power series which solves at all orders an appropriate invariance equation.

One way to obtain manifolds invariant by F is by using the parameterization method. A brief description is the following. If $E \subset \mathbb{R}^m$ is a subspace of dimension n, invariant by $DF(0)$, one can try to find an invariant manifold by F tangent to E

2010 *Mathematics Subject Classification.* Primary: 37D10.

Key words and phrases. Gevrey class, parabolic points, invariant manifolds.

I.B and P.M. have been partially supported by the Spanish MINECO-FEDER Grant MTM2015-65715-P and the Catalan Grant 2014SGR504. The work of E.F. has been partially supported by the Spanish Government grants MTM2013-41168P and MTM2016-80117-P and the Catalan Government grant 2014SGR-1145.

at the origin as an embedding $K : B_\rho \subset \mathbb{R}^n \to \mathbb{R}^m$ (here B_ρ denotes the ball of radius ρ) such that $K(0) = 0$, $DK(0)\mathbb{R}^n = E$ and a *reparameterization* $R : B_\rho \to \mathbb{R}^n$, $R(0) = 0$, satisfying the *invariance equation*

$$F \circ K = K \circ R. \tag{1}$$

Well known examples of invariant manifolds are the strong stable and unstable manifolds, which, roughly speaking, are associated to the eigenvalues λ of $DF(0)$ such that $|\lambda| > \mu > 1$ and $|\lambda| < \nu < 1$, respectively, for given constants μ and ν. See, for instance, [13, 14, 15] and [7, 12, 8] and the references therein. These manifolds are as regular as the map in a neighborhood of the fixed point. In particular, analytic if so is the map. Their expansions in power series are convergent.

When one considers invariant manifolds tangent to subspaces associated to subsets of non-resonant eigenvalues the situation becomes more interesting. The invariance equation can be solved at all orders, due to the non-resonant character of the eigenvalues. This solution provides a formal invariant manifold. In general, this formal series corresponds to a regular meaningful object if one imposes the non-resonant eigenvalues to be of modulus larger (resp. smaller) than one. That is, when the non-resonant manifolds are submanifolds of the strong unstable (resp. stable) manifold. See for instance [7]. If the map is analytic, these non-resonant manifolds are also analytic and, again, their expansions are convergent.

Here we consider the totally resonant case, that is, manifolds tangent to subspaces associated to the eigenvalue 1 and, thus, submanifolds of the center manifold. We call these manifolds *parabolic*.

When the map is tangent to the identity at the fixed point, that is, $DF(0) = \mathrm{Id}$, any subspace of $\mathbb{R}^m$ is invariant by $DF(0)$. In order to identify the subspaces which are susceptible to have an invariant manifold tangent to them it is necessary to pay attention to the first next non-vanishing terms of the Taylor expansion of F at the origin. This is the case considered in [5], when one looks for one dimensional manifolds. See also [18]. In the latter, only analytic manifolds where considered, while the former includes the case of finite differentiability. The former also includes the construction of formal solutions of the invariance equation (1). Under the conditions in [5], if the map F is analytic or $\mathcal{C}^\infty$, the parabolic invariant manifolds exist and are $\mathcal{C}^\infty$ at the fixed point. See also [11] and the survey [1] in the setting of complex dynamics.

In [2] it is studied the case of F analytic, tangent to the identity and with invariant manifolds of dimension two or greater. These manifolds are analytic in their domain, although in general the fixed point is only at their boundary. In this case, however, it is easy to see that in general there are no formal solutions (in the sense of power series) of the invariance equation (1). In the same setting, in [3] it is shown that the invariant manifolds can be approximated by sums of homogeneous functions of increasing order.

In the present paper we assume that F is an analytic local diffeomorphism in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'}$ and satisfies

$$DF(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id}_d & 0 \\ 0 & 0 & C \end{pmatrix} \tag{2}$$

with $1 \notin \operatorname{Spec} C$ and Id_d is the identity matrix in $\mathbb{R}^d$. When $d = 0$, that is, when 1 is a simple eigenvalue of $DF(0)$, this class of maps was studied in [4]. There the authors proved that if the map F has the form

$$F : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^{d'} \mapsto \begin{pmatrix} x - ax^N + z\mathcal{O}_{N-1} + \mathcal{O}_{N+1} \\ Cz + \mathcal{O}_2 \end{pmatrix},$$

where $\mathcal{O}_j$ stands for $\mathcal{O}(\|(x,z)\|^j)$, with $a \neq 0$, $N \geq 2$, the invariance equation admits a formal solution $\hat{K}(t) = \sum_{k\geq 1} K_j t^j$, $K_j \in \mathbb{R}^{1+d'}$, with some polynomial reparameterization R, and that the series is α-Gevrey with $\alpha = 1/(N-1)$, that is, there exist constants $c_1, c_2 > 0$ such that

$$\|K_j\| \leq c_1 c_2^j j!^\alpha, \qquad j \geq 0.$$

Furthermore, if $a > 0$ and $\operatorname{Spec} C \subset \{z \in \mathbb{C} \mid |z| \geq 1,\ z \neq 1\}$, there is an analytic solution K of the invariance equation, defined in some convex set V with $0 \in \partial V$, that is α-Gevrey asymptotic to $\hat{K}$, that is, there exist constants $c_1, c_2 > 0$ such that

$$\left\| K(t) - \sum_{j=1}^{n-1} K_j t^j \right\| \leq c_1 c_2^n n!^\alpha |t|^n, \qquad n \geq 2, \quad t \in V.$$

Here we generalize these results to the case $d > 0$, $d' \geq 0$. That is, if the map F has the linear part (2) and certain conditions on the nonlinear terms are met (see Theorem 3.1), the invariance equation (1) for the map F admits a formal solution $\hat{K}(t)$, which is γ-Gevrey for a precise γ (defined in (4)). We provide examples for which this value of γ is sharp, that is, $\hat{K}(t)$ is not γ'-Gevrey for any $0 \leq \gamma' < \gamma$. These conditions can be seen as non-resonances, because they allow to solve some cohomological equations (see also Claim 4.2). Also they would imply the existence of a *characteristic direction*, if the map was truly tangent to the identity at the fixed point.

Adding some additional conditions (see Theorem 3.3), we also prove that there is a true invariant manifold given by an analytic parameterization K which is γ-Gevrey asymptotic the the formal series $\tilde{K}$ in some complex convex set with 0 at its boundary. We will refer to this manifold as a parabolic manifold and we notice that the information about its internal dynamics is given by $R(t)$ which, in our case, turns out to be a polynomial. Depending on R the parabolic manifold may behave as a (weak) stable manifold (in the sense that the iterates of its points converge to de origin) or a (weak) unstable manifold. In those cases we will denote them by parabolic stable/unstable manifolds.

Of course, the conditions that allow the existence of a formal solution are weaker than the ones we need to impose in order to have a true invariant manifold. However, we prove that if the map possesses a one dimensional parabolic stable invariant manifold to the origin, tangent to a particular direction associated to an eigenvalue equal to 1, and it is non-degenerate (in the sense of Proposition 3.2), then there are suitable coordinates in which the map satisfies our conditions (listed in (3) and hypotheses below).

Our results provide upper bounds for the coefficients of the asymptotic expansion of the invariant manifold. The existence of lower bounds remains open. Although we provide examples that show the optimality of our results, we also prove that if the map is the time one map of an autonomous analytic vector field, satisfying our hypotheses, the invariant manifold, when written as a graph, extends analytically to a neighborhood of the origin (see Claim 4.2). That is, the invariant manifolds can be more regular than what we claim. This is no longer true for the stroboscopic Poincaré map of time-periodic equations (Claim 4.3). However, although obtaining lower bounds is out of the scope of the present work, we show in Proposition 3.8 that the conditions to obtain such lower bounds cannot depend on a finite number of coefficients of the Taylor expansion of the map F.

An important consequence of our present results is the Gevrey character of some invariant manifolds in some problems of Celestial Mechanics. In several instances of the restricted three body, like the Sitnikov problem or the restricted planar three body problem, the parabolic infinity is foliated by periodic orbits. The associated stroboscopic Poincaré map satisfies the conditions of our existence result (Theorem 3.3) with $d' = 0$, which implies that the manifolds are at least 1/3-Gevrey at the origin. See Section 4.3 for more details. Simó and Martínez announced in 2009 [16] that, in the case of the Sitnikov problem, the manifolds are precisely 1/3-Gevrey, which would imply the optimality of our result in the sense that these manifolds are not more regular. The numerical experiments in [17] strongly support the same claim for the restricted circular planar three body problem. These computations and the example we provide in Claim 4.3 move us to conjecture that the invariant manifolds of infinity of the restricted three body problem are exactly 1/3-Gevrey (see Conjecture 4.5 for the precise statement).

The structure of the paper is as follows. In Section 2 we introduce the definitions and notations we will use along the paper. In Section 3 we collect the main results of the paper. Section 4 is devoted to present some examples that show that the Gevrey order we find is optimal. We also show how our theorems apply to the restricted three body problem. The rest of the paper contains the proofs of the results on Section 3. In Section 5 we obtain the formal solution of the invariance equation. Its Gevrey character is studied in Section 6. The existence of the true manifold is proved in Section 7. The appendix contains the proofs of Propositions 3.2 and 3.8.

2. **Set up and notation.** Let $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'}$ be an open neighborhood of $0 = (0,0,0)$. We consider $F : \mathcal{U} \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'}$, the real analytic maps defined by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - ax^N + f_N(x,y,z) + f_{\geq N+1}(x,y,z) \\ y + x^{M-1}B_1 y + x^{M-1}B_2 z + g_M(x,y,z) + g_{\geq M+1}(x,y,z) \\ Cz + h_{\geq 2}(x,y,z) \end{pmatrix}, \tag{3}$$

where:

- $N, M \geq 2$ are integer numbers;
- the constant a is non-zero;
- $1 \notin \operatorname{Spec} C$;
- $f_N(x,y,z)$ is a homogeneous polynomial of degree N such that $f_N(x,0,0) = 0$. We introduce the notation
$$v = \frac{1}{(N-1)!}\partial_x^{N-1}\partial_y f_N(0,0,0) \in \mathbb{R}^d, \quad w = \frac{1}{(N-1)!}\partial_x^{N-1}\partial_z f_N(0,0,0) \in \mathbb{R}^{d'},$$
so that $\partial_y f_N(x,0,0) = x^{N-1}v^\top$ and $\partial_z f_N(x,0,0) = x^{N-1}w^\top$;
- $g_M(x,y,z)$ is a homogeneous polynomial of degree M such that $g_M(x,0,0) = 0$, $D_y g_M(x,0,0) = 0$ and $D_z g_M(x,0,0) = 0$;
- $f_{\geq N+1}$ has order $N+1$ (the function and its derivatives vanish up to order N at $(0,0,0)$), $g_{\geq M+1}$ has order $M+1$ and $h_{\geq 2}$ has order 2.

Since F is real analytic, it can be extended to a complex neighborhood $\mathcal{U}_{\mathbb{C}}$ of $\mathcal{U}$. For simplicity, we will denote also by F this complex extension.

We introduce the following notational conventions we use throughout the paper. We denote by $\hat{W}(t) = \sum_{k\geq 0} W_k t^k$ any formal series in t and if $W(t)$ is a map, we denote $W_k = \frac{1}{k!}D^kW(0)$, if the derivatives are defined. The expressions $W_{\le l}$, $W_{\ge l+1}$, etc. will mean $\sum_{k=0}^{l} W_k t^k$, $\sum_{k\ge l+1} W_k t^k$, etc., and we will use them without further mention. The projection over the x, y or z-component is denoted by π^x, π^y and π^z. If $W(\cdot) \in \mathbb{C}^{1+d+d'}$ (or if W is a map taking values in $\mathbb{C}^{1+d+d'}$, or a power series with coefficients in $\mathbb{C}^{1+d+d'}$), we write $W^x = \pi^x W$, $W^y = \pi^y W$ and $W^z = \pi^z W$. We also use $\pi^{x,y}W = W^{x,y} = (W^x, W^y)$, or any other combination of the variables.

We finally introduce the constants

$$\alpha = \frac{1}{N-1} \qquad \text{and} \qquad \gamma = \begin{cases} \dfrac{1}{N-1}, & \text{if } N \le M, \\ \dfrac{1}{N-M}, & \text{if } N > M, \end{cases} \tag{4}$$

which will play a capital role in our results.

3. Main results.

We start dealing with formal solutions of the invariance equation $F \circ K = K \circ R$. We provide conditions that ensure the existence of a formal solution as a power series, which turns out to be γ-Gevrey.

Theorem 3.1. *Let F be a map of the form (3). If the matrix B_1 satisfies that*

- *if $M < N$, the matrix B_1 is invertible,*
- *if $M = N$, the matrices $B_1 + la\mathrm{Id}$ are invertible for $l \ge 2$,*
- *if $M > N$, no conditions are needed for B_1,*

then there exist formal power series $\hat{R}(t) = \sum_{n\ge1} R_n t^n \in \mathbb{R}[[t]]$, $\hat{K}(t) = \sum_{n\ge0} K_n t^n \in \mathbb{R}[[t]]^{1+d+d'}$ with $K_0 = (0,0,0)$ and $K_1 = (1,0,0)^\top$ such that

$$F \circ \hat{K} = \hat{K} \circ \hat{R} \tag{5}$$

(in the sense of formal series composition).

More precisely, under these conditions, there exists a unique polynomial $R(t) = t - at^N + bt^{2N-1}$ such that for any $c \in \mathbb{R}$, there is a unique formal power series $\hat{K}(t) = \sum_{n\ge1} K_n t^n \in \mathbb{R}[[t]]^{1+d+d'}$ with $K_0 = (0,0,0)^\top$, $K_1 = (1,0,0)^\top$ and $K_N^x = c$, satisfying (5).

This expansion is γ-Gevrey, that is, there exist constants $c_1, c_2 > 0$ such that

$$\|K_n\| \le c_1 c_2^n n!^\gamma, \qquad n \ge 0,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^{1+d+d'}$.

We prove Theorem 3.1 along Sections 5 and 6. First, in Proposition 5.1 we prove the existence of the formal solution of (5) and provide formulas to compute it. Then, with the aid of some technical lemmas, we prove in Proposition 6.6 that this formal solution is γ-Gevrey.

The following proposition emphasizes the conditions on our map given by (3) are not too restrictive when considering parabolic-hyperbolic fixed points.

Proposition 3.2. *Let $\mathcal{F} : \mathcal{U} \to \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'}$ be a real analytic map of the form*

$$\mathcal{F}(x,y,z) = (x,y,Cz) + \mathcal{N}(x,y,z), \qquad \mathcal{N}(x,y,z) = \mathcal{O}(\|(x,y,z)\|^2),$$

with $1 \notin \operatorname{Spec} C$, having an invariant curve associated to the origin of the form $(y,z) = \varphi(x)$. Assume that there exist $N \ge 2$ and $a \ne 0$ such that

$$\mathcal{N}^x(x,\varphi(x)) = -ax^N + \mathcal{O}(|x|^{N+1}) \tag{6}$$

and that φ is C^r with $r \ge N$. Then, by means of changes of variables and a blow up, $\mathcal{F}$ can be expressed in the form (3) for some $M \ge 2$.

The proof of this result is elementary. We defer it to Appendix A.

The following result assures that, under additional conditions, the formal expansion $\hat{K}$ given by Theorem 3.1 is the asymptotic series of a true solution of the invariance equation, analytic in some domain with 0 at its boundary.

Theorem 3.3. *Let F be a map of the form (3). Assume that $a > 0$ and*

- *If $M \geq N$, the matrix C satisfies $\operatorname{Spec} C \subset \{z \in \mathbb{C} : |z| \geq 1\} \backslash \{1\}$.*
- *If $M < N$, the matrix C satisfies $\operatorname{Spec} C \subset \{z \in \mathbb{C} : |z| > 1\}$ and the matrix B_1 is such that $\operatorname{Spec} B_1 \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

Then, for any $0 < \beta < \alpha\pi$, there exist ρ small enough and a real analytic function K defined on the open sector

$$S = S(\beta, \rho) = \{t = re^{\mathrm{i}\varphi} \in \mathbb{C} \;:\; 0 < r < \rho,\ |\varphi| < \beta/2\} \tag{7}$$

such that K is a solution of the invariance equation $F \circ K = K \circ R$.

Moreover, K is γ-Gevrey asymptotic to the γ-Gevrey formal solution $\hat{K}$. That is, for any $0 < \bar{\beta} < \beta$ and $0 < \bar{\rho} < \rho$, there exist constants c_1, c_2 such that, for any $n \geq 2$,

$$\left\| K(t) - \sum_{j=1}^{n-1} K_j t^j \right\| \leq c_1 c_2^n n!^{\gamma} |t|^n,$$

for all $t \in \bar{S}_1 := \{t = re^{\mathrm{i}\varphi} \in \mathbb{C} \;:\; 0 < r \leq \bar{\rho},\ |\varphi| \leq \bar{\beta}/2\}$.

In particular, K can be extended to a $\mathcal{C}^\infty$ function in $[0, \rho)$.

The proof of this theorem is given in Section 7.

Now we give conditions that ensure that the manifold given by Theorem 3.3 is unique (in a suitable open set).

Theorem 3.4. *Under the same assumptions of Theorem 3.3, if the matrices C and B_1 satisfy that*

$$\operatorname{Spec} C \subset \{z \in \mathbb{C} : |z| > 1\}, \qquad \text{and} \qquad \operatorname{Spec} B_1 \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$

there exists a unique right hand side branch of a curve in the center manifold which is a parabolic stable manifold to the origin. That is, if we denote by $B(\rho) \subset \mathbb{R}^{1+d+d'}$ the open ball of radius ρ, the following local stable manifold

$$W^s_\rho = \{(x, y, z) \in B(\rho) \;:\; F^k(x, y, z) \in B(\rho) \cap \{x > 0\},\ \text{for all } k \geq 0\}$$

satisfies that $W^s_\rho = K([0, \rho))$.

This theorem is proven by using the same geometrical arguments in [4]. We omit the proof.

Remark 3.5. In the last two theorems we have assumed $a > 0$. Clearly, if $a < 0$, the map F^{-1} has the form (3) substituting a, B_1, B_2 and C by $-a, -B_1, -B_2$ and C^{-1} respectively. Therefore, if $a < 0$, we can apply (if the other conditions are satisfied) Theorems 3.3 and 3.4 to F^{-1} obtaining a local unstable parabolic invariant manifold.

A straightforward consequence of Theorem 3.3 is the following.

Corollary 3.6. *If $a \neq 0$, there exists a unique constant b such that the real analytic maps*

$$f(x) = x - ax^N + f_{\geq N+1}(x), \qquad R(x) = x - ax^N + bx^{2N-1}$$

are conjugated in a domain $S(\beta, \rho)$, with $0 < \beta < \alpha\pi$ and ρ small, by means of an analytic function $h : S(\beta, \rho) \to \mathbb{C}$ which is α-Gevrey asymptotic to a α-Gevrey formal series at 0.

In the next section we will provide examples and describe the parabolic manifolds as graphs of functions. We remark that

Remark 3.7. Let F be a map of the form (3) satisfying the hypotheses of Theorem 3.3. Then, the graph invariance equation

$$F^{y,z}(x,\Phi(x)) = \Phi(F^x(x,\Phi(x))) \tag{8}$$

with the condition $\Phi(0)=0$, $D\Phi(0)=0$, has a γ-Gevrey solution if and only if the invariance equation $F\circ K = K\circ R$ has a γ-Gevrey solution with $K(t)=(t,0)+\mathcal{O}(t^2)$ and $R(t) = t - at^N + bt^{2N-1}$. It is unique if the hypotheses of Theorem 3.4 are satisfied.

Indeed, if Φ satisfies (8), then $\tilde{K}(x)=(x,\Phi(x))$ and $\tilde{R}=F^x(x,\Phi(x))$ is a solution of $F\circ K = K\circ R$. Let h be the γ-Gevrey conjugation provided by Corollary 3.6. Then $R(t)=h^{-1}\circ\tilde{R}\circ h(t) = t-at^N+bt^{2N-1}$ and $K=\tilde{K}\circ h$ is the solution we are looking for. Reciprocally, if $F\circ K = K\circ R$, then $\Phi(x)=K^{y,z}((K^x)^{-1}(x))$ is solution of the graph invariance equation. Notice that $K^x(t)=t+O(t^2)$ is invertible around the origin and its inverse is Gevrey.

The same happens at a formal level. In this case, only the hypotheses of Theorem 3.1 are required.

The statements in this section provide upper bounds to the coefficients of the formal solution of the invariance equation. In the next section we will give examples that show that our results are sharp but also examples that show that a map satisfying our hypotheses can have an analytic invariant manifold. To provide conditions that ensure the existence of lower bounds of the coefficients remains an open problem. The following proposition shows that these conditions cannot depend only on a finite number of coefficients of the Taylor expansion of F at the origin.

Proposition 3.8. *Let F be an analytic map of the form (3) satisfying the hypotheses of Theorem 3.1 and $\hat{\varphi}(x)=\sum_{k\geq 2}\varphi_k x^k$ a formal solution of the invariance equation (8). For $p\geq 0$, let $\varphi_{\leq p}(x)=\sum_{2\leq k\leq p}\varphi_k x^k$.*

Then, for any $p\geq 2$, there exists an analytic map G such that

$$\|F(x,y,z)-G(x,y,z)\| = \mathcal{O}(\|(x,y,z)\|^{p+1})$$

with graph $\varphi_{\leq p}$ as invariant manifold to the origin. If $p\geq\max\{N,M\}$, G satisfies the hypotheses of Theorem 3.1 and, consequently, graph $\varphi_{\leq p}$ is a parabolic invariant manifold.

We defer the proof of this proposition to Appendix B.

In the following section we consider some examples. It is often easier to provide examples of maps arising from flows. The following remark is straightforward, but allows us to apply our results directly to flows.

Remark 3.9. Let $X(x,y,z,t)$ be a T-periodic vector field, $(x,y,z)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d'}$, of the form

$$X(x,y,z,t)=\begin{pmatrix} -ax^N + f_N(x,y,z,t) + f_{\geq N+1}(x,y,z,t) \\ x^{M-1}B_1 y + x^{M-1}B_2 z + g_M(x,y,z,t) + g_{\geq M+1}(x,y,z,t) \\ Dz + h_{\geq 2}(x,y,z,t)\end{pmatrix}. \tag{9}$$

Assume that the functions $f_N, f_{\geq N+1}, g_M, g_{\geq M+1}, h_{\geq 2}$ satisfy the hypotheses in Section 2 for all $t\in[0,T]$ and that $0\notin\operatorname{Spec} D$. Then, any stroboscopic Poincaré map of $\dot{\xi}=X(\xi,t)$ has the form (3) with the same a, B_1, B_2 and $C=e^{TD}$.

4. **Examples.** In this section we provide several examples. In particular we show that, under the hypotheses of Theorem 3.3, the parabolic manifold (and, consequently, the formal solution) is indeed γ-Gevrey and not more regular, that is, it is not γ'-Gevrey for $0 \le \gamma' < \gamma$.

It is more convenient to work with differential equations and manifolds represented as graphs. That is, for a given time periodic system $X(x,y,z,t)$ of the form (9), we look for formal solutions $(y,z) = \hat{\Phi}(x,t)$, depending periodically on t, of the invariance equation:

$$X^{y,z}(x,\hat{\Phi}(x,t),t) = D_x\hat{\Phi}(x,t)\big(X^x(x,\hat{\Phi}(x,t),t)\big) + \frac{\partial\hat{\Phi}}{\partial t}(x,t). \tag{10}$$

It is often useful to use the following equivalent definition of a s-Gevrey series (see [6]): a formal series $\sum_{n\ge 0} a_n z^n$ is s-Gevrey if there exist $c_1, c_2 > 0$ such that $|a_n| \le c_1 c_2^n \Gamma(1+sn)$, for all $n \ge 0$.

4.1. **Some elementary examples.** The first one is a generalization of the ones in [4]. Here we add the variables corresponding to the eigenvalue equal to 1 but still require the presence of the hyperbolic directions.

Claim 4.1. *Let $X(x,y,z)$ be the autonomous vector field*

$$X(x,y,z) = \big(-ax^N, x^{M-1}B_1 y + g(x,y,z), Cz + x^\ell c\big), \qquad (x,y,z) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d'}, \tag{11}$$

with $c \ne 0$, C an invertible matrix and $g = g_M + g_{\ge M+1}$ as in (3).

Assume that a, B_1 satisfy their corresponding conditions in Theorem 3.1. Then

- *If $M \ge N$ and $d' \ge 1$, for any $g(x,y,z) = \mathcal{O}(\|(x,y,z)\|^{M+1})$, the formal invariance equation (10) has a γ-Gevrey solution which is not γ'-Gevrey for any $0 \le \gamma' < \gamma$.*
- *If $M < N$, taking $g(x,y,z) = x^\nu b$, $b \ne 0$, with $\nu \ge M+1$, the formal invariance equation (10) has a solution $\hat{\Phi}(x) = \sum_{n\ge 1}\Phi_n x^n$ which is γ-Gevrey and is not γ'-Gevrey for any $0 \le \gamma' < \gamma$.*

Proof. We introduce $\hat{\Phi}(x) = (\hat{\varphi}(x), \hat{\psi}(x))$. We have that $\hat{\psi}(x) = \sum_{n\ge 2}\psi_n x^n$ satisfies

$$-a\sum_{n\ge 2} n\psi_n x^{n+N-1} = C\sum_{n\ge 2}\psi_n x^n + x^\ell c$$

or, equivalently,

$$\sum_{n\ge 2}\psi_n x^n = -x^\ell C^{-1}c - aC^{-1}\sum_{n\ge N+1}(n-N+1)x^n\psi_{n-N+1}.$$

Therefore $\psi_2, \dots = \psi_{\ell-1} = 0$, $\psi_\ell = -C^{-1}c$ and

$$\psi_n = -a(n-N+1)C^{-1}\psi_{n-N+1}, \qquad n \ge \ell+1. \tag{12}$$

Then $\psi_n = 0$ if $n \ne \ell + k(N-1)$ and

$$\psi_{\ell+k(N-1)} = (-1)^{k+1}a^k\prod_{i=0}^{k-1}(\ell + i(N-1))C^{-k-1}c,$$

which implies, since $N-1 \ge 1$, that $\|\psi_{\ell+k(N-1)}\| \ge \|c\|\|C\|^{-1}(a\|C\|^{-1})^k\Gamma(\ell+k)/\Gamma(\ell)$. Then, using that $\Gamma(\ell + k(N-1)\alpha) = \Gamma(\ell+k)$ we conclude that the formal series $\hat{\psi}$ is exactly of order $\alpha = 1/(N-1)$.

If $M \ge N$, $\gamma = 1/(N-1)$, then, $\hat{\psi}$ is exactly of the Gevrey order claimed in Theorem 3.1. Therefore, no matter the Gevrey order of $\hat{\varphi}$, the asymptotic series $\hat{\Phi}$ is γ-Gevrey.

Now we consider the case $M < N$. The invariance of the formal solution $\hat\varphi(x) = \sum_{n\geq 2}\varphi_n x^n$ reads

$$-a\sum_{n\geq 2} n\varphi_n x^{n+N-1} = bx^\nu + B_1\sum_{n\geq 2}\varphi_n x^{n+M-1}.$$

Since $M < N$ and B_1 is invertible, $\varphi_2 = \cdots = \varphi_{\nu-M} = 0$, $\varphi_{\nu-M+1} = -B_1^{-1}b$, and

$$\varphi_n = -a(n+M-N)B_1^{-1}\varphi_{n-N+M}, \qquad n \geq \nu - M + 1.$$

In the same way as in (12), it follows that $\hat\varphi$ is Gevrey of order $\gamma = 1/(N-M)$. □

We emphasize that, when $M < N$, the map defined by (11) has a Gevrey formal solution of order precisely $\gamma = 1/(N-M)$ even if $d' = 0$, that is, even if F is tangent to identity, but the same claim (for this particular example) only holds for $M \geq N$ if $d' \geq 1$. In the next subsection we will deal with the case $M \geq N$ and $d' = 0$, which is the relevant one in the problems of celestial mechanics we will consider in Section 4.3.

4.2. **The tangent to the identity case ($d' = 0$) when $M \geq N$.** In this section we present a family of differential equations of the form (9) having a formal solution of the invariance equation (10). We check that this formal solution is precisely γ-Gevrey. Recall that in this case $\gamma = 1/(N-1)$.

The example we will consider will be given by a non autonomous time periodic vector field. The reason is because if the vector field is autonomous, the parabolic invariant manifold is analytic (when written as a graph), as the following claim shows.

Claim 4.2. *Assume $M \geq N$. Let X be an analytic vector field of the form*

$$X(x,y) = \big(-ax^N + f(x,y), B_1x^{M-1}y + g(x,y)\big), \qquad (x,y)\in\mathbb{R}\times\mathbb{R}^d \tag{13}$$

with $f = f_N + f_{\geq N+1}$, $g = g_M + g_{\geq M+1}$ as in (3), $a \neq 0$ and B_1 satisfying the condition stated in Theorem 3.1. Then, the invariance equation (10) has a real analytic solution $\varphi : B_\rho \subset \mathbb{C} \to \mathbb{C}^d$ tangent to the x-axis at the origin.

As a consequence the real analytic maps

$$F(x,y) = (x - ax^N + \tilde f(x,y), y + B_1x^{M-1}y + \tilde g(x,y)),$$

with $a \neq 0$, which are the time 1 map of systems like (13), have an analytic solution of the graph invariance equation (8).

Proof. The one dimensional invariant manifold we are looking for is the graph of a function $y = \varphi(x)$ satisfying the equation

$$\frac{dy}{dx} = \frac{1}{-ax^N + f(x,y)}\big[B_1x^{M-1}y + g(x,y)\big]. \tag{14}$$

We introduce the new variable u by $y = xu$. The system becomes

$$\frac{du}{dx} = \frac{1}{-ax^N + f(x,xu)}\big[(ax^{N-1}\mathrm{Id} + B_1x^{M-1} - x^{-1}f(x,xu))u + x^{-1}g(x,xu)\big]. \tag{15}$$

We introduce the functions $\bar f(x,u) := f(x,xu)$ and $\bar g(x,u) := g(x,xu)$ and we notice that they satisfy $\bar f(x,xu) = |x|^N\mathcal{O}(\|(x,u)\|)$ and $\bar g(x,u) = \mathcal{O}(|x|^{M+1}) + \mathcal{O}(|x|^M\|u\|^2)$. In addition, since f and g are analytic functions at $(x,y) = (0,0)$ and $M \geq N$, so are the functions $x^{-N+1}\bar f(x,u), x^{-N}\bar f(x,u)$ and $x^{-N}\bar g(x,u)$ at $(x,u)=(0,0)$. We rewrite (15) as

$$\frac{du}{dx}=\frac{1}{-ax+x^{-N+1}\bar f(x,u)}\big[(a\mathrm{Id}+B_1x^{M-N}-x^{-N}\bar f(x,u))u+x^{-N}\bar g(x,u)\big]. \tag{16}$$

We consider now the system

$$\dot x=-ax+x^{-N+1}\bar f(x,u),\qquad \dot u=(a\mathrm{Id}+B_1x^{M-N}-x^{-N}\bar f(x,u))u+x^{-N}\bar g(x,u). \tag{17}$$

The origin is a fixed point, having a single hyperbolic direction corresponding to the eigenvalue $-a$. Indeed, when $M>N$, the linear part of the field in (17) at $(x,u)=(0,0)$ is $A=\mathrm{diag}(-a,a\mathrm{Id})$. However, when $M=N$, this linear part is

$$A=\begin{pmatrix} -a & 0\\ \partial_x[x^{-N}\bar g](0,0) & a\mathrm{Id}+B_1\end{pmatrix},$$

which may be not diagonal. Using the non-resonance condition $-a\notin\mathrm{Spec}\,(B_1+la\mathrm{Id})$ if $M=N$ and that $a\neq 0$ if $M>N$, one deduces from the theory of nonresonant invariant manifolds ([7]) that the one-dimensional invariant manifold corresponding to the eigenvalue $-a$ is the graph of a real function h, analytic at $x=0$, which is a solution of (16). Let $h(x)=cx+\mathcal{O}(x^2)$ (the constant c is 0 if either $M>N$ or $\partial_x[x^{-N}\bar g](0,0)=0$). Then $y=xh(x)=\mathcal{O}(x^2)$ is a real analytic solution of (14) tangent to the x axis. □

Claim 4.3. *Let X be the 2π-periodic vector field defined by*

$$X(x,y,t)=\big(-ax^N,bx^{N-1}y+x^{N+1}\cos t\big),\qquad (x,y)\in\mathbb{R}^2,$$

with $a,b>0$. The parabolic stable manifold has a formal Taylor expansion at 0 which is Gevrey of order exactly $\gamma=1/(N-1)$.

Proof. We first note that Theorems 3.3 and 3.4 assure the existence and uniqueness of the parabolic stable manifold when $a,b>0$.

For any initial conditions x_0,y_0,t_0, the associated flow is given by

$$x(t)=\frac{x_0}{\big(1+a(N-1)(t-t_0)x_0^{N-1}\big)^{\alpha}}$$
$$y(t)=(1+a(N-1)(t-t_0)x_0)^{\beta}\left(y_0+\int_{t_0}^{t}\frac{x_0^{N+1}\cos t}{\big(1+a(N-1)(t-t_0)x_0^{N-1}\big)^{\alpha(N+1)+\beta}}\,dt\right),$$

where we have introduced $\beta=b/a$. Since we are looking for the stable invariant manifold, we want the solution such that $(x(t),y(t))\to(0,0)$ as $t\to\infty$. Hence, since $\beta>0$, we need to impose

$$y_0=-x_0^{N+1}\int_{t_0}^{\infty}\frac{1}{\big(1+a(N-1)(t-t_0)x_0^{N-1}\big)^{\alpha(N+1)+\beta}}\cos t\,dt.$$

Therefore the stable invariant manifold is described by

$$y=\varphi(x,t)=-x^{N+1}\int_0^{\infty}\frac{1}{\big(1+a(N-1)\tau x^{N-1}\big)^{\alpha(N+1)+\beta}}\cos(t+\tau)\,d\tau.$$

Notice that φ is 2π-periodic with respect to t. Now we will prove that the series of φ at $x=0$ is Gevrey of order $\gamma=1/(N-1)$. First we introduce $\sigma=\alpha(N+1)+\beta>1$

and we decompose

$$\begin{aligned}\varphi(x,t) &= -x^{N+1}\int_0^\infty \frac{e^{i(t+\tau)}+e^{-i(t+\tau)}}{2\big(1+a(N-1)\tau x^{N-1}\big)^\sigma}\,d\tau\\ &= -\mu x^2 e^{it}\int_0^\infty \frac{e^{\mu i x^{-(N-1)}u}}{2(1+u)^\sigma}\,du - \mu x^2 e^{-it}\int_0^\infty \frac{e^{-\mu i x^{-(N-1)}u}}{2(1+u)^\sigma}\,du,\end{aligned}$$

with $\mu^{-1} = a(N-1)$. We take $\theta \in (0,\pi)$ and change the integration path in the above integrals as:

$$\varphi(x,t) = -\mu x^2 e^{it}\int_0^{\infty e^{i\theta}} \frac{e^{\mu i x^{-(N-1)}u}}{2(1+u)^\sigma}\,du - \mu x^2 e^{-it}\int_0^{\infty e^{-i\theta}} \frac{e^{-\mu i x^{-(N-1)}u}}{2(1+u)^\sigma}\,du.$$

It is well known that these integrals define the confluent hypergeometric functions Ψ (see [9, p. 280]) so that

$$\varphi(x,t) = -\frac{\mu}{2}x^2\left(e^{it}\Psi\big(1,-\sigma+2,\mu i x^{-(N-1)}\big) + e^{-it}\Psi\big(1,-\sigma+2,-\mu i x^{-(N-1)}\big)\right).$$

By [9, p. 302], an asymptotic expansion of $\Psi(a,c,z)$ for large $|z|$ is

$$\Psi(a,c,z) = \sum_{n\ge 0}(-1)^n\frac{(a)_n(a-c+1)_n}{n!}z^{-a-n},$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$. Therefore, the Taylor formal series at 0 of φ is

$$\begin{aligned}\hat\varphi(x,t) =& -\frac{\mu}{2}x^2\frac{1}{\Gamma(\sigma)}\sum_{n\ge 0}(-1)^n\Gamma(n+\sigma)\left(\big(\mu i x^{-(N-1)}\big)^{-n-1} + \big(-\mu i x^{-(N-1)}\big)^{-n-1}\right)\\ =&\mu x^2\frac{1}{\Gamma(\sigma)}\cos t\sum_{n\ge 0}\Gamma(2n+1+\sigma)\frac{x^{(N-1)(2n+2)}}{\mu^{2n+2}}\\ &-\mu x^2\frac{1}{\Gamma(\sigma)}\sin t\sum_{n\ge 0}\Gamma(2n+\sigma)\frac{x^{(N-1)(2n+1)}}{\mu^{2n+1}}.\end{aligned}$$

This formal series is Gevrey of order $\gamma = 1/(N-1)$. Indeed, comparing $\Gamma(k+\sigma)$ with $\Gamma(1+\gamma(N-1)(k+1)) = \Gamma(k+2)$ we conclude that $\hat\varphi$ is a Gevrey formal series of order exactly γ. □

4.3. **Aplications to celestial mechanics.** The three body problem describes the motion of three point bodies evolving under their mutual Newtonian gravitational attraction. The restricted three body problem is the simplification of the three body problem obtained by assuming that one of the bodies has zero mass. Consequently, the bodies with mass, usually called *primaries*, describe Keplerian orbits. See, for instance, [19].

Among the several instances of the restricted three body problem one finds the Sitnikov problem, which is the special case when the primaries move in ellipses and the massless body in the line orthogonal to the plane of the primaries through their center of mass. The relevant parameter in the Sitnikov problem is the eccentricity e of the orbits of the primaries. When $e = 0$, the Sitnikov problem is integrable. Another important subproblem is the so called restricted planar three body problem (RPTBP), when the massless body moves in the plane where the primaries lie, while the latter describe Keplerian ellipses. In this case, a relevant parameter is the mass ratio of the primeries, μ, which can be assumed to be in $[0,1/2]$. When $\mu = 0$, the RPTBP is integrable.

In both cases, the parabolic infinity can be written as

$$\begin{aligned}
\dot{x} &= -\frac{1}{4}(x+y_1)^3(x+(x+y_1)^3\mathcal{O}_0) \\
\dot{y}_1 &= \frac{1}{4}(x+y_1)^3(y_1+(x+y_1)^3\mathcal{O}_0) \\
\dot{\tilde{y}} &= \frac{1}{4}(x+y_1)^2(x-y_1)\tilde{y}+(x+y_1)^5\mathcal{O}_0 \\
\dot{t} &= 1,
\end{aligned} \tag{18}$$

where $(x, y_1, \tilde{y}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ and $\mathcal{O}_k$ stands for a function in $(x, y_1, \tilde{y}, t)$, 1-periodic with respect to t, analytic in a neighborhood of $x = y_1 = 0$, $\tilde{y} = 0$ and of order $\mathcal{O}(\|(x, y_1, \tilde{y})\|^k)$. In the case of the Sitnikov problem, $n = 0$, while $n = 2$ in the RPTBP. See [20] for the derivation of the above equations in Sitnikov problem and [10] in the planar restricted three body problem.

It is immediate to check that any stroboscopic Poincaré map of the system (18) has the form

$$\begin{pmatrix} x \\ y_1 \\ \tilde{y} \end{pmatrix} \mapsto \begin{pmatrix} x-\frac{1}{4}x^4+y_1\mathcal{O}_3+\mathcal{O}_5 \\ y_1+\frac{1}{4}y_1x^3+y_1^2\mathcal{O}_2+\mathcal{O}_5 \\ \tilde{y}+\frac{1}{4}\tilde{y}x^3+y_1\tilde{y}\mathcal{O}_2+\mathcal{O}_5 \end{pmatrix},$$

which has the form (3) with

$$d = 1+n, \quad d' = 0, \quad N = M = 4, \quad a = \frac{1}{4}, \quad B_1 = \frac{1}{4}\mathrm{Id}_{1+n}.$$

Consequently $\alpha = 1/3$. Since the eigenvalues of B_1 are positive, Theorems 3.1, 3.3 and 3.4 apply. Hence we have

Corollary 4.4. *The parabolic infinity in the Sitnikov problem (for any $e \in [0,1)$) and in the RPTBP (for any $\mu \in [0,1/2]$) possesses invariant manifolds which are $1/3$-Gevrey.*

As we have already mentioned, Theorems 3.1, 3.3 only provide upper bounds on the coefficients of the expansion of the invarariant manifold. However, in view of Martínez and Simó's numerical computations [17] and the example in Claim 4.3, where a time periodic perturbation of a system with a parabolic fixed point is considered, we present the following conjecture.

Conjecture 4.5. *The parabolic infinity in the Sitnikov problem, with $e \in (0,1)$, and in the RPTBP, with $\mu \in (0,1/2]$, possesses invariant manifolds which are* precisely *$1/3$-Gevrey, that is, they are not γ'-Gevrey for any $0 \le \gamma' < 1/3$.*

5. **Formal parameterization of the manifold.**

In this section we obtain a formal solution of the equation $F\circ K = K\circ R$, that is, a formal series which solves the equation at all orders. We will need also a precise expression of the coefficients in order to obtain Gevrey estimates for them.

We will use the following notation, that arises from the Faà-di-Bruno formula. Assuming that f and g are two $\mathcal{C}^\infty$ functions such that $f\circ g$ makes sense, $f(0) = 0$ and $g(0) = 0$, we have that $(f\circ g)_l = \frac{1}{l!}D^l(f\circ g)(0)$ satisfies

$$(f\circ g)_l = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ l_i\ge 1}} f_k[g_{l_1}, \cdots, g_{l_k}]. \tag{19}$$

Here f_k and g_k are k-multilinear symmetric maps. This expression also holds when dealing with the composition of formal power series $\hat{f}(w) = \sum_{l\geq 1} f_l w^l$ and $\hat{g}(v) = \sum_{l\geq 1} g_l v^l$. The coefficient of the l order term of the formal composition $f\circ g$ is given by (19). It depends only on $f_{\leq l}(w) = \sum_{k=1}^{l} f_k w^k$ and $g_{\leq l}(v) = \sum_{k=1}^{l} g_k v^k$. The only term of $(\hat{f}\circ\hat{g})_l$ in which f_l appears is $f_l g_1^l$, and the only term in which g_l appears is $f_1 g_l$.

We introduce the maps

$$\mathcal{L}(x,y,z) = \begin{pmatrix} x - ax^N \\ y \\ Cz \end{pmatrix}, \qquad G(x,y,z) = F(x,y,z) - \mathcal{L}(x,y,z), \tag{20}$$

for $l \geq 2$, the family of operators

$$\mathcal{A}_l = \begin{cases} -(B_1 + al\mathrm{Id})^{-1}, & \text{if} \quad N = M, \\ -B_1^{-1}, & \text{if} \quad M < N, \\ -(la)^{-1}\mathrm{Id}, & \text{if} \quad M > N \end{cases} \tag{21}$$

and

$$L = \min\{N, M\}.$$

Proposition 5.1. *There exists a unique $b \in \mathbb{R}$ such that for any $c \in \mathbb{R}$ there exists a unique formal power series $\hat{K}(t) = \sum_{l=1}^{\infty} K_l t^l$, $K_l \in \mathbb{R}^{1+d+d'}$ with $K_1 = (1,0,0)^\top$ and $K_N^x = c$, such that $R(t) = t - at^N + bt^{2N-1}$ and K satisfies the equation $F\circ K - K\circ R = 0$ formally.*

The coefficients of K and R can be given inductively. For $l > 1$ we have

$$\begin{aligned} K_l^z &= -(C - \mathrm{Id})^{-1} E_l^z, \\ K_l^y &= \begin{cases} \mathcal{A}_l E_{l+L-1}^y, & \text{if} \quad N < M, \\ \mathcal{A}_l (E_{l+L-1}^y + B_2 K_l^z), & \text{if} \quad N \geq M, \end{cases} \\ K_l^x &= \begin{cases} \dfrac{-1}{a(l-N)} (E_{l+N-1}^x + v^\top K_l^y + w^\top K_l^z), & \text{if} \quad l \neq N, \\ c, & \text{if} \quad l = N, \end{cases} \end{aligned}$$

and

$$R_{l+N-1} = \begin{cases} 0, & \text{if} \quad l > 1, \ l \neq N, \\ b = E_{2N-1}^x + v^\top K_N^y + w^\top K_l^z, & \text{if} \quad l = N; \end{cases}$$

where

$$\begin{aligned} E_{l+N-1}^x = &-a \sum_{\substack{l_1+\cdots+l_N = l+N-1 \\ 1\leq l_i \leq l-1}} \prod_{i=1}^{N} K_{l_i}^x + \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k = l+N-1 \\ 1\leq l_i\leq l-1}} G_k^x[K_{l_1}, \cdots, K_{l_k}] \\ &- \sum_{k=2}^{l-1} K_k^x \sum_{\substack{l_1+\cdots+l_k = l+N-1 \\ 1\leq l_i \leq l+N-2}} \prod_{i=1}^{k} R_{l_i}, \end{aligned} \tag{22}$$

$$E^y_{l+L-1} = \sum_{k=M}^{l+L-1} \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i \le \min\{l-1,l+L-M\}}} G^y_k[K_{l_1},\cdots,K_{l_k}] - \sum_{k=M-L+2}^{\min\{l-1,l+L-N\}} K^y_k \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^k R_{l_i} \tag{23}$$

and

$$E^z_l = \sum_{k=2}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l-1}} G^z_k[K_{l_1},\cdots,K_{l_k}] - \sum_{k=2}^{l-N+1} K^z_k \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l+N-2}} \prod_{i=1}^k R_{l_i}. \tag{24}$$

In addition, if $1\le l\le M-L+1$*,* $K^y_l = 0$.

Proof. First we prove by induction that there exist a formal series $K(t)=\sum_{n\ge1}K_n t^n$, $K_n=(K^x_n,K^y_n,K^z_n)^\top\in\mathbb{R}^{1+d+d'}$ and a polynomial $R(t)=\sum_{n\ge1}^{n_0}R_nt^n$, $R_n\in\mathbb{R}$, with as much as possible coefficients equal to 0, such that the error

$$E^l = F\circ K_{\le l} - K_{\le l}\circ R_{\le l+N-1}$$

satisfies

$$E^l(t) = \begin{cases} \big(\mathcal{O}(t^{l+N}),\mathcal{O}(t^{M+1}),\mathcal{O}(t^{l+1})\big)^\top, & \text{if } \ l<M-L+1, \\ \big(\mathcal{O}(t^{l+N}),\mathcal{O}(t^{l+L}),\mathcal{O}(t^{l+1})\big)^\top, & \text{if } \ l\ge M-L+1. \end{cases} \tag{25}$$

To deal simultaneously with both cases we introduce $P(l)$ as

$$P(l) = \begin{cases} M+1, & \text{if } \ 1\le l<M-L+1, \\ l+L, & l\ge M-L+1. \end{cases}$$

Note that $P(l-1)+1\ge P(l)$ and that $P(l)=\max\{M+1,l+L\}$.

We can write $E^l(t)=\sum_{n\ge1}E^l_nt^n$, with $E^l_n\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d'}$. We denote by $E^{l,x}_{l+N-1},E^{l,y}_{P(l)}$ and $E^{l,z}_l$ the first non-zero terms of $(E^{l,x},E^{l,y},E^{l,z})$ respectively. From the proof it will become clear that $E^{m,x}_{l+N-1},E^{m,y}_{P(l)}$ and $E^{m,z}_l$ actually do not depend on m provided $m\ge l-1$. We will simply denote them by $E^x_{l+N-1},E^y_{P(l)}$ and E^z_l respectively. These values are the ones which appear in the statement.

Taking $R(t)=t-at^N+\mathcal{O}(t^{N+1})$ and $K_1=(1,0,0)^\top$ the claim holds true for $l=1$ because

$$E^1(t) = F\circ K_{\le1}(t) - K_{\le1}\circ R_{\le N}(t) = (\mathcal{O}(t^{N+1}),\mathcal{O}(t^{M+1}),\mathcal{O}(t^2)).$$

Now, let $l\ge2$ and assume that there exist polynomials $K_{\le l-1}$ of degree at most $l-1$ and $R_{\le l+N-2}$ of degree at most $l+N-2$ such that

$$E^{l-1}(t) = \big(\mathcal{O}(t^{l+N-1}),\mathcal{O}(t^{P(l-1)}),\mathcal{O}(t^l)\big)^\top.$$

We remark that the value of the constant $b=R_{2N-1}$ will be determined at the step $l=N$.

In addition, we assume that $K^y_j=0$, $1\le j\le l-1\le M-L+1$.

We look for $K_l\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d'}$ and $R_{l+N-1}\in\mathbb{R}$ such that $K_{\le l}(t)=K_{\le l-1}(t)+K_lt^l$ and $R_{\le l+N-1}(t)=R_{\le l+N-2}(t)+R_{l+N-1}t^{l+N-1}$ satisfy (25). Defining $\Delta_l(t)=K_lt^l$, we have that

$$
\begin{aligned}
E^l = F\circ K_{\le l} - K_{\le l}\circ R_{\le l+N-1}\\
=& E^{l-1}\\
&+ F\circ K_{\le l} - F\circ K_{\le l-1} - (DF\circ K_{\le l-1})\Delta_l &(26)\\
&+ (DF\circ K_{\le l-1})\Delta_l &(27)\\
&- K_{\le l}\circ R_{\le l+N-1} - K_{\le l}\circ R_{\le l+N-2} &(28)\\
&- \Delta_l\circ R_{\le l+N-2}. &(29)
\end{aligned}
$$

By the induction hypothesis,

$$E^{l-1}(t) = (E^x_{l+N-1}t^{l+N-1}, E^y_{P(l-1)}t^{P(l-1)}, E^z_l t^l)^\top + \big(\mathcal{O}(t^{l+N}), \mathcal{O}(t^{P(l)}), \mathcal{O}(t^{l+1})\big)^\top.$$

Now we identify the lowest order terms in (26), (27), (28) and (29).

Using that $l\ge 2$ we easily estimate (26)

$$
\begin{aligned}
(F\circ K_{\le l} - F\circ K_{\le l-1} - (DF\circ K_{\le l-1})\Delta_l)(t) &= \big(\mathcal{O}(t^{2l+N-2}), \mathcal{O}(t^{2l+M-2}), \mathcal{O}(t^{2l})\big)^\top\\
&= \big(\mathcal{O}(t^{l+N}), \mathcal{O}(t^{P(l-1)+1}), \mathcal{O}(t^{l+1})\big)^\top
\end{aligned}
$$

since $2l+M-2\ge M+1+2l-3 > M+1$ and $2l+M-2\ge l+L+l-2\ge l+L$.

Concerning (27), taking into account that $K_1 = (1,0,0)^\top$,

$$((DF\circ K_{\le l-1})\Delta_l)(t) = \begin{pmatrix} (1-Nat^{N-1})t^l K^x_l + v^\top K^y_l t^{l+N-1} + w^\top K^z_l t^{l+N-1} + \mathcal{O}(t^{l+N})\\ K^y_l t^l + B_1 K^y_l t^{l+M-1} + B_2 K^z_l t^{l+M-1} + \mathcal{O}(t^{l+M})\\ CK^z_l t^l + \mathcal{O}(t^{l+1})\end{pmatrix}.$$

As for (28), taking into account that $K^y_j = 0$ if $1\le j\le l-1\le M-L+1$, which implies that

$$K_{\le l}(t) = \begin{cases} K^y_l t^l, & l-1\le M-L+1,\\ K_{M-L+1}t^{M-L+1} + O(t^{M-L+2}), & \text{otherwise},\end{cases}$$

we have that

$$(K_{\le l}\circ R_{\le l+N-1} - K_{\le l}\circ R_{\le l+N-2})(t) = \begin{pmatrix} R_{l+N-1}t^{l+N-1}\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\mathcal{O}(t^{l+N})\\ \Delta^y_l(t)\\ \mathcal{O}(t^{l+N})\end{pmatrix},$$

where

$$\Delta^y_l(t) = \begin{cases} lK^y_l O(t^{2l+N-2}), & l-1\le M-L+1,\\ O(t^{l+M+N-L}), & \text{otherwise},\end{cases}$$

In all cases $\mathcal{O}(t^{l+M+N-L}) = \mathcal{O}(t^{P(l-1)+1})$.

Finally we evaluate (29)

$$\Delta_l\circ R_{\le l+N-2}(t) = K_l(t-at^N + \mathcal{O}(t^{N+1}))^l = K_l t^l - laK_l t^{l+N-1} + \mathcal{O}(t^{l+N})K_l.$$

From the above calculations, since $l\ge 2$, we have

$$
\begin{aligned}
E^l(t) &= \begin{pmatrix} (E^x_{l+N-1} + a(l-N)K^x_l - R_{l+N-1} + v^\top K^y_l + w^\top K^z_l)t^{l+N-1}\\ E^y_{P(l-1)}t^{P(l-1)} + alK^y_l t^{l+N-1} + B_1 K^y_l t^{l+M-1} + B_2 K^z_l t^{l+M-1}\\ (E^z_l + (C-\mathrm{Id})K^z_l)t^l\end{pmatrix}\\
&+ \begin{pmatrix}\mathcal{O}(t^{l+N})\\ \mathcal{O}(t^{P(l-1)+1}) + \mathcal{O}(t^{l+N})K^y_l\\ \mathcal{O}(t^{l+1})\end{pmatrix}.
\end{aligned}
$$

This expression permits to choose (K_l^x, K_l^y, K_l^z) and R_{l+N-1} in order to E^l has the claimed order. We start dealing with the third component. We have to take

$$K_l^z = -(C - \mathrm{Id})^{-1} E_l^z.$$

When dealing with the second one we have to distinguish two cases. If $P(l-1) = M + 1$, which means that $l < M - L + 1$ we have that $l + L - 1 < M$. Then if $L = N < M$, $K_l^y = 0$. Otherwise, if $L = M \le N$, $l \le 1$ and this case is void. Now suppose $P(l-1) = l + L - 1$. If $M \le N$, we take

$$K_l^y = \mathcal{A}_l(E_{l+L-1}^y + B_2 K_l^z),$$

while, if $N < M$,

$$K_l^y = \mathcal{A}_l E_{l+L-1}^y.$$

Finally, considering the x component, if $l \ne N$, we take

$$R_{l+N-1} = 0, \qquad K_l^x = -\frac{1}{a(l-N)}\left(E_{l+N-1}^x + v^\top K_l^y + w^\top K_l^z\right),$$

otherwise,

$$R_{2N-1} = E_{2N-1}^x + v^\top K_N^y + w^\top K_N^z, \qquad K_N^x \quad \text{is free} .$$

We write $c = K_N^x$ which can be chosen arbitrarily. We recall that R_{2N-1} corresponds to the coefficient b.

Now we come to compute E_{l+N-1}^x, E_{l+L-1}^y and E_{l+L-1}^z.

By definition, E_l^z is the term of order l of $\pi^z E^{l-1} = F^z \circ K_{\le l-1} - K_{\le l-1}^z \circ R_{\le l+N-2}$, that is,

$$E_l^z = \frac{1}{l!} D^l \pi^z E^{l-1}(0).$$

By the Faà di Bruno formula (19),

$$E_l^z = \sum_{k=1}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i \le l-1}} F_k^z[K_{l_1}, \dots, K_{l_k}] - \sum_{k=1}^{l-1} K_k^z \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}. \tag{30}$$

In the first term of (30) the addend with $k = 1$ vanishes because $K_1^z = 0$. Moreover, for $k \ge 2$, $F_k^z = G_k^z$. In the second term, the addend with $k = 1$ also vanishes. Moreover, if $k > l - N + 1$, since

$$\sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}$$

is the coefficient of t^l in

$$R(t)^k = (t - at^N + bt^{2N-1})^k = t^k - akt^{k+N-1} + \mathcal{O}(t^{k+2N-2}),$$

the addend with this k vanishes because then $l < k + N - 1$ and the next non-zero term after order k is of order $k + N - 1$. This proves formula (24).

Analogously,

$$E_x^{l+N-1} = \frac{1}{(l+N-1)!} D^{l+N-1} \pi^x E^{l-1}(0), \qquad E_y^{l+L-1} = \frac{1}{(l+L-1)!} D^{l+L-1} \pi^y E^{l-1}(0).$$

Applying again Faà di Bruno's formula we obtain

$$E^y_{l+L-1} = \sum_{k=1}^{l+L-1} \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i\le l-1}} F^y_k[K_{l_1},\cdots,K_{l_k}] - \sum_{k=1}^{l-1} K^y_k \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i} \quad (31)$$

and

$$E^x_{l+N-1} = \sum_{k=1}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l-1}} F^x_k[K_{l_1},\cdots,K_{l_k}] - \sum_{k=1}^{l-1} K^x_k \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}.$$

We begin by determining the indices in (31) that provide non-zero terms in E^y_{l+L-1}. The term with $k=1$ in the first addend (31) vanishes because it would be $F^y_1 K^y_{l+L-1}$, but for E^l we are working with $K_{\le l-1}$. Moreover, since $F^y_k = 0$ if $2\le k\le M-1$, the sum must start with $k=M$. Also, $M\le k\le l_1+\cdots+l_k = l+L-1$ implies $l\ge M-L+1$. In addition, when $l = M-L+1$, we always have that

$$l_1+\cdots+l_k = M-L+1+L-1 = M.$$

Therefore, if $l = M-L+1$, $k=M$ and $l_1=\cdots=l_M=1$. Since $K_1=(1,0,0)^\top$, the corresponding term is

$$F^y_M[\overbrace{K_1,\cdots,K_1}^{M)}] = \frac{1}{M!}\partial^M_x F^y(0,0) = 0.$$

Then if $l\le M-L+1$ the first term is void. To finish with the first term, we note that for all i, using again that $k\ge M$,

$$M-1+l_i \le l_1+\cdots+l_k = l+L-1 \quad\Rightarrow\quad l_i\le l+L-M,$$

that is, the first term of (31) has the form claimed in formula (23). With respect to the second term of (31), we only need to note that $K^y_k=0$ for $1\le k\le M-L+1$, and analogously as before, that

$$\sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}$$

is the coefficient of t^{l+N-1} of $R(t)^k$. Therefore since if vanishes for $k<l+L-1<k+N-1$, we have that $l+L-1\ge k+N-1$ which implies that $k\le l+L-N$. This ends the proof of formula (23) for E^y_l. To check formula (22) for E^x_{l+N-1} we use the form of $F^x(x,y,z) = x-ax^N+G^x(x,y,z)$ and the proof follows the same lines as the one for E^y_{l+L-1}. □

6. **Gevrey estimates.** Before starting to obtain the Gevrey estimates of the formal solution K we perform two changes of coordinates. The first one is a close to the identity change that uses the $(N-1)$-degree approximation of the formal parabolic curve obtained in Proposition 5.1 to put it closer to the x-axis. In the new variables the parameterization will be the embedding to the x-axis plus terms of order at least N.

The structure of this section is quite similar to the counterpart in [4], however, there are some differences to take into account.

Lemma 6.1. *We define the change of variables*

$$(x,y,z) = \Phi(\bar{x},\bar{y},\bar{z}) := K_{\le N-1}(\bar{x}) + (0,\bar{y},\bar{z}),$$

where $K_{\le N-1}(\bar{x}) = \sum_{j=1}^{N-1} K_j \bar{x}^j$. In these new variables:

1. *$\bar{F} = \Phi^{-1}\circ F \circ \Phi$ has the same form (3) of F with the same constant a, vectors $v^\top, w^\top$ and the same matrices B_1 and C.*
2. *The formal solution $\bar{K}$ and $\bar{R}$ of $\bar{F}\circ\bar{K} - \bar{K}\circ\bar{R} = 0$ obtained applying Proposition 5.1 to $\bar{F}$ satisfies*

$$\bar{K}(t) = (t,0,0)^\top + O(t^N) \qquad \text{and} \qquad \bar{R}(t) = t - at^N + bt^{2N-1} = R(t).$$

3. *The Gevrey character is not affected by this change, i.e., if one of K or $\bar{K}$ is Gevrey of some order the other is also Gevrey of the same order.*

The proof of this lemma depends on cumbersome but straightforward computations and uses, among other properties, that $K_1^y = \cdots = K_{M-L+1}^y = 0$.

Next we perform a rescaling of parameter λ to achieve a good control on the growth of the terms K_l of the formal solution up to some suitable order l_0 so that we can start an induction procedure to estimate the terms K_j from l_0 on and obtain a significantly simpler bound from them.

Let $\mathcal{U} \subset \mathbb{C}^{1+d+d'}$ be the domain of a complex extension of $\bar{F}$. Let $B(\delta)$ be a ball of radius $\delta > 0$ such that $\overline{B(\delta)} \subset \mathcal{U}$.

Let $\bar{G} = \bar{F} - \mathcal{L}$ with $\mathcal{L}$ defined in (20). Given $\lambda \ge 1$ we introduce

$$\begin{aligned} \tilde{F}(x,y,z) &= \lambda\bar{F}(\lambda^{-1}x,\lambda^{-1}y,\lambda^{-1}z), & \tilde{G}(x,y,z) &= \lambda\bar{G}(\lambda^{-1}x,\lambda^{-1}y,\lambda^{-1}z), \\ \tilde{K}(t) &= \lambda\bar{K}(\lambda^{-1}t), & \tilde{R}(t) &= \lambda\bar{R}(\lambda^{-1}t), \end{aligned} \tag{32}$$

and for the sake of simplicity, we omit the dependence of λ on the notation.

Lemma 6.2. *Let $\bar{\delta}$ be such that $\overline{B}(\bar{\delta})$ is contained in the complex domain $\mathcal{U}$ of $\bar{F}$. Let $\bar{G} = \bar{F} - \mathcal{L}$ and $\bar{\kappa} = \max_{(x,y,z)\in\overline{B}(\bar{\delta})} \|\bar{G}(x,y,z)\|$.*

For all $\delta_0, \varepsilon, \mu_0 > 0$ and $l_0 \in \mathbb{N}$, there exists $\lambda := \lambda(\delta_0,\varepsilon,\mu_0,l_0,\bar{\delta}) \ge 1$ such that the functions $\tilde{F}$, $\tilde{G}$, $\tilde{K}$ and $\tilde{R}$ defined above in (32), satisfy the following properties:

1. *$\tilde{F}$ has the form (3), where the corresponding values of a, v, B_1, B_2 and C (which we denote with the same letter with tilde) are*

$$\tilde{a} = \lambda^{-(N-1)}a, \qquad \tilde{v} = \lambda^{-(N-1)}v, \qquad \tilde{w} = \lambda^{-(N-1)}w, \qquad \tilde{B}_i = \lambda^{-(M-1)}B_i,$$

   with $i = 1,2$ and $\tilde{C} = C$, respectively. The domain of $\tilde{F}$ contains a ball $\overline{B}(\tilde{\delta})$, with $\tilde{\delta} = \lambda\bar{\delta} > \delta_0$.
2. *$\tilde{R}(t) = t - \tilde{a}t^N + \tilde{b}t^{2N-1}$ with $\tilde{b} = \lambda^{-2N+2}b$. We further ask $|\lambda^{-1/\gamma}a| \le \varepsilon$ where γ is defined in (4). As a consequence $\tilde{a} \le \varepsilon$.*
3. *Formally we have that*

$$\tilde{F}\circ\tilde{K} - \tilde{K}\circ\tilde{R} = 0.$$

4. *Let $\tilde{\kappa} = \max\limits_{(x,y,z)\in\overline{B}(\tilde{\delta})} \|\tilde{G}(x,y,z)\|$. It is clear that $\tilde{\kappa} = \lambda\bar{\kappa}$ and hence $\|\tilde{G}_k\| \le \lambda\bar{\kappa}\tilde{\delta}^{-k}$ for all $k \ge 0$.*
5. *$\|\tilde{K}_l\| \le \mu_0 l!^\gamma$ for all $N \le l \le l_0$. We recall that $\tilde{K}_l = 0$ if $2 \le l \le N-1$.*

We remark that $\sigma := \frac{\tilde{b}}{\tilde{a}^2}$ does not depend on the rescaling parameter λ.

Lemma 6.3. *The matrices $\tilde{\mathcal{A}}_l$ defined as in (21) with $\tilde{B}_1$ instead of B_1, after the changes of variables in Lemmas 6.1 and 6.2, satisfy $\tilde{\mathcal{A}}_l = \lambda^{N-1}\mathcal{A}_l$ if $M \geq N$ and $\tilde{\mathcal{A}}_l = \lambda^{M-1}\mathcal{A}_l$ when $M < N$. Moreover, if $l \geq l_0 \geq 2|a|^{-1}\|B_1\|$,*

$$\|\tilde{\mathcal{A}}_l\| \leq 2\lambda^{N-1}(l|a|)^{-1}, \qquad \text{if} \qquad M \geq N.$$

The following technical lemmas are slight variations of lemmas in [4]. For the reader's convenience, we state and prove them.

Lemma 6.4. *Let $k,\nu \in \mathbb{N}$, $\nu \geq k$ and $\beta \geq \frac{1}{N-1}$. Let also*

$$J^1_{k,\nu} = k!^{\beta} R_{k,\nu}, \qquad \text{where} \qquad R_{k,\nu} = \sum_{\substack{l_1+\cdots+l_k=\nu \\ l_i \geq 1}} \prod_{i=1}^{k} \tilde{R}_{l_i}$$

and $\tilde{R}_l$ are the coefficients of the polynomial $\tilde{R}(t) = t - \tilde{a}t^N + \tilde{b}t^{2N-1}$. Let $\sigma = \frac{\tilde{b}}{\tilde{a}^2}$ and $m = \frac{\nu-k}{N-1}$. We have

$$\begin{cases} J^1_{k,k} = k!^{\beta} \\ |J^1_{k,\nu}| \leq (\nu - N + 1)!^{\beta}(\nu - mN + 1)|\tilde{a}|^m(1+|\sigma|)^{m/2}, & \text{if} \quad \frac{\nu-k}{N-1} \in \mathbb{N}, \\ J^1_{k,\nu} = 0, & \text{otherwise.} \end{cases}$$

Proof. Note that $R_{k,\nu}$ is the coefficient of t^{ν} of the polynomial $(t - \tilde{a}t^N + \tilde{b}t^{2N-1})^k$. Then, we can rewrite it as

$$R_{k,\nu} = \sum_{\substack{m_1+m_2+m_3=k \\ m_1+Nm_2+(2N-1)m_3=\nu}} \frac{k!}{m_1!m_2!m_3!}(-\tilde{a})^{m_2}\tilde{b}^{m_3}. \tag{33}$$

The conditions on the indices m_2, m_3 in the previous formula imply $(N-1)m_2 + 2(N-1)m_3 = \nu - k$, that is, $m_2 + 2m_3 = (\nu-k)/(N-1) \in \mathbb{N}$. Therefore $R_{k,\nu} = 0$ if $m := (\nu-k)/(N-1) \notin \mathbb{N}$. When $\nu = k$, $m_2 = m_3 = 0$ and $m = 0$. Then $R_{k,k} = 1$ and $J^1_{k,k} = k!^{\beta}$. If $m \geq 1$, we reduce (33) to a sum with a single index as

$$R_{k,\nu} = \sum_{m_3=0}^{[\frac{m}{2}]} \frac{(\nu-(N-1)m)!}{(\nu-mN+m_3)!(m-2m_3)!m_3!}(-\tilde{a})^{m-2m_3}\tilde{b}^{m_3}.$$

Using

$$\frac{(\nu-(N-1)m)!}{(\nu-Nm+m_3)!} \leq \frac{(\nu-(N-1)m)!}{(\nu-Nm)!} \leq (\nu-(N-1)m)^{m-1}(\nu-Nm+1)$$

and

$$\sum_{m_3=0}^{[\frac{m}{2}]} \frac{|\tilde{a}|^{m-2m_3}|\tilde{b}|^{m_3}}{(m-2m_3)!m_3!} \leq |\tilde{a}|^m \sum_{m_3=0}^{[\frac{m}{2}]} \frac{1}{([m/2]-m_3)!m_3!}\left|\frac{\tilde{b}}{\tilde{a}^2}\right|^{m_3} \leq \frac{1}{[\frac{m}{2}]!}|\tilde{a}|^m(1+|\sigma|)^{m/2},$$

we get

$$|R_{k,\nu}| \leq \frac{1}{[\frac{m}{2}]!}(\nu-(N-1)m)^{m-1}(\nu-Nm+1)|\tilde{a}|^m(1+|\sigma|)^{m/2}.$$

Finally, since $k = \nu - (N-1)m$, using that $m \geq 1$ and that

$$\frac{(\nu-(N-1)m)!^{\beta}(\nu-(N-1)m))^{m-1}}{(\nu-N+1)!^{\beta}} = \frac{(\nu-(N-1)m)^{m-1}}{[(\nu-N+1)\cdots(\nu-(N-1)m+1)]^{\beta}}$$

$$\le \frac{(\nu-(N-1)m)^{m-1}}{(\nu-(N-1)m+1)^{(m-1)(N-1)\beta}} \le 1,$$

we obtain

$$|J^1_{k,\nu}| \le \frac{1}{\left[\frac{m}{2}\right]!}(\nu-(N-1)m)!^\beta(\nu-(N-1)m)^{m-1}(\nu-Nm+1)|\tilde{a}|^m(1+|\sigma|)^{m/2},$$

where we use that $(N-1)\beta \ge 1$. □

The next lemma collects two technical results on bounds of some products of factorials.

Lemma 6.5. *Let $N \ge 2$, $\beta \ge \frac{1}{N-1}$ and $N_\beta = N^{\beta(N-1)}$.*

i) Let $k \ge 1$, $\nu \ge kN$ and

$$M_{k,\nu} = \sum_{\substack{l_1+\cdots+l_k=\nu \\ l_i \ge N}} (l_1! \cdot \cdots \cdot l_k!)^\beta. \tag{34}$$

If $\nu < kN$, the sum in (34) is void and we define $M_{k,\nu}=0$. We have

$$\begin{cases} M_{k,\nu} \le (\nu-k+1)!^\beta N_\beta^{k-1}, & \nu \ge kN, \\ M_{k,\nu} = 0, & \textit{otherwise.} \end{cases}$$

ii) Let $k \ge 1$, $\nu \ge k$ and

$$J^2_{k,\nu} = \sum_{\substack{l_1+\cdots+l_k=\nu \\ l_i=1 \ or \ l_i \ge N}} (l_1! \cdot \cdots \cdot l_k!)^\beta, \qquad \nu \ge k. \tag{35}$$

(If $\nu < k$ the sum in (35) is void and we define $J^2_{k,\nu}=0$.) We have

$$J^2_{k,\nu} \le \frac{(N_\beta+1)^k-1}{N_\beta}(\nu-k+1)!^\beta, \qquad \nu \ge k.$$

Proof. i) If $kN > \nu$, one has that $M_{k,\nu}=0$. Let us assume that $kN \le \nu$. One can check that, if $a,b,c \in \mathbb{N}$ with $b \le c$, then $(a+b)!c! \le b!(a+c)!$. Therefore, for $l_1, l_2, \cdots, l_k \ge N$ such that $l_1+\cdots+l_k=\nu$ one has that $l_1!l_2! \le N!(l_1+l_2-N)!$, that

$$l_1!l_2!l_3! \le N!(l_1+l_2-N)!l_3! = N!(l_1+l_2-2N+N)!l_3! \le N!^2(l_1+l_2+l_3-2N)!$$

and applying this procedure recursively we get

$$l_1!l_2! \cdot \cdots \cdot l_k! \le N!^{k-1}(l_1+\cdots+l_k-(k-1)N)! = N!^{k-1}(\nu-(k-1)N)!.$$

On the other hand it is clear that

$$\begin{aligned} \#\{l_1+\cdots+l_k=\nu,\ l_i \ge N\} &= \#\{m_1+\cdots+m_k=\nu-kN,\ m_i \ge 0\} \\ &= \binom{\nu-kN+k-1}{k-1}. \end{aligned}$$

Therefore

$$M_{k,\nu} \le N!^{\beta(k-1)}(\nu-(k-1)N)!^\beta \binom{\nu-kN+k-1}{k-1}.$$

Now we use that $N!^\beta \le N^{\beta(N-1)}$ that

$$\binom{\nu-kN+k-1}{k-1} = \frac{\nu-kN+k-1}{k-1} \cdot \cdots \cdot \frac{\nu-kN+1}{1} \le (\nu-kN+1)^{k-1}$$

and that

$$\frac{(\nu-k+1)!}{(\nu-(k-1)N)!} \geq (\nu-(k-1)N+1)^{(k-1)(N-1)}$$

to obtain

$$\begin{aligned} M_{k,\nu} &\leq N_\beta^{k-1}(\nu-(k-1)N)!^\beta(\nu-kN+1)^{k-1} \\ &\leq N_\beta^{k-1}(\nu-k+1)!^\beta \frac{(\nu-kN+1)^{k-1}}{(\nu-(k-1)N+1)^{\beta(N-1)(k-1)}}. \end{aligned}$$

Finally the bound in i) follows because $\beta(N-1) \geq 1$.

ii) For $k=\nu$, $J_{k,\nu}^2 = 1$ and the bound is obvious. Assume that $\nu > k$. Then,

$$\begin{aligned} J_{k,\nu}^2 = \sum_{i=0}^{k-1} \binom{k}{i} M_{k-i,\nu-i} &\leq (\nu-k+1)!^\beta \sum_{i=0}^{k-1} \binom{k}{i} N_\beta^{k-i-1} \\ &\leq (\nu-k+1)!^\beta \frac{(N_\beta+1)^k-1}{N_\beta} \end{aligned}$$

and the proof is complete. □

Now we are going to prove that $\tilde{K}(t) = \sum_{l\geq 1}\tilde{K}_l t^l$ is Gevrey of order γ. Recall that γ was defined in (4).

Proposition 6.6. *Let $\bar{\delta}, \bar{\kappa}$ be as in Lemma 6.2. We take $\delta_0 = 2(1+N_\gamma)$, with $N_\gamma = N^{\gamma(N-1)}$ and ε, μ_0 and l_0 according to the cases*

- *$N \leq M$,*

$$\varepsilon = \frac{1}{8(1+|\sigma|)},$$

$$\mu_0 = 1,$$

$$l_0 \geq \max\left\{6\bar{\kappa}(1+N)^M\bar{\delta}^{-M}(N|a|)^{-1}, N + \frac{4\mu_0^{N-1}}{N}(1+N)^N\left(1+\frac{2\bar{\kappa}}{|a|\bar{\delta}^N}\right)\right\}.$$

- *$M < N$,*

$$\varepsilon = \min\left\{\frac{1}{8(1+|\sigma|)}, \frac{1}{4\|B_1^{-1}\|(1+|\sigma|)^{1/2}}\right\},$$

$$\mu_0 = \min\left\{1, \left[\frac{N_\gamma}{6\bar{\kappa}\|B_1^{-1}\|}\left(\frac{\bar{\delta}}{1+N_\gamma}\right)^M\right]^{\frac{1}{M-1}}\right\},$$

$$l_0 \geq N + \frac{4\mu_0^{N-1}}{N_\gamma}(1+N_\gamma)^N\left(1+\frac{2\bar{\kappa}}{|a|\bar{\delta}^N}\right).$$

Let $\tilde{F}$ rescaled with λ depending on $\delta_0, \varepsilon, \mu_0, l_0$ as in Lemma 6.2 and $\tilde{K}$ be the formal solution $\tilde{K}(t) = \sum_{j=1}^\infty \tilde{K}_j t^j$ of equation $\tilde{F}\circ\tilde{K} - \tilde{K}\circ\tilde{R} = 0$. Then

$$\|\tilde{K}_j\| \leq \mu_0 j!^\gamma, \qquad j \geq 0.$$

Proof. We use the formulas for $\tilde{K}$ and E in Proposition 5.1 applied to $\tilde{F}$, and hence we have that $\tilde{K}_l = 0$ if $2 \leq l \leq N-1$. By Lemma 6.2 and the choice of parameters we have that $\|\tilde{K}_j\| \leq \mu_0 j!^\gamma$ for $N \leq j \leq l_0$. We will use Lemmas 6.4 and 6.5 with $\beta = \gamma$.

We assume by induction that $\|\tilde{K}_j\| \leq \mu_0 j!^\gamma$ for $1 \leq j \leq l-1$ for some $l > l_0$.

We start bounding $\tilde{K}_l^z$. We introduce

$$H_l^1 = \sum_{k=2}^{l} \sum_{\substack{l_1+\cdots+l_k=l\\1\le l_i\le l-1}} \tilde{G}_k^z[\tilde{K}_{l_1},\cdots,\tilde{K}_{l_k}], \qquad H_l^2 = \sum_{k=N}^{l-N+1} \tilde{K}_k^z \tilde{R}_{k,l}$$

so that $E_l^z = H_l^1 - H_l^2$. We have that, by 4 of Lemma 6.2, taking λ such that $\mu_0(N_\gamma+1)/\tilde{\delta} < 1/2$ and using Lemma 6.5,

$$\begin{aligned}\|H_l^1\| &\le \lambda\bar{\kappa}\sum_{k=2}^{l}\tilde{\delta}^{-k}\mu_0^k J_{k,l}^2 \le \frac{\lambda\bar{\kappa}l!^\gamma}{N_\gamma}\sum_{k=2}^{l}\left[\frac{\mu_0(N_\gamma+1)}{\tilde{\delta}}\right]^k \frac{(l-k+1)!^\gamma}{l!^\gamma}\\ &\le 2\frac{\lambda\bar{\kappa}l!^\gamma}{N_\gamma}\left[\frac{\mu_0(N_\gamma+1)}{\tilde{\delta}}\right]^2 \le \mu_0 l!^\gamma\lambda^{-1}\big[2\bar{\kappa}\bar{\delta}^{-2}(N_\gamma+1)^2N_\gamma^{-1}\big].\end{aligned}$$

Moreover, for H_l^2, it is clear that $\|H_l^2\| \le \sum_{k=N}^{l-N+1}\mu_0 k!^\gamma|\tilde{R}_{k,l}| = \mu_0\sum_{k=N}^{l-N+1}J_{k,l}^1$. Then, using Lemma 6.4 and writing $m=(l-k)/(N-1)$,

$$\begin{aligned}\|H_l^2\| &\le \mu_0(l-N+1)!^\gamma\sum_{m=1}^{[\frac{l-N}{N-1}]}(l-mN+1)|\tilde{a}|^m(1+|\sigma|)^{m/2}\\ &\le \mu_0(l-N+1)!^\gamma(l-N+1)2|\tilde{a}|(1+|\sigma|)^{1/2}\\ &\le \mu_0 l!^\gamma\lambda^{-N+1}\big[2|a|(1+|\sigma|)^{1/2}\big].\end{aligned}$$

Therefore, since $\|\tilde{K}_z^l\| \le \|(C-\mathrm{Id})^{-1}\|\|E_l^z\| \le \|(C-\mathrm{Id})^{-1}\|(\|H_l^1\|+\|H_l^2\|)$,

$$\|\tilde{K}_z^l\| \le \mu_0 l!^\gamma\lambda^{-1}\left[2\bar{\kappa}\bar{\delta}^{-2}(N_\beta+1)^2N_\beta^{-1} + \lambda^{-N+2}\big[2|a|(1+|\sigma|)^{1/2}\big]\right] \tag{36}$$

and taking λ big enough we obtain $\|\tilde{K}_z^l\| \le \mu_0 l!^\gamma$.

To bound $\tilde{K}_l^y$ we introduce H_l^3 and H_l^4 so that $E_y^{l+L-1} = H_l^3 - H_l^4$:

$$H_l^3 = \sum_{k=M}^{l+L-1} \sum_{\substack{l_1+\cdots+l_k=l+L-1\\1\le l_i\le\min\{l-1,l+L-M\}}} \tilde{G}_k^y[\tilde{K}_{l_1},\cdots,\tilde{K}_{l_k}], \quad H_l^4 = \sum_{k=N}^{\min\{l-1,l+L-N\}} \tilde{K}_k^y\tilde{R}_{k,l+L-1}.$$

We distinguish two cases, when $L=\min\{N,M\}=N$ and $L=M<N$. First we deal with the case $L=N$. In this case $\gamma=1/(N-1)$, $N_\gamma=N$ and $\mu_0=1$. By item ii) of Lemma 6.5, we have that

$$\begin{aligned}\|\tilde{\mathcal{A}}_l H_l^3\| &\le \lambda^N\bar{\kappa}\frac{1}{l|a|}\sum_{k=M}^{l+N-1}J_{k,l+N-1}^2\frac{1}{\tilde{\delta}^k} \le \lambda^N\bar{\kappa}\frac{1}{Nl|a|}\sum_{k=M}^{l+N-1}(l+N-k)!^\gamma\left(\frac{1+N}{\tilde{\delta}}\right)^k\\ &\le \lambda^N\bar{\kappa}\frac{1}{Nl|a|}(l+N-M)!^\gamma\sum_{k=M}^{l+N-1}\left(\frac{1+N}{\tilde{\delta}}\right)^k\\ &\le \lambda^N\bar{\kappa}\frac{2}{Nl|a|}(l+N-M)!^\gamma\left(\frac{1+N}{\tilde{\delta}}\right)^M,\end{aligned}$$

since we are assuming that $\frac{1+N}{\tilde{\delta}} \le \frac{1+N}{\delta_0} = 1/2$. Therefore, using that $\tilde{\delta}=\lambda\bar{\delta}$, that $M\ge N=L$ and that $l\ge l_0$,

$$\|\tilde{\mathcal{A}}_l H_l^3\| \le l!^\gamma l_0^{-1}\left[\lambda^{N-M}\bar{\kappa}\frac{2}{N|a|}\left(\frac{1+N}{\bar{\delta}}\right)^M\right] \le \frac{1}{3}l!^\gamma$$

by definition of l_0.

Now we deal with H_l^4. It is clear that $\|H_l^4\| \le \sum_{k=N}^{l-1} J^1_{k,l+N-1}$. Recall that $J^1_{k,l+N-1} \neq 0$ if and only if $k = l+N-1-m(N-1)$ for some $m \in \mathbb{N}$. If $m=1$, then $k = l+N-1-(N-1) = l$ which is a contradiction, because $k \le l-1$. Moreover, since $k \ge N$, $m \le (l-1)/(N-1)$. Therefore, using Lemmas 6.3 and 6.4:

$$\|\tilde{\mathcal{A}}_l H_l^4\| \le 2\lambda^{N-1}\frac{1}{|a|l}\sum_{m=2}^{\left[\frac{l-1}{N-1}\right]} l!^\gamma (l+N-mN)|\tilde{a}|^m(1+|\sigma|)^{m/2}.$$

Then, since $|\tilde{a}(1+|\sigma|)^{1/2}| \le \varepsilon(1+|\sigma|) \le 1/8$ and $\tilde{a} = \lambda^{-(N-1)}a$, we have that

$$\|\tilde{\mathcal{A}}_l H_l^4\| \le 4\lambda^{N-1}\frac{1}{|a|l} l!^\gamma (l-N)|\tilde{a}|^2(1+|\sigma|) \le l!^\gamma \lambda^{-(N-1)}[4|a|(1+|\sigma|)] \le \frac{1}{3}l!^\gamma$$

if λ is big enough. On the one hand, when $N<M$, $\tilde{K}_l^y = \tilde{\mathcal{A}}_l(H_l^3+H_l^4)$ and the previous bounds imply the induction result. On the other hand, when $N=M$, $\tilde{K}_l^y = \tilde{\mathcal{A}}_l(H_l^3+H_l^4+\tilde{B}_2\tilde{K}_l^z)$. By (36), the term $\|\tilde{\mathcal{A}}_l\tilde{B}_2\tilde{K}_l^z\|$ is bounded by $l!^\gamma/3$ if λ is large enough and therefore, also in this case, we are done.

Now we deal with the case $L = M < N$. Recall that $\gamma = 1/(N-M)$. Then

$$\begin{aligned}
\|\tilde{\mathcal{A}}_l H_l^3\| &\le \lambda^M \bar{\kappa}\|\tilde{B}_1^{-1}\|\sum_{k=M}^{l+M-1} J^2_{k,l+M-1}\frac{\mu_0^k}{\tilde{\delta}^k}\\
&\le \lambda^M \bar{\kappa}\|\tilde{B}_1^{-1}\|\sum_{k=M}^{l+M-1}\frac{1}{N_\gamma}(l+M-k)!^\gamma\left(\frac{\mu_0(1+N_\gamma)}{\tilde{\delta}}\right)^k\\
&\le \lambda^M \bar{\kappa}\frac{\|\tilde{B}_1^{-1}\|}{N_\gamma}l!^\gamma\sum_{k=M}^{l+M-1}\left(\frac{\mu_0(1+N_\gamma)}{\tilde{\delta}}\right)^k\\
&\le \lambda^M 2\bar{\kappa}\frac{\|\tilde{B}_1^{-1}\|}{N_\gamma}l!^\gamma\left(\frac{\mu_0(1+N_\gamma)}{\tilde{\delta}}\right)^M
\end{aligned}$$

since $\mu_0 \le 1$ and $\frac{1+N_\gamma}{\tilde{\delta}} \le \frac{1+N_\gamma}{\delta_0} = 1/2$. Moreover, using that $\tilde{\delta} = \lambda\bar{\delta}$,

$$\|\tilde{\mathcal{A}}_l H_l^3\| \le \mu_0 l!^\gamma \mu_0^{M-1}\left[2\bar{\kappa}\frac{\|\tilde{B}_1^{-1}\|}{N_\gamma}\left(\frac{1+N_\gamma}{\bar{\delta}}\right)^M\right] \le \frac{1}{3}\mu_0 l!^\gamma$$

by definition of μ_0.

Now we deal with H_l^4. By induction hypothesis, $\|H_l^4\| \le \mu_0\sum_{k=N}^{l+M-N} J^1_{k,l+M-1}$. Then, using Lemmas 6.3 and 6.4:

$$\begin{aligned}
\|\tilde{\mathcal{A}}_l H_l^4\| &\le \mu_0\lambda^{M-1}\|\tilde{B}_1^{-1}\|\sum_{m=1}^{\left[\frac{l+M-N-1}{N-1}\right]}(l+M-N)!^\gamma(l+M-mN)|\tilde{a}|^m(1+|\sigma|)^{m/2}\\
&\le 2\mu_0\lambda^{M-1}\|\tilde{B}_1^{-1}\|(l+M-N)!^\gamma(l+M-N)|\tilde{a}|(1+|\sigma|)^{1/2}\\
&\le 2\mu_0\lambda^{M-N}\|\tilde{B}_1^{-1}\||a|(1+|\sigma|)^{1/2}(l+M-N)!^\gamma(l+M-N)
\end{aligned}$$

since $|\tilde{a}|(1+|\sigma|)^{1/2} \le \varepsilon(1+|\sigma|)^{1/2} \le 1/2$. Now we stress that, since $\gamma = 1/(N-M)$,

$$(l+M-N)!^\gamma(l+M-N) \le l!^\gamma\frac{l+M-N}{(l+M-N+1)^{\gamma(N-M)}} \le l!^\gamma.$$

Therefore, by Lemma 6.2,

$$\|\tilde{\mathcal{A}}_l H_l^4\| \le \mu_0 l!^\gamma\lambda^{M-N}\left[2\|\tilde{B}_1^{-1}\|\lambda^{M-N}|a|(1+|\sigma|)^{1/2}l!^\gamma\right] \le \frac{1}{3}\mu_0 l!^\gamma,$$

if λ is big enough. Notice that, by Lemma 6.3, $\tilde{\mathcal{A}}_l \tilde{B}_2 = \mathcal{A}_l B_2$, so that by (36) we can take λ big enough such that $\|\tilde{\mathcal{A}}_l \tilde{B}_2 \tilde{K}_l^z\| \le \mu_0 l!^\gamma/3$. Therefore, we also get in this case that $\|\tilde{K}_l^y\| \le \mu_0 l!^\gamma$.

Now we deal with $\tilde{K}_l^x$. We decompose $\tilde{K}_l^x = H_l^5 + H_l^6 + H_l^7 + H_l^8$ with

$$H_l^5 = \frac{1}{l-N} \sum_{\substack{l_1+\cdots+l_N=l+N-1 \\ 1\le l_i \le l-1}} \prod_{i=1}^N \tilde{K}_{l_i}^x, \qquad H_l^6 = \frac{1}{\tilde{a}(l-N)} \sum_{k=N}^{l-1} \tilde{K}_k^x R_{k,l+N-1},$$

$$H_l^7 = \frac{1}{\tilde{a}(l-N)} \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l}} G_k^x[\tilde{K}_{l_1}, \cdots, \tilde{K}_{l_k}], \quad H_l^8 = \frac{\left(\tilde{v}^\top \tilde{K}_l^y + \tilde{w}^\top \tilde{K}_l^z\right)}{\tilde{a}(l-N)},$$

where in H_l^7 we use that $\tilde{K}_2^x = \cdots = \tilde{K}_{N-1}^x = 0$ by Lemma 6.1 and that $\tilde{K}_l^y, \tilde{K}_l^z$ are already known. We notice that, using ii) of Lemma 6.5 and that $\mu_0 \le 1$

$$\begin{aligned}
|H_l^5| &\le \mu_0^N \frac{1}{l-N} \sum_{\substack{l_1+\cdots+l_N=l+N-1 \\ l_i=1 \text{ or} N\le l_i\le l-1}} (l_1! \cdot \cdots \cdot l_N!)^\gamma \le \mu_0 l!^\gamma \frac{1}{l_0-N}\left[\frac{\mu_0^{N-1}(1+N_\gamma)^N}{N_\gamma}\right] \\
&\le \frac{1}{4}\mu_0 l!^\gamma
\end{aligned}$$

by definition of l_0. To bound $|H_l^6|$ we use Lemma 6.4 and we get

$$\begin{aligned}
|H_l^6| \le& \mu_0 \frac{1}{|\tilde{a}|(l-N)} \sum_{k=N}^{l-1} k!^\gamma R_{k,l-N+1} \\
\le& \mu_0 \frac{1}{|\tilde{a}|(l-N)} l!^\gamma \sum_{m=2}^{[\frac{l-1}{N-1}]} (l-N(m-1))|\tilde{a}|^m (1+|\sigma|)^{m/2} \\
\le& \mu_0 l!^\gamma \frac{1}{|\tilde{a}|} 2|\tilde{a}|^2(1+|\sigma|) = \mu_0 l!^\gamma \lambda^{-(N-1)}\big[2|a|(1+|\sigma|)\big] \le \frac{1}{4}\mu_0 l!^\gamma,
\end{aligned}$$

where we used that $|\tilde{a}|(1+|\sigma|) \le \varepsilon(1+|\sigma|) \le \frac{1}{2}$ and that λ is large enough. To bound H_l^7, we recall that $\frac{1+N_\gamma}{\tilde{\delta}} \le 1/2$, $\tilde{a} = \lambda^{1-N} a$ and that $\tilde{\delta} = \lambda \bar{\delta}$. Then, by definition of l_0:

$$\begin{aligned}
|H_l^7| \le& \frac{\lambda \bar{\kappa}}{|\tilde{a}|(l-N)} \sum_{k=N}^{l+N-1} \tilde{\delta}^{-k} \mu_0^k J_{k,l+N-1}^2 \\
\le& \frac{\lambda \bar{\kappa}}{|\tilde{a}|(l-N)} \sum_{k=N}^{l+N-1} (l+N-k)!^\gamma \frac{1}{N_\gamma}\left(\frac{\mu_0(1+N_\gamma)}{\tilde{\delta}}\right)^k \\
\le& \frac{2\lambda\bar{\kappa}}{|\tilde{a}|(l-N)N_\gamma} l!^\gamma \left(\frac{\mu_0(1+N_\gamma)}{\tilde{\delta}}\right)^N \le \mu_0 l!^\gamma \frac{1}{l_0-N}\left[\frac{2\bar{\kappa}\mu_0^{N-1}}{|a|N_\gamma}\left(\frac{1+N_\gamma}{\bar{\delta}}\right)^N\right] \\
\le& \frac{1}{4}\mu_0 l!^\gamma.
\end{aligned}$$

Moreover, since by Lemma 6.2, $\tilde{v} = \lambda^{-(N-1)} v$ and $\tilde{w} = \lambda^{-(N-1)} w$, H_l^8 can be made smaller than $\mu_0 l!^\gamma/4$ provided λ is big enough. Then $|\tilde{K}_l^x| \le \mu_0 l!^\gamma$. $\square$

7. **A solution of the invariance equation** $F \circ K - K \circ R = 0$**.** In this section we prove Theorem 3.3, that is, there exists a real analytic function which is a true solution of the invariance equation in an appropriate domain and it is γ-Gevrey asymptotic to the formal solution $\hat{K}$ found in the previous section.

We will use some basic properties about Gevrey functions. A summary of these properties can be found in [4]. See also [6].

We begin by applying Borel-Ritt's theorem for Gevrey functions to the formal solution $\hat{K}$ ([6, p.17]). Let $0 < \beta < \alpha\pi$ be an opening of a sector. Then there exist ρ small enough and a γ-Gevrey real analytic function, K_e, defined on the sector $S(\beta,\rho)$, which is γ-Gevrey asymptotic to the formal solution $\hat{K}$ (see (7) for the definition of the sector). Then,

$$F \circ K_e - K_e \circ R = E$$

being E a real analytic function on $S(\beta,\rho)$ γ-Gevrey asymptotic to the identically zero formal series. As a consequence, for any closed sector

$$\bar{S}_1 := \bar{S}_1(\bar{\beta},\bar{\rho}) = \{t \in \mathbb{C} : 0 < |t| \le \bar{\rho},\ |\arg(t)| \le \bar{\beta}/2\} \subset S(\beta,\rho)$$

there exist c_0, c such that

$$\|E(t)\| \le c_0 \exp\left(-c|t|^{-1/\gamma}\right), \qquad \text{if } t \in \bar{S}_1.$$

We look for a real analytic function H defined on $\bar{S}_1$ such that

$$F \circ (K_e + H) - (K_e + H) \circ R = 0, \qquad \sup_{t \in \bar{S}_1} |H(t)| \exp\left(c|t|^{-1/\gamma}\right) < +\infty. \tag{37}$$

For that we rewrite (37) as a fixed point equation. Let us introduce $\hat{C}(t)$ and $\mathcal{N}$ as:

$$\hat{C}(t) = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} + B_1[K_e^x(t)]^{M-1} & 0 \\ 0 & 0 & C \end{pmatrix}, \quad \mathcal{N}(H) = F(K_e + H) - F(K_e) - \hat{C}(t)H.$$

Then the equation (37) becomes

$$\hat{C}(t)H(t) - H \circ R(t) = -E(t) - \mathcal{N}(H)(t). \tag{38}$$

We introduce $S_1(\bar{\beta},\bar{\rho}) = \mathrm{int}(\bar{S}_1(\bar{\beta},\bar{\rho}))$ and the Banach spaces

$$\mathcal{X}_{\ell,m} = \{H : \bar{S}_1(\bar{\beta},\bar{\rho}) \cup \{0\} \to \mathbb{C}^m,\ \mathcal{C}^0,\ \text{real analytic in } S_1(\bar{\beta},\bar{\rho}) \text{ and } \|H\|_\ell < +\infty\}$$

with

$$\|H\|_\ell := \sup_{t \in \bar{S}_1} \|H(t)\| |t|^{-\ell} \exp\left(c|t|^{-1/\gamma}\right).$$

It is straightforward to check that, if H_1, H_2 are $\mathcal{C}^0$ functions in $\bar{S}(\bar{\rho},\bar{\beta}) \cup \{0\}$, satisfying that $H_1(0) = H_2(0) = 0$, then, denoting $\Delta H(t) = H_1 - H_2$,

$$\begin{aligned} \|\mathcal{N}^{x,y}(H_1)(t)\| \le{}& a_1\|H_1(t)\|^2 + |t|^{M-1}\left(a_2|t| + a_3|t|^{N-M}\right)\|H_1(t)\| \\ &+ a_4\|B_2\|\|H_1(t)\|, \\ \|\mathcal{N}^z(H_1)(t)\| \le{}& b_1\|H_1(t)\|^2 + b_2|t|\|H_1(t)\|, \\ \|\mathcal{N}^{x,y}(H_1)(t) - \mathcal{N}^{x,y}(H_2)(t)\| \le{}& a_1\|\Delta H(t)\|^2 \\ &+ |t|^{M-1}\left(a_2|t| + a_3|t|^{N-M} + a_4\|B_2\|\right)\|\Delta H(t)\|, \\ \|\mathcal{N}^z(H_1)(t) - \mathcal{N}^z(H_2)(t)\| \le{}& b_1\|\Delta H(t)\|^2 + b_2|t|\|\Delta H(t)\|. \end{aligned} \tag{39}$$

To prove the above inequalities we take into account that $K_e^y, K_e^z = \mathcal{O}(|t|^2)$ as well as the form (3) of F.

We observe that, by scaling the variable z, the norm $\|B_2\|$ is as small as we need. In addition, the matrix C does not change with this scaling.

We are forced to distinguish two cases according to the different values of M and N.

7.1. **The case $M \geq N$.** Recall that we are assuming that $\operatorname{Spec} C \subset \{z \in \mathbb{C} : |z| \geq 1\}$. In this case we reinterpret (38) as the fixed point equation

$$H(t) = \mathcal{F}(H)(t) := (\hat{C}(t))^{-1} \left[H \circ R(t) - E(t) - \mathcal{N}(H)(t)\right] \tag{40}$$

which is, essentially, the same as the one considered in [4]. A crux point is that, if $H \in \mathcal{X}_{0,1+d+d'}$, then

$$\|H \circ R\|_0 \leq e^{-\frac{\alpha}{2} c(N-1)\cos\lambda} \|H\|_0 \tag{41}$$

so that this term is contracting. Following the steps in the mentioned work, one can easily check that, taking $0 < \beta < \alpha\pi$ and ρ small enough, the fixed point equation (40) has a unique solution belonging to the Banach space $\mathcal{X}_{0,1+d+d'}$ for any $\bar{\rho}, \bar{\beta}$ such that $\bar{S}_1(\bar{\beta}, \bar{\rho}) \subset S(\beta, \rho)$. As a consequence, the invariance condition (37) can be solved and the solution $K_e + H$ is analytic in the sector $S(\beta, \rho)$ and α-Gevrey asymptotic to the formal solution $\hat{K}$.

7.2. **The case $M < N$.** When $M < N$ the strategy developed for the case $M \geq N$ can not be applied. In this case bound (41) is not longer true. Indeed, as shown in Lemma 7.1 below (see also [4]), $|R(t)| \leq |t|(1 + \nu|t|^{N-1})^{-\alpha}$ with $\nu > 0$. Then, if $H \in \mathcal{X}_{0,1+d+d'}$,

$$e^{c|t|^{-1/\gamma}} \|H \circ R(t)\| \leq \|H\|_0 e^{-c|R(t)|^{-(N-M)} - c|t|^{-(N-M)}} \leq \|H\|_0 e^{-c\frac{\alpha}{2}(N-1)\cos\lambda |t|^{M-1}}.$$

This implies that the term $H \circ R$ is not contracting. For this reason we rewrite (38) as another fixed point equation. We recall that, when $M < N$, $\gamma = 1/(N-M)$ and we are assuming that $\operatorname{Spec} B_1 \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and that $\operatorname{Spec} C \subset \{z \in \mathbb{C} : |z| > 1\}$.

First we define an appropriate norm in $\mathbb{C}^{1+d+d'}$. We take a norm in $\mathbb{C}^{d'}$ such that $\|C^{-1}\|_{d'} < 1$. Notice that, since $\operatorname{Spec} B_1 \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, there exists a norm in $\mathbb{C}^d$ such that $\|\mathrm{Id} - B_1 t^{M-1}\|_d < 1 - \mu|t|^{M-1} \leq 1$. This follows from the fact that $\mathrm{Id} - B_1 t^{M-1}$ is in Jordan form if B_1 is in Jordan form as well. Therefore, since $K_e^x(t) = t + \mathcal{O}(t^2)$, taking ρ small enough,

$$\left\|\left(\mathrm{Id} - B_1 [K_e^x(t)]^{M-1}\right)^{-1}\right\|_d \leq 1.$$

We finally define

$$\|(x, y, z)\| = \max\{|x|, \|y\|_d, \|z\|_{d'}\}.$$

If necessary, we will write $\|(x, y)\| = \max\{|x|, \|y\|_d\}$.

We rewrite equation (38) as:

$$\mathcal{G}(H)(t) = -E(t) + \mathcal{N}(H)(t) + (0, 0, H^z \circ R(t))^\top, \tag{42}$$

with $\mathcal{G}(H)(t) = \hat{C}(t)H(t) - (H^x \circ R(t), H^y \circ R(t), 0)$. The usual way to proceed is: i) to find a formal inverse, $\mathcal{S}$, of the linear operator $\mathcal{G}$, ii) to prove that $\mathcal{S}$ is continuous in appropriate Banach spaces and iii) to write equation (42) as a fixed point equation and to apply the fixed point theorem.

The formal operator $\mathcal{S} = (\mathcal{S}^x, \mathcal{S}^y, \mathcal{S}^z)$, acting on analytic functions T, defined by

$$\mathcal{S}^x(T) = \sum_{j\geq 0} T^x \circ R^j,$$
$$\mathcal{S}^y(T) = \sum_{j\geq 0} \prod_{i=0}^{j} (\mathrm{Id} + B_1[K_e \circ R^i]^{M-1})^{-1} T^y \circ R^j,$$
$$\mathcal{S}^z(T) = C^{-1} T^z$$

is the formal inverse of $\mathcal{G}$. The proof of this fact is straightforward. We (formally) rewrite equation (42) as:

$$H = \mathcal{F}(H) := -\mathcal{S}\big(E + \mathcal{N}(H) + (0, 0, H^z \circ R)^\top\big) \tag{43}$$

To obtain accurate bounds for $\mathcal{S}$, we need precise estimates on the convergence of the iterates $R^k(t)$ for $t \in S(\beta, \rho)$. Given $\nu > 0$, let $\mathcal{R}_\nu : [0, \infty) \to \mathbb{R}$ be defined by

$$\mathcal{R}_\nu(u) = \frac{u}{(1 + \nu u^{N-1})^\alpha}.$$

Lemma 7.1. *Let $R : S(\beta, \rho) \to \mathbb{C}$ be a map of the form $R(t) = t - at^N + \mathcal{O}(|t|^{N+1})$, $a > 0$. Assume that $\beta < \alpha\pi$. Then, for any $0 < \nu < a(N-1)\cos\lambda$, with $\lambda = (N-1)\beta/2$, there exists ρ small enough such that*

$$|R^k(t)| \leq \mathcal{R}_\nu^k(|t|) = \frac{|t|}{\big(1 + k\nu|t|^{N-1}\big)^\alpha}.$$

In addition, R maps $S(\beta, \rho)$ into itself.

Proof. Writing $t = |t|e^{i\theta}$, the computation of the modulus of $R(t)$ gives

$$|R(t)| = |t|\big[1 - a|t|^{N-1}\cos(N-1)\theta + \mathcal{O}(|t|^N)\big]$$

and $\mathcal{R}_\nu(u) = u(1 - \alpha\nu u^{N-1} + \mathcal{O}(u^{2N-2}))$. Therefore, since $a\cos(N-1)\beta/2 > \alpha\nu$ and ρ is small enough, $|R(t)| \leq \mathcal{R}_\nu(|t|)$, for all $t \in S(\beta, \rho)$.

Since $\mathcal{R}_\nu$ is the flow time 1 of the one dimensional equation $\dot{u} = -\alpha\nu u^N$, i.e. $\mathcal{R}_\nu(u) = \varphi(1, u)$, then $\mathcal{R}_\nu^k$ is the flow time k of the same equation, that is:

$$\mathcal{R}_\nu^k(u) = \varphi(k, u) = \frac{u}{\big(1 + k\nu u^{N-1}\big)^\alpha}.$$

Using that $\frac{d}{du}\mathcal{R}_\nu(u) > 0$, it is easy to prove by induction that $|R^k(t)| \leq \mathcal{R}_\nu^k(|t|)$.

To prove that $R(S(\beta, \rho)) \subset S(\beta, \rho)$ is straightforward, see [4]. □

Now we deal with the linear operator $\mathcal{S}$.

Lemma 7.2. *Let $0 < \beta < \alpha\pi/2$, $0 < \nu < a(N-1)\cos\lambda$ and $\ell, \ell' \in \mathbb{R}$. If ρ is small enough, then, for any $\bar{\beta} \in (0, \beta)$ and $\bar{\rho} \in (0, \rho)$, $\mathcal{S}$ is a well defined, linear and bounded operator from $\mathcal{X}_{\ell,1+d} \times \mathcal{X}_{\ell',d'}$ to $\mathcal{X}_{\ell-M+1,1+d} \times \mathcal{X}_{\ell',d'}$. In addition,*

$$\|\mathcal{S}^{x,y}(T^{x,y})\|_{\ell-M+1} \leq \frac{2}{c\alpha(N-M)\nu}\|T^{x,y}\|_\ell, \qquad \|\mathcal{S}^z(T^z)\|_{\ell'} \leq \|C^{-1}\|\|T^z\|_{\ell'}.$$

Proof. Let T be a function belonging to $\mathcal{X}_{\ell,1+d+d'}$. Since $\mathcal{S}^z(T^z) = C^{-1}T^z$, the claim is clear. We have that

$$\begin{aligned}
\|\mathcal{S}^{x,y}(T^{x,y})(t)\| &\le \sum_{j\ge 0} \|T^{x,y}(R^j(t))\| \le \sum_{j\ge 0} \|T^{x,y}\|_\ell |R^j(t)|^\ell \exp\left(-c|R^j(t)|^{-1/\gamma}\right) \\
&\le \sum_{j\ge 0} \frac{\|T^{x,y}\|_\ell |t|^\ell}{\left(1+j\nu|t|^{N-1}\right)^{\alpha\ell}} \exp\left(-\frac{c}{|t|^{1/\gamma}}\left(1+j\nu|t|^{N-1}\right)^{\alpha/\gamma}\right) \\
&\le \|T^{x,y}\|_\ell |t|^\ell \left(e^{-c|t|^{-(N-M)}} + \int_0^\infty \frac{e^{-\frac{c}{|t|^{(N-M)}}\left(1+x\nu|t|^{N-1}\right)^{\alpha(N-M)}}}{\left(1+x\nu|t|^{N-1}\right)^{\alpha\ell}}\, dx \right).
\end{aligned}$$

Let $I(t)$ be the integral in the right hand side of the last inequality. By performing the change of variables

$$\begin{aligned}
v &= \left(1+x\nu|t|^{N-1}\right)^{\alpha(N-M)}, \\
dv &= \alpha(N-M)\nu|t|^{N-1}\left(1+x\nu|t|^{N-1}\right)^{\alpha(N-M)-1} dx \\
&= \alpha(N-M)\nu|t|^{N-1} v^{-(M-1)/(N-M)} dx
\end{aligned}$$

we have that

$$\begin{aligned}
I(t) &= \frac{1}{\alpha(N-M)\nu|t|^{N-1}} \int_1^\infty e^{-\frac{c}{|t|^{N-M}}v} v^{-(\ell-M+1)/(N-M)}\, dv \\
&= \frac{|t|^{-(\ell-M+1)/(N-M)}}{\alpha(N-M)\nu|t|^{M-1}} \int_{|t|^{-(N-M)}}^\infty e^{-cw} w^{-(\ell-M+1)/(N-M)}\, dw.
\end{aligned}$$

Integrating by parts we easily obtain

$$I(t) \le \frac{1}{c\alpha(N-M)\nu|t|^{M-1}} e^{-\frac{c}{|t|^{N-M}}} + \frac{|\ell-M+1|}{N-M}|t| I(t)$$

and the claim is proven since we can take $|t| < \rho$ small enough. □

It is clear that $E \in \mathcal{X}_{0,1+d+d'}$ for any $0 < \bar\rho < \rho$ and $0 < \bar\beta < \beta < \alpha\pi$. By Lemma 7.2, $\mathcal{S}(E) \in \mathcal{X}_{-M+1,1+d} \times \mathcal{X}_{0,d'} \subset \mathcal{X}_{-M+1,1+d+d'}$. We introduce

$$\varrho = 2\|\mathcal{S}(E)\|_{-M+1}, \qquad D = \frac{2}{c\alpha(N-M)\nu}.$$

Lemma 7.3. *Let $0 < \beta < \alpha\pi$, $0 < \nu < a(N-1)\cos\lambda$ and ρ small enough such that the conclusions of Lemmas 7.1 and 7.2 hold true. For any $0 < \bar\beta < \beta$ and $0 < \bar\rho < \rho$, we introduce $\mathcal{B}_{-M+1}(\varrho) \subset \mathcal{X}_{-M+1,1+d+d'}$ the closed ball of radius ϱ.*

Then, if ρ is small enough, the fixed point equation (43), $H = \mathcal{F}(H)$, has a unique solution $H \in \mathcal{B}_{-M+1}(\varrho)$.

Proof. Let $H \in \mathcal{B}_{-M+1}(\varrho)$. We notice again that, by means of a scaling in the z-variable, $\|B_2\|$ is small enough. In addition, since $N > M$, $|t| \ge |t|^{N-M}$. By using bound (39) of $\|\mathcal{N}(H)\|$, we have that, for $|t| \le \bar\rho < \rho$,

$$\begin{aligned}
\|\mathcal{N}^{x,y}(H)\|_0 &\le \left(\varrho a_1 |t|^{-(M-1)} e^{-c|t|^{-(N-M)}} + (a_2+a_3)|t| + a_4\|B_2\|\right)\|H\|_{-M+1} \\
&\le (\rho^{1/2} + a_4\|B_2\|)\|H\|_{-M+1}, \\
\|\mathcal{N}^z(H)\|_{-M+1} &\le \left(\varrho b_1 |t|^{-(M-1)} e^{-c|t|^{-(N-M)}} + b_2|t|\right)\|H\|_{-M+1} \le \rho^{1/2}\|H\|_{-M+1}
\end{aligned}$$

if ρ is small enough. Hence, by Lemma 7.2

$$\|\mathcal{S}(\mathcal{N}(H))\|_{-M+1} \le \max\left\{D, \|C^{-1}\|\right\}(\rho^{1/2} + a_4\|B_2\|)\|H\|_{-M+1}.$$

We take ρ and $\|B_2\|$ small enough, such that

$$\max\left\{D, \|C^{-1}\|\right\}(\rho^{1/2} + a_4\|B_2\|) \le \frac{1}{2}(1 - \|C^{-1}\|). \tag{44}$$

Moreover, since $\|\mathcal{S}^z(H^z \circ R)\|_{M+1} \le \|C^{-1}\|\|H^z\|_{-M+1}$,

$$\begin{aligned}\|\mathcal{F}(H)\| &\le \|\mathcal{S}(\mathcal{N}(H))\|_{-M+1} + \|\mathcal{S}^z(H^z \circ R)\|_{-M+1}\\ &\le \frac{1}{2}(1 + \|C^{-1}\|)\|H\|_{-M+1} < \|H\|_{-M+1}.\end{aligned}$$

We have proven that $\mathcal{F}(\mathcal{B}_{-M+1}(\varrho)) \subset \mathcal{B}_{-M+1}(\varrho)$.

Now we check that $\mathcal{F}$ is contractive. Indeed, let $H_1, H_2 \in \mathcal{B}_{-M+1}(\varrho)$ be two functions in $\mathcal{B}_{-M+1}(\varrho)$. Again using (39),

$$\|\mathcal{S}(\mathcal{N}(H_1)) - \mathcal{S}(\mathcal{N}(H_2))\|_{-M+1} \le \max\left\{D, \|C^{-1}\|\right\}(\rho^{1/2} + a_4\|B_2\|)\|H_1 - H_2\|_{-M+1}$$

and, since $\|\mathcal{S}^z(H_1^z \circ R - H_2^z \circ R)\|_{M+1} \le \|C^{-1}\|\|H_1^z - H_2\|_{-M+1}$, we obtain

$$\|\mathcal{F}(H_1) - \mathcal{F}(H_2)\|_{-M+1} \le \frac{1}{2}(1 + \|C^{-1}\|)\|H_1 - H_2\|_{-M+1},$$

using (44). □

We deduce from this lemma that equation (37) is satisfied for $H \in \mathcal{X}_{-M+1,1+d+d'}$. Therefore, the function $K = K_e + H$ is a solution of $F \circ K = K \circ R$, analytic in $S(\beta, \rho)$ and with $\hat{K}$ as its asymptotic γ-Gevrey series. This proves Theorem 3.3 for the case $M < N$.

Appendix A. **Proof of Proposition 3.2.** We write $\varphi = \varphi_{\le r} + \varphi_{>r}$ being $\varphi_{\le r}$ the Taylor decomposition of φ up to order r. Note that $\varphi(0) = 0$. We also will use $\mathcal{N}_{\le r}$ and the notation introduce in Section 2.

We perform the normal form procedure (using the struture of eigenvalues) to assure that $\mathcal{N}^z(x, y, 0) = \mathcal{O}(\|(x, y)\|^N)$, the change of variables

$$(\bar{x}, \bar{y}, \bar{z}) = (x, y - \varphi^y_{\le r}(x), z - \varphi^z_{\le r}(x))$$

and the blow up

$$\bar{x} = u, \qquad \bar{y} = u^m v, \qquad \bar{z} = u^n w$$

for some $n, m \in \mathbb{N}$ to be determined later. We obtain the new map $F = (F^u, F^v, F^w)$:

$$\begin{aligned}F^u(u, v, w) =& u + \mathcal{N}^x(u, u^m v + \varphi^y_{\le r}(u), u^n w + \varphi^z_{\le r}(u)),\\ F^v(u, v, w) =& \frac{1}{\left(F^u(u, v, w)\right)^m}\left[u^m v + \mathcal{N}^y(u, u^m v + \varphi^y_{\le r}(u), u^n w + \varphi^z_{\le r}(u))\right.\\ &\left. + \varphi^y_{\le r}(u) - \varphi^y_{\le r}(F^u(u, v, w))\right],\\ F^w(u, v, w) =& \frac{1}{\left(F^u(u, v, w)\right)^n}\left[u^n w + \mathcal{N}^z(u, u^m v + \varphi^y_{\le r}(u), u^n w + \varphi^z_{\le r}(u))\right.\\ &\left. + \varphi^z_{\le r}(u) - \varphi^z_{\le r}(F^u(u, v, w))\right].\end{aligned}$$

We have that F is a real analytic map and that

$$F^u(u, v, w) = u + \mathcal{N}^x_{\le r}(u, u^m v + \varphi^y_{\le r}(u), u^n w + \varphi^z_{\le r}(u)) + o(|u|^r),$$

where $\mathcal{N}_x^{\le r}$ is a polynomial of degree r. Then,

$$F^u(u, v, w) = u + \mathcal{N}^x_{\le r}(u, \varphi^y_{\le r}(u), \varphi^z_{\le r}(u)) + \mathcal{O}(|u|^{m+1}\|v\|, |u|^{n+1}\|w\|) + o(|u|^r).$$

In addition, since

$$\mathcal{N}^x_{\le r}(u, \varphi^y_{\le r}(u), \varphi^z_{\le r}(u)) - \mathcal{N}(u, \varphi^y(u), \varphi^z(u)) = o(|u|^{r+1}),$$

taking $m+2 \ge N$ and $n+2 \ge N$ and using (6)

$$F^u(u,v,w) = u - f^u(u,v,w)u := u - \big(au^{N-1} + \mathcal{O}(|u|^N) + \mathcal{O}(|u|^m\|v\|, |u|^n\|w\|)\big)u.$$

Therefore, F^u satisfies the form of the first component in (3).

Now we deal with F^v. We first note that, since φ is invariant, we have that

$$\varphi_{\le r}(u) + \mathcal{N}^y(u, \varphi^y_{\le r}(u), \varphi^z_{\le r}(u)) - \varphi^y_{\le r}(F^u(u,0,0)) = o(|u|^r).$$

Secondly, we observe that, using the mean's value theorem

$$F^v(u,v,w) = \frac{v}{(1+f^u(u,v,w))^m} + \bar{B}_1(u,v,w)v + u^{n-m}\bar{B}_2(u,v,w)w + o(\|u|^{r-m}),$$

$\bar{B}_1, \bar{B}_2$ being matrices with every entry of order at least $\mathcal{O}(\|(u, u^m v, u^n w)\|)$. Note that,

$$\begin{aligned}\frac{v}{(1+f^u(u,v,w))^m} =& v + mau^{N-1}v + \mathcal{O}(|u|^N\|v\|) + \mathcal{O}(|u|^m\|v\|^2) + \mathcal{O}(|u|^n\|w\|^2),\\ \bar{B}_1(u, u^m v, u^n w)v =& \bar{B}_1(u,0,0)v + \mathcal{O}(|u|^m\|v\|^2) + \mathcal{O}(|u|^n\|w\|^2),\\ \bar{B}_2(u, u^m v, u^n w)w =& \bar{B}_2(u,0,0)w + \mathcal{O}(|u|^m\|v\|^2) + \mathcal{O}(|u|^n\|w\|^2).\end{aligned}$$

Let $M_1, M_2 \in \mathbb{N}$, $M_1, M_2 \ge 2$ and $\hat{B}_1, \hat{B}_2$ real matrices be such that $\bar{B}_i(u,0,0) = \hat{B}_i u^{M_i-1} + \mathcal{O}(|u|^{M_i})$, $i=1,2$. Eventually, $\hat{B}_1, \hat{B}_2$ can be the zero matrix if either $\bar{B}_1(u,0,0)$ or $\bar{B}_2(u,0,0)$ are flat at $u=0$. In this case one can take either $M_1 \ge N$ or $M_2 \ge N$.

We have then that

$$\begin{aligned}F^v(u,v,w) =& v + \big(mau^{N-1}\mathrm{Id} + u^{M_1-1}\hat{B}_1\big)v + u^{M_2-1}\hat{B}_2 w + \mathcal{O}(|u|^{M_1}\|v\|)\\ &+ \mathcal{O}(|u|^{M_2}\|w\|) + \mathcal{O}(|u|^N\|v\|) + \mathcal{O}(|u|^m\|v\|^2) + \mathcal{O}(|u|^n\|w\|^2) + o(\|u|^{r-m})\end{aligned}$$

The result follows with $M = \min\{M_1, N\}$ taking $m+2 \ge \max\{M_1, N\}$, $n = m + \max\{0, M - M_2\} \ge m$ and B_1 and B_2 adequately.

Finally we deal with F^w. We recall that $\mathcal{N}^z(x,y,0) = \mathcal{O}(\|(x,y)\|^N)$ and we notice that $\varphi^z(x) = \mathcal{O}(|x|^N)$. Indeed, φ^z satisfies

$$C\varphi^z(x) + \mathcal{N}^z(x, \varphi^y(x), \varphi^z(x)) = \varphi^z(x + \mathcal{N}^x(x, \varphi^y(x), \varphi^z(x))$$

or, taking into account condition (6),

$$\begin{aligned}\left(C - \mathrm{Id} + \int_0^1 \partial_z \mathcal{N}^z(x, \varphi^y(x), \lambda\varphi^z(x))\, d\lambda\right)\varphi^z(x) =& -\mathcal{N}^z(x, \varphi^y(x), 0) - \varphi^z(x)\\ &+ \varphi^z(x + \mathcal{N}^x(x, \varphi^y(x), \varphi^z(x)))\\ =& \mathcal{O}(|x|^N).\end{aligned}$$

Since the matrix in the left hand side is invertible if $|x|$ is small enough, $\varphi^z(x) = \mathcal{O}(|x|)^N$.

Performing analogous computations as the ones for F^v and taking into account that $\varphi^z_{\le r}(x) = \mathcal{O}(|x|^N)$ and that $\mathcal{N}^z(x,y,0) = \mathcal{O}(\|(x,y)\|^N)$ one obtains that

$$F^w(u,v,w) = Cw + \hat{C}_1 u^{N+m-n-1}v + \mathcal{O}(\|u,v,w\|^2) = Cw + \mathcal{O}(\|u,v,w\|^2)$$

and the proof is complete since $N+m-n-1 \ge 1$.

Remark A.1. As a consequence of the proof, Proposition 3.2 holds true if $\mathcal{F} \in \mathcal{C}^r$ with r big enough (including the $\mathcal{C}^\infty$ case). In addition the map of the form (3) is also $\mathcal{C}^r$.

Appendix B. **Proof of Proposition 3.8.** Since graph $\hat{\varphi}$ is the formal invariant manifold of F, it satisfies

$$\hat{\varphi}(F^x(x,\hat{\varphi}(x))) - F^{y,z}(x,\hat{\varphi}(x)) = 0. \tag{45}$$

Let $\tilde{g}(x) = \varphi_{\le p}(F^x(x,\varphi_{\le p}(x))) - F^{y,z}(x,\varphi_{\le p}(x))$. We claim that $\tilde{g}(x) = \mathcal{O}(\|x\|^{p+1})$. Indeed, using that $\varphi_1 = 0$ (since graph $\hat{\varphi}$ is tangent to the x-direction), we have that

$$\begin{aligned}\hat{\varphi}(F^x(x,\hat{\varphi}(x))) =& \varphi_{\le p}(F^x(x,\varphi_{\le p}(x))) + [\varphi_{\le p}(F^x(x,\hat{\varphi}(x))) - \varphi_{\le p}(F^x(x,\varphi_{\le p}(x)))] \\ &+ \varphi_{>p}(F^x(x,\hat{\varphi}(x))) \\ =& \varphi_{\le p}(F^x(x,\varphi_{\le p}(x))) + \mathcal{O}(\|x\|^{N+p}) + \mathcal{O}(\|x\|^{p+1})\end{aligned}$$

and

$$\begin{aligned}F^{y,z}(x,\hat{\varphi}(x)) =& F^{y,z}(x,\varphi_{\le p}(x)) + [F^{y,z}(x,\hat{\varphi}(x)) - F^{y,z}(x,\varphi_{\le p}(x))] \\ =& F^{y,z}(x,\varphi_{\le p}(x)) + \mathcal{O}(\|x\|^{p+1}).\end{aligned}$$

Hence, taking into account (45), the claim follows.

We define G by $G^x = F^x$ and $G^{y,z} = F^{y,z} + \tilde{g}$. Clearly, $F(x,y,z) - G(x,y,z) = \mathcal{O}(\|(x,y,z)\|^{p+1})$. We only need to check that graph $\varphi_{\le p}$ is invariant by G, which is true since, by the definition of $\tilde{g}$ and G,

$$\begin{aligned}\varphi_{\le p}(G^x(x,\varphi_{\le p}(x))) = \varphi_{\le p}(F^x(x,\varphi_{\le p}(x))) &= F^{y,z}(x,\varphi_{\le p}(x)) + \tilde{g}(x) \\ &= G^{y,z}(x,\varphi_{\le p}(x)),\end{aligned}$$

which concludes the proof.

Acknowledgments. I.B and P.M. have been partially supported by the Spanish Government MINECO-FEDER grant MTM2015-65715-P and the Catalan Government grant 2014SGR504. The work of E.F. has been partially supported by the Spanish Government grants MINECO MTMT2013-41168-P and MTM2016-80117-P (MINECO/FEDER, UE) and the Catalan Government grant 2014SGR-1145.

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Received December 2016; revised March 2017.

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