SIAM J. Applied Dynamical Systems
Vol. 11, No. 3, pp. 1062–1097

The Inner Equation for Generalized Standard Maps*

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Abstract. We study particular solutions of the *inner equation* associated with the splitting of separatrices on *generalized standard maps*. An exponentially small complete expression for their difference is obtained. We also provide numerical evidence that the inner equation provides quantitative information about the splitting of separatrices even in the case when the limit flow does not.

Key words. inner equation, exponentially small phenomena, splitting of separatrices, asymptotic formula, numerical computations

AMS subject classifications. 37J10, 34C37, 37C29, 34E10, 34M37

DOI. 10.1137/110859932

1. Introduction.

The phenomenon of the splitting of separatrices occurs when a dynamical system having an invariant object (a fixed point, a periodic orbit, a torus, etc.) with coincident branches of its stable and unstable invariant manifolds (a separatrix) is perturbed. Generically, a new invariant object of the perturbed system arises which still possesses stable and unstable invariant manifolds, but the latter no longer coincide.

The problem of measuring the size of this splitting is long-standing in dynamics. It is related to the existence of transversal homoclinic points and, consequently, with the nonintegrability and with the size of the stochastic zone of the system under study.

The most popular tool for measuring the splitting of separatrices is the Melnikov approach [26]. It is based on classical perturbation theory and provides a first order approximation for the splitting by using the distance between the stable and unstable invariant manifolds of the perturbed system. Nevertheless there are plenty of interesting (and in some sense generic) situations where this approach fails: when the Melnikov function does not correctly predict the size of the splitting or when no Melnikov function is available, for instance when integrable systems near simple resonances are perturbed. In this case, Poincaré already detected in [28] that the separatrix splits, but it turns out that the size of this splitting is exponentially small in the perturbation parameter, what it is usually known as a beyond-all-orders phenomenon. Consequently a direct application of a first order perturbation theory never will be able to provide a good estimation for this exponentially small splitting. There are other settings, related, for instance, to Arnold diffusion and fluid transport, when the splitting of separatrices is exponentially small in the perturbation parameter, but from now on we will restrict ourselves to the case of near identity, analytic, area-preserving maps.

*Received by the editors December 21, 2011; accepted for publication (in revised form) by E. Sander June 26, 2012; published electronically September 18, 2012.
http://www.siam.org/journals/siads/11-3/85993.html

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1.1. Exponentially small splitting of separatrices in analytic maps. Throughout this introduction we will avoid precise statements and technicalities, but we will give the main ideas about the exponentially small phenomena.

Consider an area-preserving analytic map, close to the identity, that is, a map which can be written as

$$G(z,h) = z + hg(z,h), \qquad z \in \mathbb{R}^2, \tag{1.1}$$

where h is a small parameter and $g(0,h) = 0$, so that the origin is a fixed point for any value of h. Assume also that the origin is a weakly hyperbolic fixed point. Namely, redefining the parameter h if necessary, the eigenvalues λ, λ^{-1} of $DG(0)$ are of the form $\lambda = \mathrm{e}^h = 1 + O(h)$. In this case, there exist W^{s} and W^{u}, the stable and unstable invariant manifolds of the origin, respectively. The goal is to measure the discrepancy between these invariant manifolds. Notice that, since for $h = 0$ the map $G(z,0) = z$, this is a beyond-all-orders phenomenon. The strategy is to not consider the first approximation of the map G as simply taking $h = 0$, but as the time h map of the vector field

$$z' = g(z,0). \tag{1.2}$$

It can be seen, for instance in [10], that this approximation holds under generic and checkable assumptions. If the vector field (1.2) possesses a homoclinic connection γ_0 associated with the origin (the fixed point), then one expects that the exponentially small splitting of separatrices phenomenon arises for maps of the form (1.1). In fact in [10] it is proved that, for any $p \in W^{\mathrm{s}}$,

$$\mathrm{dist}(p, W^{\mathrm{u}}) \le K_\sigma \mathrm{e}^{-2\pi\sigma/h}, \tag{1.3}$$

with $\sigma > 0$ and K_σ a constant depending on σ and p but independent of h. Nevertheless this upper bound is not useful for deciding whether the separatrix γ_0 splits or not. It turns out to be mandatory to obtain an expression for the asymptotic behavior of the splitting.

We emphasize here that, even when the distance between W^{s} and W^{u} seems a good choice for measuring the splitting, it depends on the point p. This is because this measure does not exploit the area-preserving character of our map. There are several quantities more appropriate for this task. One of them is the Lazutkin invariant (see formula (2.7) in section 2.1), which is related to the angle between W^{s} and W^{u} at a homoclinic point. An upper bound similar to (1.3) for the Lazutkin invariant can be obtained but with K_σ depending only on σ.

If the asymptotic behavior for the splitting has to be proved, the first question that arises from (1.3) is how much bigger σ could be. To find this optimal value of σ one has to know the analyticity domain of γ_0, the homoclinic connection of the vector field (1.2). It is proven in [9] that γ_0 has complex singularities; henceforth it is analytic in a maximal complex strip $\{t \in \mathbb{C} : |\mathrm{Im}t| < \sigma_0\}$. The bound (1.3) holds for any $\sigma < \sigma_0$, changing K_σ appropriately. Notice that if we take any fixed $\sigma_* < \sigma_0$, we do not obtain a sharp upper bound, simply because the result also holds for σ, with $\sigma_* < \sigma < \sigma_0$, and henceforth, taking h small enough, we get a better estimate than the previous one. As a consequence any asymptotic formula will require taking σ arbitrarily close to σ_0 as a function of h.

The key point for proving the bound (1.3) is to obtain good parameterizations for the invariant manifolds W^{s}, W^{u}, which are analytic in the complex strip $\{t \in \mathbb{C} : |\mathrm{Im}t| < \sigma\}$ with

$\sigma < \sigma_0$. The natural parameterization for the invariant manifolds is the functions $\gamma^{\mathrm{u,s}}(t)$ that satisfy

$$G(\gamma^{\mathrm{u,s}}(t), h) = \gamma^{\mathrm{u,s}}(t+h). \tag{1.4}$$

Notice that the homoclinic connection γ_0 satisfies this invariance equation for the time h flow of the vector field (1.2). As we have mentioned in the above paragraph, to obtain an asymptotic formula for the splitting it is necessary to find solutions of the invariance equation (1.4) defined for values of t arbitrarily close to σ_0 as a function of h. Since the strip is limited by the singularities of the homoclinic connection γ_0, this study becomes harder when the values of t are closer to these singularities. The *inner equation* is a suitable approximation of the invariance equation (1.4) for values of t close to these singularities.

The main goal of this paper is to derive the inner equation for a large set of area-preserving maps (the so-called *generalized standard maps*) and to obtain information about some special solutions and their difference. This is a first step in the proof of an asymptotic formula for the splitting of the invariant manifolds for these maps, but obtaining this formula is beyond the scope of this work. Nevertheless we will provide some numerical results which, combined with heuristic arguments (see section 4, especially (4.6)), support the relation between the splitting and the inner equation.

1.2. The inner equation. An overview. The study of the inner equation has been at the heart of the proof of the exponentially small splitting of separatrices in many examples, for maps [18, 23, 24] as well as for flows [20].

In the case of area-preserving analytic maps, the use of the inner equation dates back to [21], where a scheme to obtain an asymptotic formula for the splitting of separatrices of the Chirikov standard map was established. In that paper, a particular instance of the inner equation was introduced: the so-called *semistandard map*. Further development of the ideas in [21] led to the first rigorous proof of the asymptotic formula for the Chirikov standard map in [18]. A brief discussion on the splitting size of the Chirikov standard map can be found in [13]. From the same authors, the survey on exponentially small phenomena [15] introduces, among other things, the inner equations associated with *polynomial standard maps* and lists in an informal way asymptotic formulas for the splitting of separatrices in those cases. It is also remarkable that in the paper [16] resurgence theory is applied to the study of the solutions of the inner equation associated with the area-preserving Hénon map. This paper is strongly related to [14]. Also in the study of perturbation of the McMillan map [23, 24] resurgence methods were applied to studying the inner equation. Summarizing, one can find rigorous results on the inner equation in [18, 16, 24], particularly examples which are covered under our present work, which also includes and generalizes those present in [15] and the numerical study [17].

In the case of flows, the inner equation has been a successful tool for measuring the splitting of separatrices when the Melnikov function fails to predict the size of the splitting, as in the rapidly forced pendulum. (See [19, 20] or [2] for a generalization to arbitrary polynomial Hamiltonian systems of one and a half degrees of freedom, following the study on the inner equation in [1].) A different technique based on continuous averaging to study the exponentially small behavior of the splitting can be found in [30].

The purpose of the present paper is twofold, a combination of rigorous theoretical results in a general setting and numerical experiments avoiding lengthy proofs in particular examples. One of the numerical examples shows a type of behavior that is not covered by the surveys [15, 17] (see the end of this section).

We study some second order difference equations, called inner equations, which have the form either

$$\phi(z+1) - 2\phi(z) + \phi(z-1) = -\phi^n(z) + G(\phi(z))$$

or

$$\phi(z+1) - 2\phi(z) + \phi(z-1) = -\mathrm{e}^{n\phi(z)} + G(\mathrm{e}^{\phi(z)}),$$

depending on the class of maps under consideration, and where $G(w)$ is an analytic function such that $G(w) = \mathcal{O}(w^{n+1})$.

These equations appear, in particular, in the problem of exponentially small splitting of separatrices in *generalized standard maps* (see the next section for definitions), but they can appear in studies of other types of maps (with parabolic fixed points, for instance), and, with this applicability in mind, we consider them in their full generality (see (3.3) and (3.4)). In particular, our present results generalize those on the inner equations appearing in [21, 18, 13, 15, 16, 23, 24]. It is important to remark that in the previous literature on the subject the symmetries of the particular problems under consideration were exploited extensively in the proofs. Our present formulation does not rely on additional symmetries, making it suitable for applications. In particular, we provide all the technical details and complete proofs of the statements concerning the inner equations and their solutions. As a side comment for the specialists, there are several technical improvements in the proofs of our theoretical results, which we expect can be applied in other problems related to difference equations.

We describe a large set of formal solutions of these inner equations, from which some *true* solutions are obtained, and we derive a complete formula for their difference. The main results are collected in section 3, while section 2 provides a more detailed introduction of the problem and description of some of the known results. Sections 5, 6, and 7 are devoted to proving the theoretical results, while section 4 contains the numerical results with a nonrigorous exposition of their relation to the developed theory. It should be remarked that the relation between the inner equation and the actual computation of the splitting, in the particular cases where proofs are available (see [18, 23, 24]), is lengthy and full of technicalities. Our exposition here tries to give the reader an idea of the link between the inner equation and splitting size, by making very strong assumptions, in order to explain the obtained numerical results. These assumptions are fully proved in the literature for the Chirikov standard map and the McMillan map.

The numerical experiments have been conducted to test the applicability of the theoretical results. Although academic in nature, they show the relation between the splitting of separatrices and the difference between two solutions of the inner equation. Moreover, the main example exhibits a behavior that is not covered by the surveys [15, 17]. In this example, given by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \varepsilon(x - x^3) - \varepsilon^2 x^7 \\ y + \varepsilon(x - x^3) - \varepsilon^2 x^7 \end{pmatrix},$$

where ε is a small parameter, although the size of the splitting is much larger than the guess suggested by [10], the leading term of its asymptotic behavior is provided by the inner equation. As a matter of fact, the splitting size in this example behaves asymptotically when $\varepsilon \to 0$ as

$$\frac{A}{h^{10/3}}\exp\left(-\frac{\pi^2}{h}+\frac{2^{5/4}\sqrt{\pi}\Gamma(3/4)^2)}{h^{1/2}}\right)(1+\text{higher order terms}),$$

where $\varepsilon = 4\sinh^2(h/2)$ and A is a constant related to some inner equation, while the naïve guess provided by the limit flow (see section 2 for details), in this case the Duffing equation $\ddot{x} = x - x^3$, would be exponential with exponent $-\pi^2/h$. That is, the correction term is larger than any power of h. See sections 2.4 and 4.

We remark that although the computation of the actual splitting has been performed by using the multiple precision package PARI-GP, the computation of the leading term has been achieved by using the standard `long double` precision in `C`.

2. Generalized standard maps and exponentially small splitting of separatrices.

2.1. Generalized standard maps. We will say that an area-preserving map $(x^*, y^*) = F(x, y)$ is a *generalized standard map* if it can be written in the form

$$\left\{\begin{aligned} x^* &= x + y + f(x,h),\\ y^* &= y + f(x,h), \end{aligned}\right. \tag{2.1}$$

where h is a small parameter. We will assume that f depends analytically in its arguments on $|h| < h_0$, $|x| < \rho_0$, for some fixed $h_0, \rho_0 > 0$. We will be interested in the case when the origin is a fixed point of F, that is, $f(0, h) = 0$. Moreover, we will assume the origin to be weakly hyperbolic, although our study may be applied also to the case of a parabolic fixed point.

The parameter h is chosen in such a way that $\operatorname{spec} DF(0,0) = \{e^h, e^{-h}\}$. This last condition is equivalent to imposing $f'(0,h) = \frac{\partial}{\partial x}f(0,h) = \varepsilon$, with $\varepsilon = 4\sinh^2(h/2)$. We further assume that

$$f(x,h) = \sum_{k\geq 0} f_k(x)h^{k+2} = \varepsilon f_0(x) + O(h^3 x). \tag{2.2}$$

Under these conditions, the map (2.1) can be written as a close to the identity map: with the scaling $\tilde{x} = x$, $h\tilde{y} = y$, it becomes (using again x and y as variables)

$$\left\{\begin{aligned} x^* &= x + hy + O(h^2 x),\\ y^* &= y + hf_0(x) + O(h^2 x). \end{aligned}\right. \tag{2.3}$$

When h is small, the map (2.3) is well approximated by the time h map of the flow of the Hamiltonian system

$$\left\{\begin{aligned} \dot{x} &= y,\\ \dot{y} &= f_0(x). \end{aligned}\right. \tag{2.4}$$

We assume that the origin in (2.4), which is a fixed point, possesses a homoclinic connection, $\gamma_0(t) = (x_0(t), y_0(t))$. By a shift in t, we can choose γ_0 such that x_0 is an even function, that is, γ_0 intersects transversally the line $\{y = 0\}$ at $t = 0$. The invariant manifolds of the origin for the map (2.3) are close to this homoclinic connection. Hence, if h is small, by the conservation of the area, they must intersect. It is not difficult to check that the expansions in powers of h of the stable and unstable curves coincide. As a consequence, the expansion of the angle of intersection in powers of h vanishes, which, in view of the analytic nature of the problem, suggests that this angle may have an exponentially small behavior in h. In fact, Fontich and Simó, in [10], obtained an exponentially small upper bound for the angle. They showed that if γ_0 is analytic in the complex strip $\{|\mathrm{Im} t| < \sigma_0\}$ and the map F is defined around the homoclinic orbit, then, for any $0 < \sigma < \sigma_0$, the distance between the stable and the unstable manifold of the origin of (2.3) is bounded by $K_\sigma e^{-2\pi\sigma/h}$ for any $0 < h < h_\sigma$, where K_σ and h_σ are positive constants depending on σ and K_σ depends also on the point where this distance is measured. Restoring to the original variables, the same applies to the invariant manifolds of the origin of (2.1).

Equivalently, a *natural parametrization* $\gamma(t) = (x(t), y(t))$ of the invariant manifolds of the origin of (2.1), when condition (2.2) is satisfied, that is, a parametrization satisfying $F \circ \gamma(t) = \gamma(t + h)$, must be a solution of the difference equation

$$x(t + h) - 2x(t) + x(t - h) = f(x(t), h), \tag{2.5}$$

with $y(t) = x(t) - x(t - h)$. This equation implies that the curve $\gamma = (x, y)$ is invariant by F and that the action of F on γ is conjugated to the shift on the parameter t: $t \mapsto t+h$. One must supply additional conditions on γ to obtain the invariant stable and unstable curves: if γ is the unstable (resp., stable) manifold of the origin, then $\lim_{t\to-\infty} x(t) = 0$ (resp., $\lim_{t\to\infty} x(t) = 0$) is required.

Since the left-hand side of the invariance equation (2.5) is formally

$$x(t + h) - 2x(t) + x(t - h) = 4\sinh^2\left(\frac{h}{2}\frac{\partial}{\partial t}\right)(x)(t) = h^2\ddot{x}(t) + O(h^4),$$

it can be approximated, when h is small, by the second order differential equation

$$\ddot{x} = f_0(x), \tag{2.6}$$

which is nothing more than (2.4).

In order to measure the difference between the invariant manifolds, the *Lazutkin invariant* at a homoclinic point $p = \gamma^u(0) = \gamma^s(0)$,

$$\omega(p) = \det\left(\frac{d}{dt}\gamma^u(0), \frac{d}{dt}\gamma^s(0)\right), \tag{2.7}$$

is often used, where $\gamma^{u,s}(t)$ are natural parametrizations of the unstable and stable manifolds. Unlike the angle between the invariant curves, $\omega(p)$ is a symplectic invariant and depends only on the homoclinic orbit, not on the specific point p. Another symplectic invariant quantity that can be used to measure the splitting of the separatrices is the area of the lobe between two consecutive homoclinic points.

Since an upper bound of the splitting of the separatrices is known, the question of its asymptotic behavior when h tends to 0 arises. Some well-known examples in the literature where this formula is available are briefly summarized in the next subsection.

2.2. Examples of generalized standard maps with exponentially small splitting of separatrices. There are not many examples with a complete proof of an asymptotic formula for the splitting of separatrices in area-preserving maps. Here we quote two. There is a more abundant literature about splitting of separatrices in Hamiltonian systems with one and a half degrees of freedom (see [29, 8, 4, 30, 22, 27, 5]).

The first example is the Chirikov standard map, introduced by Chirikov as a basic model of the motion of a system close to a nonlinear resonance (see, for instance, [3]). It corresponds to taking $f(x, h) = \varepsilon \sin(x)$, with $\varepsilon = 4\sinh^2(h/2)$. This map is in fact defined in the annulus, and the limit flow (2.4) is a pendulum with the saddle at the origin. The separatrix of the pendulum is analytic in the strip $\{|\mathrm{Im}t| < \pi/2\}$ and has a singularity at $t = \mathrm{i}\pi/2$. The symmetries of the problem imply that there is a homoclinic point p on the line $x = \pi$.

In [18], Gelfreich proved, following the scheme developed by Lazutkin in [21], that

$$\omega(p) \asymp \frac{4\pi}{h^2} \mathrm{e}^{-\pi^2/h} \sum_{k\geq 0} h^{2k}\omega_k,$$

where the series on the right-hand side is asymptotic. In particular, the exponent in the exponential is well predicted by the Fontich–Simó theorem in [10].

The second example is the perturbed McMillan map. The McMillan map itself was introduced in [25] in the modelization of particle accelerator dynamics. In [6, 23, 24], perturbations of the McMillan family of the form

$$\begin{cases} x^* = y, \\ y^* = -x + \dfrac{2\cosh(h)y}{1+y^2} + \tilde{\varepsilon}V'(y) \end{cases} \tag{2.8}$$

are considered, with $V(y) = \sum_{k\geq 2} V_k y^{2k}$ analytic in a neighborhood of $y = 0$. In the above formula, h is the Lyapunov exponent of the origin, which is the small parameter, and $\tilde{\varepsilon}$ is independent of h and not necessarily small. The McMillan map is obtained when $\tilde{\varepsilon} = 0$ and is integrable with a polynomial first integral. See [6] for more details about the McMillan map.

With a linear change of coordinates, the map (2.8) can be written in the form (2.1) with

$$\begin{aligned} f(x,h) &= \varepsilon\frac{x-2x^3}{1+\varepsilon x^2} + \frac{\tilde{\varepsilon}}{\varepsilon^{1/2}} V'\left(\varepsilon^{1/2}x\right) \\ &= \varepsilon(x-(2-4\tilde{\varepsilon}V_2)x^3) - \varepsilon^2(x^2-(2+6\tilde{\varepsilon}V_3)x^5) + O(\varepsilon^4), \end{aligned} \tag{2.9}$$

where, again, $\varepsilon = 4\sinh^2(h/2)$. The limit flow (2.6) is the Duffing equation

$$\ddot{x} = x - (2-4\tilde{\varepsilon}V_2)x^3,$$

with homoclinic $x_0(t) = \alpha/\cosh(t)$, $\alpha = (1-2\tilde{\varepsilon}V_2)^{-1/2}$ (assuming $|\tilde{\varepsilon}| < (2V_2)^{-1}$). Its singularities closest to the real line are located at $\pm\mathrm{i}\pi/2$. In [23, 24], improving a partial result in [6], it was proven that, if $\hat{V}(2\pi) \neq 0$, where

$$\hat{V}(\zeta) = \sum_{k\geq 2} V_k \frac{\zeta^{2k-1}}{(2k-1)!}$$

is the Borel transform of V, then the invariant manifolds to the origin of (2.8) split when $\tilde{\varepsilon} \neq 0$ and the Lazutkin invariant of a particular homoclinic orbit satisfies

$$\omega \asymp \frac{4\pi\tilde{\varepsilon}}{\beta^2 h^2} e^{-\pi^2/h} \sum_{k\geq 0} h^{2k} B_k^+(\tilde{\varepsilon}),$$

where the functions B_k^+ are analytic around $\tilde{\varepsilon} = 0$, $\beta^2 = 1 - 2\tilde{\varepsilon}V_2/\cosh h$, and $B_0^+(\tilde{\varepsilon}) = 4\pi^2\hat{V}(2\pi) + O(\tilde{\varepsilon})$. If the map is written in the form (2.1), with the function f given in (2.9), the Lazutkin invariant has an additional h^2 in the denominator. Again, the exponent of the exponential is well predicted by the Fontich–Simó theorem.

2.3. Numerical studies for polynomial generalized standard maps. In [17], Gelfreich and Simó presented a detailed numerical study of the splitting of the separatrices of the generalized standard map (2.1) in the case $f(x,h) = \varepsilon p(x)$, with $p(x) = \sum_{k=1}^{n} p_k x^k$ a polynomial of degree n with $p_1 = 1$ (which implies $f'(0,h) = \varepsilon$) and $p_n < 0$. Is is also assumed that there is a homoclinic curve to the origin in the limit flow system (2.6).

Then, via numerical experiments, the authors showed that the asymptotic behavior of the Lazutkin invariant depends only on the relative position of the singularities of the homoclinic solution of (2.6), on the degree n of the polynomial p, and on the coefficient p_n:

$$\omega \asymp \frac{C_n}{|p_n|^{\nu/2} h^{\nu}} \mathrm{e}^{-2\pi\rho/h} \tilde{\omega}(h) + \cdots,$$

where $\nu = 2(n+1)/(n-1)$; ρ is the minimum distance to the real line of the singularities of the homoclinic of (2.6); $\tilde{\omega}(h) \not\equiv 0$ is either a constant, a periodic function, or a quasi-periodic function of $1/h$, depending only on the number of singularities at $|\mathrm{Im} t| = \rho$ and their relative positions; and C_n depends only on n.

Also in this case, the exponential behavior is well predicted by the Fontich–Simó theorem.

2.4. A discrepant example. Numerical observations. We introduce the generalized standard map (2.1) induced by

$$f(x,h) = \varepsilon(x - x^3) - \varepsilon^2 x^7. \tag{2.10}$$

Note that this map possesses terms in ε^2, like the McMillan map has (see (2.9)). Unlike the McMillan case, the function defining this map is entire.

The limit flow (2.6) for this map is also a Duffing equation, in this case $\ddot{x} = x - x^3$, with homoclinic $x_0(t) = \sqrt{2}/\cosh(t)$, whose singularities are located at the same place of the homoclinic of the McMillan map, $\pi/2$ being their minimum distance to the real line. Hence, one could be tempted to infer that the exponential behavior of the Lazutkin invariant is of order $e^{-\pi^2/h}$.

However, our numerical experiments suggest that the Lazutkin invariant at the *first* homoclinic point over the line $y = 0$, in the topology of the unstable manifold, behaves like

$$\omega \asymp \frac{A}{h^{10/3}} \mathrm{e}^{-2\pi\rho(h)/h} + \cdots , \tag{2.11}$$

where

$$\rho(h) = \frac{\pi}{2} - \frac{2^{1/4}\Gamma(3/4)^2}{\sqrt{\pi}} h^{1/2} + O(h^{3/2}) \tag{2.12}$$

and $A = 871.683\ldots$. In particular, the size of the Lazutkin invariant is much larger than the naïve guess, which, in turn, suggests that the approximation of the invariant manifolds provided by the limit flow (2.6) is not good enough to predict the asymptotic formula of the splitting. Section 4 is devoted to explaining these numerical experiments. In particular, we will conjecture the source of the function $\rho(h)$ and the origin and computation of the constant A.

2.5. Inner equation for generalized standard maps. In all the aforementioned examples, the constants ω_0, $B_0^+(\tilde{\varepsilon})$, C_n, and A in the leading term of the asymptotic behavior of the Lazutkin invariant are related to a suitable inner equation, whose solutions provide better approximations of the invariant manifolds for values of t in some regions of $\mathbb{C}$ than the one provided by the limit flow (2.6). Even in the case of the generalized standard map defined by (2.10), where the limit flow (2.6) does not provide enough information, the numerically evaluated constant A in (2.11) is obtained from such an inner equation.

In order to be able to construct the inner equation we will impose several conditions on the function defining the generalized standard map.

Let F be a generalized standard map of the form (2.1), induced by a function $f(x, h) = \sum_{k\geq 0} f_k(x) h^{k+2}$, satisfying the hypotheses in section 2.1. We furthermore assume the following:

(HP1) For each $k \geq 0$, $f_k(x) = \sum_{j=1}^{d_k} f_{k,j} x^j$, with $f_{k,d_k} \neq 0$.

(HP2) The function $k \mapsto (d_k - 1)/(k + 2)$ has a global maximum on $\mathbb{N}$. Let $I \subset \mathbb{N}$ be the set where this maximum is achieved.

Hypothesis (HP2) implies a restriction in the rate of growth of the degree of each of the polynomials f_k, which can be at most linear in k. We also remark that, combining hypotheses (HP1) and (HP2) with the fact that f is analytic in the bidisk $\mathbb{D}_{\rho_0} \times \mathbb{D}_{h_0}$, one obtains that the domain of analyticity with respect to x depends on h and tends to be the whole complex plane when h tends to 0.

We fix $\chi \in \mathbb{C}$. We introduce the new unknown $\phi(z)$ defined by $x(\chi + hz) = h^{-\alpha}\lambda\phi(z)$, with

$$\alpha = \frac{k+2}{d_k - 1} \quad \text{for any } k \in I$$

and λ a parameter to be determined later. Note that, by definition of I in (HP2), α is indeed independent of $k \in I$. The invariance equation (2.5) becomes

$$\phi(z+1) - 2\phi(z) + \phi(z-1) = h^{\alpha}\lambda^{-1} f(h^{-\alpha}\lambda\phi(z), h). \tag{2.13}$$

With the standing hypotheses, the right-hand side above admits an expansion of suitable positive powers of h as follows:

$$\begin{aligned} h^\alpha\lambda^{-1}f(h^{-\alpha}\lambda\phi(z),h) &= \sum_{k\geq 0}\sum_{j=1}^{d_k} f_{k,j}h^{-\alpha(j-1)}\lambda^{j-1}\phi^j(z)h^{k+2} \\ &= \sum_{k\geq 0} h^{k+2-\alpha(d_k-1)}\left(f_{k,d_k}\lambda^{d_k-1}\phi^{d_k}(z)+\sum_{j=1}^{d_k-1}h^{\alpha(d_k-j)}f_{k,j}\lambda^{j-1}\phi^j(z)\right) \\ &= \sum_{k\in I} f_{k,d_k}\lambda^{d_k-1}\phi^{d_k}+\mathcal{O}(h^{\min\{1,\alpha\}}), \end{aligned}$$

where in the last equality we have used the definitions of α and I. The inner equation is obtained by keeping only the first term in h in the right-hand side of (2.13):

$$\phi(z+1)-2\phi(z)+\phi(z-1)=\sum_{k\in I} f_{k,d_k}\lambda^{d_k-1}\phi^{d_k}. \tag{2.14}$$

Let $n=\min\{d_k : k\in I\}$. To simplify the notation we introduce the coefficients $\tilde G_k$ such that

$$\sum_{k\in I} f_{k,d_k}\lambda^{d_k-1}\phi^{d_k}=\sum_{k\geq n}\tilde G_k\lambda^{k-1}\phi^k.$$

Now we take λ such that $\lambda^{n-1}=-(\tilde G_n)^{-1}$. With this choice, the inner equation associated with the generalized standard map is

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\phi^n(z)+\sum_{k\geq n+1} G_k\phi^k(z), \tag{2.15}$$

with $G_k=\tilde G_k\lambda^{k-1}$. Notice that $G(\phi):=\sum_{k\geq n+1}G_k\phi^k$ is analytic in a neighborhood of $\phi=0$.

In the trigonometric case one can proceed analogously. Indeed, assume that $f(x,h)=\sum_{k\geq 0} f_k(x)h^{k+2}$, with f satisfying the following:

(HT1) For each $k\geq 0$, $f_k(x)=\sum_{j=-d_k}^{d_k} f_{k,j}\mathrm{e}^{\mathrm{i}jx}$ is a trigonometric polynomial of degree $d_k\geq a$, with $f_{k,d_k}\neq 0$.

(HT2) The function $k\mapsto d_k/(k+2)$ has a global maximum on $\mathbb{N}$. Let $I\subset\mathbb{N}$ be the set where this maximum is achieved.

For any $\chi\in\mathbb{C}$, we define $\phi(z)$ by $x(\chi+hz)=-\mathrm{i}\log\left(h^\alpha\lambda\right)+\mathrm{i}\phi(z)$ with

$$\alpha=\frac{k+2}{d_k}\quad\text{for any } k\in I \tag{2.16}$$

and λ a parameter. Then, the invariance equation (2.5) becomes

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\mathrm{i}f(-\mathrm{i}\log\left(h^\alpha\lambda\right)+\mathrm{i}\phi(z),h). \tag{2.17}$$

As in (2.14), the inner equation is the above equation when $h\to 0$. In this case, taking λ appropriately, one obtains

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\mathrm{e}^{(n-1)\phi(z)}+\sum_{k\geq n} G_k\mathrm{e}^{k\phi(z)}, \tag{2.18}$$

where $n-1=\min\{d_k : k\in I\}$. The discrepancy in the definition of n in both cases allows us to make a unified treatment of the problem in the next sections.

Since the original invariance equation (2.5) is autonomous, the inner equation (2.15) or (2.18) does not depend on the choice of the complex number χ introduced with the new unknown ϕ. Nevertheless, this complex number is essential when the size of the splitting of separatrices is studied and has to be well chosen. Roughly speaking, it will measure the exponential smallness of the splitting, which turns out to be $\mathcal{O}(h^\nu \mathrm{e}^{-2\pi \mathrm{Im}\chi/h})$ for some $\nu\in\mathbb{R}$. This asymptotic behavior has been proved only for particular maps (see section 2.2), but there is numerical evidence (see sections 2.3 and 2.4) that it also holds in a more general setting. We plan, in a future work, to prove it for the generalized standard maps.

In the examples presented in sections 2.2 and 2.3, χ is chosen to be the location of the singularity of the homoclinic solution γ_0 of the limit flow (2.6) that is closest to the real line. In the example in section 2.4, χ is also related to the singularities of a homoclinic solution of some flow, which is no longer (2.6) but $\ddot{x}=x-x^3-\varepsilon x^7$.

2.5.1. Some examples of the inner equation. Here we show how the inner equation is derived for some examples.

The first one is the map introduced in section 2.4. Its inner equation is

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\phi^7(z). \tag{2.19}$$

Indeed, in this case $f(x,h)=\varepsilon(x-x^3)+\varepsilon^2x^7$ with $\varepsilon=4\sinh^2(h/2)$. Therefore, $f_0(x)=x-x^3$, $f_{2k}(x)=f_{2k,1}x-f_{2k,3}x^3-f_{2k,7}x^7$, and $f_{2k-1}(x)=0$ for $k\geq 1$, which implies that $d_0=3$, $d_{2k}=7$, and $d_{2k-1}=0$ for $k\geq 1$. In this situation, it is clear that $n=7$, $\alpha=2/3$, and the set $I=\{2\}$; therefore the right-hand side of (2.14) is $f_{2,7}\lambda^6\phi^7$, and defining λ adequately, we encounter (2.19).

Now we compute the inner equation for the generalized standard map induced by $f(x,h)=\varepsilon(x-x^3)$. In this case $f_{2k}(x)=f_{2k,1}x-f_{2k,3}x^3$, $d_{2k}=3$, $f_{2k+1}(x)=0$, and $d_{2k+1}=0$ for $k\geq 0$, and this implies that $n=3$, $\alpha=1$, and the set $I=\{0\}$. Then, the right-hand side of (2.14) is $f_{0,d_0}\lambda^2\phi^3$, and we obtain the inner equation

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\phi^3(z). \tag{2.20}$$

We can also encounter inner equations having infinite terms on their right-hand sides, for instance, by considering $f(x,h)=\varepsilon\sin(x)+\sum_{k\geq 1}a_kh^{2k+2}\sin((k+1)x)$. In this case $d_{2k}=2k+2$, $d_{2k+1}=0$, $n=2$, $\alpha=2$, and $I=\{k\in\mathbb{N} : k \text{ is even}\}$, so that the inner equation is

$$\phi(z+1)-2\phi(z)+\phi(z-1)=-\mathrm{e}^{\phi(z)}+\sum_{k\geq 2}G_k(\mathrm{e}^{k\phi(z)}).$$

The main purpose of this paper is to provide some particular solutions of the *inner equation* (2.15) and (2.18) as well as to compute an explicit formula for their difference. The precise statement is placed in next section, while its proof is spread over the subsequent ones. As we have already commented in section 1.2, this computation has been at the heart of the proof of the splitting of separatrices in all the known examples, and it also gives an explanation to the numerical results concerning the example in section 2.4.

3. Main results.

We consider the linear operators

$$\Delta(\phi)(z) = \phi(z+1) - \phi(z) \tag{3.1}$$

and

$$\Delta^2(\phi)(z) = \Delta(\phi)(z) - \Delta(\phi)(z-1) = \phi(z+1) - 2\phi(z) + \phi(z-1) \tag{3.2}$$

and two types of inner equation. The first one, under the hypotheses (HP1) and (HP2), which from now on we will call polynomial case, is

$$\Delta^2(\phi) = g(\phi, \mu) := -\phi^n + G(\phi, \mu), \tag{3.3}$$

and the second one, under the hypotheses (HT1) and (HT2), which we will call trigonometric case, is

$$\Delta^2(\phi) = g(\phi, \mu) := -\mathrm{e}^{(n-1)\phi} + G(\mathrm{e}^{\phi}, \mu), \tag{3.4}$$

with G an analytic function in some open bidisk $\mathbb{D}(\varrho) \times \mathbb{D}(\mu_0) \in \mathbb{C}^2$ and such that

$$G(y, \mu) = \sum_{k \geq n+1} G_k(\mu) y^k \qquad \text{in the polynomial case,} \tag{3.5}$$

$$G(\mathrm{e}^y, \mu) = \sum_{k \geq n} G_k(\mu) \mathrm{e}^{ky} \qquad \text{in the trigonometric case.} \tag{3.6}$$

The parameter μ is included for the sake of completeness and is a regular parameter.

Remark 3.1. Let $\alpha \in \mathbb{R}$ be such that $\alpha n > 1$. If we consider inner equations of the form either $\Delta^2(\phi) = g(\phi^\alpha, \mu)$ in the polynomial case or $\Delta^2(\phi) = g(\alpha\phi, \mu)$ in the trigonometric one, the results in this section also hold true with the same proof. However, in order to avoid a new parameter, we restrict ourselves to the hypotheses above.

In this section we present the results dealing with both formal and analytic solutions of the inner equation.

Given $\nu > 0$, we will denote by

$$\mathbb{C}[[z^{-\nu}], \{\mu\}] = \left\{ \phi(z) = \sum_{k \geq 1} \frac{c_{k-1}(\mu)}{z^{\nu k}} \;\middle|\; c_{k-1} : B(\mu_0) \to \mathbb{C} \right\}$$

the space of formal power series in $z^{-\nu}$ without constant term, whose coefficients c_{k-1} depend analytically on $\mu \in B(\mu_0)$.

Proposition 3.2. *Let $n \geq 2$, $r = 2/(n-1)$.*

1. *If n is even, then (3.3) and (3.4) admit a unique formal solution $\tilde{\phi}$ such that $\tilde{\phi} \in \mathbb{C}[[z^{-r}], \{\mu\}]$ with $c_0^{n-1} = -r(r+1)$, in the case of (3.3), and, in the case of (3.4), $\tilde{\phi} - \tilde{\phi}_0 \in \mathbb{C}[[z^{-r}], \{\mu\}]$, with*

$$\tilde{\phi}_0(z) = \frac{1}{n-1} \log\left(-\frac{2}{n-1} \frac{1}{z^2} \right). \tag{3.7}$$

Moreover, any formal solution of the inner equation (3.3) belonging to $\mathbb{C}[[z^{-r/2}], \{\mu\}]$ is of the form $\tilde{\phi}(z-c, \mu)$ for some $c \in \mathbb{C}$ and c_0 such that $c_0^{n-1} = -r(r+1)$. The same applies to any formal solution ϕ of (3.4) such that $\phi - \tilde{\phi}_0 \in \mathbb{C}[[z^{-r/2}], \{\mu\}]$.

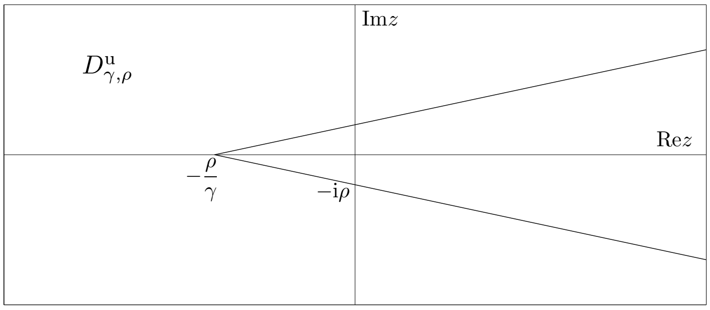

Figure 1. *Unstable domain.*

2. *If $n = 2m - 1$ with $m \geq 2$, the formal solutions are*

$$\tilde{\phi}(z,\mu) = \sum_{k\geq 1} \frac{1}{z^{kr}} \sum_{0\leq j\leq [\frac{k-1}{m-1}]} c_{k-1,j}(\mu) \log^j z, \tag{3.8}$$

in the case of (3.3), *with $c_0 = c_{0,0}$ satisfying $c_0^{n-1} = -r(r+1)$, and*

$$\tilde{\phi}(z,\mu) = \frac{1}{n-1} \log\left(-\frac{2}{n-1}\frac{1}{z^2}\right) + \sum_{k\geq 1} \frac{1}{z^{kr}} \sum_{0\leq j\leq [\frac{k}{m-1}]} c_{k-1,j}(\mu) \log^j z, \tag{3.9}$$

in the case of (3.4). *The symbol $[x]$ stands for the integer part of x. The coefficients $c_{k-1,j}$ are analytic functions in $B(\mu_0)$.*
The solution is unique, provided that $c_{m-1,0} = 0$. Any other formal solution of the form (3.8) *or* (3.9) *is obtained from these by translation.*

Now we deal with the analytic solutions of the inner equation. Let us define the complex domains where these solutions are defined. For any $\rho, \gamma > 0$, we introduce (see Figure 1)

$$D^{\mathrm{s}}_{\gamma,\rho} = \{z \in \mathbb{C} : |\mathrm{Im} z| > -\gamma \mathrm{Re} z + \rho\}, \qquad D^{\mathrm{u}}_{\gamma,\rho} = -D^{\mathrm{s}}_{\gamma,\rho}. \tag{3.10}$$

Let $\tilde{\phi}_0$ be defined by (3.7) in the trigonometric case and $\tilde{\phi}_0 \equiv 0$ in the polynomial case. Let ϕ_0 be the truncation up to order n in z^{-r} of the formal solution provided by Proposition 3.2; that is, if $n = 2m$ with $m \geq 1$,

$$\phi_0(z) = \tilde{\phi}_0(z) + \sum_{k=1}^{n} c_{k-1}(\mu) z^{-kr}, \tag{3.11}$$

and if $n = 2m - 1$ with $m \geq 2$, ϕ_0 in the polynomial case is

$$\phi_0(z) = \sum_{k=1}^{n} \frac{1}{z^{kr}} \sum_{0\leq j\leq [\frac{k-1}{m-1}]} c_{k-1,j}(\mu) \log^j z, \tag{3.12}$$

and in the trigonometric case is

$$\phi_0(z) = \tilde{\phi}_0(z) + \sum_{k=1}^{n} \frac{1}{z^{kr}} \sum_{0\le j\le[\frac{k}{m-1}]} c_{k-1,j}(\mu)\log^j z. \tag{3.13}$$

Theorem 3.3 (existence theorem). *Let $r = 2/(n-1)$ and c_0 be such that $c_0^{n-1} = -r(r+1)$. For any $\gamma > 0$ there exists ρ_0 big enough such that for any $\rho \ge \rho_0$ the inner equations* (3.3) *and* (3.4) *have two analytic solutions $\phi^{\mathrm{u,s}} : D^{\mathrm{u,s}}_{\gamma,\rho} \times B(\mu_0) \to \mathbb{C}$ such that*

$$\phi^{\mathrm{u,s}}(z,\mu) = \phi_0(z) + \psi^{\mathrm{u,s}}(z,\mu),$$

with

$$\sup_{(z,\mu)\in D^{\mathrm{u,s}}_{\gamma,\rho}\times B(\mu_0)} |z^{r+2}\psi^{\mathrm{u,s}}(z,\mu)| < +\infty.$$

Now we state the theorem for the difference $\phi^{\mathrm{u}} - \phi^s$. First we define the complex domain (see Figure 2)

$$E_{\gamma,\rho} = D^{\mathrm{u}}_{\gamma,\rho} \cap D^{\mathrm{s}}_{\gamma,\rho} \cap \{z\in\mathbb{C} : \mathrm{Im} z < 0\} \backslash \{z\in\mathbb{C} : |\mathrm{Re} z| \le 1,\ |\mathrm{Im} z| \le \rho+\gamma\}, \tag{3.14}$$

where the difference between two solutions of the inner equation (3.3), $\phi^{\mathrm{u}} - \phi^{\mathrm{s}}$, is defined.

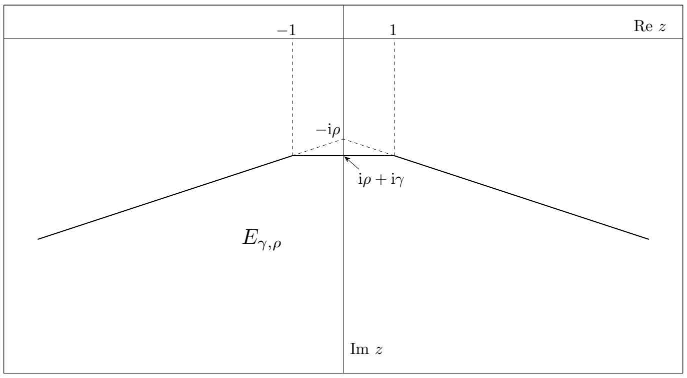

Figure 2. *Inner domain.*

To unify the notation we introduce the new parameters

$$\ell = \begin{cases} r+2 & \text{polynomial case,} \\ 2 & \text{trigonometric case,} \end{cases} \qquad d_\ell = \begin{cases} c_0 & \text{polynomial case,} \\ 1 & \text{trigonometric case.} \end{cases} \tag{3.15}$$

Theorem 3.4. *Let $\phi^{\mathrm{u,s}}$ be two analytic solutions of* (3.3) *and* (3.4) *satisfying the conditions stated in Theorem* 3.3.

Their difference $\phi^{\mathrm{u}} - \phi^{\mathrm{s}} : E_{\gamma,\rho} \times B(\mu_0) \to \mathbb{C}$ can be expressed as

$$\phi^{\mathrm{u}}(z,\mu) - \phi^{\mathrm{s}}(z,\mu) = \zeta_1(z,\mu)\sum_{k<0} p^1_k(\mu)e^{2\pi ikz} + \zeta_2(z,\mu)\sum_{k<0} p^2_k(\mu)e^{2\pi ikz}, \tag{3.16}$$

with p^1_k, p^2_k analytic functions in $B(\mu_0)$ and ζ_1, ζ_2 satisfying that

1. *their Wronskian*

$$W(\zeta_1,\zeta_2) := \begin{vmatrix} \zeta_1(z,\mu) & \zeta_2(z,\mu) \\ \zeta_1(z+1,\mu) & \zeta_2(z+1,\mu) \end{vmatrix} = 1;$$

2. *there exists a constant C such that for any $z \in E_{\gamma,\rho}$ and $\mu \in B(\mu_0)$,*

$$\left|z^{1-\ell}e^{2\pi iz}\big(\zeta_1(z,\mu) - \partial_z\phi^{\mathrm{s}}(z,\mu)\big)\right| \le C, \qquad \left|\frac{z^{-\nu}}{\log^\sigma z}\left(\zeta_2(z,\mu) - \frac{z^\ell}{rd_\ell(2\ell-1)}\right)\right| \le C,$$

with $\nu = \ell - r$, $\sigma = 0$ if $n > 3$, $\nu = \ell - 1$ if $n \le 3$, $\sigma = 0$ if $n = 2$, and $\sigma = 1$ if $n = 3$.

From now on we will skip the dependence on μ being always analytic.

4. Numerical results.

In this section we present some numerical results concerning the generalized standard map (2.1) given by the functions $f_1(x,h) = \varepsilon(x - x^3) - \varepsilon^2 x^7$ in (2.10) and $f_2(x,h) = \varepsilon(x - x^3)$. We recall here that $\varepsilon = 4\sinh^2(h/2)$.

We notice that both functions f_1, f_2 satisfy the hypotheses of section 2.5. Henceforth, as we show in section 2.5.1, we can construct the inner equation for the generalized standard map induced by them:

$$\Delta^2(\phi) = -\phi^7 \quad \text{and} \quad \Delta^2(\phi) = -\phi^3. \tag{4.1}$$

The first one corresponds to f_1 and the second one to f_2.

Let

$$\Theta := \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$$

be the difference between the two solutions of the inner equation (4.1) given by Theorem 3.3. First, in a general setting, we relate the main term of Θ to the Lazutkin invariant for the standard map (2.1) induced by f. Next, we compute the actual Lazutkin invariant for the maps defined by f_1 and f_2, which is computed numerically by using multiprecision routines. After that we summarize the method for computing the main term of the difference $\Theta := \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$, by exploiting the theoretical framework we have developed. One aspect worth noting is that these computations have been performed through standard `long double` precision arithmetic.

A similar, but more detailed, numerical comparison between the Lazutkin invariant and the difference Θ is performed in [12] for the Swift–Hohenberg equation.

4.1. The relation between the Lazutkin invariant and Θ.

For computing the first asymptotic term of Θ we now take advantage from the fact that we have an alternative expression for Θ by using the functions ζ_1 and ζ_2 given in Theorem 3.4. Indeed, we actually can write the difference Θ as

$$\Theta(z) = \zeta_1(z)\sum_{k<0} p_k^1 \mathrm{e}^{2\pi \mathrm{i}kz} + \zeta_2(z)\sum_{k<0} p_k^2 \mathrm{e}^{2\pi \mathrm{i}kz}$$

with $p_j(z) = \sum_{k<0} p_k^j \mathrm{e}^{2\pi \mathrm{i}kz}$, $j = 1,2$, 1-periodic functions. We recall that by Theorem 3.4, $W(\zeta_1,\zeta_2) = 1$, and henceforth $p_1 = W(\Theta,\zeta_2)$ and $p_2 = W(\zeta_1,\Theta)$.

On the one hand, we introduce the new quantity $\omega_{\mathrm{in}}(z)$:

$$\omega_{\mathrm{in}}(z) := -\frac{d}{dz}W(\Theta,\zeta_1)(z) = \frac{d}{dz}p_2(z) = \sum_{k<0} 2\pi \mathrm{i}kp_k^2 \mathrm{e}^{2\pi \mathrm{i}kz} \approx -2\pi \mathrm{i}p_{-1}^2 \mathrm{e}^{-2\pi \mathrm{i}z}. \tag{4.2}$$

The last equality has been deduced as $\mathrm{Im} z \to -\infty$. On the other hand, note that by using the first approximations of ζ_1 and ζ_2 in Theorem 3.4, since $\ell > 0$, $\zeta_1(z) \to 0$ as $\mathrm{Im} z \to -\infty$, and $\zeta_2(z) = \mathcal{O}(z^\ell)$, the main term of Θ is

$$\Theta(z) = \zeta_1(z)p_1(z) + \zeta_2(z)p_2(z) \approx \zeta_2(z)p_2(z) \approx z^\ell \frac{p_{-1}^2}{rd_\ell(2\ell-1)} \mathrm{e}^{-2\pi i z}.$$

We recall here that only p_{-1}^2 is unknown; the other quantities are defined in terms of the inner equation. Henceforth, both $\omega_{\mathrm{in}}(z)$ and $z^{-\ell}\Theta(z)$ are asymptotically equivalent.

In order to compare the numerical results with our theoretical framework we will gather in a rather informal way several facts, some of them not proven. In particular, to transform assumptions (A1) and (A2) below into proven facts would require involved arguments even for particular cases. For this reason, we will avoid precise statements. The chain of reasoning is a slight modification of that in [23], which also follows [21, 18].

Let f be a real analytic function satisfying the hypotheses in sections 2.1 and 2.5. We first remark that there exists a solution of the invariance equation (2.5) induced by f, $x^{\mathrm{u}}(t)$, iπ-antiperiodic, entire, and real analytic in t, such that $\lim_{\mathrm{Re} t \to -\infty} x^{\mathrm{u}}(t) = 0$ and $x^{\mathrm{u}}(0) = x^{\mathrm{u}}(-h)$ (and $x^{\mathrm{u}}(t) - x^{\mathrm{u}}(t-h) > 0$ for $t \le 0$). Then, the function $x^{\mathrm{s}}(t) = x^{\mathrm{u}}(-t)$ is also a solution of (2.5), with the same regularity, satisfying $\lim_{\mathrm{Re} t \to \infty} x^{\mathrm{s}}(t) = 0$. Hence, $\gamma^{\mathrm{u,s}}(t) = (x^{\mathrm{u,s}}(t), x^{\mathrm{u,s}}(t) - x^{\mathrm{u,s}}(t-h))$ are natural parametrizations of the invariant manifolds of the origin. We notice that $p = \gamma^{\mathrm{u}}(0) = \gamma^{\mathrm{s}}(0) = (x^{\mathrm{u,s}}(0), 0)$ is the *first* homoclinic point. Let $D(t) = x^{\mathrm{s}}(t) - x^{\mathrm{u}}(t)$.

Using the h-step Wronskian

$$W_h(u,v)(t) = \begin{vmatrix} u(t) & v(t) \\ u(t)-u(t-h) & v(t)-v(t-h) \end{vmatrix} = \begin{vmatrix} u(t) & v(t) \\ \Delta_h u(t) & \Delta_h v(t) \end{vmatrix},$$

the Lazutkin invariant (2.7) can be written as

$$\omega(p) = \det(\dot\gamma^u, \dot\gamma^s)_{|t=0} = \frac{d}{dt}\det(\dot\gamma^u, \gamma^s - \gamma^u)_{|t=0} = \frac{d}{dt} W_h(\dot x^u, D)_{|t=0}. \tag{4.3}$$

Since both x^u and x^s are solutions of the second order difference equation (2.5), their difference D also satisfies a *linear* second order equation, namely,

$$\Delta_h^2 D(t) = -\left(\int_0^1 \frac{\partial}{\partial x} f(sx^{\mathrm{s}}(t) + (1-s)x^{\mathrm{u}}(t), h)\, ds\right) D(t). \tag{4.4}$$

Notice that if x^{u} is close to x^{s}, then (4.4) is close to the linearization of the invariance equation (2.5) around x^{u}. Hence, our first assumption is that

(A1) there is a (real analytic) solution η_1 of (4.4) close to $\dot x^{\mathrm{u}}$.

Let η_2 be another (real analytic) solution of (4.4) with $W_h(\eta_1, \eta_2) = 1$, which can be obtained by the "variation of constants" method. Hence, we can write $D = c_1\eta_1 + c_2\eta_2$, where c_1 and c_2 are the h-periodic functions $c_1 = W_h(D, \eta_2)$ and $c_2 = W_h(\eta_1, D)$. Substituting this expression for D into (4.3) and using that η_1 is close to $\dot x^u$, we have that

$$\omega(p) \approx \frac{d}{dt} W_h(\eta_1, D)_{|t=0}. \tag{4.5}$$

Since f satisfies the hypotheses of section 2.5, we can construct an inner equation associated with the standard map induced by f. The second assumption is

(A2) there exists $\chi \in \mathbb{C}$ (which can depend on h) such that, for values of t satisfying $|t-\chi| = \mathcal{O}(h)$, $x^{\mathrm{u,s}}(t)$ are close to $h^{-\alpha}\lambda\phi^{\mathrm{u,s}}((t-\chi)/h)$. Here $\phi^{\mathrm{u,s}}$ are the solutions of the inner equation (3.3) given by Theorem 3.3, and α, λ are both parameters introduced in section 2.5. Since f is real analytic, one can assume that $\mathrm{Im}\chi > 0$.

As a consequence, since, by Theorem 3.4, $\zeta_1(z) = \partial_z\phi^{\mathrm{u}}(z) + \mathcal{O}(z^{r+1}\mathrm{e}^{2\pi \mathrm{i} z})$,

$$\dot{x}^{\mathrm{u}}(t) \approx h^{-\alpha-1}\lambda\frac{d}{dz}\phi^{\mathrm{u}}((t-\chi)/h) \approx h^{-\alpha-1}\lambda\zeta_1((t-\chi)/h).$$

Recall now that $p_2(z) = -W(\Theta, \zeta_1)(z)$. Hence, taking into account the scaling and assumption (A1), for values of t close to χ,

$$W_h(\eta_1, D)(t) \approx W_h(h^{-\alpha-1}\lambda\zeta_1, h^{-\alpha}\lambda\Theta)((t-\chi)/h) = h^{-2\alpha-1}\lambda^2 p_2\big((t-\chi)/h\big).$$

Then, since $W_h(\eta_1, D)(t)$ and $W(\zeta_1, \Theta)((t-\chi)/h)$ are both h-periodic and since the first one is a real analytic function, we easily have that for real t

$$\frac{d}{dt}W_h(\dot{x}^{\mathrm{u}}, D)(t) \approx 2h^{-2\alpha-2}\mathrm{Re}\left(\lambda^2\cdot\frac{d}{dz}p_2\big((t-\chi)/h\big)\right) = 2h^{-2\alpha-2}\mathrm{Re}\left(\lambda^2\cdot\omega_{\mathrm{in}}\big((t-\chi)/h\big)\right),$$

with ω_{in} defined in (4.2). Hence, evaluating at $t=0$,

$$\omega(p) \approx 2h^{-2\alpha-2}\mathrm{Re}\left(\lambda^2\cdot\omega_{\mathrm{in}}(-\chi/h)\right). \tag{4.6}$$

Our goal now is to check numerically the above formula for the maps induced by f_1 and f_2.

4.2. The limit flow and its singularities. In the cases of the Chirikov standard map and the perturbations of the McMillan map in [18] and [23], respectively, $\chi = \mathrm{i}\pi/2$ is the closest to the real line singularity of the homoclinic orbit of the limit flow (2.6). In the maps induced by $f_1(x,h) = \varepsilon(x-x^3) - \varepsilon^2x^7$ and $f_2(x,h) = \varepsilon(x-x^3)$ under consideration, the closest to the real line singularity of the homoclinic of the limit flow $\ddot{x} = x - x^3$ is also $\mathrm{i}\pi/2$ (see section 2.4). Nevertheless, our numerical computations show that this singularity is not the right guess for χ in the case of f_1. For this reason, we consider the *higher order* (in h) limit flow

$$\ddot{x} = x - x^3 - \varepsilon x^7. \tag{4.7}$$

The parametrization, $x_0(t,h)$, of the homoclinic loop to the origin such that $\dot{x}_0(0,h) = 0$ has a singularity at

$$\rho(h) = \int_{x_0(0,h)}^{+\infty}\frac{dx}{\sqrt{x^2/2 - x^4/4 - \varepsilon x^8/8}},$$

where $x_0(0,h) = \sqrt{2} + O(h^2)$ is the positive root of $x^2/2 - x^4/4 - \varepsilon x^8/8$ and the integral is computed along the real line. The other singularities can be obtained by changing the path of integration. It can be seen that

$$\rho(h) = \mathrm{i}\frac{\pi}{2} - \mathrm{i}\frac{2^{1/4}\Gamma(3/4)^2}{\sqrt{\pi}}h^{1/2} + O(h^{3/2}). \tag{4.8}$$

We remark that, although the singularities of the homoclinic of (4.7) tend to the singularities of the limit flow $\ddot{x} = x - x^3$ from (2.6) (in a rather slow way), they are of a different type: whereas the latter are poles, the former are branching points.

We choose the values $\chi = \mathrm{i}\frac{\pi}{2} - \mathrm{i}\frac{2^{1/4}\Gamma(3/4)^2}{\sqrt{\pi}}h^{1/2}$ for f_1 and $\chi = \mathrm{i}\frac{\pi}{2}$ for f_2, and we will assume that (A2) holds for them.

4.3. Numerical computations.

We now define

$$\tilde{\omega}(h) = h^{2\alpha+2}\lambda^{-2}\mathrm{e}^{2\pi|\chi|/h}\omega(p), \qquad \tilde{\omega}_{\mathrm{in}}(z) = 2\mathrm{e}^{2\pi\mathrm{i}z}\mathrm{Re}\left(\omega_{\mathrm{in}}(z)\right), \tag{4.9}$$

taking $\lambda = 1$ and, on the one hand, $\alpha = 2/3$ for f_1 and, on the other hand, $\alpha = 1$ for f_2. We note that, since χ has no real part, checking formula (4.6) is equivalent to checking that

$$\tilde{\omega}(h) \approx \tilde{\omega}_{\mathrm{in}}\big(-\chi/h\big) \Leftrightarrow \lim_{h\to 0}\tilde{\omega}(h) = \lim_{\mathrm{Im}z\to-\infty}\tilde{\omega}_{\mathrm{in}}(z).$$

First we show the results for $\tilde{\omega}(h)$. We have numerically computed this quantity by using multiprecision routines written in PARI-GP. In Figure 3 we show the computed values for $f_1(x,h) = \varepsilon(x - x^3) - \varepsilon^2 x^7$ and for the map induced by $f_2(x,h) = \varepsilon(x - x^3)$. Let us denote by $\tilde{\omega}_i(h)$ the value of $\tilde{\omega}(h)$ for the corresponding maps f_i, $i = 1, 2$. We have added a correction factor $\mathrm{esc} = 85 \cdot 10^{-4}$ in order to have the same magnitude for both values of $\tilde{\omega}(h)$.

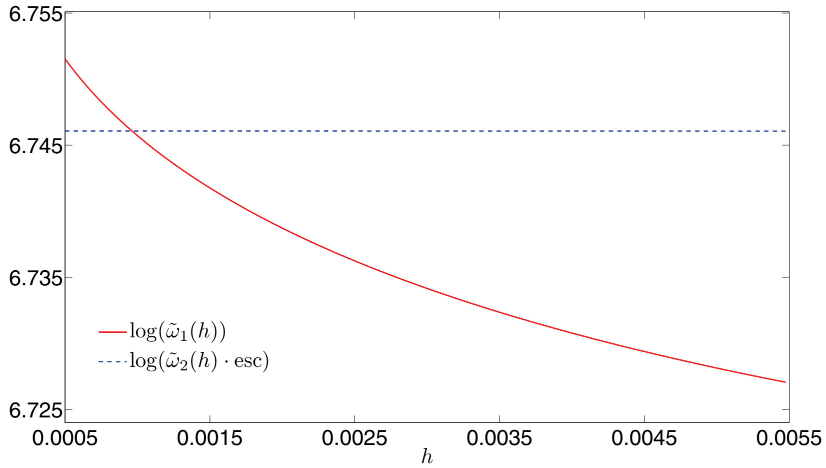

Figure 3.

These numbers have been obtained by explicitly computing $\omega(p) = \det(\dot{\gamma}^u(0), \dot{\gamma}^s(0))$, following the strategy in [7]. Due to the exponentially small behavior of this quantity, it has been necessary to compute $\dot{\gamma}^{u,s}(0)$ with increasing accuracy, thus making it impossible to achieve very small values of h.

Notice that, in the case of the map induced by $f_2(x,h) = \varepsilon(x - x^3)$, the values of $\tilde{\omega}_2(h)$ converge quite quickly, when h becomes smaller, to a constant value

$$\tilde{\omega}_2(h) \approx 1.00083 \cdot 10^5. \tag{4.10}$$

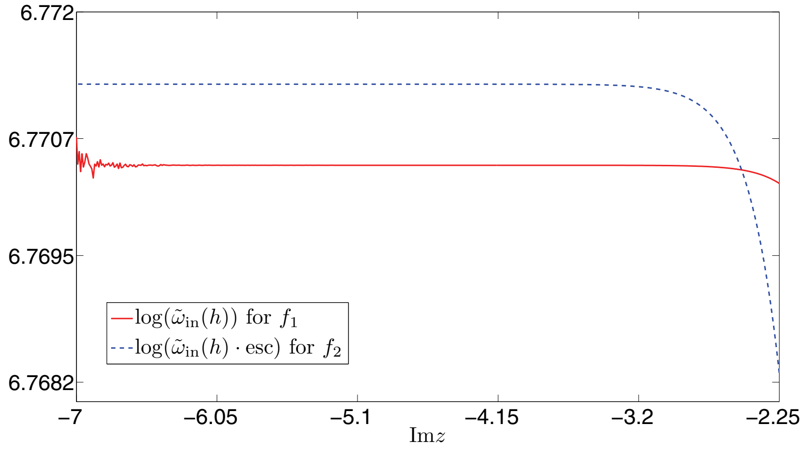

Figure 4.

In the case of the map induced by $f_1(x,h) = \varepsilon(x - x^3) - \varepsilon^2 x^7$, the convergence of the values of $\tilde\omega_1(h)$ is slower, as Figure 3 shows. However, computing $\tilde\omega_1(h)$ for $h = 1/2000 + k/40000$, $k = 0,\dots,199$, and making some assumptions on the form of the asymptotic expansion of $\tilde\omega_1(h)$ in h, it is possible to extrapolate the limit value with better accuracy.

In this way, we have obtained that

$$\tilde\omega_1(h) \approx 871.683. \tag{4.11}$$

We remark that, with the computed data, in which each value of $\tilde\omega_i(h)$, $i = 1,2$, has a few hundred correct digits, it would be possible to obtain a better approximation of this value and also to compute the coefficients of the asymptotic expansion. Since our intention was to compare the results obtained by the analysis of the solutions of the inner equation, we have not pursued this direction.

Now we compute $\tilde\omega_{\rm in}(z)$. By definition (4.2) of $\omega_{\rm in}(z)$ and (3.1) of the operator Δ,

$$\tilde\omega_{\rm in}(z) = 4\pi {\rm e}^{2\pi {\rm i} z}{\rm Re}\big(\Theta(z)\cdot\Delta(\partial_z\phi^{\rm s})(z) - \partial_z\phi^{\rm s}(z)\cdot\Delta(\Theta)(z)\big) + \mathcal{O}({\rm e}^{-2\pi {\rm i} z}z^{2r+2}, {\rm e}^{-2\pi {\rm i} z}), \tag{4.12}$$

where we have used that, by Theorem 3.4, $\zeta_1(z) - \partial_z\phi^{\rm u}(z) = \mathcal{O}({\rm e}^{-2\pi {\rm i} z}z^{r+1})$.

For symmetry reasons, we choose $z = -i\rho$ with $\rho \in [2.25, 7]$. We have used `long double` precision in `C` for calculating $\phi^{\rm s,u}(z), \partial_z\phi^{\rm s,u}(z)$. The strategy was suggested in [15]:

- First we compute the formal series $\widetilde\phi_N$ up to order N big enough. We know that the solutions $\phi^{\rm s,u}$ are close to $\widetilde\phi_N$ if $|z|$ is big enough. Analogously for $\partial_z\phi^{\rm s,u}$.
- We evaluate the formal series $\widetilde\phi_N(z \pm k)$ and $\partial_z\widetilde\phi_N(z\pm k)$ with $k \in \mathbb{N}$ big enough.
- Since both $\phi^{\rm s,u}$ satisfy the inner equation, we obtain $\phi^{\rm s,u}(z)$ and $\phi^{\rm s,u}(z+1)$ recurrently. Analogously for $\partial_z\phi^{\rm s,u}(z)$ and $\partial_z\phi^{\rm s,u}(z+1)$.

We have computed $\tilde\omega_{\rm in}(z)$ for the inner equations (4.1). Our results are given in Figure 4, where we have added the scaling factor ${\rm esc} = 871\cdot 10^{-5}$.

We can observe that, on the one hand, when ${\rm Im}z \in [-3,-2]$ the theoretical error in (4.12) is big. On the other hand, when ${\rm Im}z \in [-7,-6]$ the round-off errors (for f_1) begin to be bigger than the theoretical error, and hence the computed values have noise. Nevertheless for values of $\mathrm{Im} z \in [-5, -3]$ the computed values of $\tilde{\omega}_{\mathrm{in}}(z)$ behave like a constant. More precisely, we have found $\tilde{\omega}_{\mathrm{in}}(z) = 871.6833\ldots$ for $\Delta^2(\phi) = -\phi^7$ and $\tilde{\omega}_{\mathrm{in}}(z) = 1.000832\ldots 10^5$ for $\Delta^2(\phi) = -\phi^3$, which agree with the results for $\tilde{\omega}(h)$ given in (4.10) and (4.11).

5. Formal solutions of the inner equation.

In this section we prove the existence of formal solutions of the inner equation (3.3). The proof of the existence of formal solutions of (3.4) follows the same procedure. Hence, we skip it.

We start by defining the spaces to which these formal solutions belong. For $n \in \mathbb{N}$ and $r(n-1) = 2$, we define

$$X_r = \left\{ \phi(z) = \sum_{k \geq 1} \frac{c_{k-1}}{z^{kr}} \,\middle|\, c_k \in \mathbb{C} \right\}, \tag{5.1}$$

the space of formal power series in x^{-r} without constant term, and, if $n = 2m - 1$, $m \geq 2$, that is, $r = 1/(m-1)$, then

$$X_r^{\log} = \left\{ \phi(z) = \sum_{k \geq 1} \sum_{0 \leq j \leq \left[\frac{k-1}{m-1}\right]} c_{k-1,j} \frac{\log^j z}{z^{kr}} \,\middle|\, c_{k,j} \in \mathbb{C} \right\}, \tag{5.2}$$

where $[x]$ denotes the integer part of x, the space of formal power series in x^{-r} and $\log z$, with the power of $\log z$ bounded by the power of x^{-r}, without constant term.

We will say that $\phi = \mathcal{O}_{kr}$, with $k \in \mathbb{N}$, if and only if $z^{kr}\phi \in \mathbb{C}[[z^{-r}]]$ is a power series with terms z^{-jr} for $j \geq 0$. We will also use $\mathcal{O}_{kr,j}$ in $X_r^{\log}$, with $k \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$, meaning that $\phi = \mathcal{O}_{kr,j}$ implies that $z^{kr}(\log z)^{-j}\phi(z)$ is a formal power series with terms of the form $z^{-k'r} \log^{j'} z$, with $k' \geq 0$ and $j' \geq -j$ such that $j' \leq 0$ whenever $k' = 0$. We keep both notations in order to emphasize that $\mathcal{O}_{kr}$ is a series without logarithms, while $\mathcal{O}_{kr,0}$ is a series whose leading term does not have logarithms.

We collect several properties of these spaces in the following lemma, whose proof is straightforward.

Lemma 5.1. *Let $n \geq 2$, $r = 2/(n-1)$, and g be an analytic function around the origin with $g(y) = Ay^\ell + \mathcal{O}(y^{\ell+1})$ for some $\ell \in \mathbb{N} \cup \{0\}$. The spaces X_r, for n even, and $X_r^{\log}$, for n odd, have the following properties:*

1. *X_r and $X_r^{\log}$ are invariant by the formal differential operator $\frac{\partial^2}{\partial z^2}$. Furthermore, if $\phi \in X_r^{\log}$ (resp., X_r), then $\frac{\partial^2}{\partial z^2}\phi(z) = z^{-2}\psi(z)$ with $\psi \in X_r^{\log}$ (resp., X_r).*
2. *If $\phi(z) = az^{-r} + \tilde{\phi}(z)$, with $\tilde{\phi} = \mathcal{O}_{2r,j}$, $0 \leq j \leq [1/(m-1)]$ (resp., $\mathcal{O}_{2r}$), then $g(az^{-r} + \tilde{\phi}(z)) = Aa^\ell z^{-\ell r} + \varphi(z)$, with $\varphi = \mathcal{O}_{(\ell+1)r,j}$ (resp., $\mathcal{O}_{(\ell+1)r}$).*

Moreover, in the case $n = 2m-1$, $X_r^{\log}$ is also invariant by translation; that is, if $\phi(z) \in X_r^{\log}$, then $\phi(z - c) \in X_r^{\log}$ for any $c \in \mathbb{C}$. In the case $n = 2m$, if $\phi \in X_r$, then $\phi(z - c) \in X_{r/2}$.

We recall the function $g(y) = y^n - G(y)$. We remark that, since the operator Δ^2 can be written formally as

$$\Delta^2\phi(z) = 4\sinh^2\left(\frac{1}{2}\frac{\partial}{\partial z}\right)\phi(z) = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{12}\frac{\partial^4}{\partial z^4} + \cdots\right)\phi(z), \tag{5.3}$$

item 1 in Lemma 5.1 implies that the inner equation (3.3) is well defined in X_r and $X_r^{\log}$. We introduce

$$\epsilon(\phi) = \Delta^2(\phi) - g(\phi). \tag{5.4}$$

It is clear that $\epsilon(\phi)(z) = z^{-2}\hat{\epsilon}(z)$ with $\hat{\epsilon} \in X_r^{\log}$ (resp., X_r).

The next lemma follows directly from the definition of $X_r^{\log}$.

Lemma 5.2. *Let $n = 2m-1$, $r = 1/(m-1)$, $\phi \in X_r^{\log}$, and $\epsilon(\phi)$ be as in (5.4). If $z^2\epsilon(\phi)(z)$ has no terms of order N or smaller in z^{-r} (that is, no terms of the form $z^{-kr}\log^j z$, with $1 \le k \le N$), then $\epsilon(\phi) = \mathcal{O}_{(N+1)r+2,L}$, where $L = [N/(m-1)]$.*

Definition 5.3. *Let $n \ge 2$, $N \in \mathbb{N}$, $r = 2/(n-1)$, and $\phi \in X_r$ or $X_r^{\log}$. We will call truncated series of order N of ϕ to $\tilde{\phi}_N$ having the following form:*

1. *If n is even,*
$$\tilde{\phi}_N(z) = \sum_{k=1}^{N} \frac{c_{k-1}}{z^{kr}}.$$
2. *If $n = 2m-1$ is odd,*
$$\tilde{\phi}_N(z) = \sum_{k=1}^{m-1} \frac{c_{k-1}}{z^{kr}} + \sum_{k=m}^{N} \frac{1}{z^{kr}} \sum_{0 \le j \le \left[\frac{k-1}{m-1}\right]} c_{k-1,j} \log^j z.$$

Throughout the proof of Proposition 3.2, we will need to compute several times the formal series $g(\phi+\psi) - g(\phi)$, with different ϕ and ψ. The following lemma, which follows from the properties in Lemma 5.1, summarizes the result.

Lemma 5.4. *Let $n \ge 2$, $r = 2/(n-1)$, $N \ge 2$, $N \in \mathbb{N}$, and $\phi \in X_r$ or $X_r^{\log}$. We define $\psi_N = \tilde{\phi}_N - \tilde{\phi}_{N-1}$, where $\tilde{\phi}_N$ and $\tilde{\phi}_{N-1}$ are the truncated series of order N and $N-1$, respectively. We have the following:*

1. *If n is even,*
$$g(\tilde{\phi}_N(z)) - g(\tilde{\phi}_{N-1}(z)) = -n\frac{c_0^{n-1}}{z^2}\psi_N(z) + \mathcal{O}_{(N+1)r+2}.$$
2. *If $n = 2m-1$ is odd, writing $L = [N/(m-1)]$,*
$$g(\tilde{\phi}_N(z)) - g(\tilde{\phi}_{N-1}(z)) = -n\frac{c_0^{n-1}}{z^2}\psi_N(z) + \mathcal{O}_{(N+1)r+2,L}.$$

The following proposition implies the existence of a formal solution of the inner equation (3.3) and hence Proposition 3.2.

Proposition 5.5. *Let $n \ge 2$, $r = 2/(n-1)$, and c_0 be such that $c_0^{n-1} = -r(r+1)$. The inner equation (3.3) admits a formal solution ϕ with $z^r(\phi(z) - c_0 z^{-r}) \in X_r$ if n is even, and $z^r(\phi(z) - c_0 z^{-r}) \in X_r^{\log}$ if n is odd.*

Let $N \ge 2$ and $\tilde{\phi}_N$ be the truncated series defined as in Definition 5.3. Writing the truncation error of order N as
$$\epsilon_N := \epsilon(\tilde{\phi}_N) = \Delta^2(\tilde{\phi}_N) - g(\tilde{\phi}_N),$$
where ϵ was defined by (5.4), we have that

1. *if* $n \geq 2$ *is even,* $\epsilon_N = \mathcal{O}_{(N+1)r+2}$*;*
2. *if* $n = 2m-1 \geq 2$ *is odd and* $L = [N/(m-1)]$*, then*
   (i) *if* $1 \leq N \leq m-1$*,* $\epsilon_N = \mathcal{O}_{(N+1)r+2}$*,*
   (ii) *if* $m \leq N$*,* $\epsilon_N = \mathcal{O}_{(N+1)r+2,L}$*.*

Proof. We deal first with 1. We prove the claim by induction over N. We start by assuming $N = 1$. Let $\tilde{\phi}_1(z) = c_0 z^{-r}$. By item 2 in Lemma 5.1 and using (5.3), we have that

$$\epsilon_1(z) = \Delta^2(\tilde{\phi}_1)(z) - g(\tilde{\phi}_1(z)) = r(r+1)\frac{c_0}{z^{r+2}} + \frac{c_0^n}{z^{nr}} + \mathcal{O}_{(n+1)r}.$$

The claim for $N = 1$ follows from the facts that $r = 2/(n-1)$, which implies $\mathcal{O}_{(n+1)r} = \mathcal{O}_{2r+2}$ and $c_0^{n-1} = -r(r+1)$.

Now we assume the claim for $N-1$; that is, there exist coefficients c_k, $1 \leq k \leq N-2$, such that $\tilde{\phi}_{N-1}$ satisfies

$$\epsilon_{N-1}(z) = \epsilon(\tilde{\phi}_{N-1})(z) = \frac{A_{N-1}}{z^{Nr+2}} + \mathcal{O}_{(N+1)r+2}.$$

We look for $\tilde{\phi}_N(z) = \tilde{\phi}_{N-1}(z) + c_{N-1}z^{-Nr}$ satisfying the claim. We have that

$$\epsilon_N(z) = \epsilon_{N-1}(z) + \Delta^2\left(\frac{c_{N-1}}{z^{Nr}}\right) - g\left(\tilde{\phi}_{N-1}(z) + \frac{c_{N-1}}{z^{Nr}}\right) + g(\tilde{\phi}_{N-1}(z)).$$

By item 1 of Lemma 5.4,

$$g\left(\tilde{\phi}_{N-1}(z) + \frac{c_{N-1}}{z^{Nr}}\right) - g(\tilde{\phi}_{N-1}(z)) = -n\frac{c_0^{n-1}}{z^2}\frac{c_{N-1}}{z^{Nr}} + \mathcal{O}_{(N+1)r+2}. \tag{5.5}$$

Hence, using again (5.3),

$$\epsilon_N(z) = \frac{A_{N-1}}{z^{Nr+2}} + \lambda_N\frac{c_{N-1}}{z^{Nr+2}} + \mathcal{O}_{(N+1)r+2},$$

where the coefficient λ_N is

$$\lambda_N = Nr(Nr+1) + nc_0^{n-1} = \frac{4}{(n-1)^2}\left(N - \frac{n+1}{2}\right)(N+n). \tag{5.6}$$

Clearly, the claim follows if λ_N is different from 0, which is true since n is even and positive.

Now we assume $n = 2m-1$, $m \geq 2$. The induction process from the previous case can be used, provided that $\lambda_N \neq 0$. This is true for $N \neq m$. Hence, the claim holds for $1 \leq N \leq m-1$. Let $\tilde{\phi}_{m-1}(z) = c_0/z^r + \cdots + c_{m-2}/z^{(m-1)r}$ be the corresponding function. It satisfies

$$\epsilon_{m-1}(z) = \epsilon(\tilde{\phi}_{m-1})(z) = \frac{A_{m-1}}{z^{mr+2}} + \mathcal{O}_{(m+1)r+2}. \tag{5.7}$$

Now we consider the case $N = m$. Since $\lambda_m = 0$, this case cannot be dealt with as before. We need to include logarithms in the formal series.

Notice that, from (5.3),

$$\Delta^2\left(\frac{\log^\ell z}{z^{kr}}\right) = kr(kr+1)\frac{\log^\ell z}{z^{kr+2}} - \ell(2kr+1)\frac{\log^{\ell-1} z}{z^{kr+2}} + \ell(\ell-1)\frac{\log^{\ell-2} z}{z^{kr+2}} + \mathcal{O}_{kr+4,\ell}. \tag{5.8}$$

We look for $\tilde\phi_m = \tilde\phi_{m-1}+\psi_m$ satisfying the claim, with $\psi_m(z) = c_{m-1,1}z^{-mr}\log z + c_{m-1,0}z^{-mr}$. Hence we have that

$$\epsilon_m = \epsilon_{m-1} + \Delta^2(\psi_m) - g(\tilde\phi_{m-1}+\psi_m) + g(\tilde\phi_{m-1}).$$

From (5.8), we have that

$$\Delta^2(\psi_m)(z) = \frac{mr(mr+1)}{z^2}\psi_m(z) - (2mr+1)c_{m-1,1}\frac{1}{z^{mr+2}} + \mathcal{O}_{mr+4,1}, \tag{5.9}$$

while, from 2 in Lemma 5.4,

$$g(\tilde\phi_{m-1}(z)+\psi_m(z)) - g(\tilde\phi_{m-1}(z)) = -n\frac{c_0^{n-1}}{z^2}\psi_m(z) + \mathcal{O}_{(m+1)r+2,L}, \tag{5.10}$$

with $L = [N/(m-1)] = [m/(m-1)]$.

Hence, substituting (5.9) and (5.10) into the expression for ϵ_m above, we obtain

$$\epsilon_m(z) = \epsilon_{m-1}(z) + \frac{\lambda_m}{z^2}\psi_m(z) - (2mr+1)c_{m-1,1}\frac{1}{z^{mr+2}} + \mathcal{O}_{(m+1)r+2,L},$$

where the coefficient λ_N was introduced in (5.6) and, in fact, satisfies $\lambda_m = 0$. Since $\epsilon_{m-1}(z) = A_{m-1}z^{mr+2} + \mathcal{O}_{(m+1)r+2}$ (see (5.7)), taking $c_{m-1,1} = A_{m-1}/(2mr+1)$, we have that $\epsilon_m = \mathcal{O}_{(m+1)r+2,L}$. Notice that the coefficient $c_{m-1,0}$ is free. Hence, the claim is proven for $1 \le N \le m$.

Now, proceeding by induction, the result is proven. ∎

6. A solution of the inner equation.

The goal of this section is to prove the existence of a solution of the inner equation satisfying the properties stated in Theorem 3.3.

For any $\gamma, \rho > 0$, we recall the complex domains

$$D^{\rm s}_{\gamma,\rho} = \{z \in \mathbb{C} : |{\rm Im} z| > -\gamma {\rm Re} z + \rho\}, \quad D^{\rm u}_{\gamma,\rho} = -D^{\rm s}_{\gamma,\rho},$$

defined in (3.10) (see Figure 1). We also introduce the norms

$$\|\varphi\|^{\rm u,s}_{\nu,\gamma,\rho} = \sup_{z\in D^{\rm u,s}_{\gamma,\rho}} |z^\nu \varphi(z)|$$

and the Banach spaces

$$\mathcal{X}^{\rm u,s}_{\nu,\gamma,\rho} = \{\varphi : D^{\rm u,s}_{\gamma,\rho} \to \mathbb{C} \ \text{ such that } \ \|\varphi\|^{\rm u,s}_{\nu,\gamma,\rho} < +\infty\}.$$

We also define the functional space

$$\mathcal{X}^{\rm u,s}_{\nu,k,\gamma,\rho} = \{\varphi : D^{\rm u,s}_{\gamma,\rho} \to \mathbb{C} \ \text{ such that } \ \overline\varphi(z) := (\log z)^{-k}\varphi(z) \in \mathcal{X}^{\rm u,s}_{\nu,\gamma,\rho}\},$$

and, if there is no danger of confusion, we will simply denote them

$$\mathcal{X}_\nu = \mathcal{X}^{\mathrm{u,s}}_{\nu,\gamma,\rho}, \quad \mathcal{X}^{\log}_{\nu,k} = \mathcal{X}^{\mathrm{u,s,}}_{\nu,k,\gamma,\rho}, \quad \|\cdot\|_\nu = \|\cdot\|^{\mathrm{u,s}}_{\nu,\gamma,\rho}, \quad D_{\gamma,\rho} = D^{\mathrm{u,s}}_{\gamma,\rho}.$$

From now on we will denote by C a generic positive constant independent of γ, ρ, ν. We state (without proof) the following lemma, which will be used without mention throughout this section.

Lemma 6.1. *Let $0 < \nu_1, \nu_2$. For any $\varphi_1 \in \mathcal{X}_{\nu_1}$ and $\varphi_2 \in \mathcal{X}_{\nu_2}$,*

$$\varphi_1 \cdot \varphi_2 \in \mathcal{X}_{\nu_1+\nu_2} \quad \text{and} \quad \|\varphi_1 \cdot \varphi_2\|_{\nu_1+\nu_2} \le \|\varphi_2\|_{\nu_2} \cdot \|\varphi_1\|_{\nu_1}.$$

Also there exists $C > 0$ such that if $0 < \nu_1 < \nu_2$ and $\varphi \in \mathcal{X}_{\nu_2}$, then

$$\varphi \in \mathcal{X}_{\nu_1} \quad \text{and} \quad \|\varphi\|_{\nu_1} \le C\rho^{-(\nu_2-\nu_1)}\|\varphi\|_{\nu_2}.$$

As in the previous section, we will denote by $\mathcal{O}_\nu$ and $\mathcal{O}_{\nu,k}$ a generic function belonging to $\mathcal{X}_\nu$ and $\mathcal{X}^{\log}_{\nu,k}$, respectively.

Theorem 3.3 is rephrased in terms of the Banach spaces $\mathcal{X}^{\mathrm{u,s}}_{\nu,\gamma,\rho}$ in the following proposition.

Proposition 6.2. *Given $\gamma > 0$, there exists $\rho_0 > 0$ such that for any $\rho \ge \rho_0$ the inner equation (3.3) (polynomial case) or (3.4) (trigonometric case)*

$$\Delta^2(\phi) = g(\phi) \tag{6.1}$$

have exactly two solutions $\phi^{\mathrm{u,s}}$ of the form

$$\phi^{\mathrm{u,s}} = \phi_0 + \psi^{\mathrm{u,s}},$$

where ϕ_0 is the truncated series of order n defined in (3.11), (3.12), and (3.13), depending on the case we are dealing with, and $\psi^{\mathrm{u,s}} \in \mathcal{X}^{\mathrm{u,s}}_{r+2,\gamma,\rho}$.

The properties of ϕ_0 that we are interested in follow from Proposition 5.5.

Corollary 6.3. *Let us consider the remainder of order n:*

$$\epsilon_0 = \epsilon(\phi_0) = \Delta^2(\phi_0) - g(\phi_0),$$

where ϕ_0 is the truncated series of order n defined in (3.11), (3.12), and (3.13).

For any $\gamma > 0$ there exists ρ_0 big enough such that the following hold:

1. *If n is even, $\phi_0 = c_0 z^{-r} + \mathcal{O}_{2r}$ in the polynomial case, and $\phi_0 = \frac{r}{2}\log(-rz^{-2}) + \mathcal{O}_r$ in the trigonometric one.*
2. *If $n = 2m-1$ is odd, for the polynomial case $\phi_0 = c_0 z^{-r} + \mathcal{O}_{2r} + \mathcal{O}_{mr,1}$. Notice that, since $m \ge 2$, in particular we also have that $\phi_0 = c_0 z^{-r} + \mathcal{O}_{2r,1}$. In the trigonometric case, we have that $\phi_0 = \frac{r}{2}\log(-rz^{-2}) + \mathcal{O}_r + \mathcal{O}_{(m-1)r,1}$, which also implies that $\phi_0 = \frac{r}{2}\log(-rz^{-2}) + \mathcal{O}_{r,1}$.*
3. *For any value of n we have that $\epsilon_0 \in \mathcal{X}_{nr+2}$.*

The proof of Proposition 6.2 is performed in two steps. In section 6.1 we introduce a linear equation which is close to the first order variational equation of (6.1) with respect to ϕ_0. Such linear equations can be easily inverted in the adequate Banach spaces. Finally, in section 6.2 we look for $\psi^{\mathrm{u,s}}$ as a solution of a suitable fixed point equation.

From now on we will deal only with the $-$u$-$ case, the $-$s$-$ case being analogous. For that reason we will omit $-$u$-$ from our notation.

6.1. The linearized inner equation. We introduce the function

$$H(z) = (1+z^{-1})^\ell - 2 + (1-z^{-1})^\ell \tag{6.2}$$

for both cases, the polynomial and the trigonometric one with ℓ defined in (3.15). In this section we are going to study the following linear homogeneous second order difference equation:

$$\Delta^2(\phi)(z) = H(z)\phi(z). \tag{6.3}$$

We recall that the Wronskian of two solutions, ϕ_1, ϕ_2, of a linear difference equation is defined as

$$W(\phi_1,\phi_2)(z) = \begin{vmatrix} \phi_1(z) & \phi_2(z) \\ \phi_1(z+1) & \phi_2(z+1) \end{vmatrix}.$$

In addition, on the one hand, (6.3) has the obvious solution $\eta_2(z) = z^\ell$, and, on the other hand, it is a well-known fact that $\eta_1 = b \cdot \eta_2$ is a solution of (6.3) if and only if

$$\Delta b(z) = \frac{1}{\eta_2(z)\cdot \eta_2(z+1)} = \frac{1}{z^\ell (z+1)^\ell}.$$

One can also deduce that $W(\eta_1,\eta_2) \equiv 1$.

We will need a right inverse of the linear operator Δ defined in appropriate Banach spaces. For this reason we introduce the formal operator

$$\Delta^{-1}(h)(z) = \sum_{k\geq 1} h(z-k). \tag{6.4}$$

We emphasize that we are dealing with the unstable case.

Lemma 6.4. *Let $\alpha > 0$. For any $\gamma > 0$ there exists $\rho_0 > 0$ such that, for any $\rho \geq \rho_0$, $\Delta^{-1} : \mathcal{X}_{\alpha+1,\gamma,\rho} \to \mathcal{X}_{\alpha,\gamma,\rho}$ is a right inverse of the operator Δ defined in (3.1) with $\|\Delta^{-1}\| \leq C$.*

The proof of this lemma is straightforward and can be found in [18].

The first variational around ϕ_0 of the inner equation (6.1) is given by

$$\Delta^2(\phi) = \Delta^2(\phi) = Dg(\phi_0)\phi, \tag{6.5}$$

and we notice that

$$Dg(\phi_0) = \begin{cases} -n\phi_0^{n-1} + DG(\phi_0), & \text{polynomial case,} \\ -(n-1)\mathrm{e}^{\phi_0(n-1)} + DG(\mathrm{e}^{\phi_0})\mathrm{e}^{\phi_0}, & \text{trigonometric case.} \end{cases}$$

By using the identities $c_0^{n-1} = -r(r+1)$ and $nr = r+2$, the fact that $H(z) = (\ell-1)\ell z^{-2} + \mathcal{O}_3$, and Corollary 6.3, the result is as follows.

Lemma 6.5. *For any $\gamma > 0$ there exists $\rho_0 > 0$ big enough such that the following hold:*

1. *The function $H(z)$ satisfies $H = Dg(\phi_0) - A$, with $A \in \mathcal{X}_{r+2}$ if $n \neq 3$, and $A \in \mathcal{X}^{\log}_{r+2,1}$ if $n = 3$.*
2. *The function $\eta_2(z) = z^\ell$ is a solution of (6.3). Consequently, the function η_1 defined by*

$$\eta_1(z) = z^\ell \sum_{k>0} \frac{1}{(z-k)^\ell (z-(k+1))^\ell}$$

is also an independent solution with $W(\eta_1,\eta_2) = 1$. By Lemma 6.4, $\eta_1 \in \mathcal{X}_{\ell-1}$.

We notice that property 1 of Lemma 6.5 implies that the linear equation (6.3) is a good approximation of the first order variational with respect to ϕ_0 given in (6.5).

Finally, as we will see in the lemma below, Lemma 6.5 allows us to invert the linear operator $\mathcal{L}(\phi)(z) = \Delta^2(\phi)(z) - H(z)\phi(z)$.

Lemma 6.6. *For any $\gamma > 0$ there exists $\rho_0 > 0$ such that for any $\rho \geq \rho_0$ the operator $\mathcal{L}(\phi) = \Delta^2(\phi) - H \cdot \phi$ has right inverse $\mathcal{L}^{-1} : \mathcal{X}_{\alpha+2,\gamma,\rho} \to \mathcal{X}_{\alpha,\gamma,\rho}$ if $\alpha > \ell - 1$ and has the expression*

$$\mathcal{L}^{-1}(h) = \eta_1 \cdot \Delta^{-1}(\eta_2 \cdot h) - \eta_2 \cdot \Delta^{-1}(\eta_1 \cdot h). \tag{6.6}$$

Moreover, $\|\mathcal{L}^{-1}(h)\|_{\alpha,\gamma,\rho} \leq C\|h\|_{\alpha+2,\gamma,\rho}$, C being an independent constant of γ, ρ.

Proof. We will omit γ, ρ from the notation. On the one hand, η_1, η_2 are independent solutions of the homogeneous linear equation $\mathcal{L}(\phi) = 0$, and hence, by the variation of constants method, we obtain formula (6.6). On the other hand, if $g \in \mathcal{X}_{\alpha+2}$ with $\alpha > \ell - 1$, then $\eta_2 \cdot g \in \mathcal{X}_{\alpha+2-\ell}$ and $\eta_1 \cdot g \in \mathcal{X}_{\alpha+\ell+1}$, and by Lemma 6.4, $\eta_1 \cdot \Delta^{-1}(\eta_2 \cdot g) \in \mathcal{X}_\alpha$ and $\eta_2 \cdot \Delta^{-1}(\eta_1 \cdot g) \in \mathcal{X}_\alpha$. The bound $\|\mathcal{L}^{-1}(g)\|_\alpha \leq C\|g\|_{\alpha+2}$ is obtained by a direct application of Lemma 6.4. ■

6.2. The fixed point equation. In this section we are going to prove Proposition 6.2 about the existence and properties of solutions of the inner equations (3.3) (polynomial case) and (3.4) (trigonometric case),

$$\Delta^2(\phi) = -g(\phi),$$

of the form $\phi = \phi_0 + \psi$, with ϕ_0 given by (3.11) (n even), (3.12) (n odd, polynomial case), or (3.13) (n odd, trigonometric case).

We introduce

$$\epsilon_0 = -\Delta^2(\phi_0) + g(\phi_0), \qquad \mathcal{R}(\psi) = \psi^2 \int_0^1 D^2 g(\phi_0 + \lambda\psi)(1-\lambda)\, d\lambda, \tag{6.7}$$

and we note that if $\phi = \phi_0 + \psi$ is a solution of the inner equation, then, by 1 of Lemma 6.5, ψ has to satisfy the second order difference equation given by

$$\Delta^2(\psi) - H \cdot \psi = \epsilon_0 + A \cdot \psi + \mathcal{R}(\psi). \tag{6.8}$$

As we proved in Lemma 6.6, the linear operator $\mathcal{L}$ has a right inverse in some adequate Banach spaces. Using it, we will obtain a solution of (6.8) by using the fixed point equation given by

$$\psi = \mathcal{F}(\psi) := \mathcal{L}^{-1}(\epsilon_0) + \mathcal{L}^{-1}(A \cdot \psi) + \mathcal{L}^{-1} \circ \mathcal{R}(\psi). \tag{6.9}$$

Proposition 6.7. *Let $\gamma > 0$. There exists $\rho_1 > 0$ big enough such that, for any $\rho \geq \rho_1$, the fixed point equation* (6.9) *has a unique solution $\psi \in \mathcal{X}_{r+2,\gamma,\rho}$.*

Proof. We first note that there exists $\rho_0 > 0$ such that $\mathcal{L}^{-1}(\epsilon_0) \in \mathcal{X}_{r+2,\gamma,\rho_0}$ since, by Corollary 6.3, $\epsilon_0 \in \mathcal{X}_{r+4,\gamma,\rho_0}$ if ρ_0 is large enough. Let $\varrho_0 = 2\|\mathcal{L}^{-1}(\epsilon_0)\|_{r+2,\gamma,\rho_0}$. During the proof of this proposition we will denote by K a generic constant depending only on ϕ_0, ρ_0, and γ, and we will omit the dependence on γ and ρ in the Banach spaces and norms.

Let $\psi_1, \psi_2 \in B(\varrho_0) \subset \mathcal{X}_{r+2}$. We start by bounding the difference $\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\|_{r+2}$. By Lemma 6.5 we have, taking $\nu_r = 0$ if $n \neq 3$, and $\nu_r = r/2$ if $n = 3$, that $A \in \mathcal{X}_{r+2-\nu_r,\gamma,\rho_1}$, provided that ρ_1 is large enough. Henceforth, if $\psi \in \mathcal{X}_{r+2}$, $A \cdot \psi \in \mathcal{X}_{2r+4-\nu_r}$. Applying Lemmas 6.1 and 6.6, we can easily check that

$$\|\mathcal{L}^{-1}(A \cdot (\psi_1 - \psi_2))\|_{r+2} \le C\rho_1^{-r+\nu_r} \|A\|_{r+2-\nu_r} \cdot \|\psi_1 - \psi_2\|_{r+2}. \tag{6.10}$$

Now we deal with $\mathcal{R}(\psi_1) - \mathcal{R}(\psi_2)$. We recall that $\mathcal{R}$ was defined in (6.7). We notice that

$$\begin{aligned} \mathcal{R}(\psi_1) - \mathcal{R}(\psi_2) = \left(\psi_1^2 - \psi_2^2\right) \int_0^1 D^2 g(\phi_0 + \lambda\psi_1)(1-\lambda)\, d\lambda \\ + \psi_2^2 \int_0^1 [D^2 g(\phi_0 + \lambda\psi_1) - D^2 g(\phi_0 + \lambda\psi_2)](1-\lambda)\, d\lambda. \end{aligned} \tag{6.11}$$

We first claim that, if $\lambda \in [0, 1]$ and $z \in D_{\gamma,\rho_1}$ with ρ_1 big enough,

$$|D^2 g(\phi_0(z) + \lambda\psi_1(z))| \le K|z|^{\ell-4} \le K|z|^{-2+r}, \tag{6.12}$$

where ℓ was defined in (3.15). Indeed, we deal first with the polynomial case. In this case, by definition (3.5) of g, there exists a constant K such that $|g(y)| \le K|y|^n$. Moreover, since g is an analytic function, Cauchy's theorem implies that if $y_0 \in \mathbb{D}(\varrho/2)$,

$$|D^2 g(y_0)| \le K|y_0|^{-2} \sup_{|y-y_0| \le |y_0|/2} |g(y)| \le K|y_0|^{n-2}. \tag{6.13}$$

Also, since $\psi_1 \in B(\varrho_0) \subset \mathcal{X}_{r+2}$, there exist constants $0 < K_1 \le K_2$ and ρ_1 big enough, $K_1|z|^{-r} \le |\phi_0(z) + \lambda\psi_1(z)| \le K_2|z|^{-r} < \varrho/2$ for any $\lambda \in [0, 1]$, and $z \in D_{\gamma,\rho_1}$. Then, using $nr = r + 2$ and estimate (6.13),

$$|D^2 g(\phi_0(z) + \lambda\psi_1(z))| \le K|\phi_0(z) + \lambda\psi_1(z)|^{n-2} \le K|z|^{2r}|z|^{-rn} = K|z|^{-2+r},$$

which proves bound (6.12) in the polynomial case. The trigonometric case is easier since $|g(y)| \le K|\mathrm{e}^{y(n-1)}|$, and henceforth, a standard Cauchy estimate leads to bound (6.12). Hence, if $\psi_1, \psi_2 \in B(\varrho_0)$,

$$|D^2 g(\phi_0(z) + \lambda(\psi_1(z))) \cdot (\psi_1^2(z) - \psi_2^2(z))| \le K|z|^{-4}|\psi_1(z) - \psi_2(z)|.$$

Now we claim that, for $\lambda \in [0, 1]$ and $\psi_1, \psi_2 \in B(\varrho_0)$,

$$|[D^2 g(\phi_0(z) + \lambda(\psi_1(z))) - D^2 g(\phi_0(z) + \lambda(\psi_2(z)))]\psi_2^2(z)| \le K|z|^{-2r-4}|\psi_1(z) - \psi_2(z)|.$$

Indeed, since g is an analytic function, $D^3 g$ is bounded in $\mathbb{D}(\varrho)$, and henceforth, for any $y_1, y_2 \in \mathbb{D}(\varrho)$, $|D^2 g(y_1) - D^2 g(y_2)| \le K|y_1 - y_2|$, and the claim is proved, provided that ρ_1 is large enough to ensure that for any $z \in D_{\gamma,\rho_1}$, $\psi_1(z), \psi_2(z) \in \mathbb{D}(\varrho)$.

Finally by using the previous computations and formula (6.11), one obtains that $\mathcal{R}(\psi_1) - \mathcal{R}(\psi_2) \in \mathcal{X}_{r+6} \cup \mathcal{X}_{(2r+4)+r+2} = \mathcal{X}_{r+6} \subset \mathcal{X}_{r+4}$ and moreover

$$\|\mathcal{R}(\psi_1) - \mathcal{R}(\psi_2)\|_{r+4} \le C|\rho|^{-2}\|\psi_1 - \psi_2\|_{r+2}. \tag{6.14}$$

Then, by Lemma 6.6, $\mathcal{L}^{-1}(\mathcal{R}(\psi_1) - \mathcal{R}(\psi_2)) \in \mathcal{X}_{r+2}$ and moreover

$$\|\mathcal{L}^{-1}(\mathcal{R}(\psi_1) - \mathcal{R}(\psi_2))\|_{r+2} \leq C|\rho|^{-2}\|\psi_1 - \psi_2\|_{r+2}.$$

Using this bound, (6.10), and definition (6.9) of the operator $\mathcal{F}$, one has that, if ρ_1 is large enough and $\rho \geq \rho_1$,

$$\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\|_{r+2} \leq C\rho_1^{-r+\nu_r}\|\psi_1 - \psi_2\|_{r+2} \leq \frac{1}{2}\|\psi_1 - \psi_2\|_{r+2},$$

and hence $\mathcal{F}$ is contractive (we recall that $r - \nu_r > 0$). Moreover, if $\psi \in B(\varrho_0)$,

$$\|\mathcal{F}(\psi)\|_{r+2} \leq \|\mathcal{F}(0)\|_{r+2} + \|\mathcal{F}(0) - \mathcal{F}(\psi)\|_{r+2} \leq \|\epsilon_0\|_{r+2} + \frac{1}{2}\|\psi\|_{r+2} < \varrho_0,$$

which ends the proof of the proposition. ■

7. The difference $\phi^{\mathrm{u}} - \phi^{\mathrm{s}}$.

By Proposition 6.2 the existence of two solutions $\phi^{\mathrm{u,s}} = \phi_0 + \psi^{\mathrm{u,s}}$ of the inner equation is proved. Let us write $\Theta = \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$ and also introduce the function

$$E = -\int_0^1 (1-\lambda)D^2 g(\phi^{\mathrm{s}} + \lambda(\phi^{\mathrm{u}} - \phi^{\mathrm{s}}))\, d\lambda \cdot (\phi^{\mathrm{u}} - \phi^{\mathrm{s}}). \tag{7.1}$$

We recall that both $\phi^{\mathrm{u,s}}$ are solutions of the same nonlinear difference equation:

$$\Delta^2(\phi) = -\phi^n + G(\phi) = -g(\phi). \tag{7.2}$$

Consequently, the function Θ satisfies the linear difference equation

$$\Delta^2(\Theta) = (-Dg(\phi^{\mathrm{s}}) + E) \cdot \Theta. \tag{7.3}$$

Although we do not have a good representation of the difference $\Theta = \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$, by means of Proposition 6.2 we already know that it is well defined and some not optimal bounds for Θ are provided. This allows us to define a new linear equation (7.3) from which Θ is also a solution. In conclusion, we will use $\Theta = \phi^u - \phi^s$ both as a known function (to define $E(z)$) and as an unknown solution of the above linear equation.

The goal of this section is to prove that any analytic solution of (7.3) satisfying adequate boundary conditions has to be exponentially small, that is, of $\mathcal{O}(\mathrm{e}^{-2\pi \mathrm{i} z})$. In fact, as claimed in Theorem 3.4, we will provide an exact formula for Θ.

7.1. Notation.

Given $\rho, \gamma > 0$, let us recall the complex domain

$$E_{\gamma,\rho} = D^{\mathrm{u}}_{\gamma,\rho} \cap D^{\mathrm{s}}_{\gamma,\rho} \cap \{z \in \mathbb{C} : \mathrm{Im} z < 0\} \backslash \{z \in \mathbb{C} : |\mathrm{Re} z| \leq 1,\ |\mathrm{Im} z| \leq \rho + \gamma\}$$

defined in (3.14) (see Figure 2).

For $\nu, k \in \mathbb{R}$, we also introduce the norms

$$\|\varphi\|_{\nu,\gamma,\rho} = \sup_{z \in E_{\gamma,\rho}} |z^\nu \varphi(z)|, \qquad \|\varphi\|^{\log}_{\nu,k,\gamma,\rho} = \sup_{z \in E_{\gamma,\rho}} |z^\nu (\log z)^{-k} \varphi(z)|,$$

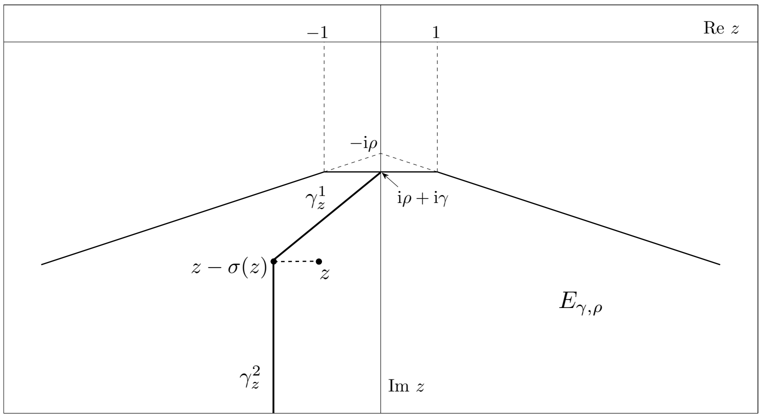

Figure 5. *Path* γ_z.

and the Banach spaces

$$\mathcal{Y}_{\nu,\gamma,\rho} = \{\varphi : E_{\gamma,\rho} \to \mathbb{C} \text{ such that } \|\varphi\|_{\nu,\gamma,\rho} < +\infty\},$$
$$\mathcal{Y}^{\log}_{\nu,k,\gamma,\rho} = \{\varphi : E_{\gamma,\rho} \to \mathbb{C} \text{ such that } \|\varphi\|^{\log}_{\nu,k,\gamma,\rho} < +\infty\}.$$

If there is no danger of confusion, we will simply denote

$$\mathcal{Y}_\nu = \mathcal{Y}_{\nu,\gamma,\rho}, \quad \|\cdot\|_\nu = \|\cdot\|_{\nu,\gamma,\rho}, \quad \mathcal{Y}^{\log}_{\nu,k} = \mathcal{Y}^{\log}_{\nu,k,\gamma,\rho}, \quad \|\cdot\|^{\log}_{\nu,k} = \|\cdot\|^{\log}_{\nu,k,\gamma,\rho}.$$

Lemma 7.1. *Let* $0 < \nu_1, \nu_2$. *For any* $f \in \mathcal{Y}_{\nu_1}$ *and* $g \in \mathcal{Y}_{\nu_2}$, *then* $f \cdot g \in \mathcal{Y}_{\nu_1+\nu_2}$ *and*

$$\|f \cdot g\|_{\nu_1+\nu_2} \leq \|f\|_{\nu_1} \cdot \|g\|_{\nu_2}.$$

Also, there exists a constant C *such that if* $0 < \nu_1 < \nu_2$ *and* $f \in \mathcal{Y}_{\nu_2}$,

$$f \in \mathcal{Y}_{\nu_1} \quad \text{and} \quad \|f\|_{\nu_1} \leq C\rho^{-(\nu_2-\nu_1)}\|f\|_{\nu_2}.$$

As in previous sections, we will denote by $\mathcal{O}_\nu$ and $\mathcal{O}_{\nu,k}$ a generic function belonging to $\mathcal{Y}_{\nu,\gamma,\rho}$ and $\mathcal{Y}^{\log}_{\nu,k,\gamma,\rho}$, respectively.

7.2. A right inverse of the operator $\Delta(\phi)(z) = \phi(z+1) - \phi(z)$. In this section we are going to construct a right inverse of the linear operator Δ,

$$\Delta(\phi)(z) = \phi(z+1) - \phi(z), \tag{7.4}$$

defined on functions belonging to $\mathcal{Y}^{\log}_{\nu,k}$ with $\nu, k \in \mathbb{R}$. We will follow the results introduced in [11] (which provide an explicit formula for Δ^{-1}), and we also give useful properties of this operator when it acts on $\mathcal{Y}^{\log}_{\nu+1,k}$.

We first notice that, since $E_{\gamma,\rho}$ is an open set, for any $z \in E_{\gamma,\rho}$ there exists $\sigma(z)$ such that $\{w \in \mathbb{C} : |z-w| < 2\sigma(z)\} \subset E_{\gamma,\rho}$. As a consequence, the complex path $\gamma_z = \gamma_z^1 \vee \gamma_z^2$ (see Figure 5),

$$\begin{aligned} &\gamma_z^1(t) = \{-i(\rho+\gamma)(1-t) + t(z-\sigma(z)), \ \ t \in [0,1)\},\\ &\gamma_z^2(t) = \{z - \sigma(z) + it, \quad t \in (-\infty, 0]\}, \end{aligned} \tag{7.5}$$

is contained in the complex set $E_{\gamma,\rho}$.

Given h an analytic function and $z \in E_{\gamma,\rho}$, we introduce the linear operators

$$\Delta_-^{-1}(h)(z) = \int_{\gamma_z} \frac{h(u)}{e^{2\pi i(u-z)} - 1}\, du \quad \text{and} \quad \Delta_+^{-1}(h)(z) = \int_{\gamma_z} \frac{h(u)}{1 - e^{-2\pi i(u-z)}}\, du. \tag{7.6}$$

Proposition 7.2. *Let $\nu, k \in \mathbb{R}$ and $\gamma > 0$. We define the linear operator*

$$\Delta^{-1} = \begin{cases} \Delta_-^{-1} & \text{if } \nu \le 0, \\ \Delta_+^{-1} & \text{if } \nu > 0. \end{cases}$$

There exists $\rho_0 > 0$ such that, for any $\rho \ge \rho_0$,

1. *if $\nu \neq 0$, $\Delta^{-1} : \mathcal{Y}^{\log}_{\nu+1,k,\gamma,\rho} \to \mathcal{Y}^{\log}_{\nu,k,\gamma,\rho}$ is a right inverse of the operator Δ;*
2. *if $\nu = 0$, $\Delta^{-1} : \mathcal{Y}^{\log}_{1,k,\gamma,\rho} \to \mathcal{Y}^{\log}_{0,k+1,\gamma,\rho}$ is a right inverse of the operator Δ.*

Moreover, in both cases, there exists a positive constant C such that $\|\Delta^{-1}\| \le C$.

Proof. Throughout this proof we will denote by K a generic constant depending only on γ and ν. We will omit γ, ρ_0, and ρ from our notation of the Banach spaces and norms.

We fix $\nu \in \mathbb{R}$, $\gamma > 0$, fulfilling the hypotheses of Proposition 7.2 and $\rho_0 \le \rho$ big enough. Let $h \in \mathcal{Y}^{\log}_{\nu+1,k}$, and introduce $\varphi = \Delta^{-1}(h)$. Our first observation is that φ is an analytic function defined in $E_{\gamma,\rho}$. Indeed, for any $\sigma_0 > 0$ we define the set

$$\Omega_{\sigma_0} = \{u \in E_{\gamma,\rho} : u - \sigma_0 \in E_{\gamma,\rho}\}.$$

We emphasize that $E_{\gamma,\rho} = \cup_{\sigma_0 > 0} \Omega_{\sigma_0}$. Moreover, we note that if $z \in \Omega_{\sigma_0}$, we can take $\sigma(z) = \sigma_0$ in the expression (7.6) of $\varphi(z)$. Henceforth, in order to deduce that φ is an analytic function in Ω_{σ_0}, we have only to study the convergence of

$$\int_{\gamma_z^2} \frac{h(u)}{e^{\mp 2\pi i(u-z)} - 1}\, du.$$

To this end, we observe that $|e^{\mp 2\pi i(\gamma_z^2 - z)} - 1| \ge |e^{\pm t 2\pi|\rho+\gamma+\mathrm{Im} z|} - 1|$ and that $|h(\gamma_z^2(t))| \le C(z)|t|^{-\nu-1}\log^k(|t|)$ for some function $C(z)$. Therefore, if $\nu \le 0$ and $t \in (-\infty, 0]$,

$$\frac{|h(\gamma_z^2(t))|}{|e^{2\pi i(\gamma_z^2 - z)} - 1|} \le 2C(z)|t|^{-\nu-1}\log^k(|t|)e^{t2\pi|\rho+\mathrm{Im} z|},$$

and we are done for the case $\nu \le 0$. The case $\nu > 0$ can be handled analogously.

Now we are going to check that Δ^{-1} is a right inverse of the operator Δ. We take into account that if $z, z+1 \in E_{\gamma,\rho}$,

$$\Delta\varphi(z) = \mp \int_{\gamma_{z+1} - \gamma_z} \frac{h(u)}{e^{\mp 2\pi i(u-z)} - 1}\, du,$$

and therefore, since the only singularity of $\mp h(u)/(e^{\mp 2\pi i(u-z)} - 1)$ is $u = z$ and it is a simple pole with residue $h(z)/2\pi i$, we have that both $\varphi_\pm$ are solutions of $\Delta(\varphi) = h$ defined in the complex domain $E_{\gamma,\rho}$. Here we have proceed exactly as in [11].

It remains only to prove that $\varphi = \Delta^{-1}(h) \in \mathcal{Y}^{\log}_{\nu,k}$, provided $h \in \mathcal{Y}^{\log}_{\nu+1,k}$. We restrict ourselves to the complex domain $\widetilde{E}_{\gamma,\rho+\gamma} \subset E_{\gamma,\rho}$ defined by

$$\widetilde{E}_{\gamma,\rho'} = D^{\mathrm{u}}_{\gamma,\rho'} \cap D^{\mathrm{s}}_{\gamma,\rho'} \cap \{z \in \mathbb{C} : \mathrm{Im} z < 0\}.$$

We notice that, if the following bounds are proved,

$$|z^{\nu}(\log z)^{-k}\varphi(z)| \le K\|h\|^{\log}_{\nu+1,k}, \quad z \in \widetilde{E}_{\gamma,\rho+\gamma}, \quad \text{if } \nu \neq 0, \tag{7.7}$$

$$|(\log z)^{-k-1}\varphi(z)| \le K\|h\|^{\log}_{1,k}, \quad z \in \widetilde{E}_{\gamma,\rho+\gamma}, \quad \text{if } \nu = 0, \tag{7.8}$$

then the same statement holds for $z \in E_{\gamma,\rho}$. Indeed, assume that bounds (7.7) and (7.8) are satisfied, and let $z \in E_{\gamma,\rho}\backslash\widetilde{E}_{\gamma,\rho,\gamma}$. We have two cases, $\mathrm{Re} z \le 0$ and $\mathrm{Re} z > 0$. On the one hand, if $\mathrm{Re} \le 0$, it is clear that $z+1 \in \widetilde{E}_{\gamma,\rho+\gamma}$ and that $\varphi(z) = \varphi(z+1) - h(z)$. On the other hand, if $\mathrm{Re} z > 0$, $z - 1 \in \widetilde{E}_{\gamma,\rho+\gamma}$, and consequently $\varphi(z) = \varphi(z-1) + h(z)$. In any case, $|\varphi(z)| \le |\varphi(z \pm 1)| + |h(z)|$. Here we have used that $\Delta(\varphi) = h$. Therefore, if $\nu \neq 0$, using bound (7.7), we obtain

$$|\varphi(z)| \le K\|h\|^{\log}_{\nu,k}\left(|z \pm 1|^{-\nu}|\log(z \pm 1)|^k + |z|^{-\nu-1}|\log z|^k\right) \le K\|h\|^{\log}_{\nu,k}|z|^{-\nu}|\log z|^k,$$

and the result is proved for $\nu \neq 0$. Analogously we check the result for $\nu = 0$.

The proof of bounds (7.7) and (7.8) is easy but requires tedious computations which will be omitted here. Nevertheless, we point out that for any fixed $z \in \widetilde{E}_{\gamma,\rho+\gamma}$ we can take $\sigma(z) = 1/2$ in the definition (7.5) of γ_z. ■

7.3. Two independent solutions of the linear equation (7.3).

We recall that $\Theta = \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$ satisfies (7.3):

$$\Delta^2\Theta = (-Dg(\phi^{\mathrm{s}}) + E)\Theta. \tag{7.9}$$

The following lemma states the properties of E that we will need. Its proof is completely analogous to that of bound (6.12), provided $\phi^{\mathrm{u}} - \phi^{\mathrm{s}} \in \mathcal{Y}_{r+2,\gamma,\rho}$.

Lemma 7.3. *Let γ and ρ satisfy the conclusions of Proposition 6.2, and E be the function defined in (7.1). We have that $E \in \mathcal{Y}_{r+6-\ell,\gamma,\rho}$.*

As we did in section 6.1, we split

$$-Dg(\phi^s) + E = H + M,$$

where H was defined in (6.3), $M \in \mathcal{Y}_{r+2}$ if $n \neq 3$, and $M \in \mathcal{Y}^{\log}_{r+2,1}$ if $n = 3$. We rewrite (7.9) as

$$\Delta^2(\Theta) - H \cdot \Theta = M \cdot \Theta.$$

A solution of the homogeneous equation $\Delta^2(\varphi) = H \cdot \varphi$ is $\eta_2(z) = z^{\ell}$. The function

$$\eta_1(z) = z^{\ell}\Delta^{-1}\left(\frac{1}{\eta_2(z+1)\eta_2(z)}\right) \in \mathcal{Y}_{\ell-1}$$

is another solution satisfying $W(\eta_1, \eta_2) = 1$.

By using these decompositions as well as Proposition 7.2 for the operator Δ^{-1}, we can obtain solutions of the nonhomogeneous linear equation $\Delta^2(\varphi) - H \cdot \varphi = h$.

Lemma 7.4. *For any $\gamma > 0$ there exists $\rho_0 > 0$ large enough such that for any $\rho \geq \rho_0$ the operator $\mathcal{L}(\varphi) = \Delta^2(\varphi) - H \cdot \varphi$ has right inverse defined in $E_{\gamma,\rho}$:*

$$\mathcal{L}^{-1}(h) = \eta_1 \cdot \Delta^{-1}(\eta_2 \cdot h) - \eta_2 \cdot \Delta^{-1}(\eta_1 \cdot h). \tag{7.10}$$

There exists $C > 0$ such that for any $\alpha \in \mathbb{R}$ and $h \in \mathcal{Y}_{\alpha+2,\gamma,\rho}$ we have the following:

1. *If $\alpha \neq \ell - 1$ and $\alpha \neq -\ell$, then $\mathcal{L}^{-1}(h) \in \mathcal{Y}_{\alpha,\gamma,\rho}$ and $\|\mathcal{L}^{-1}(h)\|_{\alpha,\gamma,\rho} \leq C\|h\|_{\alpha+2,\gamma,\rho}$.*
2. *If either $\alpha = \ell - 1$ or $\alpha = -\ell$, then $\mathcal{L}^{-1}(h) \in \mathcal{Y}^{\log}_{\alpha,1,\gamma,\rho}$ and $\|\mathcal{L}^{-1}(h)\|^{\log}_{\alpha,1,\gamma,\rho} \leq C\|h\|_{\alpha+2,\gamma,\rho}$.*

The next lemma provides a fundamental system of solutions of the linear equation (7.9).

Lemma 7.5. *Let $\gamma > 0$. There exists ρ_0 large enough such that, for any $\rho \geq \rho_0$, (7.9) has two independent solutions, $\hat{\eta}_1$ and $\hat{\eta}_2$, satisfying*

$$\hat{\eta}_1(z) = \partial_z \phi^{\mathrm{s}}(z) + \hat{\eta}_1^1(z), \qquad \hat{\eta}_1^1 \in \mathcal{Y}_{r+3,\gamma,\rho},$$
$$\hat{\eta}_2(z) = \frac{z^\ell}{r d_\ell (2\ell - 1)} + \hat{\eta}_2^1(z), \qquad \hat{\eta}_2^1 \in \mathcal{Y}^{\log}_{\nu,k,\gamma,\rho},$$

with $\nu = \min\{r - \ell, 1 - \ell\}$, $k = 0$ if $n \neq 3$, $k = 1$ if $n = 3$, and d_ℓ defined in (3.15).

Proof. First we look for $\hat{\eta}_1$. By construction, $\partial_z \phi^{\mathrm{s}}$ is a solution of the variational equation $\Delta^2 \varphi = -Dg(\phi^s)\varphi$; therefore, the equation that $\hat{\eta}_1^1$ has to satisfy is

$$\Delta^2(\varphi) - H \cdot \varphi = M \cdot \varphi + E \cdot \partial_z \phi^{\mathrm{s}}. \tag{7.11}$$

We look for $\hat{\eta}_1^1$ by means of the fixed point equation

$$\varphi = \mathcal{L}^{-1}(E \cdot \partial_z \phi^{\mathrm{s}}) + \mathcal{L}^{-1}(M \cdot \varphi). \tag{7.12}$$

We are interested in solutions belonging to $\mathcal{Y}_{r+3}$. It is enough to check that the norm of the linear operator $\mathcal{G} : \mathcal{Y}_{r+3} \to \mathcal{Y}_{r+3}$ defined by $\mathcal{G}(\varphi) = \mathcal{L}^{-1}(M \cdot \varphi)$ is less than one. This fact follows from Lemma 7.4 together with the facts that $E \in \mathcal{Y}_{r+6-\ell}$, $M \in \mathcal{Y}_{r+2}$ if $n \neq 3$, and $M \in \mathcal{Y}^{\log}_{r+2,1}$ if $n = 3$. One easily then deduces that

$$\hat{\eta}_1^1 = (\mathrm{Id} - \mathcal{G})^{-1}\big(\mathcal{L}^{-1}(E \cdot \partial_z \phi^{\mathrm{s}})\big) \in \mathcal{Y}_{r+3} \tag{7.13}$$

is a solution of (7.11).

Now we deal with the second solution of (7.9). We observe that, since $\hat{\eta}_1$ is a solution, the function $\hat{\eta}_2 = b \cdot \hat{\eta}_1$ is also a solution of the linear equation (7.9) satisfying $W(\hat{\eta}_1, \hat{\eta}_2) = 1$ if and only if b satisfies

$$\Delta(b)(z) = \frac{1}{\hat{\eta}_1(z+1)\hat{\eta}_1(z)}.$$

By Proposition 6.2, $\hat{\eta}_1 = \partial_z \phi^{\mathrm{s}} + \hat{\eta}_1^1 = \partial_z \phi_0 + \partial_z \psi^{\mathrm{s}} + \hat{\eta}_1^1$, where $\partial_z \psi^{\mathrm{s}}, \hat{\eta}_1^1 \in \mathcal{Y}_{r+3}$. Moreover, using the definitions of d_ℓ, ℓ in (3.15) and Corollary 6.3, it is a direct computation to check that b has to satisfy the linear equation

$$\Delta(b)(z) = \frac{z^{2\ell-2}}{r^2 d_\ell^2} + S(z)$$

with $S \in \mathcal{Y}_{-2\ell+3}$ if $n = 2$, $S \in \mathcal{Y}^{\log}_{-2\ell+3,1}$ if $n = 3$, and $S \in \mathcal{Y}_{-2\ell+2+r}$ if $n > 3$. We take

$$b_0(z) = \frac{z^{2\ell-1}}{r^2 d_\ell^2 (2\ell - 1)}$$

and note that $r^2 d_\ell^2 \Delta(b_0)(z) = z^{2\ell-2} + \mathcal{O}_{-2\ell+3}$. Henceforth, the difference $b_1 = b - b_0$ satisfies an equation of the form

$$\Delta(b_1) = \tilde{S}(z), \tag{7.14}$$

with $\tilde{S} \in \mathcal{Y}_{-2\ell+3}$ if $n = 2$, $\tilde{S} \in \mathcal{Y}^{\log}_{-2\ell+3,1}$ if $n = 3$, and $\tilde{S} \in \mathcal{Y}_{-2\ell+2+r}$ if $n > 3$. Applying Proposition 7.2, one has that (7.14) has a solution b_1 belonging to $\mathcal{Y}_{-2\ell+2}$ if $n = 2$, $\mathcal{Y}^{\log}_{-2\ell+2,1}$ if $n = 3$, and $\mathcal{Y}_{-2\ell+1+r}$ if $n > 3$, and the result follows. ■

7.4. A final formula for $\Theta = \phi^{\mathrm{u}} - \phi^{\mathrm{s}}$. Since Θ is a solution of the linear homogenous difference equation (7.9), the general theory allows us to write it as

$$\Theta(z) = p_1(z)\eta_1(z) + p_2(z)\eta_2(z), \tag{7.15}$$

with η_1, η_2 being two independent solutions of (7.9) and p_1, p_2 being 1-periodic analytic functions in $E_{\gamma,\rho}$. Moreover, if $W(\eta_1, \eta_2) = 1$, the functions p_1 and p_2 are determined by

$$p_1(z) = W(\Theta, \eta_2)(z), \qquad p_2(z) = -W(\Theta, \eta_1)(z). \tag{7.16}$$

Lemma 7.6. *Let $\gamma, \rho > 0$, and η_1, η_2 be two independent solutions of the linear difference equation (7.9) satisfying that $W(\eta_1, \eta_2) = 1$ and that $\eta_1 \in \mathcal{Y}_{r+1}$ and $\eta_2 \in \mathcal{Y}_{-\ell}$.*

Then there exist coefficients p_1^k, p_2^k (depending on $\eta_{1,2}$) such that

$$\Theta(z) = \eta_1(z) \sum_{k<0} p_1^k \mathrm{e}^{2\pi \mathrm{i} k z} + \eta_2(z) \sum_{k<0} p_2^k \mathrm{e}^{2\pi \mathrm{i} k z}. \tag{7.17}$$

Proof. We first point out that we already know that $\Theta = \phi^{\mathrm{u}} - \phi^{\mathrm{s}} = \psi^{\mathrm{u}} - \psi^{\mathrm{s}} \in \mathcal{Y}_{r+2,\gamma,\rho}$, provided that, by Theorem 3.3, $\psi^{\mathrm{u,s}} \in \mathcal{X}^{\mathrm{u,s}}_{r+2}$. In addition, if $h \in \mathcal{Y}_{\nu,\gamma,\rho}$, then $\Delta(h) \in \mathcal{Y}_{\nu+1,2\gamma,2\rho}$. Indeed, standard arguments can be used to prove that if $h \in \mathcal{Y}_{\nu,\gamma,\rho}$, then $\partial_z h \in \mathcal{Y}_{\nu+1,2\gamma,2\rho}$ (see, for instance, [1]). Therefore, if $z, z+1 \in E_{2\gamma,2\rho}$,

$$|h(z+1) - h(z)| \le \|\partial_z h\|_{\nu+1} \int_0^1 \frac{1}{|z+t|^{\nu+1}} \le K \|\partial_z h\|_{\nu+1} \frac{1}{|z|^{\nu+1}}.$$

Using the above property, that $\Theta \in \mathcal{Y}_{r+2}$, and formula (7.16) for p_1, p_2, one has that $p_1 \in \mathcal{Y}_1$ and $p_2 \in \mathcal{Y}_{r+4}$. In particular, $p_1, p_2 \to 0$ as $\mathrm{Im} z \to -\infty$, and since they are 1-periodic,

$$p_1(z) = \sum_{k<0} p_1^k \mathrm{e}^{2\pi \mathrm{i} k z}, \qquad p_2(z) = \sum_{k<0} p_2^k \mathrm{e}^{2\pi \mathrm{i} k z},$$

and the lemma is proved. ■

We recall that the existence of independent solutions of the linear difference equation (7.9) satisfying the hypotheses of Lemma 7.6 is guaranteed by Lemma 7.5. Hence Lemma 7.6 applied to $\hat{\eta}_1, \hat{\eta}_2$ already gives an expression of Θ which is exponentially small. Among other things, we have proved that there exist $\gamma, \rho > 0$ such that

$$|\mathrm{e}^{2\pi \mathrm{i} z} z^{-\ell}(\phi^{\mathrm{u}}(z) - \phi^{\mathrm{s}}(z))| \le K, \qquad z \in E_{\gamma,\rho}. \tag{7.18}$$

Nevertheless we have not proved Theorem 3.4 yet. We need to look for more suitable independent solutions of (7.9) to apply Lemma 7.6.

Corollary 7.7. *Let $\gamma > 0$. There exists $\rho_0 > 0$ big enough such that, for any $\rho \ge \rho_0$, (7.9) has two fundamental solutions of the form*

$$\zeta_1(z) = \partial_z \phi^{\mathrm{s}}(z) + \zeta_1^1(z),$$
$$\zeta_2(z) = \frac{z^\ell}{r d_\ell (2\ell - 1)}(z) + \zeta_2^1(z),$$

with d_ℓ defined in (3.15) and ζ_1^1 satisfying

$$\sup_{z \in E_{\gamma,\rho}} |z^{1-\ell} e^{2\pi i z} \zeta_1^1(z)| < +\infty.$$

In addition, $\zeta_2^1 \in \mathcal{Y}_{r-\ell,\gamma,\rho}$ if $n > 3$, $\zeta_2^1 \in \mathcal{Y}_{1-\ell,\gamma,\rho}$ if $n = 2$, and $\zeta_2^1 \in \mathcal{Y}^{\log}_{1-\ell,1,\gamma,\rho}$ if $n = 3$.

Proof. As in the proof of Lemma 7.5, we write $\zeta_1^1 = \zeta_1 - \partial_z \phi^{\mathrm{s}}$. We note that ζ_1^1 satisfies (7.11):

$$\Delta^2(\zeta_1^1)(z) - H(z)\zeta_1^1(z) = M(z)\zeta_1^1(z) + E(z)\partial_z\phi^{\mathrm{s}}(z).$$

We write $\zeta(z) = \mathrm{e}^{2\pi \mathrm{i} z}\zeta_1^1(z)$, and we notice that ζ has to satisfy the equation

$$\Delta^2(\zeta)(z) - H(z)\zeta(z) = M(z)\zeta(z) + \mathrm{e}^{2\pi \mathrm{i} z} E(z)\partial_z\phi^{\mathrm{s}}(z). \tag{7.19}$$

We introduce $\varphi_0(z) = \mathrm{e}^{2\pi \mathrm{i} z} E(z)\partial_z\phi^{\mathrm{s}}(z)$. We first claim that $\varphi_0 \in \mathcal{Y}_{3-\ell}$. Indeed, we note that $\partial_z\phi^{\mathrm{s}} \in \mathcal{Y}_{\ell-1}$, and we recall that

$$E(z) = -\int_0^1 (1-\lambda) D^2 g(\phi^{\mathrm{s}} + \lambda(\phi^{\mathrm{u}} - \phi^{\mathrm{s}}))\, d\lambda \big(\phi^{\mathrm{u}} - \phi^{\mathrm{s}}\big). \tag{7.20}$$

The claim follows from the facts that, by (7.18), $|z^{-\ell}\mathrm{e}^{2\pi \mathrm{i} z}(\phi^{\mathrm{u}} - \phi^{\mathrm{s}})|$ is bounded and, moreover, $|D^2 g(\phi^{\mathrm{s}} + \lambda(\phi^{\mathrm{u}} - \phi^{\mathrm{s}}))| \le K|z|^{\ell-4}$ if $z \in E_{\gamma,\rho}$ (which can be proved as in (6.12)).

It is clear that a particular solution ζ of (7.19) is given by a solution of

$$\zeta = \mathcal{L}^{-1}(\varphi_0) + \mathcal{G}(\zeta),$$

where $\mathcal{G}(\zeta) = \mathcal{L}^{-1}(M \cdot \zeta)$.

First we observe that, by Lemma 7.4, the independent term $\mathcal{L}^{-1}(\varphi_0) \in \mathcal{Y}_{1-\ell}$. Second we check that $(\mathrm{Id} - \mathcal{G})$ is invertible in $\mathcal{Y}_{1-\ell}$. Let $\psi \in \mathcal{Y}_{1-\ell}$. Since $M \in \mathcal{Y}_{\frac{r}{2}+2}$ for any $n \ge 2$, we have that $M \cdot \psi \in \mathcal{Y}_{3+\frac{r}{2}-\ell}$, and consequently, by Lemma 7.4, $\mathcal{G}(\psi) \in \mathcal{Y}_{1+\frac{r}{2}-\ell}$ and, moreover,

$$\|\mathcal{G}(\psi)\|_{1-\ell} \le \rho^{-r/2}\|\mathcal{G}(\psi)\|_{1+\frac{r}{2}-\ell} \le \rho_0^{-r/2} C\|\psi\|_{1-\ell} \le \frac{1}{2}\|\psi\|_{1-\ell}.$$

This implies that the norm of the linear operator $\mathcal{G} : \mathcal{Y}_{1-\ell} \to \mathcal{Y}_{1-\ell}$ is less than one, and therefore $\mathrm{Id} - \mathcal{G}$ is invertible. To this end, we can write ζ as

$$\zeta = \big(\mathrm{Id} - \mathcal{G}\big)^{-1}(\mathcal{L}^{-1}(\varphi_0)),$$

and we deduce that $\zeta \in \mathcal{Y}_{1-\ell}$ and $\|\zeta\|_{1-\ell} \le 2\|\mathcal{L}^{-1}(\varphi_0)\|_{1-\ell}$, which implies the result for ζ_1.

The existence and properties of ζ_2 follow from those for $\hat{\eta}_2$ in Lemma 7.5. ■

Acknowledgments. We are indebted to the anonymous referees who helped us to improve this work, especially sections 1–4.

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